SOME PROBABILISTIC TOPICS IN THE NAVIER–STOKES EQUATIONS

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ABSTRACT. We give a short overview of some topics concerning the ways randomness can be added to the three dimensional Navier–Stokes equations.

CONTENTS

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1. Introduction

The present work is a short overview of some results concerning the interactions between the analysis of the three dimensional Navier–Stokes equations and the theory of probability. Our special choice of topics does not give, by any means, a complete picture of the state of the art on the subject and several interesting matters and papers have been just outlined or even completely omitted. In the choice of topics there is definitely a bias over the work by the author of the present paper.

The main theme here is to consider the different ways randomness can be added to the Navier–Stokes equations. For some of these ways there is a reasonable physical justification, or a mathematical explanation. These reasons will be given when appropriate. Randomness can be essentially added to the data of the equations

$$\begin{aligned} &\dot u + (u\cdot\nabla)u + \nabla p = \nu\Delta u \ (+\dot\eta),\\ &\operatorname{div} u = 0, \end{aligned} \tag{1.1}$$

that is the initial condition, the external forcing and the parameters (here the viscosity). We will mainly consider the equations in dimension three and give some details for the two dimensional case when the analysis in 3D is unpractical.

In Section 2 we discuss some results obtained when randomness is added at the level of the initial condition, for instance results of almost sure global existence in super–critical spaces, the evolution of the distributions when the equations

are started with a random initial condition (statistical solutions) and invariant measures of the flow.

In Section 3 we consider the equations forced by Gaussian white noise. The literature on this subject is huge, we focus on the existence of solutions that constitute a Markov process, we discuss some topics on uniqueness and blow–up, we prove existence of densities for finite dimensional functionals of the solutions, as a probabilistic type of regularity.

Section 4 deals with invariant measures for the stochastically forced equations. In a way this should have been part of the previous section, but by importance it has deserved a section by its own. Here we discuss existence, uniqueness and convergence towards an invariant measure, as well as the existence of explicit invariant measures.

Finally, in Section 5 we consider subjects where randomness is more hidden. We consider probabilistic representation formulas for the solutions of (1.1). In the last part of the section the interest is in the interaction between the equations and statistics.

1.1. **Notation and setting.** In the rest of the paper we mainly focus on the Navier–Stokes equations with periodic boundary conditions, either without any external force or driven by Gaussian white noise. Most of the result may be or may have been already extended to other boundary conditions, external non–random forces, etc. We do not give further details and point to the references.

Consider problem (1.1) with periodic boundary conditions on the d dimensional torus $\mathbb{T}_d = [0, 2\pi]^d$ (most of the time $d = 3$, when necessary $d = 2$).

Let H be the standard space of square summable divergence free vector fields, defined as the closure of divergence free periodic smooth vector fields, with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. Define likewise V as the closure of divergence free periodic smooth vector fields with respect to the H^1 norm, with scalar product $\langle \cdot, \cdot \rangle_V$ and norm $\| \cdot \|_V$

Let Π_L be the Leray projector, $A = -\Pi_L \Delta$ the Stokes operator, and denote by $(\lambda_k)_{k \geq 1}$ and $(e_k)_{k \geq 1}$ the eigenvalues and the corresponding orthonormal basis of eigenvectors of A. Define the space $V_\alpha = D(A^{\frac{\alpha}{2}})$ for $\alpha \in \mathbf{R}$. In particular, $V_0 = H$ and $V_1 = V$.

Define the bi–linear operator $B : V \times V \to V'$ as $B(u, v) = \Pi_L (u \cdot \nabla v)$, $u, v \in V$. We recall that $\langle u_1, B(u_2, u_3) \rangle = -\langle u_3, B(u_2, u_1) \rangle$. We refer, for instance, to [Tem95] for a detailed account of the above definitions.

When appropriate, we will consider the random forcing $\dot{\eta}$ in (1.1) as $\dot{\eta} = \mathcal{S}\, dW$, where W is a cylindrical Wiener process on H (and hence $\dot{W}$ is space–time white noise), and $\mathcal{S}$ is a linear bounded operator on H. The role of $\mathcal{S}$ is to colour the noise in space, to provide some space regularity. The covariance of the driving

noise is then $\mathcal{S}\mathcal{S}^\star$. The term $\mathcal{S}\,dW$ can be always represented as

$$\mathcal{S}\,dW = \sum_n \sigma_n d\beta_n g_n,$$

where $(g_n)_{n\in\mathbf{N}}$ is an orthonormal basis of H of eigenvectors of $\mathcal{S}\mathcal{S}^\star$, $(\beta_n)_{n\in\mathbf{N}}$ are independent standard Brownian motions, and $(\sigma_n)_{n\in\mathbf{N}}$ are suitable coefficients.

2. Randomness in the initial condition

A natural way to include uncertainty in an evolution is to consider a probability distribution that weights the possible initial conditions. Moreover, the analysis of the evolution of distributions can give some knowledge of the dynamical properties of the system. For instance, a change of regularity of the measure might be interpreted in terms of the existence of different basins of attractions. In [Tao14] there is the belief that blow–up for the three dimensional Navier–Stokes might be more likely than regularity, but that carefully chosen initial distributions might avoid blow–up initial states and give only solution with global regularity. This should be an index of instability of blow–up with respect to small perturbations of the initial conditions (see also Section 3.5).

There has been recently a renewed interest in studying evolution equations with random initial condition, see for instance [BT08a, BT08b, BT14]. These ideas date back already to Bourgain [Bou96, Bou94], that considers the space–periodic non–linear Schrödinger equations in the focusing/defocusing case.

A common theme of these works is that tipically, the random initial condition may provide a short time effect of smoothing by averaging that may overcome some obstructions due to the scaling invariance of the equations. This is the case when one can prove an almost sure (with respect to the probabilistic structure given by the initial distribution) existence of a local solution when starting from a super–critical space.

2.1. The randomization.

Let us define the statistical distribution that has been used in the works we will be interested in. Let $\mathcal{H}$ be an Hilbert space and let $(e_n)_{n\ge1}$ be an orthonormal basis of $\mathcal{H}$. Consider a sequence $(\Lambda_n)_{n\ge1}$ of centred independent random variables with the property that there is $c_1>0$ such that

$$\mathbb{E}[\Lambda_n^2]\le c_1, \qquad \text{for every } n\ge1.$$

Additional uniform moments (e.g. exponential) may provide additional properties or strong estimates. In this direction, a reasonable assumption [NPS13] is

$$\mathbb{E}[\mathrm{e}^{\gamma\Lambda_n}]\le \mathrm{e}^{c_2\gamma^2}, \qquad \text{for every } \gamma\in\mathbf{R}, n\ge1. \tag{2.1}$$

This for instance provides exponential concentration around the mean of the randomization we are going to define. Fix $f \in \mathcal{H}$, the "seed", and define the *randomization* of f as

$$\Lambda f = \sum_n \Lambda_n f_n e_n,$$

where $(f_n)_{n\geq 1}$ are the Fourier coefficients of f with respect to the basis $(e_n)_{n\geq 1}$. It is immediate to see that Λf is a centred $\mathcal{H}$–valued random valued with covariance $\mathcal{U}_f^\star \mathcal{U}_f$ where $\mathcal{U}_f$ is the operator

$$x = \sum_n x_n e_n \qquad \rightsquigarrow \qquad \mathcal{U}_f x = \sum_n f_n x_n e_n.$$

If we choose, for instance, the initial random coefficients $(\Lambda_n)_{n\geq 1}$ as standard Gaussian, then Λf is also Gaussian with zero expectation and covariance $\mathcal{U}_f^\star \mathcal{U}_f$, and this characterise its distribution.

We first notice that the randomization does not give any gain in terms of smoothness. Mimicking Sobolev spaces, let us consider some compact subspace $\mathcal{H}_0$ of $\mathcal{H}$ defined by $\|x\|_{\mathcal{H}_0} = \sum \lambda_n^2 x_n^2 < \infty$, with $\lambda_n \uparrow \infty$.

Let $f \in H$ with $\|f\|_{\mathcal{H}_0} = \infty$. Let us prove that the randomization Λf cannot be in $\mathcal{H}_0$ almost surely. The proof is immediate in the Gaussian case, namely when the random variables $(\Lambda_n)_{n\geq 1}$ are standard Gaussian. Indeed,

$$\mathbb{E}[\|\Lambda f\|_{\mathcal{H}_0}^2] = \mathbb{E} \sum_n \lambda_n^2 \Lambda_n^2 f_n^2 = \sum_n \lambda_n^2 f_n^2 = \infty,$$

and Fernique's theorem [Bog98] readily implies that $\|\Lambda f\|_{\mathcal{H}_0} = \infty$, almost surely. In general, if the $(\Lambda_n)_{n\geq 1}$ are independent and uniformly not too often zero, then the same conclusion holds [BT08a, Lemma B.1]. Since a way to use the randomization is to deduce improved summability of the solution of the linear problem (see for instance Proposition 2.3), another way to look at the lack of regularization is to recall that Besov spaces, and in turn Sobolev spaces, can be characterised in terms of regularity of the caloric extension (see for instance [LR02]).

We want to study now the support of the law of Λf. We recall that for a topological space E, endowed with the Borel σ–algebra, the (topological) support of a measure μ is the set of all points x such that $\mu(A) > 0$ for each neighbourhood of x.

Lemma 2.1. *Given $f \in \mathcal{H}$, the support of the law of Λf is the whole $\mathcal{H}$ if and only if the support of the law of each Λ_n is $\mathbf{R}$ and $f_n \neq 0$ for every $n \geq 1$, where $(f_n)_{n\geq 1}$ are the Fourier coefficients of f with respect to the basis $(e_n)_{n\geq 1}$.*

Proof. Given $x \in \mathcal{H}$ and $\epsilon > 0$, we prove that $\mathbb{P}[\Lambda f \in B_\epsilon^{\mathcal{H}}(x)] > 0$. Let $N \geq 1$ and denote by $\Pi_{\leq N}$ and $\Pi_{>N}$ the projections, respectively, onto low and high modes. Choose N so that

$$\|\Pi_{>N} x\|_{\mathcal{H}} \leq \frac{\epsilon}{4}, \qquad \|\Pi_{>N} f\|_{\mathcal{H}} \leq \frac{\epsilon}{8},$$

then

$$\begin{aligned}\|\Lambda f - x\|_{\mathcal{H}} &\le \|\Pi_{\le N}(\Lambda f - x)\|_{\mathcal{H}} + \|\Pi_{>N}\Lambda f\|_{\mathcal{H}} + \|\Pi_{>N}x\|_{\mathcal{H}} \\ &\le \|\Pi_{\le N}(\Lambda f - x)\|_{\mathcal{H}} + \|\Pi_{>N}\Lambda f\|_{\mathcal{H}} + \frac{\epsilon}{4}.\end{aligned}$$

Therefore, by the above estimate and independence,

$$\begin{aligned}\mathbb{P}[\Lambda f \in B_\epsilon^{\mathcal{H}}(x)] &= \mathbb{P}[\|\Lambda f - x\|_{\mathcal{H}} \le \epsilon] \\ &\ge \mathbb{P}\Big[\|\Pi_{\le N}(\Lambda f - x)\|_{\mathcal{H}} \le \frac{\epsilon}{2}, \|\Pi_{>N}\Lambda f\|_{\mathcal{H}} \le \frac{\epsilon}{4}\Big] \\ &= \mathbb{P}\Big[\|\Pi_{\le N}(\Lambda f - x)\|_{\mathcal{H}} \le \frac{\epsilon}{2}\Big]\,\mathbb{P}\Big[\|\Pi_{>N}\Lambda f\|_{\mathcal{H}} \le \frac{\epsilon}{4}\Big] > 0,\end{aligned}$$

since, on the one hand by the Chebychev inequality,

$$\begin{aligned}\mathbb{P}\Big[\|\Pi_{>N}\Lambda f\|_{\mathcal{H}} \le \frac{\epsilon}{4}\Big] &= 1 - \mathbb{P}\Big[\|\Pi_{>N}\Lambda f\|_{\mathcal{H}} \ge \frac{\epsilon}{4}\Big] \ge \\ &\ge 1 - \frac{16}{\epsilon^2}\mathbb{E}[\|\Pi_{>N}\Lambda f\|_{\mathcal{H}}^2] = 1 - \frac{16}{\epsilon^2}\|\Pi_{>N}f\|_{\mathcal{H}}^2 \ge \frac{3}{4},\end{aligned}$$

on the other hand, by independence and the assumption on the support of the $(\Lambda_n)_{n\ge1}$,

$$\begin{aligned}\mathbb{P}\Big[\|\Pi_{\le N}(\Lambda f - x)\|_{\mathcal{H}} \le \frac{\epsilon}{2}\Big] &\ge \mathbb{P}\Big[|\Lambda_n f_n - x_n| \le \frac{\epsilon f_n}{2\|f\|_{\mathcal{H}}}, n \le N\Big] \\ &= \prod_{n\le N}\mathbb{P}\Big[|\Lambda_n f_n - x_n| \le \frac{\epsilon f_n}{2\|f\|_{\mathcal{H}}}\Big] > 0.\end{aligned}$$

To prove the converse, notice that if $f_n = 0$ for some n, then the choice $x = e_n$ yields $\|\Lambda f - x\|_{\mathcal{H}} \ge |\Lambda_n f_n - 1| = 1$. Likewise, if the support of the law of Λ_1 is not $\mathbf{R}$, then there are $x_0 \in \mathbf{R}$ and $\epsilon_0 > 0$ such that $\mathbb{P}[\Lambda_1 \in (x_0 - \epsilon_0, x_0 + \epsilon_0)] = 0$. The choice $x = x_0 f_1 e_1$ yields $\|\Lambda f - x\|_{\mathcal{H}} \ge |f_1| \cdot |\Lambda_1 - x_0| \ge \epsilon_0 |f_1|$ almost surely. □

Clearly the same proof of the lemma above holds true if $f \in \mathcal{H}_0$, for some subspace $\mathcal{H}_0$ of $\mathcal{H}$, namely in the latter case the support of the law of Λf is $\mathcal{H}_0$.

2.2. **Strong local solution with random initial condition.** We summarize how to show the existence of a local smooth solution with random initial condition in the energy space H, following [ZF11].

Let $u_0 \in H$ be the "seed", and consider the random initial condition Λu_0, using the Fourier basis of H. The main idea is that there is an immediate gain of summability from L^2 to L^3. As L^3 is a critical space for Navier–Stokes, we know by [Kat84] that there is a unique local solution.

Assume $\mathbb{E}[\Lambda_k^4] \le c$ for every $k \in \mathbf{Z}^3_s tar$. Clearly, if stronger moments (e.g. exponential) are finite, the probability estimates below are better.

As mentioned above, through randomization of $u_0 \in H$ and the fourth moment condition for the randomizing variables, it follows that

$$\mathbb{E}[\|\Lambda u_0\|_{L^3}^3] \le c_3 \|u_0\|_{L^2}^3.$$

Therefore the local existence and uniqueness of [Kat84] kicks in to give the following result.

Theorem 2.2. *Let $u_0 \in H$ then with probability one there are $T_\star = T_\star(\omega) > 0$ and a unique solution u with initial condition Λu_0 such that for all $p \geq 3$,*

- $t^{\frac{1}{2}-\frac{3}{2p}} u \in C([0,T_\star); L^p(\mathbb{T}_3))$,
- $t^{1-\frac{3}{2p}} \nabla u \in C([0,T_\star); L^p(\mathbb{T}_3))$.

The time $T_\star = \infty$ on an event Ω_∞ with $\mathbb{P}[\Omega_\infty] \geq 1 - c_4 \|u_0\|_{L^2}^3$ (esponentially close to 1 with exponential moments of Λ_k).

Under the finite sixth moment, a similar statement hold in $H^{\frac{1}{2}}$ (as in [FK64]), namely with probability one there is a unique solution u such that

$$u - \mathrm{e}^{-tA}\,\Lambda u_0 \in C([0,T_\star]; V_{\frac{1}{2}}) \cap L^2(0,T_\star; V_{\frac{3}{2}}),$$

as well as an estimate of the probability that $T_\star \geq T$ (with $T \in (0,1]$) in terms of $\|u_0\|_{L^2}$ (polynomial or exponential depending on the moments of the random coefficients of the randomization). Similar conclusions are given in [DC11a], they also prove that the solution is global if $\|u_0\|_{L^2}$ is small enough.

2.3. **Global weak solutions with random initial conditions.** The problem of finding an (interesting, see below in Section 2.5) initial distribution so that almost surely with respect to this distribution there is a unique global solution is still essentially open (but see Section 2.4 below). Clearly, there may be some "trivial" example, such as some measure concentrated on small initial conditions in, for instance, $H^{1/2}$, but this adds nothing to what we know. A well supported initial distribution that gives raise, almost surely, to global strong solutions, would suggests that a blow–up in the equation is exceptional, or "unstable", in the sense that a small variation in the initial condition might not lead to a singularity (more on this will be discussed in Section 3.5.3).

A way to obtain global *weak* solutions with no smallness assumption on the data, when starting from super–critical initial conditions has been recently proposed in [NPS13].

Here the "smoothing" effect of the randomization they use is again in terms of summability of the solution of the linear problem. The exponential tail estimate is a consequence of the assumption (2.1).

Proposition 2.3. *Let $\alpha \geq 0$, $p \geq 2$, $\sigma \geq 0$, $\gamma \in \mathbf{R}$ with $(\sigma + \alpha - 2\gamma)p < 2$, and $T > 0$, then for every $u_0 \in V_{-\alpha}$,*

$$\mathbb{P}[\|S_{\gamma,\sigma} u_0\|_{L^q(0,T;L^p(\mathbb{T}_3)} \geq \lambda] \leq c_5\, \mathrm{e}^{-c_6 \frac{\lambda^2}{c_7 \|u_0\|_{-\alpha}^2}}$$

where $S_{\gamma,\sigma} u_0(t) = t^\gamma A^{\frac{\sigma}{2}}\, \mathrm{e}^{-tA}\, \Lambda u_0$.

Let us define weak solutions in the following way.

Definition 2.4 (Definition 2.4, [NPS13]). Given $\alpha > 0$, $u_0 \in V_{-\alpha}$, a weak solution of the Navier–Stokes equations on $[0,T]$ is a vector field u such that,

- $u \in L^\infty_{\mathrm{loc}}((0,T);H) \cap L^2_{\mathrm{loc}}((0,T);V) \cap C((0,T);V^{\mathrm{weak}}_{-\alpha})$,
- $u' \in L^1(0,T;V')$,
- the equation is satisfied in V',
- $u(t) \to u_0$ weakly in $V_{-\alpha}$, as $t \to 0$.

The main theorem is as follows.

Theorem 2.5 (Theorem 2.6, [NPS13]). *If $T > 0$, $\alpha \in (0, \frac14)$ and $u_0 \in V_{-\alpha}$, then with probability one there is a global weak solution with initial condition Λu_0, of the form $u = \mathrm{e}^{-tA}\Lambda u_0 + w$, where $w \in L^\infty_{\mathrm{loc}}(0,\infty;H) \cap L^2_{\mathrm{loc}}(0,\infty;V)$.*

In short, the idea behind the theorem is that one can use the smoothing effect of the randomization of the initial condition to produce a mild solution, defined for a short time. The solution immediately enters into H and a standard weak solution can be started after a small time interval. It remains then only to show that the mild solution and the weak solution can be joined to obtain a weak solution as defined above.

This the crucial point that forces the restriction $\alpha < \frac14$ in the main theorem. Indeed to prove the equivalence between weak and mild solutions [NPS13, Lemma 4.2] (but similar assumptions are used in other crucial results of the paper) for the equation for $w = u - \mathrm{e}^{-tA}\Lambda u_0$, where terms as $B(\mathrm{e}^{-tA}\Lambda u_0, w)$ appear, one needs that, for instance $A^{\frac14}\mathrm{e}^{-tA}\Lambda u_0 \in L^{\frac83}((0,T)\times\mathbb{T}_3)$. This happens, according to the lemma, when $\alpha < \frac14$.

In the case $\alpha \in [\frac14, 1)$ something can be still said, at least in terms of mild solutions.

Theorem 2.6 ([DC11b]). *Let $\alpha \in [\frac14, 1)$ and $u_0 \in V_{-\alpha}$, then with probability one there are $T = T(\omega)$ and a unique solution u of (1.1) on $[0,T]$ with initial condition Λu_0, such that*

$$u - \mathrm{e}^{-tA}\Lambda u_0 \in L^{\frac{4}{1-s}}(0,T;L^{\frac{6}{1+s}}).$$

Again, estimates of the probability that $T > t$ are available for small t, namely if $t \in (0,1]$, there is a unique solution on $[0,t]$ on an event Ω_t with

$$\mathbb{P}[\Omega_t] \ge 1 - c_8\,\mathrm{e}^{-\frac{c_9}{t^{2\rho}\|u_0\|^2_{-s}}},$$

and $\rho = \min(\frac1m, \frac{21}{8})$.

2.4. **Fursikov's almost sure global well–posedness.** The previous section detailed results where a suitable choice of the distribution of the initial condition would allow to prove existence of a strong (or weak) solution with supercritical data. It has been well known that there exist initial distributions that ensure almost sure global well–posedness. This is a general result of Fursikov [VF88] (see

also [Fur81b, Fur83, Fur84]). Clearly, it is not difficult to provide initial distributions thtat give almost sure global well–posedness, think of the Dirac in 0, or some measure concentrated in a small ball of $V_{\frac{1}{2}}$. Indeed, the main problem of Fursikov's initial measures is that they are only characterized by their moments. It is well known that moments do not identify uniquely a measure (unless some growth condition holds). Moreover, no information on the support of these measure is available (unlike in the previous result, see Lemma 2.1). This is the reason for Fursikov to analyse the infinite dimensional system generated by the moments [Fur87, VF88].

2.4.1. *Statistical solutions.* Statistical solutions were first introduced by Foiaş in [Foi72] as a family, indexed by time, of probability measures satisfying the equations, appropriately recast (see also [FMRT01]). A different notion of statistical solution, seen as a measure on the path space (from this point of view, closer to the style of this paper, see Section 2.5 below), was formulated by [VF88]. Let us consider the latter definition.

Define, for a fixed $s \gg 0$,

$$\mathcal{S}_T = \{u : u \in L^2(0,T;V) \cap L^\infty(0,T;H), \dot{u} \in L^\infty(0,T;V_{-s})\}$$

A *space–time statistical solution* with initial condition μ, is a probability measure $\mathbb{P}$ on $\mathcal{S}_T$ such that

- $\mathbb{P}[\mathcal{S}_T] = 1$,
- the marginal of $\mathbb{P}$ at time $t = 0$ is μ,
- $\mathbb{P}$ is concentrated on solutions of (1.1),
- for every $t \in [0,T]$,

$$\mathbb{E}^{\mathbb{P}}[\|u\|^2_{L^2(0,T;V)} + \|u(t)\|^2_H + \|u\|^2_{L^\infty(0,T;H)} + \|\dot{u}\|^2_{L^\infty(0,T;V_{-s})}] \leq c_{10}(1 + \mathbb{E}^{\mu}[\|x\|^2_H]).$$

A statistical solution represents the overall distribution of the stochastic process generated by the solutions of (1.1) when the initial distribution is given by the initial measure μ.

A variant of the definition above more suited for the next section is as follows. Define

$$\mathcal{H}^{1,2}_T = \{u : u \in L^2(0,T : V_2), \dot{u} \in L^2(0,T : H)\}.$$

Notice that if $u \in \mathcal{H}^{1,2}_T$, then $u \in C([0,T];V)$. A space-time statistical solution is a probability measure $\mathbb{P}$ on $\mathcal{H}^{1,2}_T$ such that for every $z \in L^2(0,T;H)$ and $\phi \in C_b(\mathcal{H}^{1,2}_T)$,

$$\mathbb{E}^{\mathbb{P}}[\langle \dot{u} + \nu Au + B(u), z\rangle_{L^2(0,T;H)}\phi(u)] = 0.$$

2.4.2. *Statistical extremal problems and a.s. smoothness.* Given a measure μ on H such that $\mathbb{E}^{\mu}[\|x\|^k_H] < \infty$ for every $k \geq 1$, define its k^{th}–moment M_k as the element

of $\bigotimes_k H$, the tensor product of k copies of H, such that

$$\langle M_k, \phi\rangle_{H,k} = \int \langle \otimes_k u, \phi\rangle_{H,k} \mu(du), \qquad \phi \in \bigotimes_k H,$$

where $\langle\cdot,\cdot\rangle_{H,k}$ and $\|\cdot\|_{H,k}$ are the tensorizations of the scalar product and the norm of H. The existence of M is granted by the Riesz representation theorem.

Consider a measure μ on H such that $\mathbb{E}[\mathrm{e}^{\|x\|_H^2}] < \infty$ (so that all moments of μ are finite) and denote by $(m_k)_{k\ge1}$ its moments). Consider the following functional defined over probability measures on $\mathcal{H}_T^{1,2}$,

$$\mathcal{J}(\mathbb{P}) = \mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{\|u\|^2_{L^2(0,T;V_2)}}\big] + \rho \sum_{k=1}^{\infty} \frac{1}{k!} \|m_k - M_k\|_{H,k},$$

where $\rho > 0$ and $(M_k)_{k\ge1}$ are the moments of the marginal at time $t = 0$ of $\mathbb{P}$. For a measure $\mathbb{P}$ with $\mathcal{J}(\mathbb{P}) < \infty$, its moments do not grow too much (by the first term), and are not too different (at least at time 0) from the moments of μ.

The idea is to look at $\inf \mathcal{J}(\mathbb{P})$ over all statistical solutions $\mathbb{P}$ on $\mathcal{H}_T^{1,2}$. It turns out that the direct methods of calculus of variations are effective. The functional is convex, lower semi–continuous in an appropriate topology and finite in at least one measure, so that the infimum is attained and there is a unique probability measure that realizes the minimum.

If we look at the marginal ν at time 0 of the unique minimizer, then for ν–almost every initial condition (1.1) has a unique global smooth solution.

Remark 2.7. Starting from (1.1), a system of equations

$$\dot M_k + \nu A_k M_k + B_k M_{k+1} = 0,$$

for the moments of a statistical solution can be derived, where A_k and B_k are suitable tensorizations of the Stokes operator A and the Navier–Stokes non–linearity B.

Fursikov (see [VF88, Fur87] and related references) proves that there is a dense set of initial conditions for the moment system such that each of these initial conditions yields a unique solution. The problem here is that one cannot produce in general a statistical solution from moments.

Remark 2.8. In fact the first results of Fursikov [Fur80, Fur81a] in this direction deal with the set of right–hand sides yielding a global smooth solution. More precisely, set

$$N(u) = \dot u + \nu A u + B(u),$$

and solve $N(u) = f$ with a given initial condition $u(0) = u_0 \in V$. It turns out that the map $N : \mathcal{H}_T^{1,2} \to L^2([0,T];H)$ is continuous, so that the set $F_{u_0} = N(\{u \in \mathcal{H}_T^{1,2} : u(0) = u_0\})$ is exactly the set of all right–hand side $f \in L^2(0,T;H)$ such that the Navier–Stokes equations with forcing f admit a unique smooth solution.

Moreover, F_{u_0} is open in the topology of $L^2(0,T;H)$ and dense in $L^2(0,T;H)$ with in the topology $L^p(0,T;V_{-\ell})$, for suitable p and ℓ. The result can be made independent from the viscosity and can hold for the Euler equations, as long as the initial condition is smooth enough.

2.5. **Invariant measures.** The randomization of the initial condition to obtain local existence (in a super–critical space) or global existence (of a regular solution) becomes extremely useful when one knows that the system admits a (formal) invariant measure. Some explicit information is also required (the support of the measure, for instance).

So far, we do not know any explicit[1] invariant measure for the Navier–Stokes equations, and in fact we will have better luck with the randomly forced equation in Section 4.3.

We point out that a method to find invariant measures has been proposed using generalized limits of time averages [FMRT01], where a generalized limit is any linear operator that extends the ordinary notion of limit. Existence of extensions is ensured by the Hahn–Banach theorem.

Euler is a different story, and indeed explicit invariant measures can be derived. In dimension three the known conserved quantities are the kinetic energy, the helicity, namely $\int u\cdot\xi$, where $\xi = \operatorname{curl} u$ is the vorticity, the circulation, $\int_{X_t(\gamma)} u(t)\cdot dx$, where γ is a curve in physical space and X is the flow induced by u, as well as the total momentum $\int u\,dx$ and angular momentum $\int x\cdot u\,dx$. The only good candidate then is the energy, and one can consider the Gibbs–like measures

$$\mu_{E,\beta} = \frac{1}{Z_\beta}\,\mathrm{e}^{-\beta E(u)}\,du,$$

where $E(u) = \int |u|^2\,dx$ is the energy. The above measure is interpreted as usual as a Gaussian measure. The problem is that such measures are supported on fairly large spaces, as

$$\int \|x\|^2_{-\alpha}\,\mu_\beta(dx) = \infty$$

for $\alpha \le \frac{3}{2}$. The problem of the existence of a flow of solutions of Euler which leaves the above measures invariant is (yet another) open problem.

The situation is much better in dimension two, due to weaker regularity requirements, but above all due to the existence of a wealth of invariants, first of all the enstrophy $S(u) = \frac{1}{2}\int \xi^2\,dx$, as well as $\int g(\xi)\,dx$ for every reasonable g. If J is any of the above invariants, the measure $Z^{-1}_{J,\beta}\,\mathrm{e}^{-\beta J(u)}\,du$ would provide a formal invariant measure. The only reasonable measure though, those we may hope to give a sense, are given by $J = E, S$. These measure are *infinitesimally invariant*

[1]And interesting! As otherwise any steady solution $\bar u$, including $\bar u = 0$ would provide the invariant measure $\delta_{\bar u}$.

in the sense that for every smooth function F depending only on a finite number of Fourier modes, $\int B(F)\,d\mu_{J,\beta} = 0$.

The measures originating from enstrophy have a smaller support, so it is expected that it should be easier to work with them. Indeed [AC90], there exists a flow in $V_{-\alpha}$, $\alpha > \frac{1}{2}$ of Euler with invariant measure $\mu_{S,\beta}$. Measures from energy are tougher. It even happens that $\int E(u)\,\mu_{S,\beta}(du) = \infty$, but the *renormalized energy* $:E := E - \mathbb{E}^{S,\beta}[E]$ (carefully interpreted as a limit of spectral approximations) makes sense, $\mathrm{e}^{-\gamma:E:}$ is integrable with respect to $\mu_{S,\beta}$ and the measure $Z^{-1}\,\mathrm{e}^{-\beta S-\gamma:E:}$ is again invariant. We refer to [AF08] for more details. Here we raise the (philosophical) open problem of understanding the role of these invariant measures in connection with the physical phenomenon of turbulence.

We will see later that when adding a noise we will be able to find invariant measures for Navier–Stokes (in Sections 4.1 and 4.3). Ideas of *renormalization* will also play a significant role later, see Sections 4.3.1 and 4.3.2.

3. Randomness in the driving force

In the same way one can derive, at least formally, Euler equations from the Lagrangian motion of fluid particles, a version of the Navier–Stokes equations driven by a special multiplicative noise, depending on the gradient of the velocity, can be derived starting from the Lagrangian motion perturbed by noise, see [BCF91, MR04, MR05]. The presence of random forcing can also take into account all those small fluctuations that affect the motion of a fluid and that are difficult to incorporate in a robust theory. We refer for instance to [FGHR08] for the connections between the equations with random forcing and the theory of turbulence.

There is already a well–developed theory for stochastic PDEs, and in particular for equations from fluid dynamics. We refer to [FG95, Fla08, Deb13]. Here we detail a recent approach initiated by [DPD03a] that looks for solutions with additional structure, the Markov property. For well–posed problem the Markov property would not be an issue, on the other hand for problems where well–posedness is an open problem (as is in this work) extra–care is needed.

There are at least two approaches that grant existence of Markov processes solving (1.1) driven by Gaussian noise. The first [DPD03a, DO06] is essentially based on strong solutions of (1.1). In short the idea is to solve the Kolmogorov equation associated with spectral Galerkin approximations. In order to grant the existence of a limiting object of the solutions of the Kolmogorov eqution, the authors look at a Kolmogorov equation perturbed by a strong potential. The solutions of the two equations are related by a Feynamn–Kac formula. The potential, a negative exponential of the H^2 norm, does an "importance sampling" of strong solution, xince non–smooth solution would contribute with an infinite potential and hence with a null contribution in the Feynman–Kac formula.

The second approach [FR06, FR08] is based on weak solutions. The construction builds over an abstract selection principle originally due to Krylov [Kry73] (see also [SV79]). The idea is essentially to identify special classes of solutions, understood as probability measures on the space of trajectories, that are closed by conditional probability and for which weak–strong uniqueness holds. We refer also to [BFR09] for another model where this theory can be applied (see also [BR09, BR12, BR13]).

In this section we will consider (1.1) driven by a Gaussian noise, namely the noise $\dot\eta = \mathcal{S}\dot W$ in (1.1) is coloured in space by a covariance operator $\mathcal{S}^\star\mathcal{S} \in \mathscr{L}(H)$, where W is a cylindrical Wiener process (see [DPZ92] for further details). We assume that $\mathcal{S}^\star\mathcal{S}$ is trace–class and we denote by $\sigma^2 = \mathrm{Tr}(\mathcal{S}^\star\mathcal{S})$ its trace. Finally, consider the sequence $(\sigma_k^2)_{k\geq 1}$ of eigenvalues of $\mathcal{S}^\star\mathcal{S}$, and let $(q_k)_{k\geq 1}$ be the orthonormal basis in H of eigenvectors of $\mathcal{S}^\star\mathcal{S}$. For simplicity we may assume that the Stokes operator A and the covariance commute, so that

$$\dot\eta(t,y) = S\,dW = \sum_{\mathbf{k}\in\mathbf{Z}^3_\star} \sigma_{\mathbf{k}}\dot\beta_{\mathbf{k}}(t)e_{\mathbf{k}}(y).$$

3.1. **Weak and strong solutions.** Let us write (1.1), as usual, as an abstract stochastic equation,

$$du + (\nu Au + B(u))\,dt = \mathcal{S}\,dW, \tag{3.1}$$

with initial condition $u(0) = x \in H$. A weak martingale solution is a filtered probability space $(\widetilde\Omega, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}, \{\widetilde{\mathscr{F}}_t\}_{t\geq 0})$, a cylindrical Wiener process $\widetilde W$ on H and a process u with trajectories in $C([0,\infty); D(A)') \cap L^\infty_{\mathrm{loc}}([0,\infty), H) \cap L^2_{\mathrm{loc}}([0,\infty); V)$ adapted to $(\widetilde{\mathscr{F}}_t)_{t\geq 0}$ such that the above equation is satisfied with $\widetilde W$ replacing W.

Equivalently, a weak martingale solution can be described as a measure on the path space. Let $\Omega_{\mathrm{NS}} = C([0,\infty); D(A)')$ and let $\mathscr{F}^{\mathrm{NS}}$ be its Borel σ–algebra. Denote by $\mathscr{F}^{\mathrm{NS}}_t$ the σ–algebra generated by the restrictions of elements of Ω_{NS} to the interval $[0,t]$ (roughly speaking, this is the same as the Borel σ–algebra of $C([0,t]; D(A)')$). Let ξ be the canonical process, defined by $\xi_t(\omega) = \omega(t)$, for $\omega \in \Omega_{\mathrm{NS}}$

Definition 3.1 ([FR08])**.** A probability measure $\mathbb{P}$ on Ω_{NS} is a solution of the martingale problem associated to (3.1) with initial distribution μ if

- $\mathbb{P}[L^\infty_{\mathrm{loc}}(\mathbf{R}^+, H) \cap L^2_{\mathrm{loc}}(\mathbf{R}^+; V)] = 1$,
- for each $\phi \in D(A)$, the process

$$\langle \xi_t - \xi_0, \phi\rangle + \int_0^t \langle \xi_s, A\phi\rangle - \langle B(\xi_s,\phi), \xi_s\rangle\,ds$$

  is a continuous square summable martingale with quadratic variation $t\|\mathcal{S}\phi\|^2_H$ (hence a Brownian motion),
- the marginal of $\mathbb{P}$ at time 0 is μ.

The second condition in the definition above has a twofold meaning. On the one hand it states that the canonical process is a weak (in terms of PDEs) solution, on the other hand it identifies the driving Wiener process, and hence is a weak (in terms of stochastic analysis) solution.

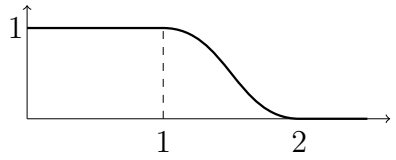
3.1.1. *Strong solutions.* It is also well–known that (3.1) admits local smooth solutions defined up to a random time (a stopping time, in fact) τ_∞ that corresponds to the (possible) time of blow–up in higher norms. To consider a quantitative version of the local smooth solutions, notice that τ_∞ can be approximated monotonically by a sequence of stopping times

$$\tau_R = \inf\{t > 0 : \|Au_R(t)\|_H \geq R\},$$

where u_R is a solution of the following problem,

$$du_R + \big(\nu Au_R + \chi(\|Au_R\|_H^2/R^2)B(u_R, u_R)\big)\,dt = \mathcal{S}\,dW,$$

with initial condition in $D(A)$, and where $\chi : [0,\infty) \to [0,1]$ is a suitable *cut–off* function, namely a non–increasing C^∞ function such that $\chi \equiv 1$ on $[0,1]$ and $\chi_R \equiv 0$ on $[2,\infty)$. The process u_R is a strong (in PDE sense) solution of the cut–off



equation. Moreover it is a strong solution also in terms of stochastic analysis, so it can be realized uniquely on any probability space, given the noise perturbation.

As it is well–known in the theory of Navier–Stokes equations, the regular solution is unique in the class of weak solutions that satisfy some form of the energy inequality. We will give two examples of such classes for the equations with noise.

Remark 3.2. The analysis of strong (PDE meaning) solutions can be done on larger spaces, up to $D(A^{1/4})$, which is a critical space with respect to the Navier–Stokes scaling. The extension is a bit technical though, see [Rom11a].

3.1.2. *Solutions satisfying the almost sure energy inequality.* An almost sure version of the energy inequality has been introduced in [Rom08, Rom10a]. Given a weak solution $\mathbb{P}$, choose $\phi = e_k$ as a test function in the second property of Definition 3.1, to get a one dimensional standard Brownian motion β^k. Since $(e_k)_{k\geq 1}$ is an orthonormal basis, the $(\beta^k)_{k\geq 1}$ are a sequence of independent standard Brownian motions. Then the process $W_{\mathbb{P}} = \sum_k \beta^k e_k$ is a cylindrical Wiener process[2] on H. Let $z_{\mathbb{P}}$ be the solution to the linearization at 0 of (3.1), namely the solution of

$$dz + Az = \mathcal{S}\,dW, \tag{3.2}$$

[2]Notice that W is measurable with respect to the solution process.

with initial condition $z(0) = 0$, and where $W = W_{\mathbb{P}}$. Finally, set $v_{\mathbb{P}} = \xi - z_{\mathbb{P}}$. It turns out that $v_{\mathbb{P}}$ is a solution of

$$\dot{v} + \nu A v + B(v + z_{\mathbb{P}}, v + z_{\mathbb{P}}) = 0, \qquad \mathbb{P} - \text{a.s.},$$

with initial condition $v(0) = \xi_0$. An energy balance functional can be associated to $v_{\mathbb{P}}$,

$$\mathcal{E}_t(v, z) = \frac{1}{2}\|v(t)\|_H^2 + \nu \int_0^t \|v(r)\|_V^2 \, dr - \int_0^t \langle z(r), B(v(r) + z(r), v(r))\rangle_H \, dr.$$

We say that a solution $\mathbb{P}$ of the martingale problem associated to (3.1) (as in Definition 3.1) satisfies the *almost sure energy inequality* if there is a set $T_P \subset (0, \infty)$ of null Lebesgue measure such that for all $s \notin T_P$ and all $t \geq s$,

$$P[\mathcal{E}_t(v, z) \leq \mathcal{E}_s(v, z)] = 1.$$

It is not difficult to check that $\mathcal{E}$ is measurable and finite almost surely.

3.1.3. *A martingale version of the energy inequality.* An alternative formulation of the energy inequality that, on the one hand is compatible with conditional probabilities, and on the other hand does not involve additional quantities (such as the processes $z_{\mathbb{P}}$ and $v_{\mathbb{P}}$) can be given in terms of super–martingales. The additional advantage is that this definition is keen to generalization to state–dependent noise.

Define, for every $n \geq 1$, the process

$$\mathscr{E}_t^1 = \|\xi_t\|_H^2 + 2\nu \int_0^t \|\xi_s\|_V^2 \, ds - 2\,\mathrm{Tr}(\mathcal{S}^\star \mathcal{S}),$$

and, more generally, for every $n \geq 1$,

$$\mathscr{E}_t^n = \|\xi_t\|_H^{2n} + 2n\nu \int_0^t \|\xi_s\|_H^{2n-2} \|\xi_s\|_V^2 \, ds - n(2n-1)\,\mathrm{Tr}(\mathcal{S}^\star \mathcal{S}) \int_0^t \|\xi_s\|_H^{2n-2} \, ds,$$

when $\xi \in L^\infty_{\mathrm{loc}}([0, \infty); H) \cap L^2_{\mathrm{loc}}([0, \infty); V)$, and ∞ elsewhere.

We say that a solution $\mathbb{P}$ of the martingale problem associated to (3.1) (as in Definition 3.1) satisfies the *super–martingale energy inequality* if for each $n \geq 1$, the process $\mathscr{E}_t^n$ defined above is $\mathbb{P}$–integrable and for almost every $s \geq 0$ (including $s = 0$) and all $t \geq s$,

$$\mathbb{E}[\mathscr{E}_t^n | \mathscr{F}_s^{\mathrm{NS}}] \leq \mathscr{E}_s^n,$$

or, in other words, if each $\mathscr{E}^n$ is an almost sure supermartingale.

3.2. **The selection principle.** In order to carry on the construction of a Markov solution, we need to start with a class of solutions satisfying some minimal properties (sort of a set–valued Markov property). Given $x \in H$, let $\mathscr{C}(x) \subset \Pr(\Omega)$ be a set of weak martingale solutions (no other requirement so far) starting at x. The three main property we shall require are:

- (disintegration) the classes $(\mathscr{C}(x))_{x\in H}$ are closed by conditional probabilities: if $\mathbb{P}\in\mathscr{C}(x)$, then the conditional probability distribution of $\mathbb{P}$ given $\mathscr{F}_t^{\mathrm{NS}}$ is in $\mathscr{C}(\omega(t))$ for $\mathbb{P}$ a.e. ω,
- (reconstruction) this is, in a way, the inverse operation of disintegration: if one has a $\mathscr{F}_t^{\mathrm{NS}}$ measurable map $x\mapsto\mathbb{Q}_x$, with $\mathbb{Q}_x\in\mathscr{C}(x)$, and $\mathbb{P}\in\mathscr{C}(x_0)$, then the probability measure given by $\mathbb{P}$ on $[0,t]$, and, conditionally on $\omega(t)$, by the values of $\mathbb{Q}_\cdot$, is an element of $\mathscr{C}(x_0)$.
- (weak–strong uniqueness) each solution coincides with the process u_R on $[0,t]$ on the event $\{\tau_R\geq t\}$.

For the construction of a Markov solution, we require the first two properties, the third one is necessary for further analysis (continuity with respect to the initial condition, see Section 3.3 below, and convergence to a unique invariant state, see Section 4.1).

The idea is to shrink each set $\mathscr{C}(x)$ to a single element by a series of reductions, while keeping the above properties. Fix a family $(\lambda_n,f_n)_{n\geq1}$ which is dense in $[0,\infty)\times C_b(D(A)')$ and consider the functionals $J_n=J_{\lambda_n,f_n}$, where $J_{\lambda,f}$ is given by

$$J_{\lambda,f}(P)=\mathbb{E}^P\Big[\int_0^\infty \mathrm{e}^{-\lambda t}\,f(\xi_t)\,dt\Big].$$

for arbitrary $\lambda>0$ and $f:D(A)'\to\mathbf{R}$ upper semi-continuous. Next, set

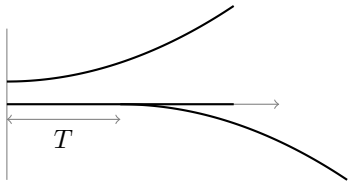
$$\mathscr{C}_0(x)=\mathscr{C}(x),\qquad \mathscr{C}_n(x)=\big\{\mathbb{P}\in\mathscr{C}_{n-1}(x):J_n(\mathbb{P})=\sup_{\mathbb{Q}\in\mathscr{C}_{n-1}(x)}J_n(\mathbb{Q})\big\}.$$

All these sets are compact and their intersection is a single element (the selection associated to this maximised sequence), $\bigcap_{n\in\mathbf{N}}\mathscr{C}_n(x)=\{\mathbb{P}_x\}$.

Example 3.3. Existence of Markov solutions holds even without noise, when the solution is suitably understood as a probability on the space of trajectories. Consider the classical non–uniqueness example $\dot X=\operatorname{sgn}(X)\sqrt{|X|}$, with initial condition in $\mathbf{R}$. The problem has a unique solution $X_x(\cdot)$ for each initial condition $x\neq0$, and two families of solutions $\{X_a^\pm=X_\star^\pm((t-a)\vee0):a\geq0\}$ for the initial condition $x=0$, where $X_\star^-,X_\star^+$ are the minimal and the maximal solutions unique solution starting at 0.

If $\mathscr{C}(x)$ denotes the set of all solutions starting at x, viewed as probability measures on the path space $C([0,\infty);\mathbf{R})$, then $\mathscr{C}(x)=\{\delta_{X_x}\}$ for $x\neq0$, where δ_{X_x} is the Dirac measure concentrated on X_x.

If $x=0$, a solution starts at 0 and stays for an arbitrary time, then follows one of the solutions $X_\star^\pm$ (suitably translated). So the *departing* time from 0 can be interpreted as a random time T whose law can be arbitrary. Therefore any selection of solutions is completely described by a random time T and on $[0,\infty]$ and a coin flip C to decide to go up or down It is easy to be convinced that a selection of solutions is Markov if and only if T is exponential (the lack of memory



plays a major role), including the degenerate cases of infinite or zero rate (namely, $T = 0$ or $T = \infty$), and T, C are independent.

Denote by $(\mathbb{P}^a_x)_{x\in[0,1]}$ the Markov families with rate a. We shall call *extremal* all those Markov solutions that can be obtained by the selection procedure. It turns out that the only extremal families are those corresponding to $a = 0$ and $a = \infty$.

In view of next Section 3.3, we notice that no solution can be continuous with respect to the initial condition.

3.3. **Continuity with respect to the initial condition.** As we shall see, Markov solutions have a good structure, good enough to ensure that solutions are continuous (in an appropriate sense) with respect to the initial condition. In a way, for well–posedness we are only missing uniqueness.

Continuity with respect to the initial condition here is understood in terms of continuity of the law, in the total variation distance, of the solution for fixed time and seen as a function of the starting point. This is a purely probabilistic notion and in fact it is ruled out for the equations without noise, as it can be easily seen by the elementary consideration shown in Figures 1 and 2. Without noise the "law"

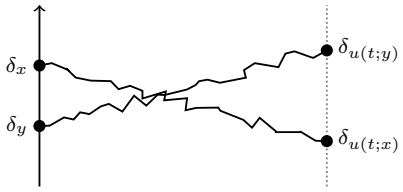


FIGURE 1. Without noise...

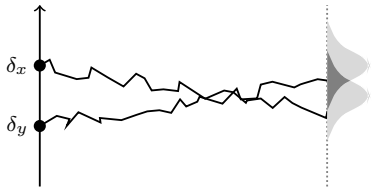


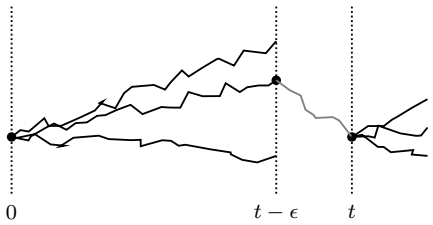
FIGURE 2. With noise...

of the solution evolves as a Dirac mass centred at the value of the solution and no possible shrinking of the total variation distance is possible (unless statistical solutions are considered, as in Section 2.4.1, but there is no smoothing effect by the noise). With noise the two distributions have a common mass for two reasons. The first reason is general: there is the diffusive effect of the Gaussian perturbation, the second reason is due to weak–strong uniqueness: there is a tiny (but non–zero) probability that τ_∞ may be large enough, so that the two laws are close.

We shall use the two facts above to show continuity. Indeed, for small times, using weak–strong uniqueness,

$$\begin{aligned}\mathbb{E}[\phi(u(\epsilon;y))]-\mathbb{E}[\phi(u(\epsilon;x))]&=\\&=\underbrace{\mathbb{E}[\phi(u(\epsilon;y))\mathbb{1}_{\{\tau_\infty>\epsilon\}}]-\mathbb{E}[\phi(u(\epsilon;x))\mathbb{1}_{\{\tau_\infty>\epsilon\}}]}_{\text{estimate with the regular solution}}+\\&\qquad+\underbrace{\mathbb{E}[\phi(u(\epsilon;y))\mathbb{1}_{\{\tau_\infty\le\epsilon\}}]-\mathbb{E}[\phi(u(\epsilon;x))\mathbb{1}_{\{\tau_\infty\le\epsilon\}}]}_{\text{short time tail of }\tau_\infty}.\end{aligned}$$

For short times the non–linearity has a small effect, so that the dynamics is essentially linear and the probability $\mathbb{P}[\tau_\infty\le\epsilon]\approx \mathrm{e}^{-1/\epsilon}$. On the other hand, when



we are below τ_∞ we can work with the strong solutions (see Section 3.1.1 and hence with the classical theory. For times of order one, the real picture is that the "uniqueness of strong solutions" argument is applied at the very last moment only, thanks to the Markov property,

$$\begin{aligned}P_t\phi(y)-P_t\phi(x)&=P_\epsilon(P_{t-\epsilon}\phi)(y)-P_\epsilon(P_{t-\epsilon}\phi)(x)\\&=o(\epsilon)+o(\|x-y\|)\\&=\mathrm{Err}(\text{non–uniqueness})+\frac{1}{\epsilon}\mathrm{Err}(x-y).\end{aligned}$$

The conclusion follows by optimizing in ϵ.

Theorem 3.4. *For any Markov family $x\mapsto\mathbb{P}_x$, the map $x\mapsto P(t,x,\cdot)$ is continuous in total variation when $x\in D(A)$.*

The restriction $x\in D(A)$ in the above theorem is due to the fact that we need to ensure the existence of smooth solutions. It can be lowered to $x\in D(A^{1/4+})$, that is up to the critical space [Rom11a].

3.4. **Some remarks on uniqueness.** For stochastic (partial) differential equations we may have different notions of uniqueness, regardless of the model we are studying (namely, without introducing any criterion originating from the physics of the problem, such as entropy solutions, etc.). On the one hand there is the notion of path–wise uniqueness, which corresponds to the standard uniqueness for ODE/PDE. On the other hand we may ask for a weaker statement, weak uniqueness, that is uniqueness of distributions.

Noise might be a promising crucial ingredient for uniqueness, see [FGP10]. A wider and deeper discussion can be found in [Fla11]. Here we only point out two simple (and uneffective so far) criteria for uniqueness and regularity from [FR08]:

- If for some (smooth) initial condition there is a smooth solution on a (possibly small but deterministic) time interval, then the problem is well posed.
- If for some initial condition uniqueness in law holds on a (possibly small but deterministic) time interval, then uniqueness in law holds for all initial conditions.

We shall see below in Section 4.2 a criterion of uniqueness in terms of invariant measures.

3.4.1. *Some examples in finite dimension.* A standard example of non–uniqueness of an elementary ODE is the equation $\dot{x} = \sqrt{|x|}$ (that we have examined in Example 3.3 and we will see again in Example 4.4). It is well–known that by adding a Gaussian perturbation $dx = \sqrt{|x|}\,dt + dB$ uniqueness (path–wise) is restored. This is part of a general phenomenon, see for instance [SV79, KR05]. Notice that we would also restore uniqueness by adding something of order one, say $\dot{x} = \sqrt{|x|} + 1$, the difference is that noise is zero plus random fluctuations.

Weak uniqueness can hold without path–wise uniqueness, as in the Tanaka equation $dx = \mathrm{sgn}(x)\,dB$. Here all solutions are Brownian motions, hence they all have the same distributions, but there is no pathwise uniqueness since, for instance, if x is a solution, then so is $-x$. To have examples of non–uniqueness of distributions we need to allow degeneracy in the noise coefficient, for instance as in the Girsanov equation $dx = |x|^\alpha\,dB$, with $\alpha < \frac{1}{2}$. This problem has a infinite dimensional counterpart, where several interesting phenomena happen [BMP10, MME12].

Anyway, in dimension $d = 1$ there is a rather complete understanding [ES85], and the Girsanov example describes a quite universal picture.

3.4.2. *About uniqueness of the martingale problem.* A way to understand uniqueness of distributions is to understand the generator of the process solution of (1.1). Formally, we expect that the generator is

$$\bar{\mathscr{L}} = \frac{1}{2}\,\mathrm{Tr}[\mathcal{S}^\star\mathcal{S}D^2] - \langle -\nu\Pi_L\Delta u + \Pi_L\big((u\cdot\nabla)u\big), D\rangle$$

where $\mathcal{S}$ is the operator colouring the noise and Π_L is the Leray projector. It turns out that each of the Markov solutions discussed in Section 3.2 is the unique solution of the so–called martingale problem associated to a suitable generator, as stated in the next theorem.

Theorem 3.5 ([Rom11b])**.** *Given a Markov solution $(\mathbb{P}_x)_{x\in H}$, there exists a unique closed linear operator $\mathscr{L} : D(\mathscr{L}) \subset C_b(D(A)) \to C_b(D(A))$ such that for all $\lambda > 0$ and $\varphi \in C_b(D(A))$,*

$$R_\lambda(\mathscr{L})\varphi(x) = \int_0^\infty \mathrm{e}^{-\lambda t}\,P_t\varphi(x)\,dt,$$

where $(P_t)_{t\ge0}$ is the transition semigroup associated to the given Markov solution, and $R_\lambda(\mathscr{L})$ is the resolvent of $\mathscr{L}$.

The previous theorem holds under the same assumptions on the covariance as Section 3.3. Similar conclusions can be drawn under the assumptions discussed in [RX11, Rom11a].

The problem here is that each operator $\mathscr{L}$ is equal to $\bar{\mathscr{L}}$ on a class of test functions (smooth functions depending on a finite number of Fourier components). This class of functions unfortunately is not sufficient to characterize the operator[3]. Preliminary computations show that an improved knowledge of the tails of the explosion time τ_∞ (see also next Section) might be promising. We refer to [Rom11b, DPD08] for further details.

3.5. **Some remarks on blow–up.** The aim of this section is to give a brief overview on blow–up and which kind of noise might be more effective to delay or even prevent emergence of singularities. To our knowledge, [FR02a] (see also the related works [FR01, FR02b, Rom06]) is the only work concerned with singularities for the Navier–Stokes with random perturbations.

Since so far we do not know if the Navier–Stokes equations develop a singularity, it is meaningful to consider simpler models, such as the one we discuss in Section 3.5.3, that keep some of the crucial characteristic of the problem (1.1) we are interested in. A recent result of Tao [Tao14] shows that the analysis of these models may rigorously shed light on the problem of blow–up of (1.1).

The results detailed below (from [Rom14e]), show that no additive noise can be expected to prevent the formation of singularities. Recent results [FGP10, Fla11] show that a careful choice of the coefficients in the case of state dependent noise might be more promising.

3.5.1. *The drift matters.* Here we focus on additive noise and we wish to understand if it may have (and possibly how) a significant effect in preventing singularities. As we shall see, the situation is deeply different with respect to the problem of uniqueness. The effect of noise is more related to the stability of blow–up. It may even happen that noise creates singularities when there is none without noise. Consider the problem $\dot x = x^2 \sin x$. Clearly there is a global bounded solution for every initial condition, and it is not difficult to see that when adding noise, solutions blow–up[4]. In two dimensions both cases may happen [Sch93], namely there are two suitable smooth fields $b : \mathbf{R}^2 \to \mathbf{R}^2$ such that if one consider the ODE $\dot x = b(x)$ and the SDE $dx = b(x)\,dt + dB$ then

[3]To have an idea, think of the Poisson equation on a bounded domain with two different boundary conditions. The smooth test functions compactly supported on the interior of the domain cannot tell the two boundary conditions apart.

[4]The role of the noise here is to help overcome the barriers created by the zeroes of sin.

- the ODE explodes for all initial conditions, the SDE has global solutions for all initial conditions with probability one, and there is even an invariant measure,
- the ODE is non–explosive for all initial conditions and the point $(0,0)$ is asymptotically stable, the SDE has explosion with positive probability.

Stability is also related to the probability of blow–up, namely if blow–up happens with positive probability or with probability one. Let us consider $dx = b(x)\,dt + dB$, where b is one of the two functions

$$b_\pm(x) = \begin{cases} x^2 & x \geq 0, \\ \pm x & x < 0. \end{cases}$$

When the drift is b_+, blow–up happens with positive probability. When the drift is b_-, blow–up happens almost surely. In view of next sections, we notice that in both cases,

- there are T_0, p_0 and a closed set B with *open interior* such that for all initial conditions in B, $\mathbb{P}[\tau_\infty \leq T_0] \geq p_0$,
- the blow–up happens only on the positive "side": for all $p \geq 1$, namely $\mathbb{E}[\sup_t |x_-|^p] < \infty$.

3.5.2. *A criterion for the a.s. blow–up.* Define the blow–up time τ_∞ of a stochastic equation (in finite or infinite dimension)

$$dx = b(x)\,dt + dB$$

as $\tau_\infty = \sup_n \tau_n$ and

$$\tau_n = \inf\{t : H(x_t) \geq n\}.$$

for some quantity of interest for the problem (for instance, $H(x) = |x|$ in finite dimension, H is some norm in a smaller space for stochastic PDEs). Define

$$\flat(t, x_0) = \mathbb{P}_{x_0}[\tau_\infty > t],$$

then clearly $\flat(0, x_0) = 1$, $\flat(\cdot, x_0)$ is non–increasing, and $t \mapsto \flat(t, x_0)$ is continuous in t (up to technical details). Set

$$\flat(x_0) = \lim_{t\to\infty} \flat(t, x_0) = \inf_t \flat(t, x_0) = \mathbb{P}_{x_0}[\tau_\infty = \infty].$$

In general we cannot claim that $\flat(x_0) \in \{0, 1\}$, as seen by the examples of previous section. On the other hand a 0–1 law still holds for a the supremum of these probabilities.

Theorem 3.6 (0–1 law for explosion)**.** *We have that* $\sup_{x_0} \flat(x_0) \in \{0, 1\}$.

The proof of the above theorem is sketched in Figure 3. The idea now is that if we can prove an upper bound for $\flat$ that keeps $\flat$ away from 1, then by the 0–1 law, $\flat \equiv 0$.

The idea for the upper bound is based on stability of blow–up and conditional recurrence. Assume there are a closed set B_∞, $p_0 \in (0, 1)$ and $T_0 > 0$ such that

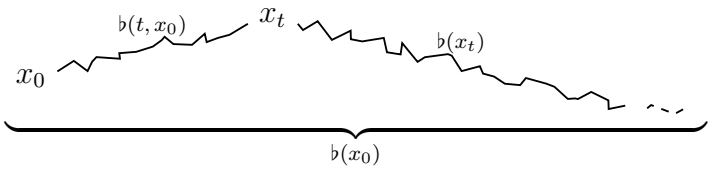


FIGURE 3. Proof of Theorem 3.6.

- $\mathbb{P}_y[\tau_\infty \le T_0] \ge p_0$ for every $y \in B_\infty$,
- $\mathbb{P}_x[\tau^{B_\infty} = \infty | \tau_\infty = \infty] = 0$ for every x,

where τ^B is the hitting time of B, then $\mathfrak{b}(x) = \mathbb{P}_x[\tau_\infty = \infty] \le \frac{1}{1+p_0}$, hence $\mathfrak{b} \equiv 0$. The heuristic idea here is that the process keeps coming back in the set where

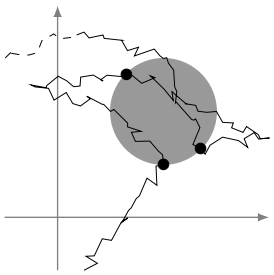


FIGURE 4. Conditional recurrence.

blow–up is likely, and once there it tries to blow up. One can look at these trials as coin tossing. Sooner or later both sides will show up. The assumptions above are needed to have a uniform control of the bias of the coin.

3.5.3. *The dyadic model of turbulence.* In this section, following [Rom14e], we apply the general criterion explained before to the viscous dyadic model driven by additive noise.

The dyadic model has been introduced in [KP05] as a simplification of the motion of energy among modes in Euler equations studied in [FP04]. The model has been the subject of further analysis in its inviscid version without noise [CFP07, CFP10, BFM10a, BFM11b, BM13a], forced by a special multiplicative noise [BFM10b, BFM11a, BM13b], and in its viscous version without noise [Che08, CF09, BMR11, BMR14] and with noise [Rom14e]. A generalized version on trees, closer to the formulation of [FP04], has been studied in [BBFM13, Bia13].

A simple derivation of the model is as follows [KP05]. The idea is to look at a solution u on $\mathbf{R}^3$ of the Euler equations and write a simplified version of the interaction of the energy packets. Consider the dyadic cubes: cubes of size 2^ℓ with vertices on $2^\ell\mathbb{Z}^3$. For a dyadic cube Q, let $\widetilde{Q}$ be the parent cube and $\widehat{Q}$ the children cubes (see Figure 5. Write the wavelet expansion of u, for an orthonormal basis $(\omega_Q)_Q$ of L^2, based on the dyadic cubes. If $u = \sum_Q u_Q \omega_Q$, the Euler non–linearity

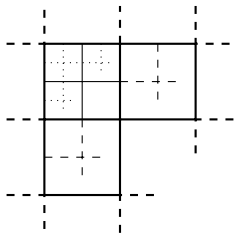


FIGURE 5. The hierarchy of dyadic cubes.

reads $(u\cdot\nabla)u = \sum_Q 2^{j(Q')}u_Q u_{Q'}\omega_Q\omega_{Q'}$. If only nearest neighbours interactions are kept,

$$[(u\cdot\nabla)u]_Q \approx B_{\mathrm{up}}(u,u) - B_{\mathrm{down}}(u,u) = 2^{\beta(j(Q)+1)}u_Q\sum u_{\widehat{Q}} - 2^{\beta j(Q)}u_{\check{Q}}^2,$$

where "up" and "down" refer to the direction of the flow of energy. The value $\beta = \frac{5}{2}$ corresponds, by scaling, to Euler/Navier–Stokes.

A further simplification is achieved by changing the set of indices from dyadic cubes to integers. By adding a viscous dissipation, we finally obtain the *viscous dyadic model*,

$$\dot{x}_n = -\nu\lambda_n^2 x_n + \lambda_{n-1}^\beta x_{n-1}^2 - \lambda_n^\beta x_n x_{n+1} + \sigma_n\dot{w}_n, \qquad n\geq 1.$$

where β measures the relative strength of the dissipative *vs* convective part. The crucial characteristics preserved by this simplification is formal balance of energy, namely

$$\frac{d}{dt}x_n^2 + 2\nu\lambda_n^2 x_n^2 = \underbrace{\lambda_{n-1}^\beta x_{n-1}^2 x_n - \lambda_n^\beta x_n^2 x_{n+1}}_{\text{telescopic!}} \quad\rightsquigarrow\quad \frac{d}{dt}\sum_{n=1}^\infty x_n^2 + 2\nu\sum_{n=1}^\infty \lambda_n^2 x_n^2 = 0.$$

We summarise some known results, first in the case without noise ($\sigma_n\equiv 0$ for all n) [Che08, BMR11],

- positive initial conditions give positive solutions.
- if $\beta\leq 2$, there is well–posedness (2DNS–regime),
- if $\beta\leq 3$, uniqueness for positive initial states,
- if $\beta\leq\beta_c\leq 3$ smoothness for positive initial state,
- if $\beta > 3$, blow–up for large enough positive initial state.

Here $\beta_c > \frac{5}{2}$, and the Navier–Stokes–like case is included. With noise, we know [Rom14e] that,

- positivity not preserved,
- if $\beta\leq 2$ trivial well-posedness (2DNSe regime),
- if $\beta\leq 3$, path–wise uniqueness,
- if $\beta\leq\beta_c$, smoothness,
- if $\beta > 3$ and $\{n : \sigma_n\neq 0\}\neq\emptyset$, then there is a.s. blow–up from any initial state.

The natural state space of the problem is $\ell^2(\mathbf{R})$, we can consider spaces of regularity (understood as decay, as in Sobolev spaces with respect to Fourier transform),

$$h_\alpha = \bigl\{x \in \ell^2 : \|x\|_\alpha^2 := \sum_n (\lambda_n^\alpha x_n)^2 < \infty\bigr\},$$

and a smooth solution is a solution in h_α (for those values of α compatible with the regularity of the driving noise).

Let us first understand the mechanism that creates blow–up in the case without noise [Che08]. Assume $\beta > 3$. There is explosion if the initial condition is *positive* and *large enough* in h_α for some suitable $\alpha > 0$. Indeed,

$$\frac{d}{dt}\|x\|_\alpha^2 + 2\nu\|x\|_{\alpha+1}^2 \approx \sum_n \lambda_n^{\beta+2\alpha} x_n^3 \approx c\|x\|_{\alpha+1}^3, \tag{3.3}$$

where the positivity of the solution makes the "$\approx$" rigorous. Hence

$$\frac{d}{dt}\|x\|_\alpha^2 \gtrsim c\|x\|_{\alpha+1}^3 - 2\nu\|x\|_{\alpha+1}^2 \gtrsim c\|x\|_\alpha^3, \tag{3.4}$$

if $\|x(0)\|_\alpha$ is large enough.

The critical exponent for the dyadic model is $\beta - 2$, and local "regular" solutions exist with initial condition in h_α, with $\alpha > \beta - 2$. Indeed, this number is the decay rate of solutions which gives the equilibrium between the linear and the non–linear part:

$$\lambda_n^2 x_n \approx \lambda_{n-1}^\beta x_{n-1}^2 - \lambda_n^\beta x_n x_{n+1} \approx \lambda_n^\beta x_n^2,$$

that is $x_n \approx \lambda_n^{2-\beta}$.

When we have at least one component forced by noise, blow–up is almost sure.

Theorem 3.7. *Assume $\sigma_n \neq 0$ for at least one index $n \geq 1$. For every $x \in h_\alpha$, $\alpha > \beta - 2$, and every martingale solution starting at x, $\mathbb{P}_x[\tau_\infty < \infty] = 1$.*

The proof of this theorem is based on the criterion explained in Section 3.5.2. The strategy is to consider a perturbation of the deterministic estimates (3.3), (3.4) in order to find a good set B_∞ where the probability of blow–up is bounded from below. The main difficulty in identifying B_∞ is that positive states are a "thin" set in ℓ^2. On the other hand, when seen at an appropriate scale, the noise perturbation is small so that, if not positive, solutions are still not too negative. This identifies B_∞ as a subset of states that are "quasi–positive" and large is some suitable norm.

To prove the conditional recurrence, we prove that it is implied by recurrence of balls in ℓ^2 (which is true by the standard energy estimate). Indeed, fix a ball

of radius M in ℓ^2, then with a probability p_2 there is contraction of negative

components (so that the solution is "quasi–positive"), while keeping the size in ℓ^2 not too larger than M. With probability p_1 it is possible to keep the negative components small, while making the positive part of the first noisy component large. The two steps together have lead the system in B_∞. The events that force the system the way we want depend on the outcomes of the noise, so are mutually independent. Hence recurrence of B_∞ is reduced to recurrence of balls in ℓ^2.

3.6. **Existence of densities for finite dimensional functionals.** There are several reasons to be interested in densities for the solutions of (1.1).

First of all, when dealing with a stochastic evolution PDE, the solution depends not only on the time and space independent variables, but also on the "chance" variable, and the existence of a density for the law of (some functionals of) the solution is thus a form of regularity with respect to this new variable.

We will be particularly interested in densities for finite dimensional (spectral) projections of the solution. By the results in Section 3.2, it is sufficient to show that the laws of two solutions agree if they agree at each time. To show that the laws at each time coincide it is sufficient to show that the finite dimensional projections are the same. Thus, the analysis of densities can be a first step towards a proof of uniqueness of the distributions[5].

An alternative proof of uniqueness, involving the invariant measures, is presented in Theorem 4.2. It is interesting, although not unexpected, that the densities of stationary processes (see Section 4 are smoother than the densities of any other solution [DR14], so that the strategy outlined above might proceed further when we consider the additional smoothing of stationary solutions together with the aforementioned theorem 4.2.

When looking for densities, we face two non–trivial problems. The first concerns the problem of a reference measure for densities. One reason to consider finite dimensional functionals is that there is no canonical reference measure in infinite dimension. Any choice should be necessary tailored to the problem at hand, and in our case we do not know enough of the problem (1.1) for this purpose (see [Rom13] for more details in this direction, see Section 4.3 for some candidates).

The second problem is related to the difficulty in proving regularity and uniqueness (either case, with or without noise). Indeed, to show existence of densities for solutions of stochastic equations a fundamental and classical tool is the Malliavin calculus, a differential calculus where the differentiating variable is the underlying noise driving the system. The Malliavin derivative $\mathcal{D}_H u(t)$, the derivative with respect to the variations of the noise perturbation, is given as

$$\mathcal{D}_H u = \lim_{\epsilon \downarrow 0} \frac{u(W + \epsilon \int H\, ds) - u(W)}{\epsilon},$$

[5]Unfortunately, we are not able to proceed beyond this first step so far.

where we have written the solution u as $u(W)$ to show the explicit dependence of u from the noise forcing. We point, for instance, to [Nua06] for further details and definitions. In dimension two the general program proceeds [MP06] providing smooth densities for the finite dimensional projections of the solution. When we turn to three dimensions, we notice that the Malliavin derivative $\mathcal{D}_H u$ of the solution u of (1.1), as a variation, satisfies the linearization around the solution, namely,

$$\frac{d}{dt}\mathcal{D}_H u - \nu\Delta\mathcal{D}_H u + (u\cdot\nabla)\mathcal{D}_H u + ((\mathcal{D}_H u)\cdot\nabla)u = SH,$$

and good estimates on $\mathcal{D}_H u(t)$ originate only from good estimates on the linearization of (1.1), which are not available so far. This settles the need of methods to prove existence and regularity of the density that do not rely on this calculus, as done in [DR14]. For other works in this direction, see for instance [DM11, BC12, KHT12, HKHY13b, HKHY13a].

3.7. **Besov bounds for the densities.** There are several possible strategies to prove the existence of the density of a random variable. We have been already convinced that Malliavin calculus does not work. In [DR14] three different strategies are presented. The first strategy is based on the idea introduced in [FR07] that, under suitable assumptions on the driving noise, Markov solutions have laws that are absolutely continuous with respect to the laws of the local strong solutions. As observed in [Rom14a] (see also [Rom14d]), the validity of this observation, when properly reformulated, goes beyond Markov solutions. A second strategy is based on the Girsanov transformation and, as the previous strategy, provides only a qualitative result, namely the existence of densities, without any further regularity property.

The third strategy, that we are going to briefly detail below, yields regularity of the density in terms of Besov spaces. Let us recall the definition of Besov spaces. The general definition is based on the Littlewood–Paley decomposition, but it is not the best suited for our purposes. We shall use an alternative equivalent definition (see [Tri83, Tri92]) in terms of differences. Define

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \qquad (\Delta_h^n f)(x) = \Delta_h^1(\Delta_h^{n-1} f)(x)$$

then the following norms, for $s>0$, $1\le p\le\infty$, $1\le q\le\infty$,

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \Big(\int_{\{|h|\le 1\}} \frac{\|\Delta_h^n f\|_{L^p}^q}{|h|^{sq}}\frac{dh}{|h|^d}\Big)^{\frac{1}{q}},$$

(with obvious modifications when $q=\infty$), where n is any integer such that $s<n$, are equivalent norms of $B^s_{p,q}(\mathbf{R}^d)$ for the given range of parameters.

The technique introduced in [DR14] is based on the following analytic lemma, which provides a quantitative integration by parts. The lemma is implicitly given in [DR14] and explicitly stated in [Rom14c].

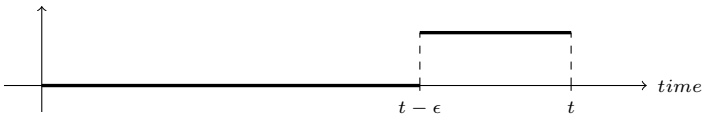
Lemma 3.8 (smoothing lemma)**.** *If μ is a finite measure on $\mathbf{R}^d$ and there are an integer $m \geq 1$, two real numbers $s > 0$, $\alpha \in (0,1)$, with $\alpha < s < m$, and a constant $c_{11} > 0$ such that for every $\phi \in C_b^\alpha(\mathbf{R}^d)$ and $h \in \mathbf{R}^d$,*

$$\left| \int_{\mathbf{R}^d} \Delta_h^m \phi(x)\, \mu(dx) \right| \leq c_{11} |h|^s \|\phi\|_{C_b^\alpha},$$

then μ has a density f_μ with respect to the Lebesgue measure on $\mathbf{R}^d$ and $f_\mu \in B^r_{1,\infty}$ for every $r < s - \alpha$. Moreover, there is $c_{12} = c_{12}(r)$ such that

$$\|f_\mu\|_{B^r_{1,\infty}} \leq c_{11} c_{12}.$$

The idea is the same used with Malliavin calculus: there is a smoothing effect (that is captured by the above lemma) and this must originate from the random perturbation. We use the random perturbation to perform the "fractional" integration by parts along the noise to be used in the above lemma. The bulk of this idea can be found in [FP10]. Our method is based on the one hand on the idea that the Navier–Stokes dynamics is "good" for short times, and on the other hand that Gaussian processes have smooth densities. When trying to estimate the Besov norm of the density, we approximate the solution by splitting the time interval in two parts.



On the first part the approximate solution u_ϵ is the same as the original solution, on the second part the non–linearity is killed. By Gaussianity this is enough to estimate the increments of the density of u_ϵ. Since u_ϵ is the one-step explicit Euler approximation of u, the error in replacing u by u_ϵ can be estimated in terms of ϵ. By optimizing the increment versus ϵ we have an estimate on the derivatives of the density.

The regularity of the density can be slightly improved from $B^{1-}_{1,\infty}$ to $B^{2-}_{1,\infty}$ if u is the stationary solution, namely the solution whose statistics are independent from time (see Section 4)

Proposition 3.9. *Given $x \in H$ and a finite dimensional subspace F of $D(A)$ generated by the eigenvectors of A, namely $F = \mathrm{span}[e_{n_1}, \ldots, e_{n_F}]$ for some arbitrary indices $n_1, \ldots, n_F$, assume that $\pi_F \mathcal{S}$ is invertible on F. Then for every $t > 0$ the projection $\pi_F u(t)$ has an almost everywhere positive density $f_{F,t}$ with respect to the Lebesgue measure on F, where u is any solution of (3.1) which is limit point of the spectral Galerkin approximations.*

Moreover, for every $\alpha \in (0,1)$, $f_{F,t} \in B^\alpha_{1,\infty}(\mathbf{R}^d)$ and for every small $\epsilon > 0$, there exists $c_{13} = c_{13}(\epsilon) > 0$ such that

$$\|f_{F,t}\|_{B^\alpha_{1,\infty}} \leq \frac{c_{13}}{(1 \wedge t)^{\alpha+\epsilon}} (1 + \|x\|_H^2).$$

The time regularity of the density can be also investigated using similar ideas. Since we are studying a stochastic evolution, it is reasonably expected that there is regularity in time and that this is "half" the regularity in space (Brownian scaling). This is confirmed by the following result.

Proposition 3.10 ([Rom14c]). *Under the same assumptions of the previous Proposition 3.9, for every $\alpha \in (0,1)$ there is $c_{14} > 0$ such that for every $s, t > 0$,*

$$\|f_{F,t} - f_{F,s}\|_{B^\alpha_{1,\infty}} \le c_{14}\big(\sup_{r\in[s,t]} \|f_{F,r}\|_{B^\alpha_{1,\infty}}\big)|t-s|^{\frac{\alpha}{2}}.$$

The above result is based again on the splitting idea we have explained. Unfortunately the splitting works only when one can exploit a sort of integration by parts. This is not possible when doing an estimate of the densities in L^1. To this purpose, in [Rom14c], the L^1 estimate of the time increments follows from (an appropriate version of) the Girsanov change of measure.

Besov bounds work well with finite dimensional projections because allow to avoid the difficulty of low regularity of solutions of Navier–Stokes in three dimension. On the other hand the method based on Markov solutions works well, at least qualitatively, for any finite dimensional functional. It may be interesting to provide some "ad–hoc" strategy to get regularity of densities of some quantities of interest for the equations. In [Rom14b] there is a proof of regularity of densities for the energies $\mathscr{E}_\alpha(t,u) = \|u(t)\|^2_\alpha$, for α negative. More precisely, the following result holds.

Proposition 3.11 ([Rom14b]). *Given a weak martingale solution u of (1.1), and a number $\alpha > \frac{3}{4}$, the real valued random variable $\mathscr{E}_{-\alpha}(t,u)$ has a density $f_{t,\alpha}$ with respect to the Lebesgue measure on $\mathbf{R}$, for every $t > 0$. Moreover,*

- *$f_{t,\alpha} \in B^r_{1,\infty}(\mathbf{R})$ for every $r < 2\alpha - \frac{3}{2}$ if $\alpha < \frac{5}{4}$,*
- *$f_{t,\alpha} \in B^r_{1,\infty}(\mathbf{R})$ for every $r < 1$ if $\alpha \ge \frac{5}{4}$.*

A similar statement can be also obtained for the α–dissipation $\mathscr{D}_\alpha(t,u) = \int_0^t \|u(s)\|^2_{1+\alpha}\,ds$ (again with α negative), as well as for the fundamental energy balance $\mathscr{E}_0(t,u) + 2\nu\mathscr{D}_0(t,u)$, namely,

$$\|u(t)\|^2_H + 2\nu\int_0^t \|u(s)\|^2_V\,ds.$$

In the latter case, the key point is that we can exploit the cancellation property of the Navier–Stokes non–linearity. For α negative but larger than the threshold of Proposition 3.11, the contribution of the non–linearity has no good control.

3.8. **Additional remarks.** An interesting question, that has been completely answered for the two–dimensional case in [MP06], concerns the existence of densities when the covariance of the driving noise is essentially non–invertible. The typical

perturbation in (1.1) we consider here is

$$\dot{\eta}(t,y) = \sum_{\mathbf{k}\in\mathcal{K}} \sigma_{\mathbf{k}} \dot{\beta}_{\mathbf{k}}(t) e_{\mathbf{k}}(y),$$

where $\mathcal{Z} \neq \mathbf{Z}^3_\star$ and is usually much smaller (finite, for instance). The idea is that the noise influence is spread, by the non–linearity, to all Fourier components. The condition that should ensure this has been already well understood [Rom04], and corresponds to the fundamental algebraic property that $\mathcal{K}$ should generate the whole group $\mathbf{Z}^3$.

It is clear that the method we have used to obtain Besov bounds cannot work in this case, because the non–linearity plays a major role. On the other hand in [Rom14a] we prove, using ideas similar to those used in the strategy with Markov solutions, the existence of a density. No regularity properties are possible, though. See also [Rom05, RX11] for other relevant results on the Navier–Stokes equations in dimension three with "hypoelliptic noise".

Another issue that is, morally, generically applicable to any statement related to the Navier–Stokes equations in 3D, is the universality of the result obtained. Since we do not know if there is a unique distribution, it may be possible that the densities of solutions obtained by different methods may have different properties. In a way this is reminiscent of the problem of suitable weak solutions introduced by [Sch77]. Only much later it has been proved that solutions obtained by the spectral Galerkin methods are suitable [Gue06] (under some non–trivial conditions though), and hence results of partial regularity are true for those solutions. In [Rom14d] we establish a "transfer principle" that, roughly speaking, states that as long as we can prove existence of a density for a finite dimensional functional of the solution and for a weak solution that satisfies *weak–strong uniqueness*, then existence of a density holds for any other solution satisfying weak–strong uniqueness and a closure property with respect to conditional probabilities.

In other words, by the transfer principle, we can prove existence of a density for solutions obtained from Galerkin approximation, and this result will extend straight away to any other solutions, for instance those produced by the Leray regularization (see for instance [Lio96]). Or we can use the special properties of Markov solutions given in [FR08, Rom10a] to prove existence of densities of a large class of finite dimensional functionals, as done in the first part of [DR14], and again this extends immediately to any solution.

An important limitation of the transfer principle is that it applies only on quantities depending only on one time. This for instance rules out the results of Proposition 3.10. Moreover, the transfer principle is qualitative in nature, as it may transfer only the existence, and in general no quantitative information can be inherited. Nevertheless, in the case of stationary solutions, the transfer principle (with some loss) can be also made quantitative.

4. Invariant measures

Existence of invariant measures for stochastic equations is a classical topic, we refer to [DPZ96] for details. In fact, it dates back to original ideas of Kolmogorov, to add noise to an equations to find a unique invariant measure (something that is way more difficult when randomness is in the initial condition, see Section 2.5), and then to study the zero–noise limit to "select" the most interesting invariant measure of the noise–less equation.

Thus the most interesting part of the analysis in this subject concerns uniqueness of invariant measures, and possibly related ergodic theorems. The theory in the two–dimensional case, starting from the first results of [FM95], is well developed, both on the side of rough noise, see for instance [DPD02], and on the side of smooth finite dimensional noise (in some way, the most physical one when turning to turbulence), see [HM06].

In dimension three the theory is less developed, and so far we only know that some special solutions (the Markov solutions of [DPD03a] or [FR08] discussed in Section 3.2) admit an invariant measure which is "unique". This uniqueness statement has to be properly and carefully understood. The main limitation in these results is in noise. Indeed, these results require that the noise acts on all modes and that the decay of noise coefficients is controlled from above and below (with a minimum of flexibility in the control, see [Rom11a]).

4.1. **Existence and uniqueness of invariant measures in 3D.** The main issue when dealing with invariant states in the three–dimensional case, hence in the case where the dynamics is not well defined, is to identify a good definition of an invariant state. A fairly general definition is a *stationary solution*. Consider as a state space the set of all trajectories, for instance $\Omega = C([0,\infty;\mathcal{V}')$ (and further requirements, if necessary in order to get compactness), where $H \subset \mathcal{V}'$ is large enough so that solutions are continuous. A stationary solution is a probability measure on the state space such that trajectories are solutions of (1.1) (with or without noise) and such that the measure is invariant with respect to the time–shift, namely with respect to the maps $\tau_t : \Omega \to \omega$ defined as

$$\tau_t(\omega)(s) = \omega(t+s), \qquad s \geq 0$$

Roughly speaking, the action of τ_t is to cut out the first part of the trajectory, thus a stationary solution depends only on the "tails" of the trajectories. The time marginal of a stationary solution is the candidate invariant measure, although there is not a well–defined flow with respect to which the measure is invariant.

The techniques of [DPD03a, FR08] are specially tailored to define a flow of solutions of (1.1). In this setting invariant measures are a meaningful notion. As already mentioned, existence of an invariant measure is straightforward, after all the dynamics without forcing has the zero as the unique globally stable point, at least in H. A bit of compactness concludes the argument.

Uniqueness is obtained in [DPD03a] as a by-product (through Doob's theorem, see [DPZ96]) of two fundamental properties: continuity of transition probabilities, discussed in Section 3.3, and the full support. Both properties rely on the strong assumptions on the noise we have mentioned.

As it regards convergence to the unique invariant measure, we know that it is exponentially fast [Oda07, Rom08]. It is worth mentioning that in [Oda07] convergence of a solution towards a stationary solution is obtained for every limit probability of Galerkin approximation. Convergence in [Rom08] holds only for Markov solutions.

A completely different approach is presented in [Bak06], based on ideas we will see later in Section 5.1.3, but is restricted to bounded (and small enough) noise, although it ensures uniqueness of a stationary solution.

In conclusion, state of the art results in this setting are still very far from the strong results obtained in the two dimensional case in [HM06]. A mild relaxation on the non–degeneracy of the noise has been given in [RX11].

4.2. **Uniqueness criterion through invariant measures.** We have already seen that, under suitable assumptions on the driving noise, every Markov solution has a unique invariant measure. As in principle there can be several different Markov solutions, so are invariant measures.

Here we wish to discuss the different notions of invariant measures we can consider and a uniqueness criterion for the law of (1.1) based on invariant measures.

Define a stationary solution $\mathbb{P}_\star$ as a probability measure on $\Omega_{\rm NS}$ that is invariant for the time shift on the path space. There are several ways to ensure existence of stationary solutions, either without noise [FR01], or with noise [Rom10b]. One way is also provided by Markov solutions since if $(\mathbb{P}_x)_{x\in H}$ any such solution and $\mu_\star$ is its invariant measure, then $\mathbb{P}_\star = \int \mathbb{P}_x\, \mu_\star(dx)$ is a stationary solution.

Define the following sets

$$\begin{aligned}
\mathscr{J} &= \{\mu \in \Pr(H) : \mu \text{ is the marginal of a stationary solution}\},\\
\mathscr{J}_m &= \{\mu \in \Pr(H) : \mu \text{ is the invariant measure of a Markov solution}\},\\
\mathscr{J}_e &= \{\mu \in \mathscr{J}_m : \mu \text{ maximizes the selection procedure}\},
\end{aligned}$$

where the selection procedure is the one outlined in Section 3.2. It is easy to check [Rom08] that $\mathscr{J}_e \subset \mathscr{J}_m \subset \mathscr{J}$, and that $\mathscr{J}$ is compact.

The set of invariant measures associated to Markov solutions has a robust structure, so that several uniqueness results and characterizations are possible. For instance, we have the following result [Rom08].

Proposition 4.1. *If $\mu_\star \in \mathscr{J}_e$, then the stationary solution $\mathbb{P}_\star$ associated to $\mu_\star$ is the unique stationary measure in $\mathscr{C}(\mu_\star)$.*

If we also assume that the covariance of the driving noise is as in [FR08] (or as in [Rom11a]), so that the dynamics is strong Feller and irreducible in a smaller

space $\mathcal{W}$ (tipically $\mathcal{W} = V_\alpha$ for a suitable $\alpha > \frac{1}{2}$), then we have the following result (see also [Kry04] for a finite dimensional analogue of this result).

Theorem 4.2. *Assume that every Markov selection is strong Feller and fully supported on $\mathcal{W}$. Let $(\mathbb{P}_x)_{x\in H}$ and $(\mathbb{P}'_x)_{x\in H}$ be two Markov solutions, with $(\mathbb{P}_x)_{x\in H}$ maximizer in one of the possible maximisation procedures of the selection. If the two families have the same invariant measure, then they coincide on $\mathcal{W}$.*

In different words, if the set $\mathscr{J}_e$ contains only one invariant measure, then the martingale problem associated to the Navier–Stokes equations (1.1) is well posed. The converse is obvious.

We have been not able to find a way to apply the criterion. On the other hand, under the same assumptions of the above theorem, we know that the measures in $\mathscr{J}_m$ are all equivalent measures [Rom08, Corollary 3.5]

Theorem 4.3. *Any μ_1, $\mu_2 \in \mathcal{I}_m$ are equivalent:*

$$\mu_2 = \frac{d\mu_2}{d\mu_1}\,\mu_1.$$

In different words: any *almost sure* event is universal and the property holds independently of the (Markov) solution. To improve the reliability of the model and give a quantitative measure of uncertainty regarding events related to the fluid motion a better understanding of the density $\frac{d\mu_2}{d\mu_1}$ is needed. In principle, by having for instance stronger summability of the density than L^1, it may be possible to ensure that

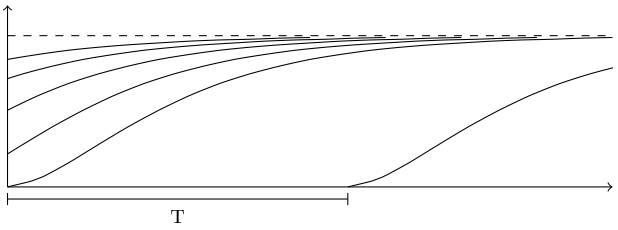
$$\mu_1[A] \approx 1 \quad \Longrightarrow \quad \mu_2[A] \approx 1,$$

so that events of high probability for one solution are strongly likely for any other, independently of uniqueness of distributions. This in a way justifies the line of research initiated in Sections 3.6–3.8.

Example 4.4 ([Rom11b]). To understand the questions of uniqueness and non–uniqueness in relation with invariant measures, let us consider a variation of the elementary example 3.3, $\dot X = -X + \sqrt{X}$, with initial condition $X(0) = x \in [0,1]$. The problem has a unique solution $X_x(\cdot)$ for $x \neq 0$ and the family of solutions $\{X^\star_a = X^\star((t-a)\vee 0) : a \geq 0\}$ for $x = 0$, where $X^\star$ is the unique solution starting at 0 such that $X^\star(t) > 0$ for all $t > 0$.

If $\mathscr{C}(x)$ denotes the set of all solutions starting at x, viewed as probability measures on the path space $C([0,\infty);[0,1])$, then $\mathscr{C}(x) = \{\delta_{X_x}\}$ for $x \in (0,1]$, where δ_{X_x} is the Dirac measure concentrated on X_x.

If $x = 0$, a solution starts at 0 and stays for an arbitrary time, then follows the solution $X^\star$ (suitably translated). So the *departing* time from 0 can be interpreted as a random time T whose law can be arbitrary. Therefore any selection of solutions is completely described by a single random variable on $[0,\infty]$. It is easy to be convinced that a selection of solutions is Markov if and only if T is exponential



(the lack of memory plays a major role), including the degenerate cases of infinite or zero rate (namely, $T = 0$ or $T = \infty$).

Denote by $(\mathbb{P}^a_x)_{x\in[0,1]}$ the Markov family with rate a. We shall call *extremal* all those Markov solutions that can be obtained by the selection procedure. It turns out that the only extremal families are those corresponding to $a = 0$ and $a = \infty$.

As it regards invariant measures, we notice that $(\mathbb{P}^a_x)_{x\in[0,1]}$ has a unique invariant measure (which is δ_1) if and only if $a < \infty$. Notice that all selections having δ_1 as their unique invariant measure coincide δ_1–almost surely.

If $a = \infty$, there are infinitely many invariant measures (the convex hull of δ_0 and δ_1). As there is no noise in this example, in general we cannot expect the invariant measures to be equivalent

4.3. **Explicit invariant measures.** With Section 2.5 in mind, let us look if we can find a suitable noise that provides an explicit expression for the invariant measure. Let us look for an invariant measure of Gibbs type, namely $\mu = \mathrm{e}^{U(x)}\, dx$, so that it has a density with respect to the (non–existent) infinite dimensional flat measure.

Formally, the Fokker-Planck equation satisfied by the infinite dimensional density $U = U(t, x)$, $t > 0$, $x \in H$, is

$$\frac{1}{2}\,\mathrm{Tr}(\mathcal{S}^\star\mathcal{S}D^2U) + \sum_n \frac{\partial}{\partial x_n}\big(U(\nu\lambda_n x_n + B_n(x))\big) = 0,$$

where $(\lambda_n)_{n\geq 1}$ are the eigenvalues of the Laplace operator, and $\mathcal{S}^\star\mathcal{S}$ is the covariance of the driving noise, and we assume for simplicity that there are numbers $\sigma_n \geq 0$ such that $Sen = \sigma_n$, that is the covariance is diagonal in the usual real Fourier basis $(e_n)_{n\geq 1}$. We postulate that

$$U = \mathrm{e}^{-\beta\sum_n \mu_n x_n^2},$$

that is we assume that the invariant measure is a Gaussian measure. We choose the coefficients $(\mu_n)_n$ to be either $\mu_n = 1$ for every n (in dimension 2 or 3), or $\mu_n = \lambda_n$ for every n (in dimension 2). With both choices we know that (formally)

$$\sum_n \mu_n x_n B_n(x) = 0.$$

With our choice of U, we have that

$$\frac{\partial U}{\partial x_n} = -2\beta\mu_n x_n U, \qquad \frac{\partial^2 U}{\partial x_n^2} = -2\beta\mu_n U + 4\beta^2\mu_n^2 x_n^2 U,$$

hence, by plugging these formulas in the Fokker–Planck equation and using the formula above for B,

$$U\sum_n(-\beta\sigma_n^2\mu_n + 2\beta^2\mu_n^2\sigma_n^2 x_n^2 - 2\nu\beta\lambda_n\mu_n x_n^2 + \nu\lambda_n) = 0$$

which yields $\nu\lambda_n - \beta\mu_n\sigma_n^2 = 0$, or

$$\sigma_n^2 = \frac{\nu\lambda_n}{\beta\mu_n}.$$

4.3.1. *Invariant measures from the enstrophy.* Let us consider first dimension 2 and $U = \mathrm{e}^{-\beta S(x)}$, where S is the enstrophy. The above computations suggest to choose the identity (on H) as covariance (that is, the driving noise is white in both space and time). Let us look first at the solution z of the linear problem (3.2). A simple computation shows that $\mathbb{E}[\|z(t)\|_H^2] = \infty$, hence $\mathbb{P}[z(t) \notin H] = 1$ by Fernique's theorem [Bog98]. Since we cannot expect that the solution to the full Navier–Stokes equations could be any smoother than z (in fact it is not), standard methods do not apply. In [DPD02] the equation is understood in terms of a renormalized non–linearity, where the square of the distribution u is understood as a Wick product, very much the same as it was done in the case of the Φ_3^4 model in the stochastic quantisation of Euclidean quantum field theory (see for instance [DPD03b, Hai14c]).

Here we give the rough idea following [BR13] (where it has been applied to a fourth order problem). Let z_N be the spectral truncation of z (namely, take only modes $|k| \le N$, then $B(z_N, z_N)$ is well defined and is a Cauchy sequence in $V_{-\alpha}$, $\alpha > 0$. To see the reason for this, let us check a simpler computation that shows that $B(z_N, z_N)$ is bounded in $V_{-\alpha}$ for $\alpha > 0$. The Cauchy sequence statement follows by similar considerations. Each component of z can be written as $z_k = \frac{k^\perp}{|k|}\zeta_k$, where

$$d\zeta_k + \nu|k|^2\zeta_k = d\beta_k,$$

where $(\beta_k)_{k\in\mathbf{Z}^2_\star}$ are independent complex–valued standard Brownian motion (but for $\beta_k = \beta_{-k}$). We first notice that $\mathbb{E}[\zeta_m\zeta_n j] = 0$ if $m \neq \pm n$ by independence. Moreover, $\mathbb{E}[\zeta_m^2] = 0$ and $\mathbb{E}[|\zeta_m|^2] = (\nu|m|^2)^{-1}$ by a direct computation, hence in the sum below, most of the terms cancel out and

$$\begin{aligned}\mathbb{E}[\|B(z_N, z_N)\|_{-\alpha}^2] &= \mathbb{E}\sum_k |k|^{-2\alpha}\Big|\sum_{m+n=k}(z_n\cdot k)\pi_k z_m\Big|^2 = \\ &= \sum_k |k|^{-2\alpha}\sum_{\substack{m_1+n_1=k\\ m_2+n_2=k}}\mathbb{E}[(z_{n_1}\cdot k)(z_{n_2}\cdot k)(\pi_k z_{m_1}\cdot\pi_k z_{m_2})].\end{aligned}$$

Consider the expectation $\mathbb{E}[(z_{n_1} \cdot k)(z_{n_2} \cdot k)(\pi_k z_{m_1} \cdot \pi_k z_{m_2})]$, this is zero unless $m_1 = m_2$ and $n_1 = n_2$, or $m_1 = n_2$ and $m_2 = n_1$ (the case $m_1 = n_1$, $m_2 = n_2$ can be neglected). In both cases the expectation is bounded by $\nu^{-2}|k|^2|m|^{-2}|n|^{-2}$ and it is easy to check (see for instance [BRT07, Lemma 2.3]) that the sum

$$\sum_k |k|^{-2\alpha+2} \sum_{m+n=k} \frac{1}{|m|^2|n|^2} \leq c_{15} \sum_k \frac{\log|k|}{|k|^{2\alpha+2}}$$

converges when $\alpha > 0$. It might be possible that a different approximation of z may lead to a different limit object $B(z,z)$ (see the discussion in [BR13]).

Once we know that $B(z,z)$ is well defined, we can look for a solution $u = z + v$, where v solves

$$\partial_t v + \nu A v + B(v,v) + B(z,v) + B(v,z) = -B(z,z).$$

It turns out that v is slightly better than u (the worst part of u is all in z), so that the products $B(v,v)$, $B(v,z)$ and $B(z,v)$ make sense without any further consideration. The above equation can be solved by a fixed point argument. The passage from local to global works using the fact that the system under consideration has an explicit invariant measure (see [DPD02] for further details).

The whole idea can be pushed even further, because one can allow for rougher covariance operators, as long as the equation for v makes sense, possibly by extending the decomposition of u to new factors, for instance at second order $u = z + w + v$, with $\partial_t w + \nu A w + B(z,z) = 0$, and v so that formally the equation for u is the correct one.

Unfortunately, the whole method finds a substantial limit, as discussed in the next section.

4.3.2. *Invariant measures from the energy?* The *naive* approach discussed in the previous section finds robust general versions in the works [GIP13] and [Hai14c] (see also [Hai14a, Hai14b]. A general rule to multiply random distributions is introduced in [GIP13] based on Littlewood–Paley decomposition. The approach of [Hai14c] is more abstract and general and aims to introduce a new kind of Taylor expansion, where the basic elements of the expansion can be tailored to the problem. Both methods have been introduced to deal with problem, such as the Kardar–Parisi–Zhang equation [Hai13], where the noise perturbation is so rough that in principle should have no meaning. Our previous section suggests the general approach of writing the solution as the contribution of several terms of growing regularity, up to the point that the "last" term is smooth enough that the non–linearity makes sense. There is in general no unique way to do this and in the most challenging problems (such as KPZ), some renormalization of these quantities may be required.

The structural limitation of the method is that there might be no gain of smoothing (or worse). For such *critical* (super–critical) problems the problem is completely open. A "rule of thumb" is suggested in [Hai14c] to grab the idea. To

simplify, let us consider our case of interest, namely the Navier–Stokes equations driven by the gradient of space–time white noise. From our formal computations in Section 4.3, this is the correct noise to ensure that the Gibbs measure associated to the energy is (formally) invariant.

Space–time white noise is worth $-\frac{d}{2}-1$ derivatives (time is worth one, each space dimension is worth a half), and we loose another derivative for the gradient. The linear dissipation allows to gain two derivatives. Then we plug the result in the non–linearity. We may look at the Navier–Stokes non–linearity as $\operatorname{div}(u\otimes u)$, which is better here because we need to take the square of a distribution rather than multiply it by its gradient. The external div is taken care by integration by parts and is less messy.

By this computation of derivatives, it turns out that in dimension two we are dealing with a critical problem (and the whole stuff is much worse in dimesnion three). Chances are that is already worse in dimension two due to some logarithmic divergence typical of the two dimensional Navier–Stokes non–linearity.

In conclusion, with the knowledge so far the existence of a Navier–Stokes flow leaving the Gibbs measure from the energy invariant is an open problem.

5. Other topics

In this last section we collect some additional topics related to the Navier–Stokes equations and probabilistic tools. We will give some detailed ideas only on the problem of probabilistic representation formulas for the solutions of the *deterministic* Navier–Stokes equations (although all formulas provided may be adapted to deal with random forcing). We conclude by recalling a few works related with the statistics on the Navier–Stokes equations with noise.

5.1. **Representation formulas.** In this last part we focus on the equations (1.1) with no random data and we are interested in finding the "hidden" stochasticity that can allow to represent the solution by a formula containing random term.

The representation of the velocity field of a fluid through characteristics is very natural in fluid dynamics, due to the possibility of describing the motion of a fluid with a Lagrangian point of view. On the other hand the idea of using random processes to give a representation of solutions is as old as Brownian motion. The two facts capture two of the main phenomena of the motion of viscous fluids, transport and diffusion. Indeed, in dimension two this is sufficient to obtain probabilistic representation formulas for the solutions [Bus99].

It turns out that in dimension three another phenomenon, vortex stretching, has to be taken into account, and this makes representation formulas more challenging. Two versions, [BFR05, CI08] detailed below, give a description in these terms.

An alternative definition, that roughly speaking focuses on the interaction of energy among modes can be given by means of branching processes [LJS97]. Here we will give a simplified idea following [BRT07]. The same approach allows to

define stationary solutions for the equations driven by bounded (non–Gaussian) stationary noise [Bak06].

Other probabilistic representation formulas, not presented here have been developed in [EMPS88, EP89, AB03, CS09].

5.1.1. *Representation through noisy Lagrangian trajectories.* We consider here results from [BFR05]. Let us start from the equation for the vorticity $\xi = \operatorname{curl} u$,

$$\partial_t \xi + (u \cdot \nabla)\xi = \nu \Delta \xi + D_u \xi$$

where $D_u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the deformation tensor, that is the symmetric part of ∇u. The "stretching" term $D_u \xi$ is the responsible for three–dimensional phenomena and is not present in the equation for the two dimensional vorticity.

Let us first consider the two dimensional vorticity in an inviscid fluid. Vorticity is transported along the flow induced by velocity, so that the quantity $\xi(t, X_t)$ is conserved when X is the trajectory of a "fluid particle", namely $\dot{X} = u(t, X)$.

When dissipation ($\nu \neq 0$) and stretching (the term $D_u \xi$) are taken into account, the whole problem is more difficult. Fluid particles keep moving following the velocity field, but in order to take into account dissipation, it is more convenient to single out the effect of diffusion, using the dynamics

$$dX = u(t, X)\, dt + \sqrt{2\nu}\, dB_t, \tag{5.1}$$

with B three dimensional standard Brownian motion. To take vortex stretching into account, we can imagine that when we compare vorticity at the two ends of a fluid particle trajectory, the cumulative effect of the deformation changes the vorticity size and direction. For instance, the vorticity is stretched when sufficiently aligned with the expanding directions of D_u. Since these directions change with time, $\xi(t, X(t))$ may undergo a complicate evolution with stretching, rotations, contractions. Heuristic reasoning and numerical experiments show a predominance of the stretching mechanism,.

In [BFR05] the three phenomena, transport, diffusion and stretching, are summarised by the following representation formula

$$\xi(t, x) = \mathbb{E}[V(t, 0)\xi(0, X_0)],$$

where $X(s)$ is the solution of (5.1) with *final* condition $X(t) = x$, and $V(r, s)$ is the solution of the 3×3 matrix equation

$$\begin{cases} \frac{d}{dr} V(r, s) = D_u(r, X(r))V(r, s), & r \in (s, t) \\ V(s, s) = I. \end{cases}$$

So far, the representation is incomplete, since V depends from the deformation tensor, hence from the velocity, that in turns can be reconstructed from the vorticity (with suitable decay). In [BFR05] the reconstruction, through the Biot–Savart

law, is also formulated as a probabilistic representation

$$u(x) = \int_0^\infty \frac{1}{2t}\mathbb{E}[\xi(x + W_t) \times W_t]\, dt,$$

where W is an additional three dimensional standard Brownian motion. The representation we have described is implicit, since the formula for vorticity is given in terms of the velocity, and the formula for the velocity depends on the vorticity. Nevertheless, this allows to formulate an alternative proof of a Beale–Kato–Majda like criterion given in [Pon85], and the local existence results for the vorticity given below.

Theorem 5.1. *Given $p \in [1, \frac{3}{2})$, $\alpha \in (0, 1)$, let $\xi_0 \in C^\alpha_b(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$. Then there are $\tau > 0$, depending only on $\|\xi_0\|_{C^\alpha_b \cap L^p}$, and a unique solution u of (1.1) given by the representation formulas above.*

Obviously, the smallness of τ can be replaced by the smallness of $\|\xi_0\|_{C^\alpha_b \cap L^p}$ to have global solutions. Also suitable external forces can be considered in the probabilistic formulation.

5.1.2. *Eulerian–Lagrangian approach.* An alternative representation formula that is based on the Eulerian–Lagrangian approach developed in [Con01a, Con01b] has been presented in [Iye06, CI08]. The formula yields directly the velocity without passing through vorticity. On the one hand the effect of stretching are less apparent, on the other there is a complete decoupling between diffusion and transport effects, namely the final formula for the viscous fluid is simply the expectation of the corresponding formula for inviscid flows.

The starting point is the *Weber formula* for an inviscid fluid [Con01a],

$$u = \Pi_L(\nabla A)^T(u_0(A)),$$

where Π_L is the Leray projector, $A_t = X_t^{-1}$ and for every $x \in \mathbf{R}^3$, $X(\cdot; x)$ is the Lagrangian trajectory started at x, namely the solution of $\dot X_t = u(t, X_t)$ with initial condition x.

In order to take the effect of viscosity into account, one can consider the diffused Lagrangian trajectories (5.1), that is $dX = u(t, X)\, dt + \sqrt{2\nu} dB$, with B three–dimensional standard Brownian motion. Define again the *back–to–label* map $A_t = X_t^{-1}$ (this time X is a stochastic flow), then the representation formula is

$$u = \mathbb{E}[\Pi_L(\nabla A)^T(u_0(A))].$$

One can immediately deduce an expectation for the vorticity,

$$\xi = \mathbb{E}[(\nabla X)\xi_0(A)],$$

and the formula can easily take an external force $f(t, x)$ into account,

$$u(t) = \mathbb{E}[\Pi_L(\nabla A_t)^T(u_0(A_t))] + \mathbb{E}\Big[\Pi_L \int_0^t (\nabla X)^T f(s, X_s)\, ds\Big].$$

Further development can be found in [CI06, IM08, Iye09].

5.1.3. *Stochastic cascades via branching processes.* [LJS97, BRT07, Bak06] Unlike the previous representation formulas, the representation that will be examined in this section is completely explicit and the hidden randomness is provided by branching processes. The idea of using branching processes as the underlying engine of probabilistic representations is not new, see for instance [Sko64, INW68, McK75] where branching is coupled with diffusion, and the stochastic representation is derived directly in the physical space, so that the linear operator is limited to generators of diffusions and the non–linearity is polynomial.

In [LJS97] the authors are able to consider the Navier–Stokes non–linearity by looking for a representation in Fourier space. Their method suggests a flow of the kinetic energy among scales governed by transition probabilities computed according to the (Fourier–transformed) non–linearity, and hence called evocatively stochastic cascades. It turns out that their method is quite general and can handle a large class of semi–linear parabolic PDEs, or systems of PDEs. Here we follow the presentation in [BRT07] and consider PDEs with periodic boundary conditions of the type

$$\partial_t u = Au + F(u) + f,$$

where A is an operator with a complete set of eigenfunctions, F is a polynomial non–linearity in u and its derivatives, and f is a given driving function. The case of full space, as in [LJS97], can be considered with similar ideas.

In short, the solution u is expanded into Fourier series with respect to the eigen–functions of A. The PDE is transformed into a system of countably many ODEs for the Fourier coefficients. The solution of the system is represented by the expectation of a recursive functional over a tree of branching particles. The rules for branching, regeneration and death probabilities of particles arise from the particular PDE studied.

One major drawback of this stochastic representation is that it often fails to exist for large times t, although the solution to the PDE may still exist. The problem is that the recursive functional may fail to be integrable at some time. The paper [BRT07] provides both a comparison equation whose finiteness implies integrability of the recursive functional, and a way to avoid non–integrability by suitably pruning the tree. A different approach, unfortunately working only for ODE, and based on resummation has been proposed in [Mor05].

Due to this problem it is not easy to tackle, by this approach, problems like the long time behaviour. This can be done only in special cases of of small initial conditions and uniformly small forcing, as in [Bak06] for the Navier–Stokes equations with small bounded forcing.

We formulate the probabilistic representation for the two dimensional Navier-Stokes in its vorticity formulation, in order to avoid the additional but harmless

difficulty of vector valued coefficients. One can proceed likewise for the three–dimensional case on the torus. In terms of Fourier series the equation reads

$$\dot\xi_k = -|k|^2\xi_k + \sum_{m+n=k} \frac{k\cdot m^\perp}{|m|^2}\xi_m\xi_n + f_k,$$

where $(\xi_k)_{k\in\mathbf{Z}^2_\star}$ are the Fourier coefficients of the vorticity with respect to the complex exponentials, $(f_k)_{k\in\mathbf{Z}^2_\star}$ the coefficients of the forcing. We assume that $\chi_0 = f_0 = 0$. We then set $\chi_k = |k|^\alpha\xi_k$ for some suitable $\alpha > \frac12$, and, for $c_{16} > 0$ we then define $\lambda_k = |k|^2$, and

$$B_{kmn}(\chi,\chi') = \frac{k\cdot m^\perp}{|k\cdot m^\perp|}\chi\chi', \qquad q_{kmn} = \frac{|k|^{\alpha-2}|k\cdot m^\perp|}{c_{16}|m|^{\alpha+2}|n|^\alpha}, \qquad \gamma_k = \frac{|k|^{\alpha-2}}{d_k}f_k,$$

for all k, m, $n\in\mathbf{Z}^2_\star$ satisfying $k\cdot m^\perp\neq 0$ and $m+n=k$ (and zero otherwise). It is elementary to show that $q_k = \sum_{mn} q_{kmn} < \infty$ and that $q_k\to 0$ as $|k|\to\infty$. By choosing c_{16} large enough we have $q_k < 1$ and we set $d_k = 1 - q_k$. The equations are recast in the following form,

$$\dot\chi_k = \lambda_k\big(-\chi_k + c_{16}\sum_{m,n\in\mathbf{Z}^2_\star} q_{kmn}B_{kmn}(\chi_m,\chi_n) + d_k\gamma_k\big),$$

or, better, in its mild form

$$\chi_k(t) = \mathrm{e}^{-\lambda_k t}\,\chi_k(0) + d_k\lambda_k\int_0^t \mathrm{e}^{-\lambda_k(t-s)}\,\gamma_k(s)\,ds + \\ + c_{16}\lambda_k\int_0^t \mathrm{e}^{-\lambda_k(t-s)}\sum_{m,n\in\mathbf{Z}^2_\star} q_{kmn}B_{kmn}(\chi_m(s),\chi_n(s))\,ds,$$

with $k\in\mathbf{Z}^2_\star$. The constants $\lambda_k > 0$ are the the rate of particle evolution), q_{kmn} and d_k are the probabilities of branching and dying.

We describe first the branching tree. Define the labels set $\mathscr{I} = \bigcup_{n=0}^\infty\{1,2\}^n$, the history of a particle $\alpha = (\alpha_1,\dots,\alpha_n)$ can be read off by interpreting $\alpha_j = 1$ (or 2) as being child 1 (or 2) in a binary branching event at generation j. Fix

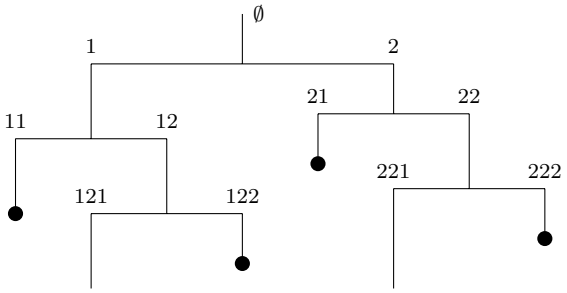


FIGURE 6. A tree with branches and deaths (•).

$k \in \mathbf{Z}^2_\star$, a tree rooted at k is a system, indexed over $\mathscr{I}$, of particle positions $\mathbf{K}_\alpha$, birth τ^B_α and death τ^D_α times, defined inductively over the length of the labels. At the root $\mathbf{K}_\emptyset = k$, $\tau^B_\emptyset = 0$, and $\tau^D_\emptyset$ is exponential with rate λ_k. Given the tree with particles with n ancestors, each of these particles, with position say k', either die with probability $d_{k'}$, or disappear giving raise to two new independent particles, with positions m and n with probability $q_{k'mn}$. The new particles will have a lifespan distributed as independent exponential random variables with rates λ_m and λ_n.

Notice that by construction, given a branching particle giving raise to two particles at positions m and n, and conditional to its genealogy, the two sub–trees generated are independent and with the same distribution of trees rooted at m and n.

To ensure that the tree has only finitely many branches before a given time t, a sufficient condition is that $q_k \leq d_k$. We define then the evaluation operator along the tree. Fix a forcing functions γ, an initial condition $\chi(0)$ and a time $t > 0$. We define an evaluation map R_t recursively backwards along the tree. For a finite tree $\mathcal{T}$ produced with the above rules, start at time $s = t$ and work back to time $s = 0$: evaluate the initial condition $\chi(0)$ at any particles that is alive at time t, evaluate the forcing function $\gamma(s)$ at any particle that dies at time $s < t$, and apply the bi–linear operators at the times of branching events.

To understand how the evaluation works, let us consider the simple example of one possible position and the ODE $\dot\chi = -\chi + \frac{1}{2}\chi^2 + \frac{1}{2}f$, with f constant in time. We have $d = q = \frac{1}{2}$, $c_{16} = 1$ and the bi–linear operator is the usual product. Hence on a tree $\mathcal{T}$ the evaluation yields $R(\mathcal{T}) = \chi(0)^{A_t} f^{D_t}$, where A_t are the particles alive at time t and D_t are the particles dead by time t.

Consider again the general system we have discussed so far. For an initial condition $\chi(0) \in \ell^\infty(\mathbf{C}^r)$, and a forcing $\gamma \in L^\infty([0,T], \ell^\infty(\mathbf{C}^r))$, the representation formula, when the expectation exists, is given by

$$\chi_k(t) = \mathbb{E}_k[R_t],$$

where $\mathbb{E}_k$ is the expectation with respect to the law of a tree rooted at k.

Unfortunately the expectation may fail to be finite, even in the seemingly simple example discussed above. The number of particles alive is Poisson, and if $|\chi(0)|$ is larger than 1, $\mathbb{E}[R_t]$ is not defined for t large enough (but possibly smaller than the existence time of the solution). A method based on pruning of trees is proposed in [BRT07] to avoid this divergence problem.

5.2. **Statistical topics.** In this last section we briefly summarise some recent works associated by the common idea of applying statistical theories to estimate the values of parameters, or the distribution of the driving forces. These results are mainly justified, at least in their spirit, by applications to weather forecasting. This also justifies (together with better analytic estimates) the fact that the analysis is centred to the two dimensional case.

In [CGH11] the main aim is to give an estimate of the viscosity in the two–dimensional Navier–Stokes equations driven by noise, with periodic or Dirichlet boundary conditions, given the full observation of a path in a time interval $[0,T]$. The infinite dimensional problem is not regular, namely the probability distributions, for different values of ν, are not mutually equivalent. The authors compute the maximum likelihood ratio estimator of spectral approximations of the problem. Galerkin approximations are needed to recover regularity. The estimators are weakly consistent and asymptotically normal, although tricky to be computed. The authors formulate two simplified estimators that are still weak consistent (although the rate of convergence is not clear) and depending only on a finite number of modes.

In [HLS13] the authors discuss how to recover the driving force and the initial conditions, given noisy observations of the fluid. They based their analysis on the Bayesian approach, looking for the maximum a posteriori estimator (minimizing a least square problem), given that the prior distributions of initial condition and forcing are Gaussian, and the forcing is white in time.

The work [BLSZ13], see also [BLL+13], deal with filtering to improve the accuracy of the estimate of the state of the system, in view of updating the posterior distribution.

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