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BOUNDED LINEAR AND COMPACT OPERATORS BETWEEN THE HAHN SPACE AND SPACES OF STRONGLY SUMMABLE AND BOUNDED SEQUENCES

EBERHARD MALKOWSKY, VLADIMIR RAKOČEVIĆ, VESNA VELIČKOVIĆ

(Presented at the 5th Meeting, held on June 16, 2021)

A b s t r a c t. *We establish the characterisations of the classes of bounded linear operators from the generalised Hahn sequence space h_d, where d is an unbounded monotone increasing sequence of positive real numbers, into the spaces w_0^p, w^p and w_∞^p of sequences that are strongly summable to zero, strongly summable and strongly bounded by the Cesàro method of order one and index p for $1 \le p < \infty$. Furthermore, we prove estimates for the Hausdorff measure of noncompactness of bounded linear operators from h_d into w^p, and identities for the Hausdorff measure of noncompactness of bounded linear operators from h_d to w_0^p. We use these results to characterise the classes of compact operators from h_d to w^p and w_0^p. Finally, we provide an example for some applications of our results and visualisations in crystallography.*

AMS Mathematics Subject Classification (2020): 46A45, 40C05, 46B45, 47H08.
Key Words: The Hahn sequence space; spaces of strongly summable and sequences; bounded bounded linear operators; Hausdorff measure of noncompactness; compact operators.

1. *Introduction and notation*

The Hahn space h was originally introduced and studied by Hahn [8], and later generalised by Goes [7]. Matrix transformations and bounded and compact operators the Hahn space have recently been studied in various papers, for instance in [17, 6, 11, 4]. A survey of these recent results can be found in [10].

We establish the characterisations of the classes $\mathcal{B}(h_d, Y)$ of bounded linear operators and their norms from the generalised Hahn space h_d into each of the spaces $Y \in \{w_0^p, w^p, w_\infty^p\}$, formulas for the Hausdorff measure of noncompactness of operators in $\mathcal{B}(h_d, w_0^p)$ and $\mathcal{B}(h_d, w^p)$, and the characterisations of their subclasses $\mathcal{K}(h_d, w_0^p)$ and $\mathcal{K}(h_d, w^p)$ of compact operators. Since the operators can be represented by infinite matrices of complex numbers, in each case, the characteristaions are expressed in terms of necessary and sufficient conditions on the entries of the matrices. Since each one of these operators can be represented by an infinite matrix of complex numbers, the mentioned characterisations are achieved by establishing necessary and sufficient conditions on the entries of the reprenting matrices to map between the respective spaces.

Measures of noncompactness are widely used in fixed point theory and applied in the study of differential and integral equations. We refer the interested reader to [1, 2, 3, 15, 22, 13]. Our results could also be used in the study of sequence spaces equations and sequence spaces inclusion relations; for related results we refer to [5].

We use the standard notations ω for the set of all complex sequences $x = (x_k)_{k=1}^\infty$, and ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences, that is, sequences terminating in zeros. We denote by $e = (e_k)_{k=1}^\infty$ and $e^{(n)} = (e_k^{(n)})_{k=1}^\infty$ $(n \in \mathbb{N})$ the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

We recall that a BK space X is a Banach sequence space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^\infty \in X$. A BK space $X \supset \phi$ is said to have AK if $x = \lim_{m\to\infty} x^{[m]}$ for all $x = (x_k)_{k=1}^\infty \in X$, where $x^{[m]} = \sum_{k=1}^m x_k e^{(k)}$.

Let $X \subset \omega$. Then the set $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k \text{ converges for all } x \in X\}$ is the *β–dual of* X. Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers, $A_n = (a_{nk})_{k=1}^\infty$ and $A^k = (a_{nk})_{n=1}^\infty$ be the sequences in the n^{th} row and the k^{th} column of A, and X and Y be subsets of ω. Then we write $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=1}^\infty$ for $x = (x_k)_{k=1}^\infty$ provided all the series converge. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the *matrix domain of* A *in* X, and (X, Y) denotes the class of all matrix transformations from X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$.

If X and Y are Banach spaces, we use the standard notation $\mathcal{B}(X, Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{|L(x)| : \|x\| = 1\}$. Also $\mathcal{K}(X, Y)$ denotes the class of all compact operators in $\mathcal{B}(X, Y)$.

For every sequence $x = (x_k)_{k=1}^{\infty} \in \omega$, let $\Delta_k x = x_k - x_{k+1}$ $(k = 1, 2, \dots)$. Goes [7] introduced and studied the generalised Hahn space h_d for arbitrary complex sequences $d = (d_k)_{k=1}^{\infty}$ with $d_k \neq 0$ for all k by

$$h_d = \left\{ x \in \omega : \sum_{k=1}^{\infty} |d_k| \cdot |\Delta_k x| < \infty \right\} \cap c_0,$$

with the norm

$$\|x\|_{h_d} = \sum_{k=1}^{\infty} |d_k| \cdot |\Delta_k x| \text{ for all } x = (x_k)_{k=1}^{\infty} \in h_d. \tag{1.1}$$

Recent research on the Hahn space and its generalisations can be found, for instance, in [19, 10, 20, 21, 4, 17, 6] and the survey paper [11].

Let $1 \leq p < \infty$

$$w_0^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\},$$

$$w^p = w_0^p \oplus e = \left\{ x \in \omega : x - \xi e \in w_0^p \text{ for some } \xi \in \mathbb{C} \right\}$$

and

$$w_\infty^p = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p < \infty \right\}$$

denote the sets of sequences that are strongly summable C_1 to zero, strongly summable C_1 and strongly bounded C_1 ([12]), with index p.

It is well-known ([13, Proposition 3.44]) that w_0^p, w^p and w_∞^p are BK spaces with

$$\|x\|_{w_\infty^p} = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}; \tag{1.2}$$

w_0^p is a closed subspace of w^p, and w^p is a closed subspace of w_∞^p; w_0^p has AK and every sequence $x = (x_k)_{k=1}^{\infty} \in w^p$ has a unique representation

$$x = \xi e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)}, \tag{1.3}$$

where ξ is the unique complex number such that $x - \xi e \in w_0^p$, the so–called w^p*–limit of* x.

2. *The classes* $\mathcal{B}(h_d, Y)$ *for* $Y \in \{w_\infty^p, w^p, w_0^p\}$

Throughout let d be an unbounded increasing sequence of positive real numbers and $1 < p < \infty$.

In this section, we are going to characterise the classes $\mathcal{B}(h_d, Y)$ and compute the operator norm of $L \in \mathcal{B}(h_d, Y)$ for $Y \in \{w_\infty^p, w^p, w_0^p\}$. We will also establish a formula for the w^p–limit of $L(x)$ when $x \in h_d$ and $L \in \mathcal{B}(h_d, w^p)$.

Since $(h_d, \|\cdot\|_{h_d})$ is a BK space with AK by [17, Proposition 2.1], and each space Y is a BK space with respect to the norm $\|\cdot\|_{w_\infty^p}$ in (1.2), each operator $L \in \mathcal{B}(h_d, Y)$ can be represented by a matrix $A \in (h_d, Y)$ by [9, Theorem 1.9], that is, there exists an infinite matrix $A \in (h_d, Y)$ such that

$$L(x) = Ax \text{ for all } x \in h_d. \tag{2.1}$$

We are going to use these facts and notations throughout the paper.

Theorem 2.1. *We have*

(a) $L \in \mathcal{B}(h_d, w_\infty^p)$ *if and only if*

$$\|A\|_{(h_d, w_\infty^p)} = \sup_{l,m} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=1}^{l} \left| \sum_{k=1}^{m} a_{nk} \right|^p \right)^{1/p} < \infty; \tag{2.2}$$

(b) $L \in \mathcal{B}(h_d, w^p)$ *if and only if* (2.2) *holds and*

$$\left\{ \begin{array}{c} \text{for each } k \in \mathbb{N}\text{, there exists } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |a_{nk} - \alpha_k|^p = 0; \end{array} \right\} \tag{2.3}$$

(c) $L \in \mathcal{B}(h_d, w_0^p)$ *if and only if* (2.2) *holds and*

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |a_{nk}|^p = 0 \text{ for each } k. \tag{2.4}$$

(d) *If* $L \in \mathcal{B}(h_d, Y)$ *for* $Y \in \{w_\infty^p, w^p, w_0^p\}$*, then*

$$\|L\| = \|A\|_{(h_d, w_\infty^p)}. \tag{2.5}$$

Proof. We write $\|A\| = \|A\|_{(h_d, w_\infty^p)}$, for short.

(a) Let $L \in \mathcal{B}(h_d, w_\infty^p)$ and A be the infinite matrix that represents L as in (2.1). Since the set

$$E = \left\{ \frac{1}{m} e^{[m]} : m \in \mathbb{N} \right\}$$

is a determining set for h_d by [17, Proposition 3.2], we have to show by [23, Theorem 8.3.4] that the following two conditions are satisfied:

(i) The columns of A belong to w_∞^p;

(ii) $L(E)$ is a bounded subset of w_∞^p.

First we show (ii).

Let $m \in \mathbb{N}$ be given and $y^{(m)} = (1/d_m)e^{[m]} \in E$. Then we have

$$A_n y^{(m)} = \sum_{k=1}^{\infty} a_{nk} y_k^{(m)} = \frac{1}{d_m} \sum_{k=1}^{m} a_{nk},$$

hence

$$\left\| A y^{(m)} \right\|_{w_\infty^p}^p = \sup_l \frac{1}{l} \sum_{n=1}^{l} \left| A_n y^{(m)} \right|^p = \sup_l \frac{1}{l} \sum_{n=1}^{l} \left(\frac{1}{d_m} \left| \sum_{k=1}^{m} a_{nk} \right| \right)^p .$$

So (2.2) is the condition in (ii).

It remains to show that the condition in (i) is redundant.

We have $|a_{nk}| = |d_k A_n y^{(k)} - d_{k-1} A_n y^{(k-1)}|$ for all n and k, hence

$$\begin{aligned} \left\| A^k \right\|_{w_\infty^p} &= \sup_l \left(\frac{1}{l} \sum_{n=1}^{l} |a_{nk}|^p \right)^{1/p} \\ &\leq \sup_l \left(\frac{1}{l} \sum_{n=1}^{l} \left| d_k A_n y^{(k)} \right|^p \right)^{1/p} + \sup_l \left(\frac{1}{l} \sum_{n=1}^{l} \left| d_{k-1} A_n y^{(k-1)} \right|^p \right)^{1/p} \\ &\leq 2 d_k \left\| A \right\| < \infty \text{ for all } k. \end{aligned}$$

This completes the proof of Part (a).

(b) and (c) Since h_d is a BK space with AK and w_0^p and w^p are closed subspaces of the BK space w_∞^p by [13, Proposition 3.44], Parts (b) and (c) follow by [23, Theorem 8.3.6].

(d) Finally we assume that $L \in \mathcal{B}(h_d, Y)$, where $Y \in \{w_0^p, w^p, w_\infty^p\}$. Then $A_n \in h_d^\beta$ for all n and by [17, Proposition 2.3]

$$h_d^\beta = bs_d = \left\{ a \in \omega : \sup_m \frac{1}{d_m} \left| \sum_{k=1}^{m} a_k \right| < \infty \right\}.$$

Writing $L_n(x) = A_n x$ $(x \in h_d)$ for all n we obtain from [6, (2.6)] and Minkowski's inequality for all $x \in h_d$ and all $l \in \mathbb{N}$

$$\begin{aligned}
\left(\frac{1}{l}\sum_{n=1}^{l}|L_n(x)|^p\right)^{1/p} &= \left(\frac{1}{l}\sum_{n=1}^{l}|A_n x|^p\right)^{1/p} \\
&\le \left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{m=1}^{\infty} d_m|\Delta_m x|\frac{1}{d_m}\left|\sum_{j=1}^{m} a_{nj}\right|\right|^p\right)^{1/p} \\
&\le \frac{1}{l^{1/p}}\sum_{m=1}^{\infty} d_m|\Delta_m x|\cdot\left(\sum_{n=1}^{l}\left(\frac{1}{d_m}\left|\sum_{j=1}^{m} a_{nj}\right|\right)^p\right)^{1/p} \\
&\le \sup_{l,m}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{j=1}^{m} a_{nj}\right|^p\right)^{1/p}\cdot\|x\|_{h_d},
\end{aligned}$$

hence

$$\|L\| \le \|A\|. \tag{2.6}$$

Now let $m \in \mathbb{N}$ be given and $x^{(m)} = (1/d_m)e^{[m]}$. Then we have

$$\|x^{(m)}\|_{h_d} = \frac{1}{d_m}\sum_{k=1}^{\infty} d_k\left|\Delta_k x^{(m)}\right| = \frac{d_m}{d_m} = 1,$$

and

$$\begin{aligned}
\left\|L(x^{(m)})\right\|_{w_\infty^p} &= \sup_l\left(\frac{1}{l}\sum_{n=1}^{l}\left|A_n x^{(m)}\right|^p\right)^{1/p} \\
&= \sup_l\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m} a_{nk}\right|^p\right)^{1/p} \le \|L\|.
\end{aligned}$$

Since $m \in \mathbb{N}$ was arbitrary, we conclude $\|A\| \le \|L\|$, and this and (2.6) imply (2.5).

Now we establish a formula for the w^p–limits of $L(x)$ and $x \in h_d$, when $L \in \mathcal{B}(h_d, w^p)$.

Theorem 2.2. *Let $L \in \mathcal{B}(h_d, w^p)$ and α_k for $k \in \mathbb{N}$ be the complex numbers in* (2.3). *Then the w^p–limit $\eta(x)$ of $L(x)$ for each sequence $x \in h_d$ is given by*

$$\eta(x) = \sum_{k=1}^{\infty} \alpha_k x_k. \tag{2.7}$$

PROOF. Let $L \in \mathcal{B}(h_d, w^p)$. We define the matrix $B = (b_{nk})_{n,k=1}^{\infty}$ by

$$b_{nk} = a_{nk} - \alpha_k$$

for all n and k, and show

$$B \in (h_d, w_0^p). \tag{2.8}$$

First we show

$$(\alpha_k)_{k=1}^{\infty} \in bs_d. \tag{2.9}$$

We have for all $l, m \in \mathbb{N}$ by Hölder's inequality

$$\begin{aligned}
\frac{1}{d_m}\left|\sum_{k=1}^{m} \alpha_k\right| &= \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} \alpha_k\right| \\
&\le \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} (a_{nk} - \alpha_k)\right| + \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} a_{nk}\right| \\
&\le \frac{1}{d_m} \sum_{k=1}^{m} \frac{1}{l} \sum_{n=1}^{l} |a_{nk} - \alpha_k| + \frac{1}{d_m} \cdot \frac{1}{l} \cdot l^{1/q} \left(\sum_{n=1}^{l} \left|\sum_{k=1}^{m} \alpha_k\right|^p\right)^{1/p} \\
&\le \frac{1}{d_m} \sum_{k=1}^{m} \left(\frac{1}{l} \sum_{n=1}^{l} |a_{nk} - \alpha_k|^p\right)^{1/p} + \|A\|.
\end{aligned} \tag{2.10}$$

Since the first term in the last inequality above tends to 0 as l tends to infinity for each fixed m by (2.3), it follows that

$$\sup_m \frac{1}{d_m}\left|\sum_{k=1}^{m} \alpha_k\right| \le \|A\|_{(h_d, w_\infty)} < \infty,$$

and so (2.9) is satisfied and $(\alpha_k)_{k=1}^{\infty} \in h_d^{\beta}$ by [17, Proposition 2.3]. Also $A \in (h_d, w^p)$ implies $A_n \in h_d^{\beta}$ for each n, and consequently $B_n = A_n - (\alpha_k)_{k=1}^{\infty} \in h_d^{\beta}$ for each n.

We obtain by (2.10)

$$\begin{aligned}\|B\| &= \sup_{l,m}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}(a_{nk}-\alpha_k)\right|^p\right)^{1/p}\\ &\le \sup_{l,m}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}a_{nk}\right|^p\right)^{1/p} + \sup_{l,m}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}\alpha_k\right|^p\right)^{1/p}\\ &\le \|A\| + \sup_{l,m}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}\alpha_k\right|^p\right)^{1/p}\\ &\le \|A\| + \sup_{m}\frac{1}{d_m}\left|\sum_{k=1}^{m}\alpha_k\right|\\ &\le 2\,\|A\|.\end{aligned}$$

Thus, $B \in (h_d, w_\infty^p)$ by Theorem 2.1 (a).

Furthermore, $\lim_{l\to\infty}(1/l)\sum_{n=1}^{l}|b_{nk}| = 0$ for each k, by definition of the matrix B, that is, the condition in (2.4) also holds, and so (2.8) is satisfied by Theorem 2.1 (c).

Finally (2.7) is an immediate consequence of (2.8).

3. *The Hausdorff measure of noncompactness of operators*

In this section, we establish an identity for the Hausdorff measure on noncompactness of operators in $\mathcal{B}(h_d, w_0^p)$ and an estimate for the Hausdorff measure of noncompactness of operators in $\mathcal{B}(h_d, w^p)$. We also characterise the classes $\mathcal{K}(h_d, w_0^p)$ and $\mathcal{K}(h_d, w^p)$.

We refer to [22, Definition II.2.1] and [16, Definition 7.11.1] for the definitions of the Hausdorff measure of compactness χ on the class $\mathcal{M}_X$ of bounded subsets of a complete metric space, and the Hausdorff measure of noncompactness $\|\cdot\|_\chi$ of operators between Banach spaces.

We need the following well–known results.

Theorem 3.1 (Goldenštein, Goh'berg, Markus [13, Theorem 2.23])**.** *Let X be a Banach space with a Schauder basis (b_n). Then the function $\mu : \mathcal{M}_X \to [0,\infty)$ defined by*

$$\mu(Q) = \limsup_{m\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_m(x)\|\right), \tag{3.1}$$

with

$$\mathcal{R}_m(x) = \sum_{n=m+1}^{\infty} \lambda_n b_n \text{ for all } x = \sum_{n=1}^{\infty} \lambda_n b_n \in X$$

satisfies the following inequality for every $Q \in \mathcal{M}_X$

$$\frac{1}{a} \cdot \mu(Q) \leq \chi(Q) \leq \mu(Q), \tag{3.2}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ *is the basis constant.*

Proposition 3.1. *Let* X *and* Y *be Banach spaces and* $L \in \mathcal{B}(X,Y)$ *and* S_X *denote the unit sphere in* X*. Then we have*

$$\|L\|_\chi = \chi(L(S_X)) \qquad \text{([16, Theorem 7.11.4])} \tag{3.3}$$

and $L \in \mathcal{K}(X,Y)$ *if and only if*

$$\|L\|_\chi = 0 \qquad \text{([16, Theorem 7.11.5]).} \tag{3.4}$$

Proposition 3.2. (a) *Let the operators* $\mathcal{R}_m : w^p \to w^p$ *for* $m \in \mathbb{N}$ *be defined by* $\mathcal{R}_m(x) = \sum_{k=m+1}^{\infty}(x_k - \xi)e^{(k)}$ *for all* $x \in w^p$*, where* ξ *is the* w^p*–limit of* x*. Then we have for all* $Q \in \mathcal{M}_{w^p}$

$$\frac{1}{2} \lim_{m\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_m(x)\|_{w^p_\infty}\right) \leq \chi(Q) \leq \lim_{m\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_m(x)\|_{w^p_\infty}\right). \tag{3.5}$$

(b) *Let the operators* $\mathcal{R}_m : w_0^p \to w_0^p$ *for* $m \in \mathbb{N}$ *be defined by*

$$\mathcal{R}_m(x) = \sum_{k=m+1}^{\infty} x_k e^{(k)}$$

for all $x = (x_k)_{k=1}^{\infty} \in w_0^p$*. Then we have for all* $Q \in \mathcal{M}_{w_0^p}$

$$\chi(Q) = \lim_{m\to\infty}\left(\sup_{x\in Q}\|\mathcal{R}_m(x)\|_{w_\infty}\right). \tag{3.6}$$

PROOF. We have by [14, Lemma 2 (a), (b)]

$$\lim_{m\to\infty}\|\mathcal{R}_m\| = \begin{cases} 2 & \text{in Part (a),} \\ 1 & \text{in Part (b),} \end{cases}$$

and (3.5) and (3.6) follow from (3.1) and (3.2).

Now we prove an estimate for $\|L\|_\chi$, if $L \in \mathcal{B}(h_d, w^p)$, and an identity $\|L\|_\chi$, if $L \in \mathcal{B}(h_d, w_0^p)$.

Theorem 3.2. (a) *Let $L \in \mathcal{B}(h_d, w^p)$. Then we have*

$$\frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_{m;l\geq r} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=r}^{l} \left| \sum_{k=1}^{m} (a_{nk} - \alpha_k) \right|^p \right)^{1/p} \right) \leq \|L\|_\chi$$

$$\leq \lim_{r\to\infty} \left(\sup_{m;l\geq r} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=r}^{l} \left| \sum_{k=1}^{m} (a_{nk} - \alpha_k) \right|^p \right)^{1/p} \right), \qquad (3.7)$$

where the complex numbers α_k are defined in (2.3).

(b) *Let $L \in \mathcal{B}(h_d, w_0^p)$. Then we have*

$$\|L\|_\chi = \lim_{r\to\infty} \left(\sup_{m;l\geq r} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=r}^{l} \left| \sum_{k=1}^{m} a_{nk} \right|^p \right)^{1/p} \right). \qquad (3.8)$$

PROOF. Let $A = (a_{nk})_{n,k=1}^\infty$ be any infinite matrix and $r \in \mathbb{N}$. We write $A^{<r>} = (a_{nk}^{<r>})_{n,k=1}^\infty$ for the matrix with the rows $A_n^{<r>} = 0$ for $1 \leq n \leq r$ and $A_n^{<r>} = A_n$ for $n \geq r+1$.

(a) Let $L \in (h_d, w^p)$, $B = (b_{nk})_{n,k=1}^\infty$ be the matrix with $b_{nk} = a_{nk} - \alpha_k$ for all n and k, and $L^{<r>} \in \mathcal{B}(h_d, w^p)$ be the operator with $L^{<r>} = \mathcal{R}_r \circ L$. We denote the unit sphere in h_d by S. Then $L^{<r>}(x) = B^{<r>}x$ for all $x \in h_d$ by (1.3) and (2.7) and we obtain by (2.5)

$$\begin{aligned}
\mu(r) &= \sup_{x\in S} \|(\mathcal{R}_r \circ L)(x)\|_{w_\infty^p} \\
&= \|B^{<r>}\|_{(h_d, w_\infty^p)} \\
&= \sup_{l,m} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=1}^{l} \left| \sum_{k=1}^{m} b_{nk}^{<r>} \right|^p \right)^{1/p} \\
&= \sup_{l,m} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=r+1}^{l} \left| \sum_{k=1}^{m} (a_{nk} - \alpha_k) \right|^p \right)^{1/p} \\
&= \sup_{l\geq r+1;m} \frac{1}{d_m} \left(\frac{1}{l} \sum_{n=r+1}^{l} \left| \sum_{k=1}^{m} (a_{nk} - \alpha_k) \right|^p \right)^{1/p}.
\end{aligned}$$

Finally, (3.3) and (3.5) imply $(1/2)\lim_{r\to\infty}\mu(r) \leq \|L\|_\chi \leq \lim_{r\to\infty}\mu(r)$, which is (3.7).

(b) The proof is similar to that of Part (a) with $\alpha_k = 0$ for all k and (3.6) instead of (3.5).

Finally the characterisations of the classes $\mathcal{K}(h_d, w^p)$ and $\mathcal{K}(h_d, w_0^p)$ are immediate consequences of (3.4) and Theorem 3.2.

Corollary 3.1. (a) *Let $L \in \mathcal{B}(h_d, w^p)$. Then $L \in \mathcal{K}(h_d, w^p)$ if and only if*

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(a_{nk}-\alpha_k)\right|^p\right)^{1/p}\right) = 0,$$

where the complex numbers α_k are defined in (2.3)*.*

(b) *Let $L \in \mathcal{B}(h_d, w_0^p)$. Then $L \in \mathcal{K}(h_d, w^p)$ if and only if*

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{d_m}\left(\frac{1}{l}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}a_{nk}\right|^p\right)^{1/p}\right) = 0.$$

We close with an application of our results.

Example 3.1. We consider the classical Hahn space $h = h_d$, where $d_k = k$ for all $k = 1, 2, \dots$, and the Cesàro matrix $C_1 = A = (a_{nk})_{n,k=1}^\infty$ of order 1, where $a_{nk} = 1/n$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ $(n = 1, 2, \dots)$.

Then we have $|\sum_{k=1}^m a_{nk}| \leq m/n$ for all m and n, hence

$$\begin{aligned} c_{lm} &= \frac{1}{m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}a_{nk}\right|^p\right)^{1/p} \\ &\leq \frac{1}{m}\left(\frac{1}{l}\sum_{n=1}^{l}\left|\frac{m}{n}\right|^p\right)^{1/p} \\ &= \left(\frac{1}{l}\sum_{n=1}^{l}\frac{1}{n^p}\right)^{1/p} \leq 1, \end{aligned}$$

and so

$$\|A\|_{(h,w_\infty^p)} = \sup_{l,m} c_{lm} \leq 1, \tag{3.9}$$

that is, the condition in (2.2) is satisfied. Furthermore, for each $k \in \mathbb{N}$,

$$0 \leq \frac{1}{l} \sum_{n=1}^{l} |a_{nk}|^p = \frac{1}{l} \sum_{n=k}^{l} \frac{1}{n} \leq \frac{1}{l} \sum_{n=1}^{l} \frac{1}{n^p}$$
$$= A_l \left(\left(\frac{1}{n^p} \right)_{n=1}^{\infty} \right) \to 0 \quad (l \to \infty),$$

since $A = C_1 \in (c_0, c_0)$. Thus the condition (2.4) is also satisfied and consequently $C_1 \in (h, w_0^p)$ by Theorem 2.1 (c).

Now $c_{11} = 1$, and so we have $\|A\|_{(h,w_\infty^p)} = \|L_{C_1}\| = 1$ by Theorem 2.1 (d) and (3.9).

Finally, we have

$$c_{lm}^{(r)} = \frac{1}{m} \left(\frac{1}{l} \sum_{n=r}^{l} \left| \sum_{k=1}^{m} a_{nk} \right|^p \right)^{1/p} \leq \frac{1}{m} \left(\frac{1}{l} \sum_{n=r}^{l} \frac{m^p}{n^p} \right)^{1/p},$$

i.e.,

$$c_{lm}^{(r)} \leq \left(\frac{1}{l} \cdot \frac{l-r+1}{r^p} \right)^{1/p} \leq \frac{1}{r}$$

for all $l \geq r$, m and r, hence

$$0 \leq \lim_{r \to \infty} \left(\sup_{m; l \geq r} c_{lm}^{(r)} \right) \leq \lim_{r \to 0} \frac{1}{r} = 0,$$

and so $L_{C_1} \in \mathcal{K}(h, w_0^p)$ by Corollary 3.1 (b).

4. *Visualisation of Wulff's crystals*

A surface energy function is a real valued function depending on a direction in three–dimensional space. We visualise the surface energy functions given by the norms of w_∞^p and h_d and the correponding Wulff's crystals which are uniquely determined by their surface energy functions according to Wulff's principle [24]. Our figures are created by our own software package.

Let

$$S = \left\{ \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|\vec{x}\|_2 = \left(\sum_{k=1}^{3} x_k^2 \right)^{1/2} = 1 \right\}$$

and $F : S \to \mathbb{R}$. Writing

$$\vec{e} = \vec{e}(u_1, u_2) = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1)$$

for

$$(u_1, u_2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$$

we consider the so–called *potential surface* with a parametric representation

$$PS = \left\{\vec{x} = F(\vec{e}(u_1, u_2))\vec{e}(u_1, u_2) : (u_1, u_2) \in \mathbb{R}\right\}$$

as a representation of the surface energy function F.

The following result is known.

Proposition 4.1 ([18, Corollary 5.5]). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^3$ and, for each $\vec{v} = (v_1, v_2, v_3) \in S$, let $\phi_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\phi_{\vec{v}}(x) = \sum_{k=1}^{3} v_k x_k$ for all $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then the boundary $\partial C_{\|\cdot\|}$ of Wulff's crystal determined by the surface energy function $F = \|\cdot\|$ is given by*

$$\partial C_{\|\cdot\|} = \left\{\vec{x} = \frac{1}{\|\phi_{\vec{e}}\|} \cdot \vec{e} : \vec{e} \in S\right\},$$

where $\|\phi_{\vec{e}}\|^*$ is the norm of the functional $\phi_{\vec{e}}$, that is, $\|\cdot\|^*$ is the dual norm of $\|\cdot\|$.

In the following visualisations, we identify (x_1, x_2, x_3) with the following sequence $(x_1, x_2, x_3, 0, \dots)$.

Example 4.1. We consider the space w_∞^p with the block norm $\|\cdot\|_b$ defined by

$$\|x\|_b = \sup_{\nu \geq 0} \left(\frac{1}{2} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p\right)^{1/p} \qquad (x \in w_\infty^p)$$

which is equivalent to $\|\cdot\|_{w_\infty^p}$ by [13, Proposition 3.44]. The dual norm $\|\cdot\|_b^*$ is given by ([13, Proposition 3.47])

$$\|a\|_b^* = \sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^q\right)^{1/q} \qquad (a \in (w_\infty^p)^\beta)$$

(see Figure 1).

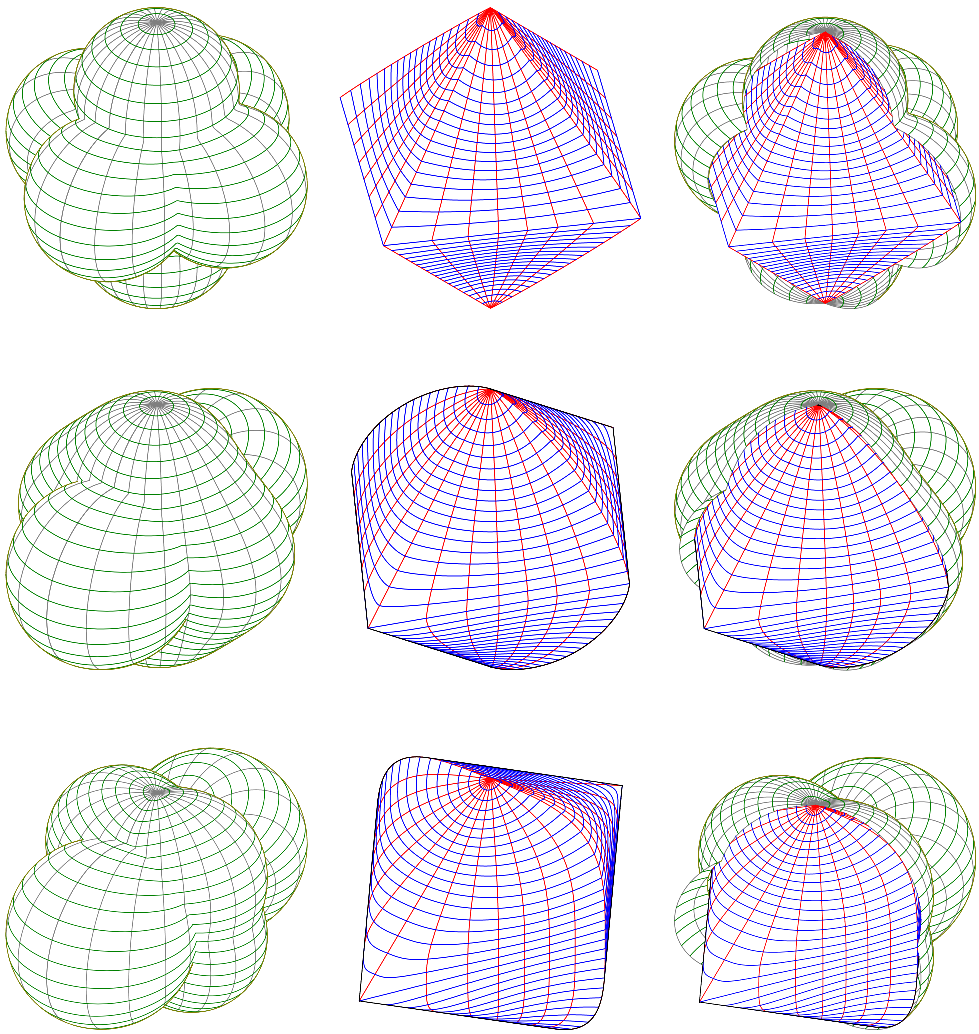

Figure 1: From left to right: Potential surfaces for $\|\cdot\|_{w^p_\infty}$, corresponding Wulff's crystal and both for top: $q = p/(p-1) = 1.025$, middle $q = 1.5$, bottom $q = 3.5$

Example 4.2. We consider the generalised Hahn space $(h_d, \|\cdot\|_{h_d})$. Then the dual norm $\|\cdot\|_{h_d} = \|\cdot\|_{bs_d}$, where by [17, Proposition 2.3]

$$\|a\|_{bs_d} = \sup_m \frac{1}{m}\left|\sum_{k=1}^{m} a_k\right| \qquad (a \in bs_d)$$

(see Figure 2).

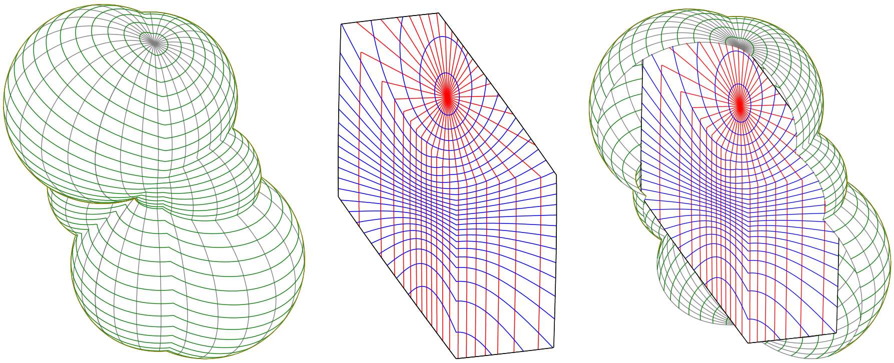

Figure 2: From left to right: Potential surface for $\|\cdot\|_{h_d}$, corresponding Wulff's crystal and both for $d_1 = 1$, $d_2 = 2$, $d_3 = 3$

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