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MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR $(p,2)-$EQUATIONS WITH ASYMMETRIC RESONANT REACTION

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Abstract. We consider a nonlinear nonhomogeneous Dirichlet problem driven by the sum of a $p-$Laplacian and a Laplacian (a $(p,2)-$ equation). The reaction is the sum of two competing terms, a parametric $(p-1)-$sublinear term and an asymmetric $(p-1)-$linear perturbation which is resonant at $-\infty$. Using variational methods together with truncations and comparison techniques and Morse theory (critical groups), we prove two multiplicity theorems which provide sign information for all the solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper we study the following nonlinear, nonhomogeneous Dirichlet problem

$$(P_\lambda) \qquad \begin{cases} -\triangle_p u(z) - \triangle u(z) = \lambda g(z,u(z)) + f(z,u(z)) \text{ in } \Omega, \\ u\mid_{\partial\Omega}= 0,\ 2 < p < \infty,\ \lambda > 0. \end{cases}$$

Here for any $q \in (0,\infty)$ by $\triangle_q$ we denote the $q-$Laplace differential operator defined by

$$\triangle_q u = div\left(|Du|^{q-2} Du\right), \text{ for all } u \in W^{1,q}(\Omega),$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^N$.

When $q = 2$, we have the usual Laplacian denoted by $\triangle$. So, in problem (P_λ), the differential operator (left hand side of (P_λ)) is the sum of a $p-$Laplacian (with $p > 2$) and of a Laplacian (a $(p,2)-$equation). Such a differential operator is not homogeneous and this is a source of difficulties in the analysis of problem (P_λ).

In the reaction (right hand side of (P_λ)), we have two terms, namely the functions $\lambda g(z,x)$ and $f(z,x)$ with $\lambda > 0$ being a parameter. Both functions are Carathéodory (that is, both are measurable in $z \in \Omega$ and continuous in $x \in \mathbb{R}$). The function $g(z,.)$ is strictly $(p-1)-$sublinear near $\pm\infty$, while $f(z,.)$ is $(p-1)-$linear near $\pm\infty$. However, $f(z,.)$ exhibits asymmetric behavior as $x \to \pm\infty$. More precisely, the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ stays above $\widehat{\lambda}_1(p) > 0$ (the principal eigenvalue of $\big(-\triangle_p, W_0^{1,p}(\Omega)\big)$) as $x \to +\infty$ and only partial interaction is allowed with $\widehat{\lambda}_1(p)$ (nonuniform nonresonance). In the negative direction (that is, as $x \to -\infty$),

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the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ stays below $\widehat{\lambda}_1(p)$ and complete interaction (resonance) with $\widehat{\lambda}_1(p)$ is allowed. The resonance occurs from the left of $\widehat{\lambda}_1(p)$ in the sense that

$$\widehat{\lambda}_1(p)|x|^p - p[\lambda G(z,x) + F(z,x)] \overset{x\to-\infty}{\longrightarrow} +\infty \text{ uniformly, for } a.a. z \in \Omega, \tag{1.1}$$

with

$$G(z,x) = \int_0^x g(z,s)\,ds \text{ and } F(z,x) = \int_0^x f(z,s)\,ds.$$

This makes the negative truncation of the corresponding energy functional coercive and so the direct method of the calculus of variations is available in the search for negative solutions of (P_λ).

Using variational methods, together with truncations and comparison techniques and Morse theory (critical groups), we show that for $\lambda > 0$ small, problem (P_λ) has at least five nontrivial solutions with sign information (namely, we have two positive solutions, one negative solution and two nodal (sign changing) solutions).

We mention that $(p,2)$ −equations arise in problems of mathematical physics (see Cherfils-Ilyasov [8]), and recently there have been some existence and multiplicity results for such equations. We mention the works of Aizicovici-Papageorgiou-Staicu [2]-[4], Cingolani-Degiovanni [9], He-Guo-Huang-Lei [15], Papageorgiou-Radulescu [21], [22], Papageorgiou-Vetro-Vetro [23], Sun [25], Sun-Zhang-Su [26]. Closer to our work here is the paper of Papageorgiou-Radulescu [22]. In [22] the authors deal with a nonparametric $(p,2)$ −equation and the reaction exhibits an asymmetric behavior as $x \to \pm\infty$. Under more restrictive conditions on the data of the problem, they prove a multiplicity theorem producing three nontrivial solutions, but they do not provide sign information for all of them.

2. Mathematical Background-Hypotheses

Let $(X, \|\cdot\|)$ be a Banach space and X^* be its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ will designate weak convergence in X.

A map $A : X \to X^*$ is said to be of type $(S)_+$ if for every sequence $\{x_n\}_{n\geq1} \subseteq X$ such that $x_n \xrightarrow{w} x$ and

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0,$$

one has

$$x_n \to x \text{ in } X \text{ as } n \to \infty.$$

A map $g : X \to X^*$ is said to be completely continuous if for every sequence $\{x_n\}_{n\geq1} \subseteq X$ such that $x_n \xrightarrow{w} x$ one has

$$g(x_n) \to g(x) \text{ in } X^* \text{ as } n \to \infty.$$

Let $\varphi \in C^1(X,\mathbb{R})$. The *Cerami condition* (*C-condition*, for short) plays a central role in critical point theory. It is a compactness-type condition on φ, namely:

"*every sequence* $\{u_n\}_{n\geq1} \subseteq X$ *such that* $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ *is bounded and*

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{as } n \to \infty$$

admits a strongly convergent subsequence".

This condition leads to a deformation theorem from which the minimax theory of the critical values of φ follows. We recall one such minimax theorem, known in the literature as the "mountain pass theorem". It will be used in the analysis of problem (P_λ).

Theorem 2.1. *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the $C-$condition $u_0,\ u_1 \in X$ and $\rho > 0$ are such that $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0),\varphi(u_1)\} < \inf\{\varphi(u) : \|u-u_0\| = \rho\} =: m_\rho,$$

and

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

then $c \geq m_\rho$ and c is a critical value of φ (i.e., there exists $\widehat{u} \in X$ such that $\varphi'(\widehat{u}) = 0$ and $\varphi(\widehat{u}) = c$).

We mention that, if $\varphi \in C^1(X,\mathbb{R})$ is coercive and

$$\varphi' = A + g$$

with $A,\ g : X \to X^*$, where A is of type $(S)_+$ and g is completely continuous, then φ satisfies the $C-$condition (see Marano-Papageorgiou [19]). This is the situation in our setting here.

In the analysis of problem (P_λ) we will mainly use the following two spaces: the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1\left(\overline{\Omega}\right) : u(z) = 0 \text{ for all } z \in \partial\Omega\right\}.$$

By $\|.\|$ we will denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincare inequality (see, e. g., Brezis [6], p. 290), we have

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

where $\|.\|_p$ stands for the L^p-norm. The space $C_0^1\left(\overline{\Omega}\right)$ is an ordered Banach space with a positive (order) cone given by

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$int\ C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n} = (Du,n)_{\mathbb{R}^N} < 0 \text{ on } \partial\Omega\right\},$$

where $n(.)$ is the outward unit normal on $\partial\Omega$.

For $q \in (1,\infty)$, by $A_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*$ $(\frac{1}{q}+\frac{1}{q'} = 1)$ we denote the nonlinear map defined by

$$\langle A_q(u),h\rangle = \int_\Omega |Du|^{q-2}(Du,Dh)_{\mathbb{R}^N}\,dz \text{ for all } u,\ h \in W_0^{1,q}(\Omega). \tag{2.1}$$

The properties of A_q are summarized below. See, e. g., Motreanu-Motreanu-Papageorgiou ([20], p.40).

Proposition 2.2. *The map* $A_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ *defined by* (2.1) *is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too), and of type* $(S)_+$.

If $q = 2$, then $A_2 = A \in \mathcal{L}\left(H_0^1(\Omega), H_0^{-1}(\Omega)\right)$.

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfies

$$|f_0(z,x)| \leq a_0(z)\left(1 + |x|^{r-1}\right) \text{ for } a.a. z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, and $1 < r \leq p^*$, where p^* is the critical Sobolev exponent corresponding to p, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and introduce the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From Motreanu-Motreanu-Papageorgiou ([20], p.409) we have:

Proposition 2.3. *If* $u_0 \in W_0^{1,p}(\Omega)$ *is a local* $C^1\left(\overline{\Omega}\right)-$ *minimizer of* φ_0, *that is, there exists* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in C_0^1\left(\overline{\Omega}\right) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}\left(\overline{\Omega}\right)$ *for some* $\alpha \in (0,1)$ *and it is also a local* $W_0^{1,p}(\Omega)-$ *minimizer of* φ_0, *that is, there exists* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in W_0^{1,p}(\Omega) \textit{ with } \|h\| \leq \rho_1.$$

Remark. The relation between Holder and Sobolev local minimizers was first proved for semilinear Dirichlet problems by Brezis-Nirenberg [7].

In the analysis of (P_λ) we will use the spectra of the Dirichlet $p-$Laplacian and of the Dirichlet Laplacian. So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\triangle_p u(z) = \widehat{\lambda}\,|u(z)|^{p-2}\,u(z) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue for problem (2.2), if there exists a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$, known as an *eigenfunction* corresponding to $\widehat{\lambda}$.

Problem (2.2) admits a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which has the following properties:

- $\widehat{\lambda}_1(p)$ is isolated (that is, there exists $\varepsilon > 0$ such that $\left(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon\right)$ contains no eigenvalues;
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are two eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u} = \theta\widehat{v}$ with $\theta \in \mathbb{R}\backslash\{0\}$);

- One has

$$\widehat{\lambda}_1(p) = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.3}$$

In (2.3) the infimum is achieved on the corresponding one dimensional eigenspace. From the above properties it follows that the elements of this eigenspace do not change sign. Moreover, the nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski-Papageorgiou [13], pp.737-738) imply that the nontrivial elements of this eigenspace belong to *int* C_+ or $-$*int* C_+.

In what follows, by $\widehat{u}_1(p)$ we denote the positive L^p- normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p)$. We know that $\widehat{u}_1(p) \in$ *int* $C_+ \setminus \{0\}$.

All these properties lead to the following lemma (see Motreanu-Motreanu-Papageorgiou ([20], p.305).

Lemma 2.4. *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \widehat{\lambda}_1(p)$ for a.a.$z \in \Omega$, and the inequality is strict on a set of positive measure, then*

$$\|Du\|_p^p - \int_\Omega \theta(z)\,|u(z)|^p\,dz \geq C_1\,\|u\|^p \text{ for all } u \in W_0^{1,p}(\Omega) \text{ and some } C_1 > 0.$$

The Ljusternik-Shnirelmann minimax scheme generates, in addition to $\widehat{\lambda}_1(p)$, a whole strictly increasing sequence $\left\{\widehat{\lambda}_k(p)\right\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k(p) \to +\infty$ as $k \to \infty$. It is not known if this sequence exhausts the spectrum of (2.2). We know that if $\widehat{u}$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$, then $\widehat{u} \in C_0^1\left(\overline{\Omega}\right)$ (nonlinear regularity theory) and $\widehat{u}$ is nodal.

We will also encounter a weighted version of (2.2). So, let $m \in L^\infty(\Omega)$, $m(z) \geq 0$ for a.a.$z \in \Omega$, $m \neq 0$, and consider the following weighted version of (2.2):

$$\begin{cases} -\triangle_p u(z) = \widetilde{\lambda} m(z)\,|u(z)|^{p-2}\,u(z) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

Problem (2.4) has a smallest eigenvalue $\widetilde{\lambda}_1(p,m) > 0$ which has the same properties as $\widehat{\lambda}_1(p) = \widetilde{\lambda}_1(p,1) > 0$. In this case, the variational characterization of $\widetilde{\lambda}_1(p,m)$ has the following form

$$\widetilde{\lambda}_1(p,m) = \inf\left\{\frac{\|Du\|_p^p}{\int_\Omega m(z)\,|u(z)|^p\,dz} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}.$$

All the properties listed for the eigenvalues and eigenfunctions of (2.2) remain valid for the corresponding items of (2.4). So, we are led to the following monotonicity property of the map $m \to \widetilde{\lambda}_1(p,m)$ (see [20], p.250).

Lemma 2.5. *If $m_1,\ m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ for a.a.$z \in \Omega$ and the two inequalities are strict on sets (in general distinct) of positive measure, then*

$$\widetilde{\lambda}_1(p,m_2) < \widetilde{\lambda}_1(p,m_1).$$

For the linear eigenvalue problem (that is, $p = 2$), we have complete knowledge of the spectrum, which is a sequence $\{\widetilde{\lambda}_k(2)\}_{k\geq 1}$ of distinct eigenvalues such that $\widetilde{\lambda}_k(2) \to \infty$ as $k \to \infty$.

For every $k \in \mathbb{N}$, by $E\big(\widehat{\lambda}_k(2)\big)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$.

Standard regularity theory implies that $E\big(\widehat{\lambda}_k(2)\big) \subseteq C_0^1(\overline{\Omega})$. Also each such eigenspace has the so-called "unique continuation property" (UCP for short). This means that, if $u \in E\big(\widehat{\lambda}_k(2)\big)$ for $k \in \mathbb{N}$ and u vanishes on a set of positive measure, then $u \equiv 0$.

We have

$$H_0^1(\Omega) = \overline{\bigoplus_{k\geq 1} E\left(\widehat{\lambda}_k(2)\right)}.$$

For every $m \in \mathbb{N}$, we define

$$\overline{H}_m = \bigoplus_{k=1}^{m} E\left(\widehat{\lambda}_k(2)\right) \text{ and } \widehat{H}_m = \overline{\bigoplus_{k=m+1}^{\infty} E\left(\widehat{\lambda}_k(2)\right)}.$$

Hence we have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{H}_m \bigoplus \widehat{H}_m.$$

All the eigenvalues admit variational characterizations:

$$\widehat{\lambda}_1(2) = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right\}, \tag{2.5}$$

$$\begin{aligned} \widehat{\lambda}_k(2) &= \inf\left\{\tfrac{\|Du\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_{k-1},\ u \neq 0\right\} \\ &= \sup\left\{\tfrac{\|Du\|_2^2}{\|u\|_2^2} : u \in \overline{H}_k,\ u \neq 0\right\}. \end{aligned} \tag{2.6}$$

In (2.5) and (2.6) the infimum and supremum are achieved on $E\big(\widehat{\lambda}_k(2)\big)$.

Using these variational characterizations and the UCP, we infer the following inequalities (see Gasinski-Papageorgiou [14], p.870).

Lemma 2.6. *(a) If $\theta \in L^\infty(\Omega)$, $\theta(z) \geq \widehat{\lambda}_k(2)$ for a.a.$z \in \Omega$ and the inequality is strict on a set of positive measure, then*

$$\|Du\|_2^2 - \int_\Omega \theta(z)\, u^2(z)\, dz \leq -C_2 \|u\|^2 \text{ for all } u \in \overline{H}_k,\ \text{some } C_2 > 0.$$

(b) If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \widehat{\lambda}_k(2)$ for a.a.$z \in \Omega$ and the inequality is strict on a set of positive measure, then

$$\|Du\|_2^2 - \int_\Omega \theta(z)\, u^2(z)\, dz \geq C_3 \|u\|^2 \text{ for all } u \in \widehat{H}_{k-1},\ \text{some } C_3 > 0.$$

Next let us recall some basic facts about critical groups. For details we refer to Motreanu-Motreanu-Papageorgiou ([20].

Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \le c\},$$
$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\} \text{ (the critical set of } \varphi),$$

and

$$K_\varphi^c = \{u \in K_\varphi : \varphi(x) = c\}.$$

Consider a topological pair (Y_1, Y_2) with $Y_2 \subset Y_1 \subset X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k^{th}- relative singular homology group for the topological pair (Y_1, Y_2) with integer coefficients. Then the *critical groups of* φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0.$$

Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the choice of the *isolating* neighborhood U.

Suppose that $\varphi \in C^1(X,\mathbb{R})$ satisfies the $C-$condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the *critical groups of* φ *at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \in \mathbb{N}_0.$$

By the second deformation theorem, this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Indeed, if $c' < c < \inf \varphi(K_\varphi)$, then by the second deformation theorem $\varphi^{c'}$ is a strong deformation retract of φ^c and so,

$$H_k(X, \varphi^c) = H_k\left(X, \varphi^{c'}\right) \text{ for all } k \in \mathbb{N}_0,$$

(see Motreanu-Motreanu-Papageorgiou [20], p. 145).

Now suppose that $\varphi \in C^1(X)$ satisfies the C-condition and K_φ is finite. We define

$$M(t,u) = \sum_{k \in \mathbb{N}_0} rank\ C_k(\varphi, u)\, t^k \text{ for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$

and

$$P(t,\infty) = \sum_{k \in \mathbb{N}_0} rank\ C_k(\varphi, \infty)\, t^k \text{ for all } t \in \mathbb{R}.$$

The *Morse relation* says that

$$\sum_{u \in K_\varphi} M(t,u) = P(t,\infty) + (1+t)\,Q(t) \text{ for all } t \in \mathbb{R}, \tag{2.7}$$

where $Q(t) = \sum_{k \ge 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k, $k \in \mathbb{N}_0$.

Next, let us finalize our notation. Given $x \in \mathbb{R}$, we set

$$x^{\pm} = \max\{\pm x, 0\}.$$

Then for $u \in W_0^{1,p}(\Omega)$, we define $u^{\pm}(.) = u(.)^{\pm}$. We have

$$u^{\pm} \in W_0^{1,p}(\Omega),\ u = u^+ - u^- \text{ and } |u| = u^+ + u^-,$$

Also, if $u,\ v \in W_0^{1,p}(\Omega)$ and $v(z) \leq u(z)$ for a.a.$z \in \Omega$, then

$$[v,u] := \left\{ y \in W_0^{1,p}(\Omega) : v(z) \leq y(z) \leq u(z) \text{ for } a.a.z \in \Omega \right\}.$$

Moreover, by $int_{\ C_0^1(\overline{\Omega})}[v,u]$ we denote the interior in $C_0^1\left(\overline{\Omega}\right)$ of the set $[v,u] \cap C_0^1\left(\overline{\Omega}\right)$.

For $k,\ m \in \mathbb{N}_0$, by $\delta_{k,m}$ we denote the Kronecker symbol, that is

$$\delta_{k,m} = \begin{cases} 1 & \text{if} \quad k = m, \\ 0 & \text{if} \quad k \neq m. \end{cases}$$

Finally, for $h,\ \widehat{h} \in L^\infty(\Omega)$, we write $h \preceq \widehat{h}$ if for every $K \subseteq \Omega$ compact we can find $c_K > 0$ such that

$$0 < c_K \leq \widehat{h}(z) - h(z) \text{ for } a.a.z \in \Omega.$$

Now we are ready to introduce the conditions on the functions $g(z,x)$ and $f(z,x)$, namely:

$(\mathbf{H}_g)$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a.$z \in \Omega$ $g(z,\cdot)$ is nondecreasing and

(*i*) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|g(z,x)| \leq a_\rho(z) \text{ for } a.a.z \in \Omega, \text{ all } x \in \mathbb{R} \text{ with } |x| \leq \rho;$$

(*ii*)

$$\lim_{x \to \pm\infty} \frac{g(z,x)}{|x|^{p-2}x} = 0 \text{ uniformly for a.a.} z \in \Omega;$$

(*iii*)

$$\lim_{x \to 0} \frac{g(z,x)}{x} = 0 \text{ uniformly for a.a.} z \in \Omega.$$

$(\mathbf{H}_f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

(*i*) for every $\rho > 0$ there exists $\widehat{a}_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leq \widehat{a}_\rho(z) \text{ for } a.a.z \in \Omega, \text{ all } x \in \mathbb{R} \text{ with } |x| \leq \rho;$$

(*ii*) there exist $\eta_+ \in L^\infty(\Omega)$ and $C_4 > 0$ such that

$$\eta_+(z) \geq \widehat{\lambda}_1(p) \text{ for } a.a.z \in \Omega, \ \eta_+ \neq \widehat{\lambda}_1(p),$$
$$\eta_+(z) \leq \liminf_{x \to \infty} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x \to \infty} \frac{f(z,x)}{x^{p-1}} \leq C_4$$
$$\text{uniformly for } a.a.z \in \Omega;$$

(*iii*) there exists $C_5 > 0$ such that

$$-C_5 \leq \liminf_{x \to -\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \limsup_{x \to -\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \widehat{\lambda}_1(p)$$
$$\text{for } a.a.z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iv) for a.a.$z \in \Omega,\ f(z,.)$ is differentiable at $x=0$ and there exists $m \in \mathbb{N}$, $m \geq 2$ such that

$$\widehat{\lambda}_m(2) \leq f_x'(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \leq \widehat{\lambda}_{m+1}(2)$$
$$\text{uniformly for } a.a.z \in \Omega,$$
$$f_x'(.,0) \neq \widehat{\lambda}_m(2),\ \ f_x'(.,0) \neq \widehat{\lambda}_{m+1}(2);$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a.$z \in \Omega$ the function

$$x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2} x$$

is nondecreasing on $[-\rho, \rho]$.

$(\mathbf{H}_0)$: For every $\lambda > 0$:

$$\lambda g(z,x)x + f(z,x)x - p[\lambda G(z,x) + F(z,x)] \to \infty \text{ as } x \to -\infty,$$
$$\text{uniformly for } a.a.z \in \Omega.$$

Remarks: Hypothesis $(\mathbf{H}_g)(ii)$ dictates a strictly $(p-1)-$sublinear growth for $g(z,\cdot)$ near $\pm\infty$. Near zero, $g(z,\cdot)$ is strictly sublinear. On the other hand, hypotheses $(\mathbf{H}_f)(ii),(iii)$ impose a $(p-1)-$linear growth for $f(z,\cdot)$ near $\pm\infty$ and a linear growth near 0. So, in the reaction of (P_λ) we have the competing effect of two terms with different asymptotic behavior as $x \to \pm\infty$ and as $x \to 0$. Note that hypotheses $(\mathbf{H}_f)(ii),(iii)$ describe an asymmetric behavior for $f(z,.)$ as $x \to \pm\infty$. It is clear from $(\mathbf{H}_f)(iii)$ that asymptotically at $-\infty$ we can have resonance with respect to $\widehat{\lambda}_1(p) > 0$- Hypothesis $(\mathbf{H}_0)$ guaranties that this resonance occurs from the left of $\widehat{\lambda}_1(p) > 0$ (see (1.1)). This way, the negative truncation of the energy functional is coercive and so the direct method of the calculus of variations can be used to generate negative solutions. Hypothesis $(\mathbf{H}_f)(v)$ is satisfied if, for example, for a.a.$z \in \Omega,\ f(z,.)$ is differentiable and for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that

$$f_x'(z,x)x^2 \geq -\widehat{\xi}_\rho |x|^p \text{ for } a.a.z \in \Omega,\ \text{all } |x| \leq \rho.$$

Examples: The following functions satisfy $(\mathbf{H}_g)$, $(\mathbf{H}_f)$ and $(\mathbf{H}_0)$. For the sake of simplicity we drop the z dependence.

$$g(x) = \begin{cases} 2|x|^{r-2}x & \text{if} \quad |x| \leq 1, \\ |x|^{q-2}x + |x|^{\tau-2}x & \text{if} \quad |x| > 1, \end{cases}$$

and

$$f(x) = \begin{cases} \widehat{\lambda}_1(p)|x|^{p-2}x - \xi|x|^{\mu-2}x & \text{if} \quad x < -1, \\ \theta x + C|x|^{r-2}x & \text{if} \quad |x| \leq 1, \\ \eta x^{p-1} - \widehat{C} & \text{if} \quad 1 < x, \end{cases}$$

where $1 < q < \tau < \mu < p < r,\ p > 2,\ \eta = \widehat{\lambda}_1(p) + \widehat{C} - \xi,\ \theta = \widehat{\lambda}_1(p) - C - \xi$, $\widehat{C} > \xi > 0,\ C < 0,\ \theta \in \left(\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)\right),\ m \geq 2.$

3. Solutions of Constant Sign

For $\lambda > 0$, we consider the C^1-functionals $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \lambda G\left(z, \pm u^\pm\right) + F\left(z, \pm u^\pm\right) dz$$
$$\text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.1. *If hypotheses* $(\mathbf{H}_g), (\mathbf{H}_f)$ *hold and* $\lambda > 0$, *then the functional* φ_λ^+ *satisfies the* $C-$*condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $\left\{\varphi_\lambda^+(u_n)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\left(\varphi_\lambda^+\right)'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.1}$$

From (3.1) we have

$$\begin{aligned} &\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \textstyle\int_\Omega \left[\lambda g\left(z, u_n^+\right) + f\left(z, u^+\right)\right] h dz\right| \\ &\leq \tfrac{\varepsilon_n \|h\|}{1+\|u_n\|} \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n' \to 0^+. \end{aligned} \tag{3.2}$$

In (3.2) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\left\|Du_n^-\right\|_p^p + \left\|Du_n^-\right\|_2^2 \leq \varepsilon_n \text{ for all } n \in \mathbb{N},$$

hence

$$u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega). \tag{3.3}$$

Using (3.3) in (3.2) we obtain

$$\begin{aligned} &\left|\langle A_p(u_n^+), h\rangle + \langle A(u_n^+), h\rangle - \textstyle\int_\Omega \left[\lambda g\left(z, u_n^+\right) + f\left(z, u^+\right)\right] h dz\right| \\ &\leq \varepsilon_n' \|h\| \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \text{with } \varepsilon_n' \to 0^+. \end{aligned} \tag{3.4}$$

Suppose that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that

$$\left\|u_n^+\right\| \to \infty, \text{ as } n \to \infty.$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1, \; y_n \geq 0, \text{ for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as n} \to \infty, \, y \geq 0. \tag{3.5}$$

From (3.4) we have

$$\begin{aligned} &\left|\langle A_p(y_n), h\rangle + \tfrac{1}{\|u_n^+\|^{p-2}}\langle A(y_n), h\rangle - \textstyle\int_\Omega \tfrac{\lambda g\left(z, u_n^+\right) + f\left(z, u_n^+\right)}{\|u_n^+\|^{p-1}} h dz\right| \\ &\leq \varepsilon_n' \tfrac{\|h\|}{\|u_n^+\|^{p-1}} \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{3.6}$$

It is clear from hypotheses $(\mathbf{H}_g), (\mathbf{H}_f)$ that we have

$$|\lambda g(z,x)x + f(z,x)| \leq C_6\left[1+|x|^{p-1}\right] \text{ for } a.a. z \in \Omega,$$
$$\text{all } x \in \mathbb{R}, \text{ some } C_6 > 0,$$

hence

$$\left\{\frac{\lambda g(\cdot, u_n^+(\cdot)) + f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{3.7}$$

From (3.7) and hypotheses $(\mathbf{H}_g)(ii), (\mathbf{H}_f)(ii)$, at least for a subsequence, we have

$$\frac{\lambda g(\cdot, u_n^+(\cdot)) + f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \eta\, y^{p-1} \text{ in } L^{p'}(\Omega), \tag{3.8}$$

with

$$\eta_+(z) \leq \eta(z) \leq C_4 \text{ for } a.a. z \in \Omega,$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

In (3.6) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.8) (also recall that $\|u_n^+\| \to \infty$ and $2 < p$). We obtain

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

hence

$$y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } \|y\| = 1,\ y \geq 0 \tag{3.9}$$

(see Proposition 2.2). We return to (3.6), pass to the limit as $n \to \infty$ and use (3.8). Then

$$\langle A_p(y), h\rangle = \int_\Omega \eta(z)\, y^{p-1} h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

therefore

$$-\triangle_p y(z) = \eta(z)\, y(z)^{p-1} \text{ for } a.a. z \in \Omega,\ y\,|_{\partial\Omega} = 0. \tag{3.10}$$

From (3.6) and Lemma 2.5, we have

$$\widetilde{\lambda}_1(p,\eta) \leq \widetilde{\lambda}_1(p,\eta_+) < \widetilde{\lambda}_1\left(p, \widehat{\lambda}_1(p)\right) = 1. \tag{3.11}$$

Combining (3.10) and (3.11), we infer that y is nodal or zero, a contradiction (see (3.9)). Therefore

$$\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$

and consequently

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (3.3)).}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as n} \to \infty. \tag{3.12}$$

In (3.2) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.12). Then

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\right] = 0,$$

hence

$$\limsup_{n\to\infty}\left[\langle A_p(u_n), u_n-u\rangle + \langle A(u), u_n-u\rangle\right] \le 0,$$

(from the monotonicity of $A(\cdot)$), therefore

$$\limsup_{n\to\infty}\langle A_p(u_n), u_n-u\rangle \le 0 \text{ (see (3.12))},$$

and by Proposition 2.2 and (3.12) we obtain that

$$u_n \to u \text{ in } W_0^{1,p}(\Omega),$$

and conclude that φ_λ^+ satisfies the $C-$condition. □

Proposition 3.2. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold and* $\lambda>0$, *then the functional* φ_λ^- *is coercive.*

Proof. On account of hypothesis $(\mathbf{H}_0)$, given any $\mu>0$, we can find $M_1=M_1(\mu)>0$ such that

$$\begin{aligned}\lambda g(z,x)x + f(z,x)x - p\left[\lambda G(z,x)+F(z,x)\right] &\ge \mu \\ \text{for } a.a. z\in\Omega, \text{ all } x &\le -M_1.\end{aligned} \tag{3.13}$$

In the sequel, for notational economy, we set

$$e_\lambda(z,x) = \lambda g(z,x) + f(z,x) \text{ and } E_\lambda(z,x) = \textstyle\int_0^x e_\lambda(s,z)\,ds.$$

We have

$$\begin{aligned}\frac{d}{dx}\left[\frac{E_\lambda(z,x)}{|x|^p}\right] &= \frac{e_\lambda(z,x)|x|^p - pE_\lambda(z,x)|x|^{p-2}x}{|x|^{2p}} \\ &= \frac{e_\lambda(z,x)x - pE_\lambda(z,x)}{|x|^p x} \\ &\le \frac{\mu}{|x|^p x} \text{ for } a.a. z\in\Omega, \text{ all } x\le -M_1,\end{aligned}$$

(see (3.13)), hence

$$\begin{aligned}\frac{E_\lambda(z,v)}{|v|^p} - \frac{E_\lambda(z,y)}{|y|^p} &\ge \frac{\mu}{p}\left[\frac{1}{|y|^p} - \frac{1}{|v|^p}\right] \\ \text{for } a.a. z\in\Omega, \text{ all } v &\le y\le -M_1.\end{aligned} \tag{3.14}$$

Hypotheses $(\mathbf{H}_g)(ii)$ and $(\mathbf{H}_f)(iii)$ imply that we can find $C_7>0$ such that

$$\begin{aligned}-C_7 \le \liminf_{x\to-\infty}\frac{E_\lambda(z,x)}{|x|^p} &\le \limsup_{x\to-\infty}\frac{E_\lambda(z,x)}{|x|^p} \le \frac{\widehat{\lambda}_1(p)}{p} \\ &\text{uniformly for } a.a. z\in\Omega.\end{aligned} \tag{3.15}$$

So, if in (3.14) we let $v\to-\infty$ and use (3.15), then we obtain

$$\widehat{\lambda}_1(p)|y|^p - pE_\lambda(z,y) \ge \mu \text{ for } a.a. z\in\Omega, \text{ all } y\le -M_1,$$

hence

$$\widehat{\lambda}_1(p)|y|^p - pE_\lambda(z,y) \to +\infty \text{ uniformly for } a.a. z\in\Omega, \tag{3.16}$$

as $y\to-\infty$.

Using (3.16), we will show that φ_λ^- is coercive.

Arguing by contradiction, suppose that φ_λ^- is not coercive. Then we can find a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ and $M_2 > 0$ such that

$$\|u_n\| \to +\infty \text{ and } \varphi_\lambda^-(u_n) \leq M_2 \text{ for all } n \in \mathbb{N}. \tag{3.17}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1 \text{ for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as n} \to \infty. \tag{3.18}$$

We have

$$\begin{aligned} &\tfrac{1}{p}\|Dy_n\|_p^p + \tfrac{1}{2\|u_n\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \tfrac{E_\lambda(z,-u_n^-)}{\|u_n\|^p}dz \\ &\leq \tfrac{M_2}{\|u_n\|^p} \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{3.19}$$

(see (3.17)). Recall that

$$|E_\lambda(z,x)| \leq C_8[1+|x|^p] \text{ for a.a. } \; z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } C_8 > 0,$$

hence

$$\left\{\frac{E_\lambda(\cdot,-u_n^-(\cdot))}{\|u_n\|^p}\right\}_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

Hence, by the Dunford-Pettis theorem and hypotheses $(\mathbf{H}_g)(ii)$, $(\mathbf{H}_f)(iii)$, at least for a subsequence, we have

$$\frac{E_\lambda(\cdot,-u_n^-(\cdot))}{\|u_n\|^p} \xrightarrow{w} \frac{1}{p}\theta\left(y^-\right)^p \text{ in } L^1(\Omega) \text{ as n} \to \infty, \tag{3.20}$$

with

$$-C_5 \leq \theta(z) \leq \widehat{\lambda}_1(p) \text{ for a.a. } \; z \in \Omega.$$

See [1]. Therefore, if in (3.19) we pass to the limit as $n \to \infty$ and use (3.18) and (3.20), then

$$\|Dy\|_p^p \leq \int_\Omega \theta(z)\left(y^-\right)^p dz, \tag{3.21}$$

(recall that $\|u_n\| \to \infty$ and $2 < p$).

First assume that the inequality $\theta(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$ (see (3.20)) is strict on a set of positive measure. Then from (3.19) and Lemma 2.4, we have

$$C_9\left\|y^-\right\|^p \leq \left\|Dy^-\right\|_p^p - \int_\Omega \theta(z)\left(y^-\right)^p dz \leq 0 \text{ for some } C_9 > 0,$$

hence

$$y \geq 0, \tag{3.22}$$

and in view of (3.21) we conclude that

$$y = 0.$$

Therefore from (3.19) and (3.20), we have

$$y_n \longrightarrow y \text{ in } W_0^{1,p}(\Omega),$$

a contradiction since $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Now assume that $\theta(z) = \widehat{\lambda}_1(p)$ for a.a.$z \in \Omega$. Then from (3.21) and (2.3) we have

$$\left\|Dy^-\right\|_p^p = \widehat{\lambda}_1(p)\left\|y^-\right\|_p^p,$$

hence

$$y^- = \tau\widehat{u}_1(p) \in int\ C_+ \text{ with } \tau \geq 0.$$

If $\tau = 0$, then $y \geq 0$ and so, as above (see the argument after (3.22)), we reach a contradiction.

If $\tau > 0$, then $y^-(z) > 0$ for all $z \in \Omega$ and so, $y(z) < 0$ for all $z \in \Omega$.

It follows that

$$u_n(z) \to -\infty \text{ for } a.a.z \in \Omega,$$

hence

$$\widehat{\lambda}_1(p)\left[u_n^-(z)\right]^p - pE_\lambda\left(z, -u_n^-(z)\right) \to +\infty \text{ for } a.a.z \in \Omega,$$

(see (3.16)), hence

$$\int_\Omega \left[\widehat{\lambda}_1(p)\left[u_n^-(z)\right]^p - pE_\lambda\left(z, -u_n^-(z)\right)\right] dz \to +\infty$$

(by Fatou lemma, see (3.16)), therefore

$$p\varphi_\lambda^-(u_n) \to +\infty$$

which contradicts (3.17). We conclude that φ_λ^- is coercive. □

Consequently (see the remarks following Theorem 2.1) we arrive at:

Corollary 3.3. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold and* $\lambda > 0$, *then the functional* φ_λ^- *satisfies the* $C-$*condition.*

Next we will show that the functional φ_λ^+ satisfies the mountain pass geometry (see Theorem 2.1) when $\lambda > 0$ is small.

Proposition 3.4. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$ *hold, then there exists* $\lambda^* > 0$ *such that for every* $\lambda \in (0, \lambda^*)$ *we can find* $\rho_\lambda > 0$ *for which*

$$\inf\left\{\varphi_\lambda^+(u) : \|u\| = \rho_\lambda\right\} = m_\lambda^+ > 0.$$

Proof. Hypotheses $(\mathbf{H}_g)$ imply that given $\varepsilon > 0$, there is a $C_{10} > 0$ such that

$$G(z,x) \leq \varepsilon x^p + C_{10}x^2 \text{ for } a.a.z \in \Omega, \text{ all } x \geq 0. \tag{3.23}$$

Also, given $r > p$, from hypotheses $(\mathbf{H}_f)(ii)$, (iv) it follows that we can find $C_{11} > 0$ such that

$$F(z,x) \leq C_{11}\left(x^r + x^2\right) \text{ for } a.a.z \in \Omega, \text{ all } x \geq 0. \tag{3.24}$$

Using (3.23) and (3.24) we obtain

$$\begin{aligned} \varphi_\lambda^+(u) &\geq \left(\frac{1}{p} - \frac{\varepsilon}{\widehat{\lambda}_1(p)}\right)\|Du\|_p^p - \lambda C_{12}\|u\|^2 - C_{13}\|u\|^r \\ &\quad \text{for all } u \in W_0^{1,p}(\Omega), \text{ some } C_{12}, C_{13} > 0 \\ &\geq \left[C_{14} - \lambda C_{12}\|u\|^{2-p} - C_{13}\|u\|^{r-p}\right]\|u\|^p \end{aligned}$$

for some $C_{14} > 0$.

Consider the function

$$\xi_\lambda(t) = \lambda C_{12} t^{2-p} + C_{13} t^{r-p} \text{ for all } t > 0.$$

Evidently $\xi_\lambda \in C^1(0,\infty)$ and since $2 < p < r$, we have

$$\xi_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 > 0$ such that

$$0 < \xi_\lambda(t_0) = \min\{\xi_\lambda(t) : \ t > 0\}.$$

Then

$$\xi_\lambda'(t_0) = 0$$

hence

$$\lambda(p-2) C_{12} t_0^{2-p-1} = (r-p) C_{13} t_0^{r-p-1},$$

therefore

$$t_0 = t_0(\lambda) = \left[\frac{\lambda(p-2) C_{12}}{(r-p) C_{13}}\right]^{\frac{1}{r-2}}.$$

Then we have

$$\xi_\lambda(t_0) = \lambda^{\frac{r-p}{r-2}} C_{12} \left[\frac{(r-p) C_{13}}{(p-2) C_{12}}\right]^{\frac{p-2}{r-2}} + C_{13} \left[\frac{\lambda(p-2) C_{12}}{(r-p) C_{13}}\right]^{\frac{r-p}{r-2}},$$

hence

$$\xi_\lambda(t_0) \to 0^+ \text{ as } \lambda \to 0^+.$$

So, we can find $\lambda^* > 0$ such that

$$\xi_\lambda(t_0) < C_{14} \text{ for all } \lambda \in (0,\lambda^*),$$

therefore

$$\inf\left\{\varphi_\lambda^+(u) : \|u\| = \rho_\lambda := t_0(\lambda)\right\} = m_\lambda^+ > 0 \text{ for all } \lambda \in (0,\lambda^*).$$

$\square$

Proposition 3.5. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$ *hold and* $\lambda > 0$ *then*

$$\varphi_\lambda^+(t\widehat{u}_1(p)) \to -\infty \textit{ as } t \to +\infty.$$

Proof. Hypotheses $(\mathbf{H}_g)(i),(ii)$ imply that given $\varepsilon > 0$, there exists $C_{15} > 0$ such that

$$G(z,x) \geq -\frac{\varepsilon}{p} x^p - C_{15} \text{ for } a.a. z \in \Omega, \text{ all } x \geq 0. \tag{3.25}$$

Similarly hypotheses $(\mathbf{H}_f)(i),(ii)$ imply that given $\varepsilon > 0$, we can find $C_{16} > 0$ such that

$$F(z,x) \geq -\frac{1}{p}\left[\eta_+(z) - \varepsilon\right] x^p - C_{16} \text{ for } a.a. z \in \Omega, \text{ all } x \geq 0. \tag{3.26}$$

Then for all $t > 0$ we have

$$\begin{aligned} \varphi_\lambda^+ \left(t\widehat{u}_1 (p)\right) &\le \tfrac{t^p}{p} \left\| D\widehat{u}_1 (p)\right\|_p^p + \tfrac{t^2}{2} \left\| D\widehat{u}_1 (p)\right\|_2^2 \\ &\quad - \tfrac{t^p}{p} \textstyle\int_\Omega \left[\eta_+ (z) - 2\varepsilon\right] \widehat{u}_1 (p)^p \, dz + C_{17} \\ &= \tfrac{t^p}{p} \left[\textstyle\int_\Omega \left[\widehat{\lambda}_1 (p) - \eta_+ (z)\right] \widehat{u}_1 (p)^p \, dz - 2\varepsilon \right] \\ &\quad + \tfrac{t^2}{2} \left\| D\widehat{u}_1 (p)\right\|_2^2 + C_{17} \end{aligned} \tag{3.27}$$

for some $C_{17} > 0$ (recall that $\left\| D\widehat{u}_1 (p)\right\|_p = 1$).

Since $\widehat{u}_1 (p) \in int\ C_+$ we see that

$$\xi^* = \int_\Omega \left[\eta_+ (z) - \widehat{\lambda}_1 (p)\right] \widehat{u}_1 (p)^p \, dz > 0.$$

Choosing $\varepsilon \in \left(0, \frac{\xi^*}{2}\right)$, from (3.27) we have

$$\varphi_\lambda^+ \left(t\widehat{u}_1 (p)\right) \le -C_{18} t^p + C_{19} t^2 + C_{17} \text{ for some } C_{18},\ C_{19} > 0,$$

hence

$$\varphi_\lambda^+ \left(t\widehat{u}_1 (p)\right) \to -\infty \text{ as } t \to \infty$$

(recall that $p > 2$). $\square$

Now using variational arguments, we will produce two positive solutions for (P_λ) when $\lambda \in (0, \lambda^*)$. Here and in what follows $\lambda^* > 0$ is the critical parameter generated in Proposition 3.4.

Proposition 3.6. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$ *hold and* $\lambda \in (0, \lambda^*)$, *then problem* (P_λ) *admits at least two positive solutions* $u_0,\ \widehat{u} \in int\ C_+$.

Proof. Propositions 3.1, 3.4 and 3.5 permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\varphi_\lambda^+} \text{ and } \varphi_\lambda^+ (0) = 0 < m_\lambda^+ \le \varphi_\lambda^+ (u_0) \tag{3.28}$$

(see Proposition 3.4). From (3.28) we see that $u_0 \ne 0$, and we have

$$\left(\varphi_\lambda^+\right)' (u_0) = 0,$$

hence

$$\left\langle A_p (u_0), h\right\rangle + \left\langle A (u_0), h\right\rangle = \int_\Omega \left[\lambda g\left(z, u_0^+\right) + f\left(z, u_0^+\right)\right] h dz \tag{3.29}$$
$$\text{for all } h \in W_0^{1,p}(\Omega).$$

In (3.29) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\left\| Du_0^-\right\|_p^p + \left\| Du_0^-\right\|_2^2 = 0,$$

hence

$$u_0 \ge 0,\ u_0 \ne 0.$$

Then, from (3.29) we obtain

$$\begin{cases} -\triangle_p u_0 (z) - \triangle u_0 (z) = \lambda g (z, u_0 (z)) + f (z, u_0 (z)) \text{ in } \Omega, \\ u_0 (z) = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.30}$$

Theorem 8.4, p.204 of Motreanu-Motreanu-Papageorgiou [20] implies that

$$u_0 \in L^{\infty}(\Omega).$$

So, we can apply Theorem 1 of Lieberman [17] and conclude that

$$u_0 \in C_+ \backslash \{0\}.$$

Since $g(z,x) \geq 0$ for $(z,x) \in \Omega \times \mathbb{R}_+$, from (3.30) it follows

$$\triangle_p u_0(z) + \triangle u_0(z) + f(z, u_0(z)) \leq 0 \text{ for } a.a. z \in \Omega. \tag{3.31}$$

Hypotheses $(\mathbf{H}_f)$ imply that

$$f(z,x) \geq -C_{20}\left[x + x^{p-1}\right] \text{ for } a.a. z \in \Omega, \text{ all } x \geq 0.$$

So, using Theorem 5.4.1, p.111 of Pucci-Serrin [24], from (3.31) we infer that

$$u_0(z) > 0 \text{ for all } z \in \Omega.$$

Then the boundary point lemma of Pucci-Serrin ([24], p.120) yields

$$u_0 \in int\ C_+.$$

Hypothesis $(\mathbf{H}_f)(iv)$ implies that given $\varepsilon > 0$, one can find $\delta > 0$ such that

$$\frac{1}{2}\left[f'_x(z,0) - \varepsilon\right] x^2 \leq F(z,x) \text{ for } a.a. z \in \Omega, \text{ all } |x| \leq \delta. \tag{3.32}$$

Recall that $\widehat{u}_1(2) \in int\ C_+$. So, there is a $t \in (0,1)$ small, such that $t\widehat{u}_1(2)(z) \in (0, \delta]$ for all $z \in \Omega$.

We have

$$\begin{array}{c} \varphi^+_\lambda(t\widehat{u}_1(2)) \\ \leq \frac{t^p}{p}\|D\widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2)\|\widehat{u}_1(2)\|_2^2 - \frac{t^2}{2}\int_\Omega\left[f'_x(z,0) - \varepsilon\right]\widehat{u}_1(2)^2 dz \\ \text{(see (3.29) and recall } G \geq 0) \\ \leq C_{21}t^p + \frac{t^2}{2}\left[\int_\Omega\left(\widehat{\lambda}_1(2) - f'_x(z,0)\right)\widehat{u}_1(2)^2 dz + \varepsilon\right] \\ \text{(recall that } \|\widehat{u}_1(2)\|_2 = 1). \end{array} \tag{3.33}$$

Hypothesis $(\mathbf{H}_f)(iv)$ implies that

$$\xi_0 := \int_\Omega\left[f'_x(z,0) - \widehat{\lambda}_1(2)\right]\widehat{u}_1(2)^2 dz > 0.$$

So, choosing $\varepsilon \in (0, \xi_0)$ in (3.33), we arrive at

$$\varphi^+_\lambda(t\widehat{u}_1(2)) \leq C_{21}t^p - C_{22}t^2 \text{ for some } C_{22} > 0.$$

Since $p > 2$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\varphi^+_\lambda(t\widehat{u}_1(2)) < 0, \ \ \|t\widehat{u}_1(2)\| \leq \rho_\lambda \tag{3.34}$$

(see Proposition 3.4). Let

$$\overline{B}_\lambda = \left\{u \in W^{1,p}_0(\Omega) : \|u\| \leq \rho_\lambda\right\}.$$

We have

$$\inf\left\{\varphi^+_\lambda(u) : u \in \overline{B}_\lambda\right\} = \mu^+_\lambda < 0, \tag{3.35}$$

(see (3.34)). Since φ_λ^+ is sequentially weakly lower semicontinuous and $\overline{B}_\lambda$ is sequentially weakly compact (by the James and Eberlein-Smulian theorem), we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(\widehat{u}) = \inf\left\{\varphi_\lambda^+(u) : u \in \overline{B}_\lambda\right\} = \mu_\lambda^+ < 0 = \varphi_\lambda^+(0), \tag{3.36}$$

hence

$$\widehat{u} \neq 0.$$

Moreover, from (3.28) and (3.35) we infer that

$$\varphi_\lambda^+(\widehat{u}) < 0 < m_\lambda^+ \leq \varphi_\lambda^+(u_0), \tag{3.37}$$

hence

$$\widehat{u} \neq u_0.$$

From (3.37) and Proposition 3.4 it follows that

$$0 < \|\widehat{u}\| < \rho_\lambda,$$

hence

$$\widehat{u} \in K_{\varphi_\lambda^+} \text{ (see (3.36))},$$

therefore

$$\widehat{u} \geq 0,\ \widehat{u} \notin \{0, u_0\}.$$

We conclude that $\widehat{u} \in C_0^1\left(\overline{\Omega}\right)$, $\widehat{u} \notin \{0, u_0\}$ is the second positive solution of (P_λ), for $\lambda \in (0, \lambda^*)$. ☐

Using Proposition 3.2 and the direct method of calculus of variations we can produce a negative solution of (P_λ), for all $\lambda > 0$.

Proposition 3.7. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold and* $\lambda > 0$, *then problem* (P_λ) *admits a negative solution* $v_0 \in -int\ C_+$.

Proof. By Proposition 3.2, φ_λ^- is coercive. Also, using the Sobolev embedding theorem we see that φ_λ^- is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^-(v_0) = \inf\left\{\varphi_\lambda^-(v) : v \in W_0^{1,p}(\Omega)\right\}. \tag{3.38}$$

Reasoning as in the proof of Proposition 3.6, we show that for $t \in (0,1)$ small, we have

$$\varphi_\lambda^-(t(-\widehat{u}_1(2))) < 0,$$

hence

$$\varphi_\lambda^-(v_0) < 0 = \varphi_\lambda^-(0),$$

(see (3.38)), therefore

$$v_0 \neq 0.$$

Also, from (3.38) we have

$$\left(\varphi_\lambda^-\right)'(v_0) = 0,$$

hence

$$\langle A_p(v_0), h\rangle + \langle A(v_0), h\rangle = \int_\Omega \left[\lambda g\left(z, -v_0^-\right) + f\left(z, -v_0^-\right)\right] h dz \tag{3.39}$$

$$\text{for all } h \in W_0^{1,p}(\Omega).$$

In (3.39) we choose $h = v_0^+ \in W_0^{1,p}(\Omega)$. Then

$$\left\|Dv_0^+\right\|_p^p + \left\|Dv_0^+\right\|_2^2 = 0,$$

hence

$$v_0 \le 0, \ v_0 \ne 0.$$

From (3.39) it follows that

$$\begin{cases} -\triangle_p v_0(z) - \triangle v_0(z) = \lambda g(z, v_0(z)) + f(z, v_0(z)) \text{ for a.a.} z \in \Omega, \\ v_0(z) = 0 \text{ on } \partial\Omega. \end{cases}$$

hence

$$\triangle_p(-v_0(z)) + \triangle(-v_0(z)) - f(z, v_0(z)) \le 0 \text{ for } a.a.z \in \Omega.$$

As in the proof of Proposition 3.6, using the results of Pucci-Serrin ([24], pp. 111, 120), we conclude that

$$v_0 \in -int\ C_+.$$

□

4. Extremal constant sign solutions

In this section we produce extremal constant sign solutions for problem (P_λ). That is, we show that for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has a smallest positive solution $\overline{u}_\lambda \in int\ C_+$ (i. e., if u is a positive solution of (P_λ), then $\overline{u}_\lambda \le u$) and for all $\lambda > 0$, problem (P_λ) has a biggest negative solution $\overline{v}_\lambda \in -int\ C_+$ (i. e., if v is a negative solution of (P_λ), then $v \le \overline{v}_\lambda$). Using $\overline{u}_\lambda$ and $\overline{v}_\lambda$, in the next section, we will produce nodal (that is, sign changing) solutions.

Hypotheses $(\mathbf{H}_g)$, $(\mathbf{H}_f)$ imply that given $\varepsilon > 0$, we can find $C_{23} = C_{23}(\varepsilon) > 0$ such that

$$\lambda g(z,x)x + f(z,x)x \ge [f_x'(z,0) - \varepsilon]x^2 - C_{23}|x|^p \quad \text{for } a.a.z \in \Omega, \ \text{all } x \in \mathbb{R}. \tag{4.1}$$

Motivated by this unilateral growth estimate for the reaction of (P_λ) $(\lambda > 0)$, we consider the following auxiliary Dirichlet $(p,2)-$equation

$$\begin{cases} -\triangle_p u(z) - \triangle u(z) = [f_x'(z,0) - \varepsilon]u(z) \\ \qquad -C_{23}|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ u(z) = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Proposition 4.1. *For all $\varepsilon > 0$ small, problem* (4.2) *has a unique positive solution $u_* \in int\ C_+$, and (since problem* (4.2) *is odd), $v_* = -u_* \in -int\ C_+$ is the unique negative solution of* (4.2).

Proof. First we show that problem (4.2) has a positive solution. To this end, let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega \left[f_x'(z,0) - \varepsilon\right]\left(u^+(z)\right)^2 dz + \frac{C_{23}}{p}\left\|u^+\right\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have

$$\sigma_+(u) \geq \frac{1}{p}\|u\|_p^p - C_{24}\|u\|^2 \text{ for some } C_{24} > 0, \text{ all } u \in W_0^{1,p}(\Omega),$$

hence

$$\sigma_+(\cdot) \text{ is coercive}$$

(recall that $p > 2$). Also $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(u_*) = \inf\left\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.3}$$

Let $u = t\widehat{u}_1(2) \in int\ C_+$ with $0 < t < 1$. Then

$$\begin{aligned}\sigma_+(t\widehat{u}_1(2)) = {} & \frac{t^p}{p}\|D\widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega\left(\widehat{\lambda}_1(2) - f_x'(z,0)\right)(\widehat{u}_1(2))^2\,dz + \varepsilon\right] \\ & + \frac{C_{23}t^p}{p}\|\widehat{u}_1(2)\|_p^p,\end{aligned}$$

(recall that $\|\widehat{u}_1(2)\|_2 = 1$). Since $m \geq 2$ (see hypothesis $(\mathbf{H}_f)\,(iv)$), we have

$$\beta_0 = \int_\Omega\left[f_x'(z,0) - \widehat{\lambda}_1(2)\right](\widehat{u}_1(2))^2\,dz > 0.$$

So, if we choose $\varepsilon \in (0, \beta_0)$, then

$$\sigma_+(t\widehat{u}_1(2)) \leq C_{25}t^p - C_{26}t^2 \text{ for some } C_{25}, C_{26} > 0, \text{ all } 0 < t < 1.$$

But $p > 2$. So, choosing $t \in (0,1)$ small, we have

$$\sigma_+(t\widehat{u}_1(2)) < 0,$$

hence

$$\sigma_+(u_*) < 0 = \sigma_+(0)$$

(see (4.3)), therefore

$$u_* \neq 0.$$

From (4.3) it follows

$$\sigma_+'(u_*) = 0,$$

hence

$$\begin{aligned}\langle A_p(u_*), h\rangle + \langle A(u_*), h\rangle = {} & \int_\Omega\left[f_x'(z,0) - \varepsilon\right]u_*^+ h\,dz \\ & - C_{23}\int_\Omega\left(u_*^+\right)^{p-1} h\,dz \text{ for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{4.4}$$

In (4.4) we choose $h = -u_*^- \in W_0^{1,p}(\Omega)$. Then

$$\left\|Du_*^-\right\|_p^p \leq 0,$$

hence

$$u_* \geq 0,\ u_* \neq 0.$$

So, from (4.4) we infer that u_* is a positive solution of (4.2).

As before, the nonlinear regularity theory implies that $u_* \in C_+ \setminus \{0\}$.

Let $a:\mathbb{R}^N\to\mathbb{R}^N$ be defined by

$$a(y)=|y|^{p-2}y+y \text{ for all } y\in\mathbb{R}^N.$$

Since $p>2$, we deduce that $a\in C^1\left(\mathbb{R}^N,\mathbb{R}^N\right)$ and

$$\nabla a(y)=|y|^{p-2}\left[I+(p-2)\frac{y\otimes y}{|y|^2}\right]+I \text{ for all } y\neq 0.$$

Note that

$$\operatorname{div} a(Du)=\triangle_p u+\triangle u \text{ for all } u\in W_0^{1,p}(\Omega).$$

Moreover, we see that

$$(\nabla a(y)\xi,\xi)_{\mathbb{R}^N}\geq|\xi|^2>0 \text{ for all } y,\ \xi\in\mathbb{R}^N,\ \xi\neq 0.$$

Therefore, the tangency principle of Pucci-Serrin ([24], p. 35) implies that

$$u_*(z)>0 \text{ for all } z\in\Omega.$$

Then, the boundary point lemma of Pucci-Serrin ([24], p. 120) yields

$$u_*\in int\ C_+.$$

Next we show that this positive solution is unique. For this purpose, we introduce the integral functional $j:L^1(\Omega)\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ defined by

$$j(u)=\begin{cases}\frac{1}{p}\left\|Du^{\frac{1}{2}}\right\|_p^p+\frac{1}{2}\left\|Du^{\frac{1}{2}}\right\|_2^2 & \text{if } u\geq 0,\ u^{\frac{1}{2}}\in W_0^{1,p}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Invoking Lemma 1 of Diaz-Saa [10] (see also Lemma 5 of Benguria-Brezis-Lieb [5]) we see that $j(\cdot)$ is convex.

Let $\widehat{u}_*\in W_0^{1,p}(\Omega)$ be another positive solution of problem (4.2). Again we have

$$\widehat{u}_*\in int\ C_+.$$

Then $u_*^2,\ \widehat{u}_*^2\in dom\ j:=\left\{u\in L^1(\Omega):j(u)<+\infty\right\}$ (the effective domain of $j(\cdot)$). Given any $h\in C_0^1\left(\overline{\Omega}\right)$, for $|t|<1$ small, we have

$$u_*^2+th\in dom\ j,\ \widehat{u}_*^2+th\in dom\ j.$$

It is easily seen that $j(\cdot)$ is Gateaux differentiable at u_*^2 and at $\widehat{u}_*^2$ in the direction $h\in C_0^1\left(\overline{\Omega}\right)$. Using the chain rule and the nonlinear Green's identity (see, for example, Gasinski-Papageorgiou ([13], p. 210), we have

$$j'\left(u_*^2\right)(h)=\frac{1}{2}\int_\Omega\frac{-\triangle_p u_*-\triangle u_*}{u_*}hdz,$$

$$j'\left(\widehat{u}_*^2\right)(h)=\frac{1}{2}\int_\Omega\frac{-\triangle_p \widehat{u}_*-\triangle \widehat{u}_*}{\widehat{u}_*}hdz.$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$\begin{aligned}0&\leq\int_\Omega\left(\frac{-\triangle_p u_*-\triangle u_*}{u_*}-\frac{-\triangle_p \widehat{u}_*-\triangle \widehat{u}_*}{\widehat{u}_*}\right)\left(u_*^2-\widehat{u}_*^2\right)dz\\&=C_{23}\int_\Omega\left(\widehat{u}_*^{p-2}-u_*^{p-2}\right)\left(u_*^2-\widehat{u}_*^2\right)dz,\end{aligned}$$

therefore

$$u_* = \widehat{u}_*,$$

(since $p > 2$). This proves the uniqueness of the positive solution $u_* \in int\ C_+$ for problem (4.2) when $\varepsilon \in (0, \beta_0)$.

Since problem (4.2) is odd, we see that $v_* = -u_* \in -int\ C_+$ is the unique negative solution of (4.2). □

We introduce the following two sets:

$$\mathcal{S}_+(\lambda) = \{u : u \text{ is a positive solution of } (P_\lambda)\} \text{ when } \lambda \in (0, \lambda^*);$$

$$\mathcal{S}_-(\lambda) = \{u : u \text{ is a negative solution of } (P_\lambda)\} \text{ for all } \lambda > 0.$$

Propositions 3.5, 3.6 and their proofs imply that

$$\varnothing \neq \mathcal{S}_+(\lambda) \subseteq int\ C_+ \text{ and } \varnothing \neq \mathcal{S}_-(\lambda) \subseteq -int\ C_+.$$

Moreover, from Filippakis-Papageorgiou [12], we know that:

- $\mathcal{S}_+(\lambda)$ is downward directed (that is, if $u_1, u_2 \in \mathcal{S}_+(\lambda)$, then we can find $u \in \mathcal{S}_+(\lambda)$ such that $u \leq u_1$, $u \leq u_2$).
- $\mathcal{S}_-(\lambda)$ is upward directed (that is, if $v_1, v_2 \in \mathcal{S}_-(\lambda)$, then we can find $v \in \mathcal{S}_-(\lambda)$ such that $v_1 \leq v$, $v_2 \leq v$).

In what follows $u_* \in int\ C_+$ and $v_* \in -int\ C_+$ are the unique constant sign solutions of (4.2) produced in Proposition 4.1 for $\varepsilon \in (0, \beta_0)$.

Proposition 4.2. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold, then:*
(a) *for all* $\lambda \in (0, \lambda^*)$ *and all* $u \in \mathcal{S}_+(\lambda)$, *we have* $u_* \leq u$;
(b) *for all* $\lambda > 0$ *and all* $u \in \mathcal{S}_-(\lambda)$, *we have* $v \leq v_*$.

Proof. (a) Let $\lambda \in (0, \lambda^*)$ and let $u \in \mathcal{S}_+(\lambda)$. We consider the Catatheodory function $k_+(z, x)$ defined by

$$k_+(z,x) = \begin{cases} 0 & \text{if } \ x < 0, \\ [f'_x(z,0) - \varepsilon]\, x - C_{23} x^{p-1} & \text{if } \ 0 \leq x \leq u(z), \\ [f'_x(z,0) - \varepsilon]\, u(z) - C_{23} u(z)^{p-1} & \text{if } \ u(z) < x. \end{cases} \tag{4.5}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)\, ds$ and introduce the C^1-functional $\gamma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(\widehat{u}) = \frac{1}{p}\|D\widehat{u}\|_p^p + \frac{1}{2}\|D\widehat{u}\|_2^2 - \int_\Omega K_+(z, \widehat{u})\, dz \text{ for all } \widehat{u} \in W_0^{1,p}(\Omega).$$

From (4.5) it is clear that γ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So we can find $\widehat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\gamma_+(\widehat{u}_*) = \inf\left\{\gamma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.6}$$

As before (see the proof of Proposition 4.1), we have

$$\gamma_+(\widehat{u}_*) < 0 = \gamma_+(0),$$

hence

$$\widehat{u}_* \neq 0.$$

From (4.6) we have

$$\gamma'_+(\widehat{u}_*) = 0,$$

hence

$$\left\langle A_p\left(\widehat{u}_*\right),h\right\rangle+\left\langle A\left(\widehat{u}_*\right),h\right\rangle=\int_\Omega k_+\left(z,\widehat{u}_*\right)hdz \text{ for all } h\in W_0^{1,p}\left(\Omega\right). \tag{4.7}$$

In (4.7) first we choose $h=-\widehat{u}_*^-\in W_0^{1,p}\left(\Omega\right)$. Then

$$\left\|D\widehat{u}_*^-\right\|_p^p+\left\|D\widehat{u}_*^-\right\|_2^2=0,$$

(see (4.5)), hence

$$\widehat{u}_*\geq 0,\ \widehat{u}_*\neq 0.$$

Next in (4.7) we choose $h=\left(\widehat{u}_*-u\right)^+\in W_0^{1,p}\left(\Omega\right)$. Then

$$\begin{aligned}
&\left\langle A_p\left(\widehat{u}_*\right),\left(\widehat{u}_*-u\right)^+\right\rangle+\left\langle A\left(\widehat{u}_*\right),\left(\widehat{u}_*-u\right)^+\right\rangle\\
&=\int_\Omega\left(\left[f_x'\left(z,0\right)-\varepsilon\right]u-C_{23}u^{p-1}\right)\left(\widehat{u}_*-u\right)^+dz \text{ (see (4.5))}\\
&\leq\int_\Omega\left[\lambda g\left(z,u\right)+f\left(z,u\right)\right]\left(\widehat{u}_*-u\right)^+dz \text{ (see (4.8))}\\
&=\left\langle A_p\left(u\right),\left(\widehat{u}_*-u\right)^+\right\rangle+\left\langle A\left(u\right),\left(\widehat{u}_*-u\right)^+\right\rangle \text{ (since } u\in\mathcal{S}_+\left(\lambda\right)\text{)},
\end{aligned}$$

therefore

$$\widehat{u}_*\leq u.$$

So, we have proved that

$$\widehat{u}_*\in\left[0,u\right],\ \widehat{u}_*\neq 0. \tag{4.8}$$

From (4.5), (4.7) and (4.8), it follows that $\widehat{u}_*$ is a positive solution of (4.2), hence

$$\widehat{u}_*=u_*\in int\ C_+ \text{ (see Proposition 4.1)}$$

therefore

$$\widehat{u}_*\leq u \text{ for all } u\in\mathcal{S}_+\left(\lambda\right),$$

(see (4.8)).

(b) The argument is similar. Let $\lambda>0$ and $v\in\mathcal{S}_-\left(\lambda\right)$. We consider the Catatheodory function $k_-\left(z,x\right)$ defined by

$$k_-\left(z,x\right)=\begin{cases}\left[f_x'\left(z,0\right)-\varepsilon\right]v\left(z\right)-C_{23}\left|v\left(z\right)\right|^{p-2}v\left(z\right) & \text{if } x<v\left(z\right),\\ \left[f_x'\left(z,0\right)-\varepsilon\right]x-C_{23}\left|x\right|^{p-2}x & \text{if } v\left(z\right)\leq x\leq 0,\\ 0 \quad \text{if } 0<x.\end{cases} \tag{4.9}$$

We set $K_-\left(z,x\right)=\int_0^x k_-\left(z,s\right)ds$ and introduce the C^1-functional $\gamma_-:W_0^{1,p}\left(\Omega\right)\to\mathbb{R}$ defined by

$$\gamma_-\left(\widehat{v}\right)=\frac{1}{p}\left\|D\widehat{v}\right\|_p^p+\frac{1}{2}\left\|D\widehat{v}\right\|_2^2-\int_\Omega K_-\left(z,\widehat{v}\right)dz \text{ for all } \widehat{v}\in W_0^{1,p}\left(\Omega\right).$$

As in part (a), via the direct method of calculus of variations, we show that

$$v\leq\widehat{v}_* \text{ for all } v\in\mathcal{S}_-\left(\lambda\right),\ \text{all } \lambda\geq 0.$$

$\square$

Now we are ready to produce extremal constant sign solutions for problem $\left(P_\lambda\right)$.

Proposition 4.3. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold, then:*

(a) *for every* $\lambda \in (0, \lambda^*)$ *we can find* $u_+ \in \mathcal{S}_+(\lambda) \subseteq int\ C_+$ *such that* $u_+ \leq u$ *for all* $u \in \mathcal{S}_+(\lambda)$;

(b) *for every* $\lambda > 0$ *there exists* $v_- \in \mathcal{S}_-(\lambda) \subseteq -int\ C_+$ *such that* $v \leq v_-$ *for all* $v \in \mathcal{S}_-(\lambda)$.

Proof. (a) Invoking Lemma 3.10 of Hu-Papageorgiou ([16], p.178) we can find $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_+(\lambda) \subseteq int\ C_+$ decreasing (recall that $\mathcal{S}_+(\lambda)$ is downward directed) such that

$$\inf \mathcal{S}_+(\lambda) = \inf_{n\geq 1} u_n.$$

For every $n \in \mathbb{N}$ we have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega [\lambda g(z, u_n) + f(z, u_n)]\, hdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \tag{4.10}$$

and

$$0 \leq u_n \leq u_1. \tag{4.11}$$

If in (4.9) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (4.11), then

$$\|Du_n\|_p^p + \|Du_n\|_2^2 \leq (\lambda + 1) C_{27}, \text{ for some } C_{27} > 0, \text{ all } n \in \mathbb{N},$$

hence

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Hence, we may assume that

$$u_n \xrightarrow{w} u_+ \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_+ \text{ in } L^p(\Omega) \text{ as n} \to \infty. \tag{4.12}$$

In (4.9) we choose $h = u_n - u_+ \in W_0^{1,p}(\Omega)$, pass to the limit as n$\to \infty$ and use (4.12). Then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u_+\rangle + \langle A(u_n), u_n - u_+\rangle] = 0,$$

hence

$$\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u_+\rangle + \langle A(u_+), u_n - u_+\rangle] \leq 0,$$

(since $A(\cdot)$ is monotone), therefore

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u_+\rangle \leq 0 \text{ (see (4.12))},$$

and it follows that

$$u_n \longrightarrow u_+ \text{ in } W_0^{1,p}(\Omega), \tag{4.13}$$

(see Proposition 2.2). From Proposition 4.2 we have

$$u_* \leq u_n \text{ for all } n \in \mathbb{N},$$

therefore

$$u_* \leq u_+. \tag{4.14}$$

Passing to the limit as $n \to \infty$ in (4.9) and using (4.13) and (4.14), we conclude that

$$u_+ \in \mathcal{S}_+(\lambda) \text{ and } u_+ = \inf \mathcal{S}_+(\lambda).$$

(b) Similarly, using this time an increasing sequence

$$\{v_n\}_{n\geq 1} \subseteq \mathcal{S}_-(\lambda) \subseteq -int\ C_+,$$

(recall hat $\mathcal{S}_-(\lambda)$ is upward directed) such that

$$\sup \mathcal{S}_-(\lambda) = \sup_{n\geq 1} v_n.$$

Then, as in part (a), and since

$$v_n \leq v_* \in -int\ C_+,$$

we generate $v_- \in \mathcal{S}_-(\lambda) \subseteq -int\ C_+$, such that $v \leq v_-$ for all $v \in \mathcal{S}_-(\lambda)$. ☐

5. Nodal solutions

In this section, using the two extremal constant sign solutions $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ produced in Proposition 4.3, we can generate nodal solutions for problem (P_λ) $(\lambda \in (0, \lambda^*))$.

Proposition 5.1. *If hypotheses* $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ *hold and* $\lambda \in (0,\lambda^*)$, *then problem* (P_λ) *has a nodal solution*

$$y_0 \in int_{C_0^1(\overline{\Omega})}[v_-, u_+].$$

Proof. Let $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ be the two extremal constant sign solutions produced in Proposition 4.3. Let $\widehat{\beta}_\lambda(z,x)$ be the Caratheodory function defined by

$$\widehat{\beta}_\lambda(z,x) = \begin{cases} \lambda g(z, v_-(z)) + f(z, v_-(z)) & \text{if } \ x < v_-(z), \\ \lambda g(z,x) + f(z,x) & \text{if } \ v_-(z) \leq x \leq u_+(z), \\ \lambda g(z, u_+(z)) + f(z, u_+(z)) & \text{if } \ u_+(z) < x. \end{cases} \tag{5.1}$$

Also, we consider the positive and the negative truncations of $\widehat{\beta}_\lambda(z,\cdot)$, namely the Carathéodory functions $\widehat{\beta}_\lambda^\pm(z,x)$ defined by

$$\widehat{\beta}_\lambda^\pm(z,x) = \widehat{\beta}_\lambda\left(z, x^\pm\right) \text{ for all } (z,x) \in \Omega \times \mathbb{R}. \tag{5.2}$$

We set $\widehat{B}_\lambda(z,x) = \int_0^x \widehat{\beta}_\lambda(z,x)\,ds$ and $\widehat{B}_\lambda^\pm(z,x) = \int_0^x \widehat{\beta}_\lambda^\pm(z,x)\,ds$ and introduce the C^1-functionals $\widehat{\psi}_\lambda$, $\widehat{\psi}_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{B}_\lambda(z, u(z))\,dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

$$\widehat{\psi}_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{B}_\lambda^\pm(z, u(z))\,dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: $K_{\widehat{\psi}_\lambda} \subseteq [v_-, u_+] \cap C_0^1\left(\overline{\Omega}\right)$, $K_{\widehat{\psi}_\lambda^+} = \{0, u_+\}$, $K_{\widehat{\psi}_\lambda^-} = \{v_-, 0\}$.

Let $u \in K_{\widehat{\psi}_\lambda}$. Then

$$\langle A_p(u), h\rangle + \langle A(u), h\rangle = \int_\Omega \widehat{\beta}_\lambda(z,x)\, h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{5.3}$$

In (5.3) first we choose $h = (u - u_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\left\langle A_p(u), (u-u_+)^+\right\rangle + \left\langle A(u), (u-u_+)^+\right\rangle \\
&= \int_\Omega \left[\lambda g(z, u_+(z)) + f(z, u_+(z))\right](u-u_+)^+ \ \text{ (see (5.1))} \\
&= \left\langle A_p(u_+), (u-u_+)^+\right\rangle + \left\langle A(u_+), (u-u_+)^+\right\rangle \ \text{ (since } u_+ \in \mathcal{S}_+(\lambda)\text{)},
\end{aligned}$$

hence

$$u \leq u_+.$$

Similarly, if in (5.3) we choose $h = (v_- - u)^+ \in W_0^{1,p}(\Omega)$, then we obtain

$$v_- \leq u.$$

So, we have proved that $u \in [v_-, u_+]$. Moreover, the nonlinear regularity theory (see Lieberman [17], Theorem 1), implies that

$$u \in [v_-, u_+] \cap C_0^1\left(\overline{\Omega}\right).$$

So, we have

$$K_{\widehat{\psi}_\lambda} \subseteq [v_-, u_+] \cap C_0^1\left(\overline{\Omega}\right).$$

In a similar fashion, we also show that

$$K_{\widehat{\psi}_\lambda^+} \subseteq [0, u_+] \cap C_+, \ K_{\widehat{\psi}_\lambda^-} = [v_-, 0] \cap (-C_+).$$

Then, the extremality of the solutions u_+ and v_- implies that

$$K_{\widehat{\psi}_\lambda^+} = \{0, u_+\}, \ K_{\widehat{\psi}_\lambda^-} = \{v_-, 0\}.$$

This proves Claim 1.

Claim 2: $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ are local minimizers of $\widehat{\psi}_\lambda$.

From (5.1) and (5.2) it is clear that $\widehat{\psi}_\lambda^+$ is coercive. Also it is sequentially weakly lower semicontinuous. So there exists $\widehat{u}_+ \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\lambda^+(\widehat{u}_+) = \inf\left\{\widehat{\psi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{5.4}$$

Recall that $\widehat{u}_+ \in int\ C_+$. So, we can find $t \in (0,1)$ small such that $t\widehat{u}_1(2) \leq u_+$ (see Marano-Papageorgiou [19], Proposition 2.1). On account of hypotheses $(\mathbf{H}_g)(iii)$, $(\mathbf{H}_f)(iv)$, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$G(z,x) \geq -\frac{\varepsilon}{2}x^2 \text{ for } a.a. z \in \Omega, \text{ all } |x| \leq \delta, \tag{5.5}$$

$$F(z,x) \geq -\frac{1}{2}\left[f_x'(z,0) - \varepsilon\right]x^2 \text{ for } a.a. z \in \Omega, \text{ all } |x| \leq \delta. \tag{5.6}$$

Then for $t \in (0,1)$ small, we have

$$\widehat{\psi}_\lambda^+(t\widehat{u}_1(2)) = \frac{t^p}{p}\|D\widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega \left(\widehat{\lambda}_1(2) - f_x'(z,0)\right)\widehat{u}_1(2)^2 dz + 2\varepsilon\right]$$

(see (5.5) and (5.6) and recall that $\|\widehat{u}_1(2)\|_2 = 1$), hence

$$\widehat{\psi}_\lambda^+ (t\widehat{u}_1(2)) < 0 = \widehat{\psi}_\lambda^+(0) \text{ for } t \in (0,1) \text{ small}$$

(choose $\varepsilon > 0$ small, see $(\mathbf{H}_f)(iv)$, Lemma 2.6 and recall that $2 < p$). Hence

$$\widehat{\psi}_\lambda^+(\widehat{u}_+) < 0 = \widehat{\psi}_\lambda^+(0) \text{ (see (5.4))}$$

therefore

$$\widehat{u}_+ \neq 0. \tag{5.7}$$

From (5.4) we have

$$\widehat{u}_+ \in K_{\widehat{\psi}_\lambda^+},$$

hence

$$\widehat{u}_+ = u_+$$

(see Claim 1 and (5.7)).

Since

$$\widehat{\psi}_\lambda|_{C_+} = \widehat{\psi}_\lambda^+|_{C_+},$$

(see (5.2)), it follows that $u_+ \in int\ C_+$ is a $C_0^1(\overline{\Omega})-$local minimizer of $\widehat{\psi}_\lambda$, hence $u_+ \in int\ C_+$ is a $W_0^{1,p}(\Omega)-$local minimizer of $\widehat{\psi}_\lambda$ (see Proposition 2.3).

Similarly for $v_- \in -int\ C_+$, working this time with the functional $\widehat{\psi}_\lambda^-$. This proves Claim 2.

Without any loss of generality, we may assume that

$$\widehat{\psi}_\lambda(v_-) \le \widehat{\psi}_\lambda(u_+). \tag{5.8}$$

The reasoning is similar if the opposite inequality holds.

We assume that $K_{\widehat{\psi}_\lambda}$ is finite. Otherwise, on account of Claim 1 and the extremality of u_+ and v_-, we already have an infinity of nodal solutions, and so we are done. Then, using Claim 2 we see that we can find $\rho \in (0,1)$ small such that

$$\begin{aligned} &\widehat{\psi}_\lambda(v_-) \le \widehat{\psi}_\lambda(u_+) < \inf\left\{\widehat{\psi}_\lambda(u) : \|u - u_+\| = \rho\right\} = \widehat{m}_\rho^+, \\ &\|v_- - u_+\| > \rho, \end{aligned} \tag{5.9}$$

(see (5.8) and Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

From (5.1) it follows that $\widehat{\psi}_\lambda$ is coercive. Therefore

$$\widehat{\psi}_\lambda \text{ satisfies the } C-\text{condition}, \tag{5.10}$$

(see Section 2). Then (5.9), (5.10) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\psi}_\lambda} \subseteq [v_-, u_+] \cap C_0^1(\overline{\Omega}) \text{ (see Claim 1)}, \ \widehat{m}_\rho^+ \le \widehat{\psi}_\lambda(y_0). \tag{5.11}$$

From (5.9) and (5.11) it follows that

$$y_0 \notin \{v_-, u_+\}. \tag{5.12}$$

So, if we can show that $y_0 \neq 0$, then y_0 will be nodal (see (5.11), (5.12) and recall the extremality of u_+ and v_-). Since y_0 is a critical point of $\widehat{\psi}_\lambda$ of mountain pass type, we have

$$C_1\left(\widehat{\psi}_\lambda, y_0\right) \neq 0 \tag{5.13}$$

(see Motreanu-Motreanu-Papageorgiou [20], p.176).

Consider the the C^1-functional $\widehat{\psi}_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0)\, u^2 dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We introduce the homotopy

$$h(t,u) = t\widehat{\psi}_\lambda(u) + (1-t)\widehat{\psi}_0(u) \text{ for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \text{ in } [0,1],\ u_n \to 0 \text{ in } W_0^{1,p}(\Omega),\ h_u'(t_n,u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{5.14}$$

From the equality in (5.14) we have

$$\begin{array}{c}\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = t_n \int_\Omega \widehat{\beta}_\lambda(z,u_n)\, h dz \\ +(1-t_n)\int_\Omega f_x'(z,0)\, u_n h dz \text{ for all } h \in W_0^{1,p}(\Omega),\ \text{all } n \in \mathbb{N}.\end{array} \tag{5.15}$$

hence

(5.16)

$$\left\{\begin{array}{l} -\triangle_p u_n(z) - \triangle u_n(z) = t_n\widehat{\beta}_\lambda(z,u_n(z)) + (1-t_n) f_x'(z,0)\, u_n(z) \text{ for a.a.} z \in \Omega, \\ u_n(z) = 0 \text{ on } \partial\Omega. \end{array}\right.$$

By (5.14), (5.16) and Corollary 8.6, p.208 of Motreanu-Motreanu-Papageorgiou [20], there exists $M_3 > 0$ such that

$$\|u_n\|_\infty \leq M_3 \text{ for all } n \in \mathbb{N}.$$

Applying Theorem 1 of Lieberman [17], we can find $\alpha \in (0,1)$ and $M_4 > 0$ such that

$$u_n \in C_0^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_4 \text{ for all } n \in \mathbb{N}. \tag{5.17}$$

From (5.17), (5.14) and the compact embedding of $C_0^{1,\alpha}\left(\overline{\Omega}\right)$ into $C_0^1\left(\overline{\Omega}\right)$, we have

$$u_n \to 0 \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ and so } u_n \in [v_-, u_+] \text{ for all } n \geq n_0. \tag{5.18}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so by passing to a subsequence if necessary we may assume

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as n} \to \infty. \tag{5.19}$$

From (5.15), we have

$$\begin{array}{c}\|u_n\|^{p-2}\langle A_p(y_n), h\rangle + \langle A(y_n), h\rangle = t_n \int_\Omega \frac{\widehat{\beta}_\lambda(z,u_n(z))}{\|u_n\|} h dz \\ +(1-t_n)\int_\Omega f_x'(z,0)\, y_n h dz \text{ for all } h \in W_0^{1,p}(\Omega),\ \text{all } n \in \mathbb{N},\end{array} \tag{5.20}$$

hence, for all $n \in \mathbb{N}$,

$$\begin{cases} -\|u_n\|^{p-2} \triangle_p y_n(z) - \triangle y_n(z) = t_n \frac{\widehat{\beta}_\lambda(z,u_n(z))}{\|u_n\|} \\ \qquad\qquad + (1-t_n) f'_x(z,0)\, y_n(z) \text{ for a.a.} z \in \Omega, \\ y_n(z) = 0 \text{ on } \partial\Omega. \end{cases}$$

We know that $y_n \in C_0^1(\overline{\Omega})$ for all $n \in \mathbb{N}$. Also, from (5.19) and Corollary 8.6, p. 108, of Motreanu-Motreanu-Papageorgiou [20], we have

$$\|y_n\|_\infty \le M_4 \text{ for some } M_5 > 0, \text{ all } n \in \mathbb{N}.$$

Invoking the nonlinear regularity theory of Lieberman ([18], p.320), we can find $\alpha \in (0,1)$ and $M_6 > 0$ such that

$$y_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } \|y_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le M_6 \text{ for all } n \in \mathbb{N}.$$

As before, using the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (5.19), we have

$$y_n \to y \text{ in } C_0^1(\overline{\Omega}). \tag{5.21}$$

Hypotheses $(\mathbf{H}_g)(iii)$, $(\mathbf{H}_f)(iii)$ and (5.1) imply that

$$\left|\widehat{\beta}_\lambda(z,x)\right| \le C_{28}|x| \text{ for } a.a.z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } C_{28} > 0,$$

hence

$$\left\{\frac{\widehat{\beta}_\lambda(\cdot,u_n(\cdot))}{\|u_n\|}\right\}_{n\ge1} \subseteq L^{p'}(\Omega) \text{ is bounded,}$$

(see (5.19) and recall that $2 < p$). So, by passing to a subsequence if necessary and using (5.18) and hypotheses $(\mathbf{H}_g)(iii)$, $(\mathbf{H}_f)(iii)$, we obtain

$$\frac{\widehat{\beta}_\lambda(\cdot,u_n(\cdot))}{\|u_n\|} \xrightarrow{w} f'_x(\cdot,0)\, y \text{ in } L^{p'}(\Omega) \text{ as n} \to \infty, \tag{5.22}$$

(see Aizicovici-Papageorgiou-Staicu ([1], proof of Proposition 14).

In (5.19) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (5.18), (5.21), (5.22). Then we obtain

$$\langle A(y), h\rangle = \int_\Omega f'_x(z,0)\, y dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

hence

$$-\triangle y(z) = f'_x(z,0)\, y(z) \text{ for } a.a.z \in \Omega, \ y\,|_{\partial\Omega} = 0. \tag{5.23}$$

From hypothesis $(\mathbf{H}_f)(iii)$ and Lemma 2.5, we have

$$\begin{gathered} \widetilde{\lambda}_m(2, f'_x(\cdot,0)) < \widetilde{\lambda}_m\left(2, \widehat{\lambda}_m(2)\right) = 1 \text{ and} \\ 1 = \widetilde{\lambda}_{m+1}\left(2, \widehat{\lambda}_{m+1}(2)\right) < \widetilde{\lambda}_{m+1}(2, f'_x(.,0)). \end{gathered} \tag{5.24}$$

From (5.23) and (5.24) it follows that

$$y = 0.$$

But from (5.21) we see that $\|y\| = 1$, a contradiction. We conclude that (5.14) cannot occur. Therefore, using the homotopy invariance property of critical groups (see, for example Gasinski-Papageorgiou [14], Theorem 5.125, p. 836), we obtain

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = C_k\left(\widehat{\psi}_0, 0\right) \text{ for all } k \in \mathbb{N}_0. \tag{5.25}$$

Let $\overline{\psi}_0 : H_0^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\overline{\psi}_0(u) = \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0)\, u^2 dz \text{ for all } u \in H_0^1(\Omega).$$

Hypothesis $(\mathbf{H}_f)(iii)$, through Lemma 2.6 implies that $u = 0$ is a nondegenerate critical point of $\overline{\psi}_0$ with Morse index

$$d_m = \dim \bigoplus_{k=1}^m E\left(\widehat{\lambda}_k(2)\right) \geq 2.$$

Hence we have

$$C_k\left(\overline{\psi}_0, 0\right) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \tag{5.26}$$

(see Motreanu-Motreanu-Papageorgiou [20], Theorem 6.51, p. 135).

Let $\widetilde{\psi}_0 = \overline{\psi}_0 \mid_{W_0^{1,p}(\Omega)}$ (recall that $2 < p$). Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, we have

$$C_k\left(\widetilde{\psi}_0, 0\right) = C_k\left(\overline{\psi}_0, 0\right) \text{ for all } k \in \mathbb{N}_0,$$

hence

$$C_k\left(\widetilde{\psi}_0, 0\right) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (5.26))}. \tag{5.27}$$

Note that

$$\left|\widehat{\psi}_0(u) - \widetilde{\psi}_0(u)\right| = \frac{1}{p}\|Du\|_p^p, \tag{5.28}$$

hence

$$\left|\left\langle \widehat{\psi}_0'(u) - \widetilde{\psi}_0'(u), h\right\rangle\right| = |\langle A_p(u), h\rangle| \leq \|Du\|_p^{p-1}\|h\|$$
$$\text{for all } h \in W_0^{1,p}(\Omega),$$

therefore

$$\left\|\widehat{\psi}_0'(u) - \widetilde{\psi}_0'(u)\right\|_{W^{-1,p'}} \leq \|u\|^{p-1}. \tag{5.29}$$

From (5.28), (5.29) and the C^1-continuity of critical groups (see, for example, Gasinski-Papageorgiou [14], Theorem 5.126, p. 836), we have

$$C_k\left(\widehat{\psi}_0, 0\right) = C_k\left(\widetilde{\psi}_0, 0\right) \text{ for all } k \in \mathbb{N}_0,$$

hence

$$C_k\left(\widehat{\psi}_0, 0\right) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (5.27))},$$

therefore

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (5.25))}. \tag{5.30}$$

Comparing (5.30) and (5.13) we infer that

$$y_0 \neq 0.$$

So, $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is a nodal solution of (P_λ) for $\lambda \in (0,\lambda^*)$ (see (5.12)).

Recall that

$$y_0 \in [v_-, u_+] \cap C_0^1\left(\overline{\Omega}\right).$$

As before (see the proof of Proposition 4.1), via the tangency principle of Pucci-Serrin ([24], p.35), we have

$$v_-(z) < y_0(z) < u_+(z) \text{ for all } z \in \Omega. \tag{5.31}$$

Let $\rho = \max\{\|v_-\|_\infty, \|u_+\|_\infty\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $(\mathbf{H}_f)(v)$. For $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$, we have

$$\begin{cases} -\triangle_p y_0(z) - \triangle y_0(z) + \widetilde{\xi}_\rho |y_0(z)|^{p-2} y_0(z) \\ = \lambda g(z, y_0(z)) + f(z, y_0(z)) + \widehat{\xi}_\rho |y_0(z)|^{p-2} y_0(z) \\ \qquad\qquad + \left(\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right) |y_0(z)|^{p-2} y_0(z) \\ \leq \lambda g(z, u_+(z)) + f(z, u_+(z)) + \widehat{\xi}_\rho u_+(z)^{p-1} \\ \qquad\qquad + \left(\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right) u_+(z)^{p-1} (\text{since } y_0 \leq u_+) \\ = -\triangle_p u_+(z) - \triangle u_+(z) + \widetilde{\xi}_\rho u_+(z)^{p-1} \text{ for a.a.} z \in \Omega. \end{cases} \tag{5.32}$$

On account of (5.31), we can say that

$$\left(\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right) |y_0|^{p-2} y_0 \preceq \left(\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right) u_+^{p-1}.$$

Then from (5.32) and Proposition 2.4 of Filippakis-O'Regan-Papageorgiou [11], we have

$$u_+ - y_0 \in int\ C_+.$$

In a similar fashion we show that

$$y_0 - v_- \in int\ C_+.$$

Therefore we conclude that

$$y_0 \in int_{C_0^1(\overline{\Omega})} [v_-, u_+].$$

□

So, we can state our first multiplicity result for problem (P_λ).

Theorem 5.2. *If hypotheses $(\mathbf{H}_g)$, $(\mathbf{H}_f)$, $(\mathbf{H}_0)$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$, problem (P_λ) has at least four nontrivial solutions*

$$u_0,\ \widehat{u} \in int\ C_+,\ v_0 \in -int\ C_+ \textit{ and } y_0 \in C_0^1\left(\overline{\Omega}\right),\ \textit{nodal.}$$

If we strenghten the conditions on the functions $g(z,\cdot)$ and $f(z,\cdot)$ we can produce a second nodal solution, for a total of five nontrivial solutions to problem (P_λ) for $\lambda \in (0,\lambda^*)$.

So, we introduce the following extra condition:

$(\mathbf{H}_1)$: for a.a.$z \in \Omega$, $g(z,\cdot)$, $f(z,\cdot) \in C^1(\mathbb{R})$ and hypotheses $(\mathbf{H}_g)(i)$, $(\mathbf{H}_f)(i)$ are replaced by

$$\left|g_x'(z,x)\right|, \ \left|f_x'(z,x)\right| \le a(z)\left[1+|x|^{r-2}\right] \text{ for } a.a.z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)$, $p \le r < p^*$.

Proposition 5.3. *If hypotheses* $(\mathbf{H}_g)(ii),(iii)$, $(\mathbf{H}_f)(ii)-(iv)$, $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *hold and* $\lambda \in (0,\lambda^*)$, *then problem* (P_λ) *has a second nodal solution* $\widehat{y} \in int_{C_0^1(\overline{\Omega})}[v_-,u_+]$.

Proof. Let $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ be the two extremal constant sign solutions (see Proposition 4.3). From Proposition 5.1 it follows that there exists a nodal solution

$$y_0 \in int_{C_0^1(\overline{\Omega})}[v_-,u_+]. \tag{5.33}$$

We know that $y_0 \in K_{\widehat{\psi}_\lambda}$ and is of mountain pass type. Then by hypothesis $(\mathbf{H}_1)$, (5.33), and Aizicovici-Papageorgiou-Staicu ([3], proof of Theorem 3) we infer that

$$C_k\left(\widehat{\psi}_\lambda, y_0\right) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{5.34}$$

Recall that $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ are local minimizers of $\widehat{\psi}_\lambda$ (see the proof of Proposition 5.1, Claim 2). Therefore

$$C_k\left(\widehat{\psi}_\lambda, u_+\right) = C_k\left(\widehat{\psi}_\lambda, v_-\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{5.35}$$

From (5.30) we have

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{5.36}$$

Finaly recall that $\widehat{\psi}_\lambda$ is coercive. Therefore

$$C_k\left(\widehat{\psi}_\lambda, \infty\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \tag{5.37}$$

(see Motreanu-Motreanu-Papageorgiou [20], p.161).

Suppose that $K_{\widehat{\psi}_\lambda} = \{y_0, u_+, v_-, 0\}$. Then from (5.34), (5.35), (5.36), (5.37) and the Morse relation (see (2.7)) with $t=-1$, we have

$$(-1)^1 + 2(-1)^0 + (-1)^{d_m} = (-1)^0,$$

hence $(-1)^{d_m} = 0$, a contradiction. Therefore there exists $\widehat{y} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{y} \notin \{y_0, u_+, v_-, 0\}, \ \widehat{y} \in K_{\widehat{\psi}_\lambda} \subseteq [v_-,u_+] \cap C_0^1\left(,\overline{\Omega}\right)$$

hence $\widehat{y}$ is the second nodal solution of (P_λ). Moreover, as in the proof of Proposition 5.1, using the strong comparison principle from [24], we have

$$\widehat{y} \in int_{C_0^1(\overline{\Omega})}[v_-,u_+].$$

□

So, we can state our second multiplicity theorem for problem (P_λ), where $\lambda \in (0,\lambda^*)$:

Theorem 5.4. *If hypotheses* $(\mathbf{H}_g)\,(ii)\,,(iii)\,,\ (\mathbf{H}_f)\,(ii)-(iv)\,,\ (\mathbf{H}_0)\,,\ (\mathbf{H}_1)$ *hold, then there exists* $\lambda^*>0$ *such that for all* $\lambda\in(0,\lambda^*)\,,$ *problem* (P_λ) *has at least five nontrivial solutions*

$$u_0,\ \widehat{u}\in int\ C_+,\ v_0\in -int\ C_+,\ y_0,\ \widehat{y},\ nodal.$$

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