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A SOBOLEV TYPE EMBEDDING THEOREM FOR BESOV SPACES DEFINED ON DOUBLING METRIC SPACES

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ABSTRACT. We obtain a Sobolev type embedding result for Besov spaces defined on a doubling measure metric space.

1. INTRODUCTION

The classical theory of Besov spaces has been recently extended to the setting of metric spaces and various results from the classical theory have their abstract variants, however abstract versions of the Sobolev embedding theorem are only available for $Q-$regular metric spaces. The purpose of this paper is to obtain a Sobolev type embedding result for the Besov spaces defined on doubling metric spaces.

There are several equivalent ways to define Besov spaces in the setting of a doubling metric space (see for example [9], [10], [13], [14], [30], [31], [39] and the references therein), in this paper, we use the approach based on a generalization of the classical the L^p-modulus of smoothness introduced in [9].

Assume that $\Omega = (\Omega, d)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ, for which the measure of every ball is positive and finite. Given $t > 0$, $0 < p < \infty$ and $f \in L^p_{loc}(\Omega)$, L^p-modulus of smoothness is defined by

$$E_p(f,t) = \left(\int_\Omega \left(⨍_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right) d\mu(x) \right)^{1/p},$$

where $⨍_B f(x) d\mu(x) := \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$.

For $0 < s < \infty$, the homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ consists of functions $f \in L^p_{loc}(\Omega)$ for which the seminorm

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(\frac{E_p(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{-s} E_p(f,t), & q = \infty, \end{cases}$$

is finite.

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Remark 1. *(See [9]) $E_p(f,t)$ is equivalent to the classical $L^p(\mathbb{R}^n)-$modulus of smoothness of a function $f \in L^p_{loc}(\mathbb{R}^n)$, indeed*

$$\begin{aligned} E_p(f,t) &= \left(\int_{\mathbb{R}^n} \left(\fint_{B(x,t)} |f(x)-f(y)|^p \, dy \right) dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} \left(\fint_{B(0,t)} |f(x+h)-f(x)|^p \, dh \right) dx \right)^{1/p} \\ &\simeq \sup_{|h|\le t} \|f(x+h)-f(x)\|_{L^p(\mathbb{R}^n)} := \omega_p(f,t), \end{aligned}$$

therefore, $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)$ coincides with the classical Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$.

In the Euclidean setting, the Sobolev embedding theorem states that (see for example [33]) if $0 < s < \frac{n}{p}$, then

$$\|f\|_{L^{p^*,q}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)},$$

where $p^* = np/(n-sp)$, and the Lorentz space $L^{p,q}(\mathbb{R}^n)$, consists of measurable functions f of finite norm

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{L^q([0,\infty))},$$

(f^* denotes the decreasing rearrangement of f, see section 2.1 below).

The abstract variant for metric spaces is just known in the following particular case (see [9] and [14]):

Theorem 2. *Let Ω be a $Q-$regular metric space, i.e. there exists $Q \ge 1$ and constant $c_Q \ge 1$ such that*

$$c_Q^{-1} r^Q \le \mu(B(x,r)) \le c_Q r^Q$$

for each $x \in X$, and for all $0 < r < diam\ \Omega$ (here diam Ω is the diameter of Ω). Suppose that $0 < s < 1$ and $1 \le q \le \infty$. If $1 < p < Q/s$ and Ω supports a $(1,p)-$Poincaré inequality ([9, Thm. 5.1]*) or, $0 < p < Q/s$, $0 < q \le \infty$ and Ω is geodesic (*[14, Thm. 4.4]*), then*

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset L^{p(Q),q}(\Omega)$$

where $p(Q) = Qp/(Q-sp)$.

The purpose of this paper is to obtain a Sobolev type embedding result for the Besov spaces defined on a doubling metric space. This will be done by obtaining pointwise estimates between the special difference[1] $f^{**}_\mu(t) - f^*_\mu(t)$ (called the oscillation of f^*_μ) and the $X-$modulus of smoothness defined by

$$E_X(f,r) := \left\| \fint_{B(x,r)} |f(x)-f(y)| \, d\mu(y) \right\|_X .$$

(here f^*_μ is the decreasing rearrangement of f, $f^{**}_\mu(t) = \frac{1}{t}\int_0^t f^*_\mu(s)ds$, for all $t > 0$ and X is a rearrangement invariant space on Ω, see sections 2 and 3).

The paper is organized as follows. In Section 2, we introduce the notation and the standard assumptions used in the paper. In Section 3, we will see that a

[1] Estimates of this type are very powerful and arise in connection with embeddings of Sobolev type (see [2],[19], [22],[23],[24], [25], [27],).

Sobolev type embedding of $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ into a rearrangement invariant space X implies a lower bound for the measure of the balls. We introduce the notion of $X-$modulus of smoothness and prove some estimates of the oscillation of f^*_μ in terms of the $X-$modulus. In Section 4, we define general Besov type spaces on doubling measure metric spaces and use oscillation inequalities obtained in the previous sections to derive embedding Sobolev theorems for our generalized Besov spaces. In Section 5 we obtain generalized uncertainty Sobolev inequalities in the context of Besov spaces. In Section 6 we obtain a criteria for essential continuity and the embedding into BMO. Finally in Section 7, we consider in detail the case $\dot{\mathcal{B}}^s_{p,q}(\Omega)$.

Throughout the paper, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\,f$, while the symbol $f \preceq g$ means that $f \leq c\,g$.

2. NOTATION AND PRELIMINARIES

A measure metric space (Ω, d, μ) will be a separable metric space (Ω, d) equipped with a Borel measure μ. We start with some definitions.

2.1. **Background on Rearrangement Invariant Spaces.** For measurable functions $f : \Omega \to \mathbb{R}$, the distribution function of f is given by

$$\mu_f(t) = \mu\{x \in \Omega : |f(x)| > t\} \qquad (t > 0).$$

The **decreasing rearrangement** f^*_μ of f is the right-continuous non-increasing function from $[0, \infty)$ into $[0, \infty]$ which is equimeasurable with f. Namely,

$$f^*_\mu(s) = \inf\{t \geq 0 : \mu_f(t) \leq s\}.$$

It is easy to see that for any measurable set $E \subset \Omega$

$$\int_E |f(x)|\, d\mu \leq \int_0^{\mu(E)} f^*_\mu(s) ds. \tag{1}$$

In fact, the following stronger property holds (cf. [3]),

$$\sup_{\mu(E)=t} \int_E |f(x)|\, d\mu = \int_0^t f^*_\mu(s) ds. \tag{2}$$

Since f^*_μ is decreasing, the function f^{**}_μ, defined by

$$f^{**}_\mu(t) = \frac{1}{t}\int_0^t f^*_\mu(s) ds,$$

is also decreasing and, moreover,

$$f^*_\mu \leq f^{**}_\mu.$$

The oscillation of f is defined by

$$O_\mu(f, t) := f^{**}_\mu(t) - f^*_\mu(t), \quad 0 < t < \mu(\Omega).$$

Remark 3. *An elementary computation shows that*

$$\frac{\partial}{\partial t} f^{**}_\mu(t) = -\frac{O_\mu(f,t)}{t}$$

and that the function $t \to tO_\mu(f,t)$ in increasing.

Conditions like $f^*_\mu(\infty) = 0$ will appear often. The following result (see [21, Proposition 2.1]) clarifies the meaning of such equality.

Proposition 4. *If $\mu(\Omega)=\infty$, then $f_{\mu}^{*}(\infty)=0$ if, and only if, $\mu_f(t)$ is finite for any $t>0$.*

We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces and refer the reader to [3], [20], for a complete treatment. We say that a Banach function space $X=X(\Omega)$ on (Ω,d,μ) is rearrangement-invariant (r.i.) space, if $g\in X$ implies that all $\mu-$measurable functions f with the same rearrangement function with respect to the measure μ, i.e. such that $f_{\mu}^{*}=g_{\mu}^{*}$, also belong to X, and, moreover, $\|f\|_X=\|g\|_X$.

For any r.i. space $X(\Omega)$ we have

$$L^{\infty}(\Omega)\cap L^{1}(\Omega)\subset X(\Omega)\subset L^{1}(\Omega)+L^{\infty}(\Omega), \tag{3}$$

with continuous embedding.

Typical examples of r.i. spaces are the $L^p(\Omega)$-spaces, Lorentz spaces, Lorentz-Zygmund spaces and Orlicz spaces.

A useful property of r.i. spaces states that if

$$\int_0^r f_{\mu}^{*}(s)ds\leq\int_0^r g_{\mu}^{*}(s)ds,$$

holds for all $r>0$, then, for any r.i. space $X=X(\Omega)$,

$$\|f\|_X\leq\|g\|_X.$$

The associated space $X'(\Omega)$ of $X(\Omega)$ is the r.i. space of all measurable functions h for which the r.i. norm given by

$$\|h\|_{X'(\Omega)}=\sup_{g\neq 0}\frac{\int_{\Omega}|g(x)h(x)|\,d\mu}{\|g\|_{X(\Omega)}} \tag{4}$$

is finite. Note that by the definition (4), the generalized Hölder inequality

$$\int_{\Omega}|g(x)h(x)|\,d\mu\leq\|g\|_{X(\Omega)}\|h\|_{X'(\Omega)} \tag{5}$$

holds.

The **fundamental function** of X is defined by

$$\phi_X(s)=\|\chi_E\|_X,$$

where E is any measurable subset of Ω with $\mu(E)=s$. We can assume without loss of generality that ϕ_X is concave. Moreover,

$$\phi_{X'}(s)\phi_X(s)=s. \tag{6}$$

Associated with an r.i. space X there are some useful Lorentz and Marcinkiewicz spaces, defined by the quasi-norms

$$\|f\|_{M(X)}=\sup_t f_{\mu}^{**}(t)\phi_X(t),\qquad\|f\|_{\Lambda(X)}=\int_0^{\mu(\Omega)}f_{\mu}^{*}(t)d\phi_X(t). \tag{7}$$

Notice that

$$\phi_{M(X)}(t)=\phi_{\Lambda(X)}(t)=\phi_X(t),$$

and that

$$\Lambda(X)\subset X\subset M(X) \tag{8}$$

and the embedding has norm 1.

2.1.1. *Indices.* Let $D_{\frac{1}{s}}f(t) = f_\mu^*\left(\frac{t}{s}\right), s > 0$, be the dilation operator, and let $h_X(s)$ its norm, i.e. The upper and lower Boyd indices associated with a r.i. space X are defined by

$$\overline{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s}. \tag{9}$$

It is also useful sometimes to consider a slightly different set of indices obtained by means of replacing $h_X(s)$ in (9) by

$$M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \; s > 0.$$

The corresponding indices are denoted $\overline{\beta}_X$, $\underline{\beta}_X$, and will be referred to as the upper and lower fundamental indices of X. Actually, the relationship between $M_X(s)$ and $h_X(s)$ is that the computation of the former is exactly the computation of the latter but done only over functions of the form $f = \chi_{(0,a)}$. Therefore we have (cf. [3])

$$0 \leq \underline{\alpha}_X \leq \underline{\beta}_X \leq \overline{\beta}_X \leq \overline{\alpha}_X \leq 1.$$

We shall usually formulate conditions on r.i spaces in terms of the Hardy operators defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(x)dx; \qquad Qf(t) = \int_t^\infty f(x)\frac{dx}{x}.$$

In particular, it is well known (cf. [3]) that if X is a r.i. space, $P : X(0,\infty) \to X(0,\infty)$ (resp. Q) is bounded if and only if $\overline{\alpha}_X < 1$ (resp. $0 < \underline{\alpha}_X$).

2.2. **Doubling measures.** Given a ball $B(x,r)$ in Ω we set $V_\mu(x,r) = \mu(B(x,r))$.

A metric measure space is called doubling if there exists a constant $C_D > 1$, such that

$$0 < V_\mu(x,2r) \leq C_D V_\mu(x,r) < \infty \tag{10}$$

for all $x \in \Omega$ and $r > 0$.

Obviously $Q-$regular spaces are doubling.

Remark 5. *Given $x \in \Omega$, the function $r \to \mu(B(x,r))$ is (usually) not continuous, thus given $t > 0$ does not necessarily exist a ball $B(x)$ centered at x such that $\mu(B(x)) = t$, however there is a ball $B(x)$ centered at x such that $t/C_D \leq \mu(B(x)) \leq t$. Indeed, consider $r_0 = \sup\left\{r : V_\mu(x,r) < t/C_D\right\}$, then*

$$V_\mu(x,r_0) \leq t/C_D \leq V_\mu(x,2r_0) \leq C_D V_\mu(x,r_0) \leq t.$$

Where C_D denotes the $\mu-$doubling constant.

Following the proofs of [37, Theorem 1] and [35, Theorem 1.4] we easily obtain the following result:

Lemma 6. *If (Ω,d,μ) is doubling, then for all bounded $A \subset \Omega$ with $\mu(A) > 0$, $x \in A$ and $0 < r < diam(A)$, we have*

$$\frac{V_\mu(x,r)}{\mu(A)} \geq 2^{-s}\left(\frac{r}{diam(A)}\right)^m \tag{11}$$

where $m = \log_2 C_D$ *and* $diam(A) = \sup_{x,y\in A} d(x,y)$[2].

A metric space (Ω, d) is called uniformly perfect (with constant a), if it is not a singleton, and if there exists a constant $a > 1$ such that

$$\Omega\backslash B(x;r) \neq \varnothing \Rightarrow B(x;r)\backslash B(x;r/a) \neq \varnothing$$

for all $x \in X$ and $r > 0$. Connected spaces are uniformly perfect, and so are many classical totally disconnected fractals; for instance, the Cantor ternary set is uniformly perfect. It is also easy to see that $Q-$regular spaces are uniformly perfect (see [15, Chapter 11]).

Now we can state the opposite inequality in Lemma 6.

Lemma 7. *Let (Ω, d, μ) be doubling and uniformly perfect. Then there exist constants $D \geq 1$ and $k > 0$, depending only on the doubling constant C_D and the uniform perfectness constant a, so that*

$$\frac{V_\mu(x,r)}{V_\mu(x,R)} \leq D\left(\frac{r}{R}\right)^k, \tag{12}$$

for all $x \in A$ and $0 < r \leq R < diam(\Omega)$[3].

Combining the previous two lemmas, the following is true in doubling uniformly perfect measure metric spaces: There exist positive constants c_0, C_0, k, m $(k \leq m)$ depending only on the doubling constant of the measure and the uniform perfectness constant of the space (Ω, d, μ) such that

$$c_0 \min(r^k, r^m) V_\mu(x,1) \leq V_\mu(x,r) \leq C_0 \max(r^k, r^m) V_\mu(x,1), \tag{13}$$

for all $x \in \Omega$ and $0 < r < \infty$.

Notice that if $diam(\Omega) < \infty$, from (11) and (12) it follows that there exist constants c_1, C_1 such that

$$c_1 r^m \leq V_\mu(x,r) \leq C_1 r^k, \tag{14}$$

for all $x \in X$ and $0 < r < diam(\Omega)$.

Definition 8. *Let $0 < k \leq m$. Let (Ω, d, μ) be a measure metric space (Ω, d, μ).*

1. *(Ω, d, μ) will be called a $(k,m)-$**space,** if inequality (13) holds*[4].
2. *A $(k,m)-$space will be called **uniform,** if there are constants $c, C > 0$, such that*

$$c \min(r^k, r^m) \leq V_\mu(x,r) \leq C \max(r^k, r^m). \tag{15}$$

3. *A $(k,m)-$space will be called **bounded from below,** if there are constants $d, D > 0$ such that*

$$d \min(r^k, r^m) \leq V_\mu(x,r) \leq D \max(r^k, r^m) V_\mu(x,1). \tag{16}$$

[2]This inequality is actually equivalent to the doubling property of the measure taking $B(x,2r)$ as the set A.

[3]Notice that if some measure satisfies the above inequality, with some constants $D \geq 1$ and $k > 0$, then by choosing $r < D^{1/k}R$ in the above inequality we have that the space is uniformly perfect with any constant bigger than $D^{1/k}$.

[4]In fact (see [39]) (Ω, d, μ) is a $(k,n)-$space if, and only if, it is doubling and uniformly perfect.

Remark 9. *It follows from (13) that a* $(k,m)-$*space is a uniform (resp. bounded from below) if, and only if,* $0<\inf_{x\in\Omega}V_\mu(x,1)\leq\sup_{x\in\Omega}V_\mu(x,1)<\infty$, *(resp.* $0<\inf_{x\in\Omega}V_\mu(x,1)$*).*

Remark 10. *From (14), we have that doubling uniformly perfect measure metric spaces with* $diam(\Omega)<\infty$, *are uniform* $(k,m)-$*spaces.*

Notation 11. *Let* (Ω,d,μ) *be a* $(k,m)-$*space. In the rest of the paper we shall use the following notation:*

(1) *For* $t>0$,

$$R(t)=\max\left(t^{m/k},t^{k/m}\right),\ r(t)=\max\left(t^{1/k},t^{1/m}\right).$$

(2) *If* (Ω,d,μ) *is a uniform, we denote*

$$\kappa_0=2C_D/c$$

where c is the same constant as in (15).

(3) *If* (Ω,d,μ) *is bounded from below, we denote*

$$\kappa_1=2C_D/d$$

where d is the same constant as in (16).

Given (Ω,d,μ) a $(k,m)-$space, we associate to the measure μ a new measure $\tilde{\mu}$ defined in the following way,

$$\tilde{\mu}(E)=\int_E\frac{d\mu(x)}{V(x,1)},$$

for all borelian sets $E\subset\Omega$.

In the following lemma we collect some properties for the measure $\tilde{\mu}$.

Lemma 12. *Let* (Ω,d,μ) *be a* $(k,m)-$*space. Let* f *be a* μ*-measurable function. Then:*

(1) *For all* $r>0$*, we have that*

$$\min(r^k,r^m)\fint_{B(x,r)}|f(y)|\,d\mu(y)\preceq\int_{B(x,r)}|f(y)|\,d\tilde{\mu}(y)\preceq\max(r^k,r^m)\fint_{B(x,r)}|f(y)|\,d\mu(y).\tag{17}$$

Thus f *is* $\mu-$*locally integrable, if and only if,* f *is* $\tilde{\mu}-$*locally integrable. Moreover,* $(\Omega,d,\tilde{\mu})$ *is a uniform* $(k,m)-$*space.*

(2) *If* (Ω,d,μ) *is uniform, for all measurable* $E\subset\Omega$*, we have that*

$$\tilde{\mu}(E)\simeq\mu(E).$$

(3) *If* (Ω,d,μ) *is bounded from below, then for all* $f\in L^1_{\tilde{\mu}}(\Omega)+L^\infty_{\tilde{\mu}}(\Omega)$

$$f^*_{\tilde{\mu}}(t)\leq f^*_\mu(dt),\quad(t>0),$$

where d is the same constant that appears in (16).

Proof. (1) Using the doubling property and the fact that $B(x,r) \subset B(y,2r)$ whenever $y \in B(x,r)$, we get

$$\begin{aligned}\int_{B(x,r)} |f(y)|\, d\tilde{\mu}(y) &= \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,1)} = \int_{B(x,r)} |f(y)| \frac{V_\mu(y,r)}{V_\mu(y,1)} \frac{d\mu(y)}{V_\mu(y,r)} \\ &\le C_0 \max(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,r)} \text{ (by (13))} \\ &\le C_D C_0 \max(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,2r)} \text{ (by (10))} \\ &\le C_D C_0 \max(r^k, r^m) \fint_{B(x,r)} |f(y)|\, d\mu(y).\end{aligned}$$

Similarly, if $y \in B(x,r)$ then $B(y,r) \subset B(x,2r)$, and thus

$$\begin{aligned}\int_{B(x,r)} |f(y)|\, d\tilde{\mu}(y) &= \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,1)} = \int_{B(x,r)} |f(y)| \frac{V_\mu(y,r)}{V_\mu(y,1)} \frac{d\mu(y)}{V_\mu(y,r)} \\ &\ge c_0 \min(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,r)} \\ &\ge c_0 \min(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(x,2r)} \\ &\ge \frac{c_0}{C_D} \min(r^k, r^m) \fint_{B(x,r)} |f(y)|\, d\mu(y).\end{aligned}$$

Taking $f = 1$ in (17) we obtain that $(\Omega, d, \tilde{\mu})$ is a uniform $(k,m)-$space.
(2) It is obvious.
(3) From (16) we get,

$$\tilde{\mu}_f(y) = \int_{\{x\in\Omega:|f(x)|>y\}} \frac{d\mu(y)}{V(y,1)} \le \frac{1}{d} \int_{\{x\in\Omega:|f(x)|>y\}} d\mu(y) = \frac{\mu_f(y)}{d}.$$

Therefore, we get

$$\mu_f(y) \le dt \Rightarrow \tilde{\mu}_f(y) \le t;$$

and thus

$$f^*_{\tilde{\mu}}(t) = \inf\{y : \tilde{\mu}_f(y) \le t\} \le \inf\{y : \mu_f(y) \le dt\} = f^*_\mu(dt).$$

□

We end this section giving some examples of spaces introduced in definition 8.

2.3. **Examples.**

2.3.1. *Closed Subsets of* $\mathbb{R}^n$ *(see* [17]*).* We denote by m_n the $n-$dimensional Lebesgue measure on $\mathbb{R}^n$ and by d_n the $n-$Euclidean distance.

(1) Consider $F \subset \mathbb{R}^2 = \{(x_1, x_2)\}$ defined by $F = F_1 \cup F_2$, where $F_1 = \{(x_1+1)^2 + x_2^2 \le 1\}$ and $F_2 = \{0 \le x_1 \le 2,\ x_2 = 0\}$. Let m_n denote the $n-$dimensional Lebesgue measure, for $n = 1$ distributed over the x_1-axes, and put $d\lambda = x_1 dm_1$. Put $\mu = m_{2|F_1} + \lambda_{|F_2}$, then (F, d_2, μ) is $(1,2)-$uniform space.
(2) Let $F \subset \mathbb{R}^2$ be the set $F = \{0 \le x_1 \le 1,\ 0 \le x_2 \le x_1^\gamma\}$ where $\gamma > 1$, and $d\nu = x_1^{1-\gamma} dm_2$ and $\mu = \nu_{|F}$, then (F, d_2, μ) is $(1,2)-$uniform space.

(3) (See [17, Proposition 1]) For every closed subset $F \subset \mathbb{R}^n$, there is a measure μ with support F satisfying

$$\mu(B(x,r)) \le c_\mu r^n \mu(B(x,1)) \text{ and } c_1 < \mu(B(x,1)) < c_2, \quad x \in F.$$

Thus is F is uniform uniformly perfect, then there is $k > 0$, depending only of c_μ and of the uniform perfectness constant of F such that (F, d_n, μ) is $(k,n)-$uniform space.

2.3.2. *Muckenhoupt weights.* A weight is a positive, locally integrable function on $\mathbb{R}^n$. For a given subset E of $\mathbb{R}^n$, let $w(E) := \int_E w(x)dx$ and $|E| := \int_E dx$. A weight w on $\mathbb{R}^n$ is said to belong to the Muckenhoupt class A_p, $1 \le p < \infty$, (see [31]) if

(18)
$$[w]_{A_p} := \begin{cases} \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w(x)}\right)^{\frac{1}{p-1}} dx\right)^{p-1} < \infty, & \text{if } 1 < p < \infty, \\ \sup_B \frac{\frac{1}{|B|}\int_B w(x)dx}{ess\ \inf_{x\in B} w(x)} < \infty, & \text{if } p = 1, \end{cases}$$

where the supremum is over all balls $B \subset \mathbb{R}^n$. For $p = \infty$, we define $A_\infty = \cup_{1\le p<\infty} A_p$. Given $w \in A_\infty$, we define

$$[w]_{A_\infty} := \sup_B \frac{1}{w(B)} \int_B M(w\chi_B)(x)dx$$

where M denotes the usual uncentered Hardy-Littlewood maxinal operator. It is known that there is a positive dimensional constant c_n such that $[w]_{A_\infty} \le c_n [w]_{A_p}$

Given $w \in A_p$, it follows easily from (18) that if there exists $M > 0$ such that $ess \inf_{|x|\ge M} w(x) = 0$, then $\inf_{x\in\mathbb{R}^n} V_\mu(x,1) = 0$. Similarly, if $ess \sup_{|x|\ge M} w(x) = \infty$, then $\sup_{x\in\mathbb{R}^n} V_\mu(x,1) = \infty$.

Proposition 13. *Given $w \in A_p$, $p \ge 1$, $(\mathbb{R}^n, d_n, w)$ is a $\left(\frac{n}{2^{n+1}[w]_{A_\infty}}, pn\right)-$space.*

Proof. Since $w \in A_p$, by [16, Theorem 2.3], we have that

$$\frac{1}{|B|}\int_B w^r(x)dx \le 2\left(\frac{1}{|B|}\int_B w(x)dx\right)^r$$

where $r = 1 + \frac{1}{2^{n+1}[w]_{A_\infty}-1}$, therefore (see [8]) there exist constants $c, C > 0$ such that

$$c\left(\frac{|E|}{|B|}\right)^p \le \frac{w(E)}{w(B)} \le C\left(\frac{|E|}{|B|}\right)^{(r-1)/r} \tag{19}$$

for any measurable set E of the ball B. Considering in (19) $E = B(x,r) \subset B(x,1) = B$ if $r < 1$, or $E = B(x,1) \subset B(x,r) = B$ if $r > 1$, and elementary computation shows

$$\min\left(r^{\frac{n}{2^{n+1}[w]_{A_\infty}}}, r^{pn}\right) w(B(x,1)) \le w(B(x,r)) \le \max\left(r^{\frac{n}{2^{n+1}[w]_{A_\infty}}}, r^{pn}\right) w(B(x,1)).$$

□

Example 14. *Let $1 \le p < \infty$, $-n < \alpha \le \beta < n(p-1)$*

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \textit{if } |x| \le 1, \\ |x|^\beta & \textit{if } |x| > 1. \end{cases}$$

Then $w_{\alpha,\beta}(x) \in A_p$, and

(1) *If* $-n < \beta < 0$, *then* $\inf_{x\in\mathbb{R}^n} V_\mu(x,1) = 0$.
(2) *If* $\beta = 0$, *then* $\sup_{x\in\mathbb{R}^n} V_\mu(x,1) < \infty$.
(3) *If* $0 < \beta < n(p-1)$, *then* $\sup_{x\in\mathbb{R}^n} V_\mu(x,1) = \infty$.

2.3.3. *Carnot-Carathéodory spaces (see* [13]*).*

(1) (Compact case). If X is a compact $n-$dimensional Carnot-Carathéodory space with the distance associated to the vector fields and endowed with any fixed smooth measure μ with strictly positive density, then (X,d,μ) is a uniform $(n,nm)-$space.
(2) (Noncompact case). Let G be a connected Lie group and fix a left invariant Haar measure μ on G. We assume that G has polynomial volume growth, that is, if U is a compact neighborhood of the identity element e of G, then there is a constant $C>0$ such that $\mu(U^n) \preceq n^C$ for all $n\in\mathbb{N}$. Then, there is a nonnegative integer n_∞ such that $\mu(U^n) \simeq n^{n_\infty}$ as $n\to\infty$. Let $X_1,...,X_n$ be left invariant vector fields on G that satisfy Hörmander's condition, that is, they together with their successive Lie brackets $[X_{i_1},[X_{i_2},[\ldots,X_{i_k}]\ldots]$ span the tangent space of G at every point of G. Let d be the associated control metric. Then this metric is left invariant and compatible with the topology on G; and there is $n_0\in\mathbb{N}$, independent of x, such that $\mu(B(x,r)) \simeq r^{n_0}$ when $0<r\le 1$, and $\mu(B(x,r)) \simeq r^{n_0}$ when $r\ge 1$. From this, it follows that (G,d,μ) is a uniform $(\min\{n_0,n_\infty\},\max\{n_0,n_\infty\})-$space.

3. Symmetrization inequalities for moduli of smoothness

Let us start proving that the boundedness from below is necessary in order to obtain Sobolev type embedding result for Besov spaces (see [12] and [18] for some related results).

Theorem 15. *Let* (Ω,d,μ) *be a doubling metric space. Let* X *be a rearrangement invariant space with* $1/p > \overline{\beta}_X$. *Assume that the following embedding holds*

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset X.$$

Then

$$\inf_{x\in\Omega} V_\mu(x,1) > 0. \tag{20}$$

In particular, $\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset L^{p^*,q}(\Omega)$ *for some* $p^* > p$ *, implies (20).*

Proof. We claim that conditions on indices imply that for $1/p > \varepsilon\overline{\beta}_X$, and t sufficiently small

$$\frac{t^{1/p}}{\varphi_X(t)} \preceq t^{\frac{1}{p}-\overline{\beta}_X-\varepsilon}. \tag{21}$$

Indeed, let $s,t>0$. Then

$$\frac{t^{1/p}}{\varphi_X(t)} = \frac{t^{1/p}}{\varphi_X(st)}\frac{\varphi_X(st)}{\varphi_X(t)} \le \frac{t^{1/p}}{\varphi_X(st)} M_X(s).$$

Thus, for $s = 1/t$, we get

$$\frac{t^{1/p}}{\varphi_X(t)} \le \frac{t^{1/p}}{\varphi_X(1)} M_X(\frac{1}{t}).$$

Let $1/p > \varepsilon > \overline{\beta}_X$. Then (see [20, p. 54]) for t sufficiently small

$$\begin{aligned}\frac{t^{1/p}}{\varphi_X(t)} &\leq \frac{t^{1/p}}{\varphi_X(1)} M_X(\frac{1}{t})\\ &\leq \frac{t^{1/p}}{\varphi_X(1)}\left(\frac{1}{t}\right)^{\overline{\beta}_X+\varepsilon}\\ &= \frac{t^{\frac{1}{p}-\overline{\beta}_X-\varepsilon}}{\varphi_X(1)},\end{aligned}$$

as we wanted to see.

For a fixed $x_0 \in \Omega$, we define the Lipschitz function

$$u_{x_0}(y) := \begin{cases} (2-d(x_0,y)) & \text{if } y \in B(x_0,2)\backslash B(x_0,1) \\ 1 & \text{if } y \in B(x_0,1) \\ 0 & \text{if } y \in \Omega\backslash B(x_0,2). \end{cases}$$

It is easily seen that

$$g_{x_0}(y) = \chi_{B(x_0,2)}(y)$$

is a generalized gradient, i.e.

$$|u_{x_0}(x) - u_{x_0}(y)| \leq d(x,y)\,|g_{x_0}(x) + g_{x_0}(y)|\,. \tag{22}$$

By Fubini theorem

$$\begin{aligned}E_p(u_{x_0},t)^p &\leq 2^p \int_\Omega |u_{x_0}(x)|^p\, d\mu(x) + 2^p \int_\Omega ⨍_{B(x,t)} |u_{x_0}(y)|^p\, d\mu(y) d\mu(x)\\ &\leq 2^p \|u_{x_0}\|_p^p + 2^p \int_\Omega |u_{x_0}(y)|^p \left(\int_{B(y,t)} \frac{1}{\mu(B(x,t))} d\mu(x)\right) d\mu(y)\\ &\preceq \|u_{x_0}\|_p^p\,,\end{aligned} \tag{23}$$

the last estimate follows from the doubling property of μ and since $B(y,t) \subset B(x,2t)$ whenever $x \in B(y,t)$.

By (22) and using a similar argument as in (23), we get

$$\begin{aligned}E_p(u_{x_0},t)^p &= \int_\Omega \left(⨍_{B(x,t)} |u_{x_0}(x) - u_{x_0}(y)|^p\, d\mu(y)\right) d\mu(x)\\ &\leq \int_\Omega \left(⨍_{B(x,t)} d(x,y)^p\, |g_{x_0}(x) + g_{x_0}(y)|^p\, d\mu(y)\right) d\mu(x)\\ &\preceq t^p \left(\int_\Omega |g_{x_0}(x)|^p\, d\mu(x) + \int_\Omega ⨍_{B(x,t)} |g_{x_0}(y)|^p\, d\mu(y) d\mu(x)\right)\\ &\preceq t^p \|g_{x_0}\|_p^p\,.\end{aligned} \tag{24}$$

Thus, combining (23) and (24) with the doubling property, we get

$$\begin{aligned}E_p(u_{x_0},t) &\preceq \min(\|u_{x_0}\|_p\,, t\,\|g_{x_0}\|_p)\\ &\preceq \min(V_\mu(x_0,2)^{1/p}, tV_\mu(x_0,2)^{1/p})\\ &\preceq \min(1,t)V_\mu(x_0,1)^{1/p}.\end{aligned}$$

Therefore,

$$\|u_{x_0}\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \preceq V_\mu(x_0,1)^{1/p}.$$

Since
$$\|u_{x_0}\|_X \geq \varphi_X(V_\mu(x_0,1))$$
from hypothesis, we have that
$$1 \leq \frac{V_\mu(x_0,1)^{1/p}}{\varphi_X(V_\mu(x_0,1))}. \tag{25}$$
If $\inf_{x\in\Omega} V_\mu(x,1) = 0$, we can select a sequence $V_\mu(x_n,1) \to 0$, thus for n large enough, (25) and (21) imply
$$1 \leq V_\mu(x_n,1)^{\frac{1}{p}-\overline{\beta}_X-\varepsilon},$$
which is impossible since $\frac{1}{p} - \overline{\beta}_X - \varepsilon > 0$. □

Recall that our aim is to obtain embedding results for Besov spaces built on doubling measure spaces. Therefore, in view of Theorem 2 it is reasonable to assume that Ω is uniformly perfect (since $Q-$regular spaces are uniformly perfect). Moreover, if we assume the additional hypothesis (i.e. Ω supports a $(1,p)-$Poincaré inequality or Ω geodesic), then Ω is connected and therefore uniformly perfect. Taking into account these considerations and the previous theorem, our framework in what follows will be a $(k,m)-$space bounded from below.

3.1. **Pointwise estimates for the rearrangement.** Let (Ω,d,μ) be a $(k,m)-$space with[5] $\mu(\Omega)=\infty$. For $f\in L^1(\Omega)+L^\infty(\Omega)$, and X a r.i. space on Ω, we define:

(1) The gradient at scale r
$$(\nabla_r^\mu f)(x) = \fint_{B(x,r)} |f(x)-f(y)|\, d\mu(y), \qquad (r>0).$$
(2) The $X-$modulus of continuity $E_X : (0,\infty)\times X \to [0,\infty)$,
$$E_X(f,r) := \|(\nabla_r^\mu f)\|_X .$$

Remark 16. *If $1\leq p<\infty$ and $X=L^p$, then by Hölder inequality, we get*
$$\begin{aligned} E_{L^p}(f,r) &= \left(\int_\Omega \left(\fint_{B(x,r)} |f(x)-f(y)|\, d\mu(y)\right)^p d\mu(x)\right)^{1/p} \\ &\leq \left(\int_\Omega \left(\fint_{B(x,r)} |f(x)-f(y)|^p\, d\mu(y)\right) d\mu(x)\right)^{1/p} \\ &= E_p(f,r) \end{aligned}$$

The aim of this section is to obtain pointwise estimates for the oscillation $O_\mu(f,t)$ in terms of the functional $E_X(f,t)$, (see [23], [27]). The next lemma will be useful in what follows:

Lemma 17. *Let $f\in L^1(\Omega)+L^\infty(\Omega)$. Let $x\in\Omega$ and $t>0$ be such that there is a ball $B_t(x)$ centered at x with $\mu(B_t(x))=t$. Then*
$$f_\mu^{**}(t/2) - f_\mu^{**}(t) \leq \left(\delta_t^\mu f\right)_\mu^{**}(t/2),$$
where
$$\left(\delta_t^\mu f\right)(x) = \frac{1}{t}\int_{B_t(x)} |f(x)-f(y)|\, d\mu(y).$$

[5]All the results given in this section also hold in the case that $\mu(\Omega)<\infty$. We leave the details to the reader.

Proof. Since

$$|f(x)|\chi_{B_t(x)}(y) \leq |f(x)-f(y)|\chi_{B_t(x)}(y) + |f(y)|\chi_{B_t(x)}(y),$$

integrating with respect to $d\mu(y)$, we have that

$$\begin{aligned} |f(x)|t &\leq \int_{B_t(x)} |f(x)-f(y)|d\mu(y) + \int_{B_t(x)} |f(y)|d\mu(y) \\ &\leq \int_{B_t(x)} |f(x)-f(y)|d\mu(y) + \int_0^t f_\mu^*(s)ds \quad \text{(by (1))}. \end{aligned}$$

Now integrating with respect to $d\mu(x)$ over a subset $E \subset \Omega$ with $\mu(E) = t/2$, we get

$$\begin{aligned} \int_E |f(x)|d\mu(x) &\leq \int_E \frac{1}{t}\int_{B_t(x)} |f(x)-f(y)|d\mu(y)d\mu(x) + \int_E \frac{1}{t}\left(\int_0^t f_\mu^*(s)ds\right)d\mu(x) \\ &= \int_E \left(\delta_t^\mu f\right)(x)d\mu(x) + \frac{1}{2}\int_0^t f_\mu^*(s)ds \end{aligned}$$

By (2), taking the supremum over all such sets E, we obtain

$$\int_0^{t/2} f_\mu^*(s)ds \leq \int_0^{t/2} \left(\delta_t^\mu f\right)_\mu^*(s)ds + \frac{1}{2}\int_0^t f_\mu^*(s)ds,$$

or equivalently

$$f_\mu^{**}(t/2) - f_\mu^{**}(t) \leq \left(\delta_t^\mu f\right)_\mu^{**}(t/2).$$

□

Theorem 18. *Let $f \in L^1(\Omega) + L^\infty(\Omega)$. Let X be an r.i. space on Ω.*

(1) *If (Ω, d, μ) is uniform, then for all $t > 0$, we get*

$$O_\mu(f,t) \leq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t)). \tag{26}$$

(2) *If (Ω, d, μ) is bounded form below, then for all $t > 0$, we get*

$$O_{\tilde{\mu}}(f,t) \leq \frac{1}{\kappa_1 1t}\frac{R(\kappa_1 t)}{\phi_X(\kappa_1 t)}E_X(f, r(\kappa_1 t)).$$

Proof. (1) Given $x \in \Omega$, and $t > 0$, by Remark 29 there is a ball $B(x)$ centered at x such that $t/C_D \leq \mu(B(x)) \leq t$. We denote by z the measure of this ball, i.e. $\mu(B_z(x)) = z$, with $t/C_D \leq z \leq t$. From (15) it follows that

$$\begin{cases} \mu(B_z(x)) \leq t \leq V_\mu(x, (t/c)^{1/m}) \leq C\,(t/c)^{k/m} & \text{if } t < c, \\ \mu(B_z(x)) \leq t \leq V_\mu(x, (t/c)^{1/k}) \leq C\,(t/c)^{m/k} & \text{if } t \geq c, \end{cases}$$

i.e.

$$\mu(B_z(x)) \leq t \leq V_\mu(x, r(t/c)) \leq CR(t/c). \tag{27}$$

Obviously, $B_z(x) \subset B\left(x, r\left(t/c\right)\right)$, and thus

$$\begin{aligned}
\left(\delta_z^\mu f\right)(x) &= \frac{1}{z} \int_{B_z(x)} |f(x) - f(y)|\, d\mu(y) \\
&\leq \frac{C_D}{t} \int_{B(x, r(t/c))} |f(x) - f(y)|\, d\mu(y) \\
&\leq C C_D \frac{R\left(t/c\right)}{t} \fint_{B(x, r(t/c))} |f(x) - f(y)|\, d\mu(y) \quad \text{(by (27))} \\
&= C C_D \frac{R\left(t/c\right)}{t} \left(\nabla_{r(t/c)}^\mu f\right)(x).
\end{aligned}$$

Taking rearrangements, we get

$$\left(\delta_z^\mu f\right)_\mu^*(s) \leq C C_D \frac{R\left(t/c\right)}{t} \left(\nabla_{r(t/c)}^\mu f\right)_\mu^*(s), \ s > 0,$$

which implies

$$\left(\delta_z^\mu f\right)_\mu^{**}(s) \leq C C_D \frac{R\left(t/c\right)}{t} \left(\nabla_{r(t/c)}^\mu f\right)_\mu^{**}(s), \ s > 0.$$

On the other hand

$$\begin{aligned}
\left(\nabla_{r(t/c)}^\mu f\right)_\mu^{**}(s) &\leq \frac{1}{\phi_X(s)} \sup_s \left(\phi_X(s) \left(\nabla_{r(t/c)}^\mu f\right)_\mu^{**}(s)\right) \\
&= \frac{1}{\phi_X(s)} \left\| \left(\nabla_{r(t/c)}^\mu f\right) \right\|_{M(X)} \quad \text{(by (7))} \\
&\leq \frac{1}{\phi_X(s)} \left\| \left(\nabla_{r(t/c)}^\mu f\right) \right\|_X \quad \text{(by (8))} \\
&= \frac{1}{\phi_X(s)} E_X(f, r(t/c)).
\end{aligned}$$

Combining this inequality and Lemma 17, we obtain

$$f_\mu^{**}(z/2) - f_\mu^{**}(z) \leq \left(\delta_z^\mu f\right)_\mu^{**}(z/2) \leq C C_D \frac{R\left(t/c\right)}{t \phi_X(z/2)} E_X(f, r(t/c)).$$

By Remark 3, we get

$$f_\mu^{**}(z/2) - f_\mu^{**}(z) = \int_{z/2}^z \left(f_\mu^{**}(s) - f_\mu^*(s)\right) \frac{ds}{s} \geq \frac{f_\mu^{**}(z/2) - f_\mu^*(z/2)}{2}.$$

In summary,

$$O_\mu\left(f, z/2\right) \leq 2 C C_D \frac{R\left(t/c\right)}{t \phi_X(z/2)} E_X(f, r(t/c)). \tag{28}$$

Finally, using that $t/C_D \leq z \leq t$, we get

$$\begin{aligned}
\frac{t}{2C_D} O_\mu\left(f, \frac{t}{2C_D}\right) &\leq \frac{z}{2} O_\mu\left(f, \frac{z}{2}\right) \text{ (by Remark 3)} \\
&\leq 2CC_D \frac{z/2}{\phi_X(z/2)} \frac{R(t/c)}{t} E_X(f, r(t/c)) \text{ (by (28))} \\
&\leq 4CC_D \frac{R(t/c)}{\phi_X(t/2)} E_X(f, r(t/c)) \text{ (since } \frac{s}{\phi_X(s)} \text{ increases)} \\
&\leq 4CC_D \left(\sup_{t>0} \frac{\phi_X(2t/c)}{\phi_X(t/2)}\right) \frac{R(t/c)\, E_X(f, r(t/c))}{\phi_X(t/c)} \\
&= 4CC_D M_X(2/c) \frac{R(t/c)\, E_X(f, r(t/c))}{\phi_X(t/c)}
\end{aligned}$$

which implies (26).

(2) By Lemma 12, we get $L^1_\mu(\Omega) + L^\infty_\mu(\Omega) \subset L^1_{\tilde{\mu}}(\Omega) + L^\infty_{\tilde{\mu}}(\Omega)$.

Since $\tilde{\mu}$ is doubling, by Remark 29, given $x \in \Omega$ and $t > 0$ there is a ball $B_z(x)$ a ball centered at x such that $t/C_D \leq \tilde{\mu}(B_z(x)) = z \leq t$. Then

$$\begin{aligned}
\left(\delta_z^{\tilde{\mu}} f\right)(x) &\leq \frac{C_D}{t} \int_{B(x, r(t/d))} |f(x) - f(y)| \frac{d\mu(y)}{V(y,1)} \\
&\leq C_D D \frac{R(t/d)}{t} \fint_{B(x, r(t/d))} |f(x) - f(y)|\, d\mu(y) \quad \text{(by (17))} \\
&= C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)(x).
\end{aligned}$$

Taking rearrangement with respect to $\tilde{\mu}$, we have that for all $s > 0$

$$\begin{aligned}
\left(\delta_z^{\tilde{\mu}} f\right)^*_{\tilde{\mu}}(s) &\leq C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^*_{\tilde{\mu}}(s) \\
&\leq C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^*_{\mu}(sd) \text{ (by Lemma 12).}
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\delta_z^{\tilde{\mu}} f\right)^{**}_{\tilde{\mu}}(s) &= \frac{1}{s} \int_0^s \left(\delta_z^{\tilde{\mu}} f\right)^*_{\tilde{\mu}}(y) dy \\
&\leq C_D D \frac{R(t/d)}{t} \frac{1}{s} \int_0^s \left(\nabla^\mu_{r\left(\frac{t}{d}\right)} f\right)^*_\mu (yd) dy \\
&= C_D D \frac{R(t/d)}{t} \frac{1}{sd} \int_0^{sd} \left(\nabla^\mu_{r\left(\frac{t}{d}\right)} f\right)^*_\mu (y) dy \\
&= C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^{**}_\mu (sd).
\end{aligned}$$

Now we finish the proof as in part 1. □

Remark 19. *These estimates are abstract variants of known estimates via the classical modulus of smoothness. For example if $X = L^p(\mathbb{R}^n)$, we obtain*

$$f^{**}(t) - f^*(t) \leq 2^{1/p} \frac{\omega_p(f, t^{1/n})}{t^{1/p}}.$$

See [19], [23] *and the references quoted therein.*

4. A Sobolev type embedding theorem for Besov spaces

Let (Ω,d,μ) be a (k,m) −space. Let X be a r.i space on Ω, let Y be a r.i. space over $[0,\infty)$ with respect to the Lebesgue measure and let $0<s<1$. We define the **Besov space** $\mathring{\mathrm{B}}^{s}_{(k,m),X,Y}(\Omega)$ as the set of those functions in $L^{1}(\Omega)+L^{\infty}(\Omega)$ for which the semi-norm $\|f\|_{\mathring{\mathrm{B}}^{s}_{(k,m),X,Y}(\Omega)}$ is finite. Here

$$\|f\|_{\mathring{\mathrm{B}}^{s}_{(k,m),X,Y}(\Omega)}:=\left\|\frac{r(t)^{-s}E_X(f,r(t))}{\phi_Y(t)}\right\|_Y.$$

In the Euclidean case, there are several examples of generalized Besov spaces, for instance if Y is a Lorentz-Zygmund space and $X=L^{p}$, we obtain the Besov spaces of generalized smoothness (see [7], [28] and the references quoted therein). If X is an Orlicz spaces and $Y=L^{q}$, we obtain the Besov-Orlicz spaces (see [1], [6], [38], [36] and the references quoted therein). Examples of Besov involving two r.i. norms can be found in [11] and [32].

Remark 20. *If $X=L^{p}(\Omega)$ and $Y=L^{q}([0,\infty))$, $(1\leq p<\infty,\ 1\leq q\leq\infty)$, we write $\mathring{B}^{s}_{(k,m),p,q}(\Omega)$ instead of $\mathring{B}^{s}_{(k,m),L^{p}(\Omega),L^{q}}(\Omega)$. In this case, $\phi_Y(t)=t^{1/q}$, and thus*

$$\begin{aligned}
\|f\|_{\mathring{B}^{s}_{(k,m),p,q}(\Omega)} &\leq \left\|\frac{E_p(f,r(t))}{(r(t))^{s}\,t^{1/q}}\right\|_{L^q} \qquad \textit{(by Remark 16)}\\
&=\left(\int_0^\infty\left(\frac{E_p(f,\max\left(t^{1/k},t^{1/m}\right))}{\max\left(t^{1/k},t^{1/m}\right)^{s}}\right)^{q}\frac{dt}{t}\right)^{1/q}\\
&=\left(\int_0^1\left(\frac{E_p(f,t^{1/m})}{t^{s/m}}\right)^{q}\frac{dt}{t}+\int_1^\infty\left(\frac{E_p(f,t^{1/k})}{t^{s/k}}\right)^{q}\frac{dt}{t}\right)^{1/q}\\
&\simeq\left(\int_0^1\left(\frac{E_p(f,t)}{t^{s}}\right)^{q}\frac{dt}{t}+\int_1^\infty\left(\frac{E_p(f,t)}{t^{s}}\right)^{q}\frac{dt}{t}\right)^{1/q}\\
&=\left(\int_0^\infty\left(\frac{E_{L^q}(f,t)}{t^{s}}\right)^{q}\frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

Therefore,

$$\dot{\mathcal{B}}^{s}_{p,q}(\Omega)\subset\mathring{B}^{s}_{(k,m),p,q}(\Omega).$$

Similarly,

$$\dot{\mathcal{B}}^{s}_{p,\infty}(\Omega)\subset\mathring{B}^{s}_{(k,m),p,\infty}(\Omega).$$

4.1. **Some new function spaces.**

Following [23], we shall now construct the range spaces for our generalized Besov-Sobolev embedding theorem.

Let (Ω,d,μ) be a (k,m) −space. Given $s\in\mathbb{R}$, we define

$$v_s(t):=\frac{t}{R(t)r(t)^{s}}=\min\left(t^{1-\frac{m+s}{k}},t^{1-\frac{k+s}{m}}\right)$$

and

$$S^{X,Y}_{\mu}(v_s)=\left\{f:\|f\|_{S^{X,Y}_{\mu}(v_s)}=\left\|v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}O_\mu(f,t)\right\|_Y<\infty\right\},$$

where ϕ_X is the fundamental function of X, a r.i space on Ω, and Y is a r.i. space on $[0,\infty)$ with respect to the Lebesgue measure.

Note that these spaces are not necessarily linear and, in particular, $\|.\|_{S^{X,Y}_{\tilde{\mu}}(v_s)}$ is not usually a norm.

Given a r.i. space X we shall say that Y satisfies the $Q(s,(k,m),X)-$condition if there exists a constant $C>0$ such that

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}Qf(t)\right\|_Y \leq C\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}f(t)\right\|_Y.$$

The following lemmas will be useful in what follows. A consequence of our first lemma is that if Y satisfies the $Q(s,(k,m),X)-$condition then $S^{X,Y}_{\mu}(v)$ is a Banach space.

Lemma 21. *Let X,Y be two r.i. spaces. If Y satisfies the $Q(s,(k,m),X)-$condition, then for all $f^*_\mu(\infty)=0$,*

$$\|f\|_{S^{X,Y}_{\mu}(v)} \simeq \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}f^{**}_\mu(t)\right\|_Y, \tag{29}$$

with constants of equivalence independent of f.

Proof. Obviously,

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}O_\mu(f,t)\right\|_Y \leq \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}f^{**}_\mu(t)\right\|_Y.$$

Conversely, from $\frac{d}{dt}f^{**}_\mu(t)=-\frac{f^{**}_\mu(t)-f^*_\mu(t)}{t}$ and the Fundamental Theorem of Calculus, we have

$$f^{**}_\mu(t)=\int_t^\infty \left(f^{**}_\mu(s)-f^*_\mu(s)\right)\frac{ds}{s}=Q\left(f^{**}_\mu-f^*_\mu\right)(t),$$

and the result follows by the $Q(s,(k,m),X)-$condition. □

The next result gives a useful criteria to check the validity of a $Q(s,(k,m),X)-$condition.

Lemma 22. *Let X,Y be two r.i. spaces. Suppose that*

$$\int_1^\infty t^{\frac{m+s}{k}-1}h_Y(1/t)M_X(1/t)M_Y(t)\frac{dt}{t}<\infty. \tag{30}$$

Then Y satisfies the $Q(s,(k,m),X)-$condition.

Proof. Let us write $v_s:=v$. We have

$$\begin{aligned}
v(t)\frac{\phi_X(t)}{\phi_Y(t)}Qf(t) &= \int_t^\infty v(t)\frac{\phi_X(t)}{\phi_Y(t)}f(x)\frac{dx}{x} = \int_1^\infty v(t)\frac{\phi_X(t)}{\phi_Y(t)}f(tx)\frac{dx}{x}\\
&\leq \int_1^\infty v(tx)f(tx)\frac{\phi_X(xt)}{\phi_Y(xt)}\sup_{t>0}\frac{v(t)}{v(tx)}\sup_{t>0}\frac{\phi_X(t)}{\phi_X(xt)}\sup_{t>0}\frac{\phi_Y(xt)}{\phi_Y(t)}\frac{dx}{x}\\
&= \int_1^\infty v(tx)f(tx)\frac{\phi_X(xt)}{\phi_Y(xt)}\sup_{t>0}\frac{v(t)}{v(tx)}M_X(1/x)M_Y(x)\frac{dx}{x}.
\end{aligned}$$

Applying Minkowski's inequality, we obtain

$$\begin{aligned}
\left\| v(t)\frac{\phi_X(t)}{\phi_Y(t)}Qf(t)\right\|_Y &\leq \int_1^\infty \left\| v(tx)f(tx)\frac{\phi_X(xt)}{\phi_Y(xt)}\right\|_Y \sup_{t>0}\frac{v(t)}{v(tx)}M_X(1/x)M_Y(x)\frac{dx}{x}\\
&\leq \int_1^\infty \left(\sup_{t>0}\frac{v(t)}{v(tx)}\right)h_Y(1/x)M_X(1/x)M_Y(x)\frac{dx}{x}\left\| v(t)f(t)\frac{\phi_X(t)}{\phi_Y(t)}\right\|_Y.
\end{aligned}$$

Finally an elementary computation shows that, if $x > 1$, then

$$\sup_{t>0} \frac{v(t)}{v(tx)} = x^{\frac{m+s}{k}-1}.$$

□

Remark 23. *In terms of indices, it is easy to see that (30) is equivalent to the inequality*

$$\frac{m+s}{k} - 1 < \underline{\alpha}_Y - \overline{\beta}_Y + \underline{\beta}_X.$$

From Theorem 18 we immediately get the following generalization of the Sobolev embedding theorem for Besov spaces.

Theorem 24. *Let (Ω, d, μ) be a $(k,m)-$space, X, Y r.i. spaces and $0 < s < 1$. Then*

(1) *If (Ω, d, μ) is uniform,*

$$\overset{\circ}{B}{}^{s}_{(k,m),X,Y}(\Omega) \subset S^{X,Y}_{\mu}(v_s).$$

Moreover if Y satisfies the $Q(s,(k,m),X)$, then for all $f^*_\mu(\infty) = 0$

$$\left\| v_s(t) \frac{\phi_X(t)}{\phi_Y(t)} f^{**}_\mu(t) \right\|_Y \preceq \|f\|_{\overset{\circ}{B}{}^{s}_{(k,m),X,Y}(\Omega)}.$$

(2) *If (Ω, d, μ) is bounded from below,*

$$\overset{\circ}{B}{}^{s}_{(k,m),X,Y}(\Omega) \subset S^{X,Y}_{\tilde{\mu}}(v_s).$$

Moreover if Y satisfies the $Q(s,(k,m),X)$, then for all $f^*_{\tilde{\mu}}(\infty) = 0$

$$\left\| v_s(t) \frac{\phi_X(t)}{\phi_Y(t)} f^{**}_{\tilde{\mu}}(t) \right\|_Y \preceq \|f\|_{\overset{\circ}{B}{}^{s}_{(k,m),X,Y}(\Omega)}.$$

Proof. (1) Let $f \in L^1(\Omega) + L^\infty(\Omega)$. Then from (26) we know that there is a constant $\kappa > 0$ such that

$$O_\mu(f,t) \preceq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)), \quad t > 0.$$

Thus,

$$\begin{aligned}
\|f\|_{S^{X,Y}_\mu(v_s)} &\preceq \left\| \frac{t}{R(t) r(t)^s} \frac{\phi_X(t)}{\phi_Y(t)} \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&= \left\| \frac{R(\kappa_0 t)\, r(\kappa_0 t)}{R(t) r(t)^s} \frac{\phi_X(t)\phi_Y(\kappa_0 t)}{\phi_X(\kappa_0 t)\phi_Y(t)} \frac{1}{\kappa_0} \frac{r(\kappa_0 t)^{-s}}{\phi_Y(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&\preceq \sup_{t>0} \left(\frac{R(\kappa_0 t)\, r(\kappa_0 t)}{R(t) r(t)^s} \frac{\phi_X(t)\phi_Y(\kappa_0 t)}{\phi_X(\kappa_0 t)\phi_Y(t)} \frac{1}{\kappa_0} \right) \left\| \frac{r(\kappa_0 t)^{-s}}{\phi_Y(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&\preceq h_Y(1/\kappa_0) \left\| \frac{r(t)^{-s}}{\phi_Y(t)} E_X(f, r(t)) \right\|_Y \\
&\preceq \|f\|_{\overset{\circ}{\mathrm{B}}{}^{s}_{(k,m),X,Y}(\Omega)}.
\end{aligned}$$

Part (2) is analogous. □

5. Uncertainty type inequalities

The purpose of this section is to extend the generalized uncertainty Sobolev inequalities obtained in [26] to the context of Besov spaces.

Definition 25. *Let (Ω, d, μ) be a $(k,m)-$space, X, Y r.i. spaces and $0 < s < 1$. We will say that a $\mu-$measurable function $w : \Omega \to (0,\infty)$ is a $(s,(k,m),X,Y)-$admissible weight if*

$$[w] := \sup_{t>0} \left(\left(\left(\frac{1}{w} \right)^*_\mu (t) \right)^s \frac{1}{v_s(t) \frac{\phi_X(t)}{\phi_Y(t)}} \right) < \infty.$$

Theorem 26. *Let (Ω, d, μ) be a uniform $(k,m)-$space, let X, Y be r.i. spaces, $0 < s < 1$ and w a $(s,(k,m),X,Y)-$admissible weight. Assume that Y satisfies the $Q(s,(k,m),X)-$condition. Let $\alpha > 0$. Then for all $f \in L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ such that $f^*_\mu(\infty) = 0$ we have that*

$$\|f\|_Y \preceq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\mathring{B}^s_{(k,m),X,Y}(\Omega)} \left\| w^{\alpha s} f \right\|^{\frac{1}{\alpha+1}}_Y . \tag{31}$$

Proof. Since $f^*_\mu(\infty) = 0$, by the Fundamental theorem of Calculus and (26), we get

$$\begin{aligned} f^{**}_\mu(t) &= \int_t^\infty \left(f^{**}_\mu(s) - f^*_\mu(s) \right) \frac{ds}{s} \\ &\preceq \int_t^\infty \frac{1}{\kappa_0 s} \frac{R(\kappa_0 s)}{\phi_X(\kappa_0 s)} E_X(f, r(\kappa_0 s)) \frac{ds}{s}. \end{aligned} \tag{32}$$

Then

$$\begin{aligned} \|f\|_Y &= \left\| f \left(\frac{w}{w} \right)^s \right\|_Y \le \left\| f \left(\frac{w}{w} \right)^s \chi_{\{w \le r^{1/s}\}} \right\|_Y + \left\| f \left(\frac{w}{w} \right)^s \chi_{\{w > r^{1/s}\}} \right\|_Y \\ &\le r \left\| f \left(\frac{1}{w} \right)^s \right\|_Y + \left\| f \left(\frac{w}{w} \right)^{s\alpha} \chi_{\{w > r^{1/s}\}} \right\|_Y \\ &\le r \left\| f \left(\frac{1}{w} \right)^s \right\|_Y + r^{-\alpha} \left\| w^{\alpha s} f \right\|_Y . \end{aligned}$$

Now we estimate the first term,

$$\begin{aligned} \left\| f \left(\frac{1}{w} \right)^s \right\|_Y &\le \left\| f^*_\mu(t) \left(\left(\frac{1}{w} \right)^*_\mu (t) \right)^s \frac{v_s(t) \frac{\phi_X(t)}{\phi_Y(t)}}{v_s(t) \frac{\phi_X(t)}{\phi_Y(t)}} \right\|_Y \\ &\le [w] \left\| f^*_\mu(t) v_s(t) \frac{\phi_X(t)}{\phi_Y(t)} \right\|_Y \\ &\le [w] \left\| f^{**}_\mu(t/2) v_s(t) \frac{\phi_X(t)}{\phi_Y(t)} \right\|_Y \\ &\preceq [w] \left\| v_s(t) \frac{\phi_X(t)}{\phi_Y(t)} \int_t^\infty \frac{1}{\kappa_0 s} \frac{R(\kappa_0 s)}{\phi_X(\kappa_0 s)} E_X(f, r(\kappa_0 s)) \frac{ds}{s} \right\|_Y \quad \text{(by (32))} \\ &\preceq [w] \left\| \frac{r(t)^{-s} E_X(f, r(t))}{\phi_Y(t)} \right\|_Y \quad \text{(by the } Q(s,(k,m),X) - \text{condition)} \\ &\preceq [w] \|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)} . \end{aligned}$$

In summary, we have proved that there is an absolute constant $A > 0$, such that

$$\|f\|_Y \leq A[w]\, r\, \|f\|_{\mathring{\mathrm{B}}^s_{(k,m),X,Y}(\Omega)} + r^{-\alpha} \left\|w^{\alpha s} f\right\|_Y . \tag{33}$$

Selecting the value $r = \left(\frac{\|w^{s\alpha} f\|_Y}{2A[w]\|f\|_{\mathring{\mathrm{B}}^s_{(k,m),X,Y}(\Omega)}}\right)^{\frac{1}{1+\alpha}}$ to compute (33) balances the two terms and we obtain the multiplicative inequality (31). □

Remark 27. *The connection with the isoperimetric weight introduced in* [26] *is the following: consider the case* $(\Omega, d, \mu) = \mathbb{R}^n$. *Obviously,* (Ω, d, μ) *is a uniform* $(k,m)-$*space with* $k = m = \frac{1}{n}$. *Let* $X = Y = L^q$, *then*

$$[w] := \sup_{t>0} \left(\left(\left(\frac{1}{w}\right)^*_\mu (t) \right)^s t^{\frac{s}{n}} \right) = \sup_{t>0} \left(\left(\left(\frac{1}{w}\right)^*_\mu (t) \right) t^{\frac{1}{n}} \right)^s .$$

Thus w *is admissible if, and only, if* $\frac{1}{w} \in L^{n,\infty}$ *(i.e.* w *is an isoperimetric weight).*

Let $\alpha > 0$, $1 \leq q < \infty$, $0 < s < 1$, *with* $s < n/q$. *By Remark 23,* L^q *satisfies the* $Q(s, (\frac{1}{n}, \frac{1}{n}), L^q)-$*condition. Then by Theorem 26, if* $\frac{1}{w} \in L^{n,\infty}$ *we have that*

$$\|f\|_q \preceq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\mathring{B}^s_{q,q}(\mathbb{R}^n)} \left\|w^{\alpha s} f\right\|^{\frac{1}{\alpha+1}}_q ,$$

where $\mathring{B}^s_{q,q}(\mathbb{R}^n)$ *is the classical Euclidean Besov space.*

6. Embedding into BMO and Essential continuity

Theorem 28. *Let* (Ω, d, μ) *be a uniform* $(k,m)-$*space. Let* X *be a r.i. space on* Ω. *Then*

$$\|f\|_{BMO(\Omega)} \preceq \sup_{t>0} \frac{R(t)}{t\phi_X(t)} E_X(f, r(t)).$$

Proof. Let $B = B(x)$ be a ball centered at x. Since (Ω, d, μ) is uniform, we have that

$$\mu(B) \leq \mu(B(x, r(\mu(B)/c)) \leq CR(\mu(B)/c).$$

Then

$$\begin{aligned}
I &:= \int_B \left| f(y) - \fint_{B(x)} f(s) d\mu(s) \right| d\mu(y) \\
&\leq \fint_B \left(\int_{B(x)} |f(y) - f(s)|\, d\mu(s) \right) d\mu(y) \\
&\leq \fint_B \int_{B(x, r(\mu(B)/c))} |f(y) - f(s)|\, d\mu(s) d\mu(y) \\
&\leq C \frac{R(\mu(B)/c)}{\mu(B)} \int_B \left(\fint_{B(x, r(\mu(B)/c))} |f(s) - f(y)|\, d\mu(s) \right) d\mu(y) \\
&\leq C \frac{R(\mu(B)/c)}{\mu(B)} E_X(f, r(\mu(B)/c)) \|\chi_B\|_{X'} \quad \text{(by (5))} \\
&= C \frac{R(\mu(B)/c)}{\phi_X(\mu(B))} E_X(f, r(\mu(B)/c)).
\end{aligned}$$

Using this estimate and Remark 29, we get

$$\begin{aligned}
\|f\|_{BMO(\Omega)} &= \sup_B \fint_B \left| f(y) - \fint_B f(s)d\mu(s) \right| d\mu(y) \\
&\leq \sup_{\mu(B)} C \frac{R(\mu(B)/c)}{\mu(B)\phi_X(\mu(B))} E_X(f, r(\mu(B)/c)) \\
&\leq \sup_{t>0} \sup_{t/C_D \leq \mu(B) \leq t} C \frac{R(\mu(B)/c)}{\mu(B)\phi_X(\mu(B))} E_X(f, r(\mu(B)/c)) \\
&\leq \sup_{t>0} \frac{R(t/c)}{t\phi_X(C_D t)} E_X(f, r(t/c)) \\
&\leq \sup_{t>0} \frac{R(t)}{t\phi_X(t)} E_X(f, r(t)).
\end{aligned}$$

□

6.1. **Essential continuity.** We are going to obtain conditions for the essential continuity of functions in Besov spaces (see [25] for some related results).

Let f be a $\mu-$measurable function on Ω. The **signed decreasing rearrangement**[6] $f_\mu^\star : [0, \infty] \to \mathbb{R}$ of f defined by

$$f_\mu^\star(s) = \inf\{t \in \mathbb{R} : \mu_f^\star(t) \leq s\},$$

where $\mu_f^\star(s) = \mu\{x \in \Omega : f(x) > s\}$. It follows readily from the definition that

$$f_\mu^\star(0^+) = ess\sup f \quad \text{and} \quad f_\mu^\star(\infty) = ess\inf f. \tag{34}$$

If f is $\mu-$integrable on Ω, the **signed maximal function** is defined by

$$f_\mu^\star(t) := \frac{1}{t}\int_0^t f_\mu^\star(s)ds = \frac{1}{t}\sup\left\{\int_E f(s)d\mu : \mu(E) = t\right\},$$

moreover it is subadditive (i.e. $(f+g)_\mu^{\star\star}(t) \leq f_\mu^{\star\star}(t) + g_\mu^{\star\star}(t)$).

Note that for positive functions

$$f_\mu^\star(t) = f_\mu^*(t),$$

and, moreover, that for $c \in \mathbb{R}$,

$$(f+c)_\mu^\star(t) = f_\mu^\star(t) + c. \tag{35}$$

The functions f and $f_\mu^\star$ are equimeasurable (that is, they have the same distribution function).

Theorem 29. *Let (Ω, d, μ) be a uniform $(k,m)-$space. Let X be a r.i. space on Ω. Then. Let $f \in L^1(\Omega) + L^\infty(\Omega)$, such that*

$$\int_0^\infty \frac{R(t)}{t\phi_X(t)} E_X(f, r(t)) \frac{dt}{t} < \infty$$

then f is $\mu-$locally essentially continuous.

Proof. If $f \geq 0$, by Theorem 18, we get

$$f_\mu^{\star\star}(t) - f_\mu^\star(t) = f_\mu^{**}(t) - f_\mu^*(t) \leq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)).$$

[6] We refer the reader to [34] and the references quoted therein for a complete treatment.

If f is bounded from below and $c = \inf(f)$, then $f - c \geq 0$, and therefore

$$\begin{aligned} O_{\mu}(f-c)(t) &\preceq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f-c, r(\kappa_0 t)) \\ &\preceq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t)). \end{aligned}$$

By (35),

$$f_{\mu}^{\star\star}(t) - f_{\mu}^{\star}(t) = (f-c)_{\mu}^{**}(t) - (f-c)_{\mu}^{*}(t),$$

and thus

$$f_{\mu}^{\star\star}(t) - f_{\mu}^{\star}(t) \preceq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t)).$$

Let $f \in L^1(\Omega) + L^{\infty}(\Omega)$, and let B be a ball. Given $n \in \mathbb{N}$, we consider $f_n = \max(f\chi_B, -n)$. Since f_n is bounded from below, we get

$$\begin{aligned} (f_n)_{\mu}^{\star\star}(t) - (f_n)_{\mu}^{\star}(t) &\preceq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f_n, r(\kappa_0 t)) \\ &\preceq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t)). \end{aligned}$$

Let $0 < a < \mu(B)$. By the fundamental Theorem of Calculus

$$\int_a^{\mu(B)} \left((f_n)_{\mu}^{\star\star}(t) - (f_n)_{\mu}^{\star}(t)\right)\frac{dt}{t} = \frac{1}{a}\int_0^a (f_n)_{\mu}^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f_n)_{\mu}^{\star}(t)dt.$$

Since $f_n(z) \to f\chi_B(z)$ μ−a.e, and $|f_n| \leq |f\chi_B|$ we have

$$\frac{1}{a}\int_0^a (f_n)_{\mu}^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f_n)_{\mu}^{\star}(t)dt \underset{n\to\infty}{\to} \frac{1}{a}\int_0^a (f\chi_B)_{\mu}^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f\chi_B)_{\mu}^{\star}(t)dt.$$

Letting $a \to 0$, we get

$$\begin{aligned} (f\chi_B)_{\mu}^{\star\star}(0) - (f\chi_B)_{\mu}^{\star\star}(\mu(B)) &\preceq \int_0^{\mu(B)} \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t))\frac{dt}{t} \\ &\preceq \int_0^{k\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}. \end{aligned}$$

By (34)

$$\begin{aligned} & ess\sup f\chi_B - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f\chi_B)_{\mu}^{\star}(t)dt \\ & \preceq \int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}. \end{aligned}$$

Similarly, considering $-f\chi_B$, instead of $f\chi_B$, we obtain

$$\begin{aligned} & \frac{1}{\mu(B)}\int_0^{\mu(B)} (-f\chi_B)_{\mu}^{\star}(s)ds - ess\inf(f\chi_B) \\ & \preceq \int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}. \end{aligned}$$

Since $f\chi_B$ and $-f\chi_B$ are both supported on B, we have that

$$\int_0^{\mu(B)} (f\chi_G)_{\mu}^{\star}(s)ds = \int_B f d\mu \quad \text{and} \quad \int_0^{\mu(B)} (-f\chi_G)_{\mu}^{\star}(s)ds = -\int_B f d\mu.$$

Adding these results, we have that for $\mu-$almost every $x, y \in B$

$$\begin{aligned} |f(x)-f(y)| &\leq ess\sup(f\chi_B) - ess\inf(f\chi_B) \\ &\leq 2\int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)} E_X(f,r(t))\frac{dt}{t}, \end{aligned}$$

and $\mu-$locally essentially continuity follows. $\square$

7. Sobolev type embeddings for homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(\Omega)$

In this Section we are going to consider in detail Sobolev type embeddings for homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ where $0<p<\infty$, $0<q\leq\infty$.

First of all, notice that an elementary computation (see Remark 20) shows that

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \simeq \begin{cases} \left(\int_0^\infty \left(\frac{E_p(f,r(t))}{r(t)^s}\right)^q \frac{dt}{t}\right)^{1/q}, & 0<q<\infty, \\ \sup_{t>0} r(t)^{-s}E_p(f,r(t)), & q=\infty. \end{cases}$$

In case that $1\leq p<\infty$ and $1\leq q\leq\infty$, our results will be a direct consequence of Theorem 24, however if $0<p,q<1$ then $L^p(\Omega)$ and $L^q([0,\infty))$ are not Banach spaces, thus the duality arguments used in the previous theory cannot be applied.

Lemma 30. *Let $0<p<1$. Let $f\in L^p(\Omega)+L^\infty(\Omega)$, then:*

(1) *If (Ω,d,μ) is uniform, then for all $t>0$, we have that*

$$O_\mu(|f|^p,t) \leq \frac{R(\kappa_0 t)}{(\kappa_0 t)^2} E_p(f,\kappa_0 t)^p.$$

(2) *If (Ω,d,μ) is bounded from below, then for all $t>0$, we have that*

$$O_{\tilde{\mu}}(|f|^p,t) \leq \frac{R(\kappa_1 t)}{(\kappa_1 t)^2} E_p(f,r(\kappa_1 t))^p.$$

Proof. Let $B=B(x)$ be a ball centered at x, since $0<p<1$, we have that

$$|f(x)|^p\chi_{B(x)}(y) \leq |f(x)-f(y)|^p\chi_{B(x)}(y) + |f(y)|^p\chi_{B(x)}(y).$$

Integrating with respect to $d\mu(y)$, we have that

$$\begin{aligned} |f(x)|^p\mu(B) &\leq \int_{B(x)} |f(x)-f(y)|^p d\mu(y) + \int_{B(x)} |f(y)|^p d\mu(y) \\ &\leq \int_{B(x)} |f(x)-f(y)|^p d\mu(y) + \int_0^{\mu(B)} \left(|f|^p\right)^*_\mu(s)ds \quad \text{(by (1))}. \end{aligned}$$

Now integrating with respect to $d\mu(x)$ over a subset $E\subset\Omega$ with $\mu(E)=\mu(B)/2$, we get

$$\begin{aligned} \int_E |f(x)|^p d\mu(x) &\leq \int_E \fint_{B(x)} |f(x)-f(y)|^p d\mu(y)d\mu(x) + \int_E \frac{1}{\mu(B)}\left(\int_0^{\mu(B)} f^*_\mu(s)ds\right)d\mu(x) \\ &\leq \int_\Omega \fint_{B(x)} |f(x)-f(y)|^p d\mu(y)d\mu(x) + \frac{1}{2}\int_0^{\mu(B)} f^*_\mu(s)ds. \end{aligned}$$

By (2), taking the supremum over all such sets E, we obtain

$$\int_0^{\mu(B)/2} \left(|f|^p\right)^*_\mu(s)ds \leq \int_\Omega \fint_{B(x)} |f(x)-f(y)|^p d\mu(y)d\mu(x) + \frac{1}{2}\int_0^{\mu(B)} \left(|f|^p\right)^*_\mu(s)ds.$$

Equivalently,

$$\left(|f|^p\right)_\mu^{**}(\mu(B)/2)-\left(|f|^p\right)_\mu^{**}(\mu(B))\leq\frac{1}{\mu(B)}\int_\Omega\fint_{B(x)}|f(x)-f(y)|^p d\mu(y)d\mu(x).$$

Now (1) and (2) follow in the same way as Theorem 18. □

Definition 31. *Let $0<p<\infty$, $0<q\leq\infty$ and let v be a weight on $(0,\infty)$, the space $S_\mu^{p,q}(v)$ is the collection of all $\mu-$measurable functions such that $\|f\|_{S_\mu^{p,q}(v)}<\infty$, where*

$$\|f\|_{S_\mu^{p,q}(v)}=\left(\int_0^\infty O_\mu(|f|^p,t)^{\frac{q}{p}}v(t)dt\right)^{1/q}.$$

Remark 32. *For $p=1$ the spaces $S_\mu^{1,q}(v)$ were introduced in* [5]*. Notice that, if $1\leq p<\infty$, $1\leq q\leq\infty$, then*

$$S_\mu^{L^p,L^q}(v_s)=S_\mu^{1,q}(v_s).$$

Corollary 33. *Let (Ω,d,μ) be a $(k,m)-$space. Let $0<s<1$ and $0<p<\infty$, $0<q\leq\infty$. Let*

$$v(t)=\min\left(t^{1+\frac{1}{\max(1,p)}-\frac{m+s\min(1,p)}{k}},t^{1+\frac{1}{\max(1,p)}-\frac{k+s\min(1,p)}{m}}\right)^{\frac{q}{\min(1,p)}}\frac{1}{t}.$$

(1) *If (Ω,d,μ) is uniform, then*
$$\dot{\mathcal{B}}_{p,q}^s(\Omega)\subset S_\mu^{\min(1,p),q}(v).$$
(2) *If (Ω,d,μ) is bounded from below, then*
$$\dot{\mathcal{B}}_{p,q}^s(\Omega)\subset S_{\tilde{\mu}}^{\min(1,p),q}(v).$$

Proof. Part (1) In the case $1\leq p<\infty$ the proof given in Theorem 24 works. In case that $0<p<1$, then by Lemma 30 it follows that

$$\left(\frac{t^2}{t^{p/q}R(t)\,r(t)^{sp}}O_\mu(|f|^p,t)\right)^{1/p}\preceq\frac{r(\kappa_0 t)^{-s}E_p(f,r(\kappa_0 t))}{(\kappa_0 t)^{1/q}},$$

and the result is obtained by taking $L^q([0,\infty))$-(quasi) norm in both sides.

Part (2) can be proved in the same way. □

The following lemmas will be useful in what follows:

Lemma 34. *(see* [2, Lemma 5.4]*) Let $1\leq p<\infty$, and suppose that (w,v) is a pair of weights satisfying the following condition: there exists $C>0$ such that for all $0<t<1$,*

$$\left(\int_0^t w(s)ds\right)^{1/q}\left(\int_t^1\frac{v(s)^{\frac{-1}{q-1}}}{s^{\frac{q}{q-1}}}ds\right)^{(q-1)/q}\leq C.$$

Then

$$\left(\int_0^1 f_\mu^{**}(s)^q w(s)ds\right)^{1/q}\leq\left(\int_0^1\left(f_\mu^{**}(s)-f_\mu^*(s)\right)^q v(s)ds\right)^{1/q}$$
$$+\left(\int_0^1 w(s)ds\right)^{1/q}\int_0^1 f_\mu^*(t)ds.$$

Lemma 35. *Let $0 < q < 1$ and $b > 0$, then*

$$\left(\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t}\right)^{1/q} \preceq \left(\int_0^1 t^b \left(f_\mu^{**}(t) - f_\mu^*(t)\right)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1).$$

Proof. We integrate by parts and obtain,

$$\begin{aligned}\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t} &= \frac{1}{b}\left[t^b f_\mu^{**}(t)^q\right]_0^1 + \frac{q}{b}\int_0^1 t^b f_\mu^{**}(t)^{q-1}\left(f_\mu^{**}(t) - f_\mu^*(t)\right)\frac{dt}{t} \\ &\le \frac{1}{b}\left[t^b f_\mu^{**}(t)^q\right]_0^1 + \frac{q}{b}\int_0^1 t^b \left(f_\mu^{**}(t) - f_\mu^*(t)\right)^q \frac{dt}{t} \text{ (since } q < 1).\end{aligned}$$

Since

$$\left[t^b f_\mu^{**}(t)^q\right]_0^b = f_\mu^{**}(1)^q - \lim_{t\to 0} t^b f_\mu^{**}(t)^q.$$

To finish the proof we need to see that the previous limit is finite. If $f_\mu^{**}(0) < \infty$ there is nothing to prove. If $f_\mu^{**}(0) = \infty$, taking into account that $tO_\mu(f,t)$ is increasing, we get

$$\begin{aligned}t\left(f_\mu^{**}(t) - f_\mu^*(t)\right)\left(\int_t^1 s^{b-1-q}\right)^{1/q} &\le \left(\int_t^1 s^q \left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q s^{b-1-q}\right)^{1/q} \\ &= \left(\int_t^1 s^b \left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.\end{aligned}$$

If $t < 1/2$, then

$$\left(\int_t^1 s^{b-1-q}\right)^{1/q} \ge \left(\int_t^{2t} s^{b-1-q}\right)^{1/q} \simeq t^{b/q-1}, \text{ if } b \neq q,$$

and

$$\left(\int_t^1 s^{b-1-q}\right)^{1/q} \ge \left(\int_t^{2t} \frac{1}{s}\right)^{1/q} \simeq 1, \text{ if } b = q.$$

Thus

$$t^{b/q}\left(f_\mu^{**}(t) - f_\mu^*(t)\right) \preceq \left(\int_t^1 s^b \left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.$$

Finally, by L'Hôpital's rule,

$$\begin{aligned}\lim_{t\to 0} t^{b/q} f_\mu^{**}(t) &= \lim_{t\to 0} \frac{f_\mu^{**}(t)}{t^{-b/q}} = \lim_{t\to 0} \frac{-(f_\mu^{**}(t) - f_\mu^*(t))/t}{-\frac{b}{q}t^{-b/q-1}} \\ &= \lim_{t\to 0} \frac{t^{b/q}(f_\mu^{**}(t) - f_\mu^*(t)}{b/q} \preceq \left(\int_0^1 s^b \left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.\end{aligned}$$

□

Lemma 36. *Given $a < b < \infty$, we define*

$$v(t) = \frac{\min(t^a, t^b)}{t}.$$

Let $0 < q \le \infty$ and $f \in L^1(\Omega) + L^\infty(\Omega)$, with $f_\mu^*(\infty) = 0$.

(1) *If $0 < a < b < \infty$, then*

$$\|f\|_{S_\mu^{1,q}(v)} \simeq \left(\int_0^\infty f_\mu^{**}(t)^q v(t)dt\right)^{1/q}. \tag{36}$$

(2) *If $a \leq 0$, then*

$$\left(\int_0^1 f_\mu^{**}(t)^q v(t)dt\right)^{1/q} \preceq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

(3) *If $b = 0$ and $q > 1$, then*

$$\left(\int_0^\infty \left(\frac{f_\mu^{**}(t)^q}{1+\ln\frac{1}{t}}\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

(4) *If $b = 0$ and $q \leq 1$ or $b < 0$ and $0 < q \leq \infty$, then*

$$\|f\|_\infty \preceq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

Proof. (1) By [5, Corollary 4.3.], (36) holds if, and only if,

$$\int_0^r v(t)dt \preceq r^q \int_r^\infty \frac{v(t)}{t^q}dt, \quad r > 0.$$

Pick $0 < \varepsilon < a$, then

$$\begin{aligned}
\int_0^r v(t)dt &= \int_0^r \min(t^{a-\varepsilon}, t^{b-\varepsilon})\frac{dt}{t^{1-\varepsilon}} \leq \min(r^{a-\varepsilon}, r^{b-\varepsilon})\int_0^r \frac{dt}{t^{1-\varepsilon}} \\
&\preceq \min(r^a, r^b) \preceq \min(r^a, r^b)\frac{r^q}{r^q} \\
&\preceq \min(r^a, r^b) r^q \int_r^{2r} \frac{dt}{s^{q+1}} \preceq r^q \int_r^{2r} v(t)\frac{dt}{s^q} \\
&\preceq r^q \int_r^\infty \frac{v(t)}{t^q}dt.
\end{aligned}$$

(2) By Lemma 35

$$\left(\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t}\right)^{1/q} \preceq \left(\int_0^1 t^b O_\mu(f,t)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1)$$

(3) By Lemma 34 with $w(t) = \left(\frac{1}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{1}{s}$ and $v(t) = \frac{1}{t}$, we get

$$\begin{aligned}
\left(\int_0^1 \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} &\preceq \left(\int_0^1 O_\mu(f,t)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1) \\
&\preceq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1),
\end{aligned}$$

and,

$$\left(\int_1^\infty \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq f_\mu^{**}(1)\left(\int_1^\infty \left(\frac{1}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq f_\mu^{**}(1).$$

(4) If $b = 0$ and $q = 1$, then

$$\|f\|_\infty = f_\mu^{**}(0) = \int_0^1 O_\mu(f,t)\frac{dt}{t} + f_\mu^{**}(1) \leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

If $b = 0$ and $0 < q < 1$, let $0 < r < 1$, then

$$
\begin{aligned}
(37) \qquad f_\mu^{**}(r) - f_\mu^{**}(1) &= \int_r^1 O_\mu(f,t)\frac{dt}{t} = f_\mu^{**}(r)^{1-q}\int_r^1 \frac{O_\mu(f,t)}{f_\mu^{**}(r)^{1-q}}\frac{dt}{t} \\
&\le f_\mu^{**}(r)^{1-q}\int_r^1 \frac{O_\mu(f,t)}{f_\mu^{**}(t)^{1-q}}\frac{dt}{t} \\
&\le f_\mu^{**}(r)^{1-q}\int_r^1 \frac{O_\mu(f,t)}{O_\mu(f,t)^{1-q}}\frac{dt}{t} \\
&= f_\mu^{**}(r)^{1-q}\int_r^1 O_\mu(f,t)^q\frac{dt}{t} \\
&\le (1-q)f_\mu^{**}(r) + q\left(\int_0^1 O_\mu(f,t)^q\frac{dt}{t}\right)^{1/q}
\end{aligned}
$$

thus

$$
qf_\mu^{**}(r) \le q\left(\int_0^1 O_\mu(f,t)^q\frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1)
$$

which implies

$$
\|f\|_\infty = f_\mu^{**}(0) \le \left(\int_0^1 O_\mu(f,t)^q\frac{dt}{t}\right)^{1/q} + \frac{f_\mu^{**}(1)}{q}.
$$

If $b < 0$ and $1 \le q \le \infty$, then

$$
\begin{aligned}
f_\mu^{**}(0) - f_\mu^{**}(1) &= \int_0^1 O_\mu(f,t)\frac{dt}{t} \\
&\le \left(\int_0^1 \left(t^b O_\mu(f,t)\right)^q\frac{dt}{t}\right)^{1/q}\left(\int_0^1 \left(t^{-b}\right)^{\frac{q}{q-1}}\frac{dt}{t}\right)^{\frac{q-1}{q}} \\
&\preceq \left(\int_0^1 \left(t^b O_\mu(f,t)\right)^q\frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

If $b < 0$ and $0 < q < 1$, then

$$
f_\mu^{**}(0) - f_\mu^{**}(1) = \int_0^1 O_\mu(f,t)\frac{dt}{t} \le \int_0^1 t^b O_\mu(f,t)\frac{dt}{t}
$$

and we finish the proof in the same way as in (37). □

Now we are ready to establish our Sobolev embedding Theorem for homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(\Omega)$. Motivated by the classical theory we will distinguish three cases: The subcritical case when an embedding into a Lorentz type spaces holds, the critical case if $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ is embedded into a logarithmic Lorentz space and the supercritical case if the Besov space is embedded into L^∞.

Theorem 37. *Let (Ω, d, μ) be a uniform $(k,m)-$space. Let $0 < s < 1$, $0 < p < \infty$, $0 < q \le \infty$ and $f \in L^{\min(1,p)}(\Omega) + L^\infty(\Omega)$, with $\left(|f|^{\min(1,p)}\right)^*_\mu(\infty) = 0$.*

(1) ***Subcritical case:***
   (a) *If $s\min(1,p) < k(1+\frac{1}{\max(1,p)}) - m$, then*

$$
\|f\|_{L^{\alpha,q}(\Omega)+L^{\beta,q}(\Omega)} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)},
$$

where

$$\frac{\min(1,p)}{\alpha} = 1 + \frac{1}{\max(1,p)} - \frac{k + s\min(1,p)}{m},$$
$$\frac{\min(1,p)}{\beta} = 1 + \frac{1}{\max(1,p)} - \frac{m + s\min(1,p)}{k}.$$

(b) *If* $k(1 + \frac{1}{\max(1,p)}) - m < s\min(1,p) < m(1 + \frac{1}{\max(1,p)}) - k$, *then*

$$\left(\int_0^1 \left(t^{\frac{1}{\alpha}} f_\mu^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}, \tag{38}$$

where $\frac{\min(1,p)}{\alpha} = 1 + \frac{1}{\max(1,p)} - \frac{m+s\min(1,p)}{k}$.

(2) ***Critical case:***
If $s\min(1,p) = m(1 + \frac{1}{\max(1,p)}) - k$, *then*

(a) *If* $q > 1$, *we get*

$$\left(\int_0^\infty \left(\frac{f_\mu^{**}(t)}{1 + \ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

(b) *If* $0 < q \leq 1$, *we get*

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

(3) ***Supercritical case:***
If $s\min(1,p) > m(1 + \frac{1}{\max(1,p)}) - k$, *then*

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

Proof. The proof follows from Lemma 36. Let us see (38), if $0 < p < 1$. Then

$$v(t) = \min\left(t^{2-\frac{m+sp}{k}}, t^{2-\frac{k+sp}{m}}\right)^{\frac{q}{p}} \frac{1}{t}$$

with $2 - \frac{m+sp}{k} \leq 0$, by Lemma 36, applied to $|f|^p$ and q/p we have that

$$\left(\int_0^1 \left(\frac{1}{t}\int_0^t f_\mu^*(s)^p ds\right)^{q/p} t^{\left(2-\frac{k+sp}{m}\right)\frac{q}{p}} \frac{dt}{t}\right)^{p/q} \preceq \left\||f|^p\right\|_{S_\mu^{1,q/p}(v)} + \left(|f|^p\right)_\mu^{**}(1).$$

Obviously

$$\left(\int_0^1 \left(f_\mu^*(s) t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{p/q} \leq \left(\int_0^1 \left(\frac{1}{t}\int_0^t f_\mu^*(s)^p ds\right)^{q/p} t^{\left(2-\frac{k+sp}{m}\right)\frac{q}{p}} \frac{dt}{t}\right)^{p/q}.$$

Since

$$\begin{aligned}\left\||f|^p\right\|_{S_\mu^{1,q/p}(v)} &= \left(\int_0^\infty O_\mu(|f|^p, t)^{\frac{q}{p}} v(t) dt\right)^{p/q} = \|f\|^p_{S_\mu^{p,q}(v)} \\ &\preceq \|f\|^p_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \text{ (by Corollary 33)},\end{aligned}$$

we have that

$$\left(\int_0^1 \left(f_\mu^*(s) t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{p/q} \preceq \|f\|^p_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \left(|f|^p\right)_\mu^{**}(1),$$

thus

$$\left(\int_0^1 \left(f_\mu^*(s)t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \left(\left(|f|^p\right)^{**}_\mu(1)\right)^{1/p}$$
$$= \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^p(\Omega)+L^\infty(\Omega)}.$$

All other cases can be proved in the same way. □

With the same proof as Theorem 37 we obtain:

Theorem 38. *Let (Ω, d, μ) be a $(k, m)-$space bounded from below. Let $f \in L^1(\Omega)+L^\infty(\Omega)$. Then Theorem 37 holds, considering $f_{\tilde{\mu}}^{**}$ and $f_{\tilde{\mu}}^*$ instead of f_μ^{**} and f_μ^*.*

By Theorems 28, 29 and 26, we obtain:

Corollary 39. *Let (Ω, d, μ) be a uniform $(k, m)-$space. Let $1 \le p < \infty$ and $0 < s < 1$. Then*

(1)
$$\|f\|_{BMO(\Omega)} \preceq \sup_{0<t<1} t^{k-m(1+/p)} E_p(f,t) + \sup_{t>1} t^{m-k(1+/p)} E_p(f,t).$$

(2) If
$$\int_0^1 t^{k-m(1+/p)} E_p(f,t)\frac{dt}{t} + \int_1^\infty t^{m-k(1+/p)} E_p(f,t)\frac{dt}{t} < \infty,$$
then, f is $\mu-$locally essentially continuous.

(3) Let $s < k(1+\frac{1}{p}) - m$, and let $w > 0$ be such that
$$[w] := \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)^*_\mu(t)\right)^s \frac{1}{\min\left(t^{1-\frac{m+s}{k}}, t^{1-\frac{k+s}{m}}\right)t^{\frac{1}{p}-\frac{1}{q}}}\right) < \infty.$$
Then, for all $\alpha > 0$ and $q \ge 1$, we have that
$$\|f\|_{L^q} \preceq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \left\|w^{\alpha s} f\right\|^{\frac{1}{\alpha+1}}_{L^q}.$$

We finish the paper by collecting all our results in the particular case that (Ω, d, μ) is $Q-$regular.

Theorem 40. *Let (Ω, d, μ) be a $Q-$regular. Let $0 < p < \infty$, $0 < q \le \infty$, and $0 < s < 1$ and $f \in L^{\min(1,p)}(\Omega) + L^\infty(\Omega)$, with $\left(|f|^{\min(1,p)}\right)^*_\mu(\infty) = 0$. Then:*

(1) ***Subcritical case**, $s < \frac{Q}{p}$:*
$$\|f\|_{L^{p(Q),q}(\Omega)} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)}$$
where $p(Q) = Qp/(Q - sp)$.

Moreover, let $w > 0$ be such that
$$[w] := \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)^*_\mu(t)\right)^s \frac{1}{t^{\frac{1}{p}-\frac{1}{q}-\frac{s}{Q}}}\right) < \infty.$$
If $1 \le p, q < \infty$, then for all $\alpha > 0$, we have that
$$\|f\|_{L^q} \preceq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \left\|w^{\alpha s} f\right\|^{\frac{1}{\alpha+1}}_{L^q}.$$

(2) ***Critical case**, $s = \frac{Q}{p}$:*

(a) *If* $q > 1$, *then*

$$\left(\int_0^\infty \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

(b) *If* $0 < q \leq 1$, *then*

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^{\frac{Q}{p}}_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

(c) *If* $p \geq 1$, *we get:*

(i)

$$\|f\|_{BMO(\Omega)} \preceq \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,\infty}(\Omega)}.$$

(ii) *If* $f \in \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,1}(\Omega)}$, *then* f *is* $\mu-$*locally essentially continuous.*

(3) ***Supercritical case***, $s > \frac{Q}{p}$:

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^{s}_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}(\Omega)+L^\infty(\Omega)}.$$

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