ψ-Hilfer fractional relaxation-oscillation equation*

N. Vieira[‡], M. Ferreira[§,‡], and M.M. Rodrigues[‡]

[‡]CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
Emails: nloureirovieira@gmail.com; mferreira@ua.pt; mrodrigues@ua.pt

[§]School of Technology and Management
Polytechnic of Leiria
Campus 2 - Morro do Lena, Alto do Vieiro,
P-2411-901, Leiria, Portugal.
E-mail: milton.ferreira@ipleiria.pt

August 8, 2022

Abstract

In this work, we solve the ψ-Hilfer fractional relaxation-oscillation equation with a force term, where the time-fractional derivatives are in the ψ-Hilfer sense. The solution of the equation is presented in terms of bivariate Mittag-Leffler functions. An asymptotic analysis of the solution of the associated homogeneous equation is performed.

Keywords: Fractional relaxation-oscillation equation; ψ-Hilfer derivative; Bivariate Mittag-Leffler function.

MSC 2010: 35R11; 26A33; 34C26; 35B05.

August 8, 2022

1 Introduction

The relaxation and oscillation processes are of great relevance in physics. From a mathematical point of view, they are modelled by linear differential equations of first and second orders in time. In [2] the fractional relaxation and oscillation equations with Caputo derivatives were studied separately. The simultaneous consideration of time-fractional derivatives or first and second orders leads to the so-called fractional relaxation-oscillation phenomena, that we study in this paper.

2 Preliminaries

In this section, we recall some basic definitions about ψ-Hilfer fractional derivatives, special functions, and the ψ-Laplace transform, that are necessary for this work.

Definition 2.1 *(cf. [4, Def. 4]) Let (a,b) be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha > 0$. Also let ψ be a monotone increasing and positive function on (a,b), having a continuous derivative ψ' in (a,b). The left Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is given by*

$$\left(I_{a^+}^{\alpha,\psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w)\, \left(\psi(t) - \psi(w)\right)^{\alpha-1} f(w)\, dw, \qquad t > a. \tag{1}$$

*The final version is published in *International Conference on Mathematical Analysis and Applications in Science and Engineering (ICMA^{2}SC'22) - Book of Extended Abstracts*, Eds: C.M.A. Pinto, J. Mendonça, L. Babo, and D. Baleanu, (2022), 163–166. It as available via the website `https://doi.org/10.34630/20734`

Next, we give the definition of the so-called ψ-Hilfer fractional derivative of a function f with respect to another function.

Definition 2.2 *(cf. [4, Def. 7]) Let $\alpha > 0$ and $m = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. Let also $I = [a, b]$ be a finite or infinite interval on the real line and $f, \psi \in C^m [a, b]$ two functions such that ψ is a positive monotone increasing function and $\psi'(t) \neq 0$, for all $t \in I$. The left ψ-Hilfer fractional derivative ${}^H D_{a^+}^{\alpha,\mu;\psi}$ of order α and type $\mu \in [0, 1]$ is defined by*

$$\left({}^H D_{a^+}^{\alpha,\mu;\psi} f \right)(t) = I_{a^+}^{\mu(m-\alpha),\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^m I_{a^+}^{(1-\mu)(m-\alpha),\psi} f(t). \tag{2}$$

We observe that when $\mu = 0$ we recover the left Riemann-Liouville fractional derivative of a function with respect to ψ (see [4, Def. 5]) and when $\mu = 1$ we obtain the left Caputo fractional derivative of a function with respect to ψ (see [4, Def. 6]). In Section 5 of [4] is presented a list of several fractional integrals and fractional derivatives that can be obtained from (1) and (2), respectively, for different choices of μ and ψ. The solution of the ψ-fractional relaxation-oscillation equation is presented in terms of the bivariate Mittag-Leffler function which has the following double series representation:

$$E_{(a_1,a_2),b}(z_1, z_2) = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \frac{(l_1 + l_2)!}{l_1!\, l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(b + a_1 l_1 + a_2 l_2)}. \tag{3}$$

When $z_1 = -c_1 t^{a_1}$, $z_2 = -c_2 t^{a_2}$, with $a_1, a_2, c_1, c_2, b, t > 0$, we have the following asymptotic expansions near the origin and at the infinity:

$$E_{(a_1,a_2),b}(-c_1 t^{a_1}, -c_2 t^{a_2}) \sim \frac{1}{\Gamma(b)} - \frac{c_1 t^{a_1}}{\Gamma(b + a_1)} - \frac{c_2 t^{a_2}}{\Gamma(b + a_2)}, \quad t \to 0^+, \tag{4}$$

$$E_{(a_1,a_2),b}(-c_1 t^{a_1}, -c_2 t^{a_2}) \sim \frac{t^{-a_1}}{c_1 \Gamma(b - a_1)}, \quad b \neq a_1,\ t \to +\infty. \tag{5}$$

The ψ-Laplace transform of a real valued function $f(t)$ with respect to ψ is defined by (see [3, Def. 13])

$$\mathcal{L}_\psi \{f(t)\}(\mathbf{s}) = \widetilde{f}_\psi(\mathbf{s}) = \int_0^{+\infty} e^{-\mathbf{s}\,\psi(t)} \psi'(t)\, f(t)\, dt, \quad \mathrm{Re}(\mathbf{s}) \in \mathbb{C},$$

where ψ is a non negative monotone increasing function in $\mathbb{R}_0^+$ and such that $\psi(0) = 0$. The ψ-Laplace transform may be written as the following composition operator involving the classical Laplace transform: $\mathcal{L}_\psi = \mathcal{L} \circ Q_{\psi^{-1}}$ where $\left(Q_{\psi^{-1}} f\right)(t) = f\left(\psi^{-1}(t)\right)$ (cf. [3, Thm. 4]). As a consequence of the previous relation, if f is a function whose classical Laplace transform is $\widetilde{f}$, the ψ-Laplace transform of $f(\psi(t))$ is also $\widetilde{f}(\mathbf{s})$ (see [3, Cor. 2]), that is,

$$\mathcal{L}\{f(t)\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}) \quad \Rightarrow \quad \mathcal{L}_\psi \{f(\psi(t))\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}).$$

We observe that the definition of the ψ-Laplace can be adapted fon any interval $[a, +\infty[\subseteq \mathbb{R}_0^+$ with ψ satisfying $\psi(a) = 0$. This is important in our work in order to the ψ-Hilfer derivative encompasses the largest number of fractional derivatives. When the ψ-Laplace transform is applied to the ψ-Hilfer derivative we obtain (see [3, Thm. 6])

$$\mathcal{L}_\psi \left\{ {}^H D_{a^+}^{\alpha,\mu;\psi} f(t) \right\}(\mathbf{s}) = \mathbf{s}^\alpha \widetilde{f}_\psi(\mathbf{s}) - \sum_{j=0}^{m-1} \mathbf{s}^{m-\mu(m-\alpha)-1-j} \left(I_{t,a^+}^{(1-\mu)(m-\alpha)-j;\psi} f \right)(a^+), \tag{6}$$

where $m = [\alpha] + 1$ and the initial-value terms $\left(I_{a^+}^{(1-\mu)(m-\alpha)-j,\psi} f \right)(a^+)$ are evaluated in the limit $t \to a^+$. The ψ-Laplace convolution operator of two functions is defined by (see [3, Def. 15])

$$(f *_\psi g)(t) = \int_0^t f\left(\psi^{-1}(\psi(t) - \psi(w))\right) \psi'(w)\, g(w)\, dw, \quad t \in \mathbb{R}^+, \tag{7}$$

and the correspondent Convolution Theorem is (see [3, Thm. 8])

$$\mathcal{L}_\psi \{(f *_\psi g)(t)\}(\mathbf{s}) = \mathcal{L}_\psi \{f\}(\mathbf{s})\, \mathcal{L}_\psi \{g\}(\mathbf{s}). \tag{8}$$

Moreover, from relation (17.6) in [1] we have that

$$\mathcal{L}_{\psi}\left\{\psi(t)^{\alpha-\gamma}\sum_{p=0}^{+\infty}\left(-a\,\psi(t)^{\alpha-\beta}\right)^{p}E_{\alpha,\alpha+(\alpha+\beta)p-\gamma+1}^{p+1}\left(-b\,\psi(t)^{\alpha}\right)\right\}(\mathbf{s})=\frac{\mathbf{s}^{\gamma-1}}{\mathbf{s}^{\alpha}+a\mathbf{s}^{\beta}+b},\tag{9}$$

where $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma)\in\mathbb{R}^{+}$, $\left|\frac{a\mathbf{s}^{\beta}}{\mathbf{s}^{\alpha}+\beta}\right|<1$, and provided that the series in (9) is convergent.

3 ψ-Hilfer fractional relaxation-oscillation equation

In this section, we solve the ψ-Hilfer fractional forced damped oscillator modelled by the following fractional differential equation

$$c_2\,{}^{H}D_{a^+}^{\alpha_2,\mu_2;\psi}u(t)+c_1\,{}^{H}D_{a^+}^{\alpha_1,\mu_1;\psi}u(t)+d^2\,u(t)=q(t),\tag{10}$$

and subject to the following initial conditions

$$\left(I_{t,a^+}^{(1-\mu_1)(1-\alpha_1);\psi}u\right)(a^+)=\kappa_1,\ \left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi}u\right)(a^+)=\kappa_2,\ \frac{d}{dt}\left[\left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi}u\right)\right](a^+)=\kappa_3,\tag{11}$$

which are evaluated in the limit $t\to a^+$. Moreover, $c_2,c_1,d,\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$, $c_2\neq 0$, $t\in I$, with $I=[a,b]$ being a finite or infinite interval on $\mathbb{R}^+$, the partial time-fractional derivatives of orders $\alpha_1\in]0,1]$ and $\alpha_2\in]1,2]$, and types $\mu_1,\mu_2\in[0,1]$, respectively, are the ψ-Hilfer derivatives given by (2), q belongs to $L_1(I)$ (when $q(t)=0$ the solution of equation (10) corresponds to an unforced damped oscillator). We look for solutions u of our problem in the space $C^2(a,b)$.

When $\psi(t)=t$, with $t\in\mathbb{R}^+$, $\mu_1=\mu_2=1$, and $c_2=0$ or $c_1=0$ in equation (10), we obtain, respectively, the time-fractional relaxation/oscillation equations with Caputo fractional derivatives. These two equations were studied separately in [2]. Moreover, equation (10) is a particular case of the time-fractional telegraph equation with ψ-Hilfer derivatives studied in [5].

Now, we solve our relaxation-oscillation problem. Applying the ψ-Laplace transform to (10) and taking into account (11), we get

$$\left(c_2\,\mathbf{s}^{\alpha_2}+c_1\,\mathbf{s}^{\alpha_1}+d^2\right)\widetilde{u}_{\psi}(\mathbf{s})-c_1\,\kappa_1\,\mathbf{s}^{-\mu_1(1-\alpha_1)}-c_2\,\kappa_2\,\mathbf{s}^{1-\mu_2(2-\alpha_2)}-c_2\,\kappa_3\,\mathbf{s}^{-\mu_2(2-\alpha_2)}=\widetilde{q}_{\psi}(\mathbf{s}).\tag{12}$$

Solving the above equation in order to $\widetilde{u}_{\psi}$, we obtain:

$$\widetilde{u}_{\psi}(\mathbf{s})=\frac{\frac{c_1\,\kappa_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}+\kappa_2\,\mathbf{s}^{1-\mu_2(2-\alpha_2)}+\kappa_3\,\mathbf{s}^{-\mu_2(2-\alpha_2)}}{\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}+\frac{d^2}{c_2}}+\frac{1}{c_2}\,\widetilde{q}_{\psi}(\mathbf{s})\,\frac{1}{\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}+\frac{d^2}{c_2}}.\tag{13}$$

Inverting the ψ-Laplace transform and taking into account (9), we have

$$\begin{aligned}u(t)=&\frac{c_1\,\kappa_1}{c_2}\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}\right)\\&+\kappa_2\,\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}\right)\\&+\kappa_3\,\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_2(2-\alpha_2)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}\right)\\&+\frac{1}{c_2}\,q(\psi(t))*_{\psi}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}\psi(t)^{\alpha_2-1}\,E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p}^{p+1}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}\right),\end{aligned}\tag{14}$$

where the ψ-convolution is given by (7). From the definition of the bivariate Mittag-Leffler function (see (3)) we can rewrite (14) as

$$
\begin{aligned}
u(t) = {} & \frac{c_1\,\kappa_1}{c_2}\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2+\mu_1(1-\alpha_1)}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2},-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right) \\
& + \kappa_2\,\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2-1+\mu_2(2-\alpha_2)}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2},-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right) \\
& + \kappa_3\,\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2+\mu_2(2-\alpha_2)}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2},-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right) \\
& + \frac{1}{c_2}\,q(\psi(t)) *_\psi \left[\psi(t)^{\alpha_2-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2},-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\right]. \qquad (15)
\end{aligned}
$$

Therefore, the solution involves series of three-parameter Mittag-Leffler functions of one variable or just bivariate Mittag-Leffler functions.

For the unforced case, the solution can be written as $u_h(t) = u_1(t) + u_2(t) + u_3(t)$, where u_1, u_2, u_3 corresponds to the first three terms in (15). These constitute a set of fundamental solutions of the homogeneous equation. Let us study the behaviour of u_h when $t \to a^+$ and $t \to +\infty$. From (4) we have the following asymptotic behaviour near the starting point $t = a$

$$
u_h(t) \sim \frac{\kappa_2}{\Gamma(\alpha_2 + \mu_2(2-\alpha_2) - 1)}\,\psi(t)^{\alpha_2+\mu_2(2-\alpha_2)-2}, \quad t \to a^+.
$$

Moreover, from (5) we have the following asymptotic behaviour for large values of t

$$
u_h(t) \sim \frac{\kappa_1}{\Gamma(\alpha_1 + \mu_1(1-\alpha_1))}\,\psi(t)^{\mu_1(1-\alpha_1)-1}, \quad t \to +\infty,
$$

whenever $\mu_2(2-\alpha_2) - \mu_1(1-\alpha_1) < 0$, for $\mu_1, \mu_2 \in [0,1]$, $\alpha_1 \in]0,1]$, and $\alpha_2 \in]1,2]$.

4 Conclusions

In this work, we solved the ψ-Hilfer fractional relaxation-oscillation equation and we showed that the solution can be expressed in terms of bivariate Mittag-Leffler functions. We studied the asymptotic behaviour of the solution of the associated homogeneous equation. This is important to understand and classify the relaxation-oscillation phenomena. Our results generalise those presented in Section 3 of [2].

Acknowledgments

The work of the authors was supported by Portuguese funds through CIDMA-Center for Research and Development in Mathematics and Applications, and FCT–Fundação para a Ciência e a Tecnologia, within projects UIDB/04106/2020 and UIDP/04106/2020. N. Vieira was also supported by FCT via the 2018 FCT program of Stimulus of Scientific Employment - Individual Support (Ref: CEECIND/01131/2018).

References

[1] H.J. Haubold, A.M. Mathai, and R.K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math. **2011** (2011), Article No.298628 (51pp).

[2] F. Mainardi, *Fractional relaxation-oscillation and fractional diffusion-wave phenomena*, Chaos Solitons Fractals **7**-No.9 (1996), 1461–1477.

[3] J.V.C. Sousa, R.F. Camargo, E.C. Oliveira, and G.S.F. Frederico, *Pseudo-fractional differential equations and generalized g-Laplace transform*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article No.44 (27pp.).

[4] J.V.C. Sousa and E. C. Oliveira, *On the ψ-Hilfer derivative*, Commun. Nonlinear Sci. Numer. Simulat. **60** (2018), 72–91.

[5] N. Vieira, M. Ferreira, and M.M. Rodrigues, *Time-fractional telegraph equation with ψ-Hilfer derivatives*, Chaos Solit. Fractals **162** (2022), Article No.112276 (26pp.).