On mixed boundary conditions, function spaces, and Kato's square root property

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

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	Prof. Dr. Pierre Portal
Tag der Einreichung:	29. März 2021
Tag der mündlichen Prüfung:	24. Juni 2021

Darmstadt 2021

Bechtel, Sebastian: On mixed boundary conditions, function spaces, and Kato's square root property
Darmstadt, Technische Universität Darmstadt,
Jahr der Veröffentlichung der Dissertation auf TUprints: 2022
Tag der mündlichen Prüfung: 24.06.2021

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Preface

Taming non-smoothness has been a major theme in the analysis of partial differential equations and other branches of analysis in the last decades. Momentarily, we will review some aspects of this journey and describe the contributions of this thesis to the field. We will keep the focus on the general picture; for a more thorough orientation concerning a particular topic in this thesis, the reader is advised to consult the introduction to the respective chapter.

Rough geometry

Let us begin with a glance at non-smooth geometry. A first fundamental question is the following: How could one measure the smoothness of an open set? Certainly, there is no single answer to this question. For instance, one may take into consideration the regularity of its boundary, either considered as being locally the graph of a function or as the boundary of a manifold with boundary [51]. Also, there are purely measure-theoretic concepts: For the set itself, a common condition is the *interior thickness condition*, which is tightly connected to the study of Sobolev spaces on an open set [55], but there are also conditions for the boundary itself, for instance the notion of *Ahlfors–David regularity*. Besides that, there are involved metric conditions like the ε-cigar condition of Jones [65], corkscrew conditions and many more.

As was just mentioned, there is a deep connection between concepts in rough geometry and the theory of Sobolev spaces. In the smooth case, many properties and constructions can be performed by "flattening the boundary" and working in the regular configuration of a halfspace, where simple reflection arguments are feasible. In the non-smooth case, a considerably more involved usage of the geometry is needed. Examples of this can be found in the works of Calderón [27] and Jones [65] on extension operators. The work of Jones already allows one to treat very irregular configurations like the von Koch snowflake [94]. But there are limits, for instance an open set has at least to be interior thick to allow for the construction of an extension operator for Sobolev spaces [55].

To establish further results in non-smooth geometry and in the theory of Sobolev spaces, it is often handy to have notions of (fractal) dimension at hand. In fact, there is a whole zoo of such dimensions, including that of Aikawa, Assouad, Hausdorff, and many more [43]. They are of different nature, for instance the Aikawa dimension is Euclidean and the Assouad dimension is purely metric. Furthermore, there are different purely metric dimensions which emphasize different aspects, and hence might not coincide for certain sets. But there are also prominent examples where different notions do coincide [71]. Often, these bridges lead to deep insights! And even if different notions are not equivalent, they occasionally obey interesting relations, for example the relation between the concept of porosity and the dimension of Assouad [73]: A set is porous if and only if its Assouad codimension is strictly positive.

Fractional dimensions are also tied to the study of Hardy's inequality [40, 54, 70] and fractional variations thereof [34, 38], the study of characteristic functions as pointwise multipliers [44,88] and their regularity as functions [89], or the existence of traces of almost everywhere defined functions [2, 66].

Boundary conditions

All of the three mentioned tools – Hardy's inequality, characteristic functions as pointwise multipliers, and trace operators – can be used to introduce homogeneous *Dirichlet boundary conditions*, which is to say that a function vanishes "in some sense" on the boundary, and we will give more details concerning this point in a moment. Observe that for the existence of a trace operator, some regularity of the function and the boundary are *a priori* needed, whereas it is always possible to write down a Hardy's term and ask for its finiteness. This already highlights that different concepts for the treatment of boundary conditions might not even be comparable, yet coincide in general, and each of them has advantages and disadvantages.

In fact, all three methods will be used in the interpolation of Sobolev spaces with boundary conditions in Chapter 2. The boundary conditions of the spaces under consideration there are always formulated using a trace operator. We will see that this allows one to apply simple functorial arguments in the treatment of these spaces. The usage of characteristic functions yields the definition of so-called *bullet spaces* in Section 2.4.2. They are a powerful tool in the theory of function spaces with boundary condition, and the interpolation behavior of Sobolev spaces with pure Dirichlet boundary conditions in a rough setting follows almost immediately from properties of these spaces. Finally, Hardy's inequality is a handy way to encode a vanishing trace condition in a manner that is accessible to direct computations. This is exploited in Section 2.6, where a very general result concerning real interpolation of Sobolev spaces with boundary condition is shown. Another example is the "special" Calderón-Zygmund decomposition shown and used in Chapter 6. In Chapter 4, a fractional Hardy term is even turned into the definition of a "vanishing trace" for functions with a fractional order of Sobolev regularity, and a very general extension result without usage of localization techniques is established.

Besides homogeneous Dirichlet boundary conditions, there are other boundary conditions that one could impose. For example, one could demand that a function does not vanish at the boundary, but attains a prescribed function defined on the boundary, which corresponds to non-homogeneous Dirichlet boundary conditions. One could also require that the gradient of the function satisfies some condition on the boundary. As with the trace operator, it is again a non-trivial question how such a condition even has to be understood. In the further course of this thesis, questions of this kind will not be addressed. Instead, a third type of boundary condition is in the spotlight: We speak of *mixed boundary conditions* if a homogeneous Dirichlet boundary condition is imposed on a portion of the boundary and *natural boundary conditions* are imposed on the rest of the boundary, which just means that we do not prescribe *any* boundary behavior on this complementary part of the boundary. In fact, mixed boundary conditions are the driving motive of this work and play a role in every chapter of this thesis.

A very useful geometric framework for the treatment of mixed boundary conditions was introduced by Gröger in [52]. In there, Gröger investigated the existence of weak solutions to an elliptic problem subject to mixed boundary conditions in L^p for the situation $p \neq 2$. Later, complex interpolation was shown in this setting [49]. With these tools available, this geometric constellation was used very successfully for the analysis of physically relevant models [32, 60, 79]. The case of real interpolation is much easier to treat [38], although the authors there have only treated the case $p = 2$. Much later, a different framework based on Jones' ideas was proposed by Brewster, Mitrea, Mitrea, and Mitrea in [26]. It includes a rich theory of extensions and traces and applications to the mixed problem in this setting.

We have already started to discuss Chapter 2 above, which is all about the interpolation of Sobolev spaces with mixed boundary conditions. In there, we suggest two geometric constellations. One is an extension of the result concerning real interpolation in [38]. In Section 2.6, we extend this result to unbounded and unconnected sets and to the case $p \neq 2$. However, the heart of this chapter is an extension of Gröger's result. *Gröger regular* sets are based on charts in which the interface between the two boundary parts is a Lipschitz submanifold of the boundary. The result in Section 2.4 is much more flexible. For instance, charts in any sense are only assumed to exist around the *Neumann part*, which is to say, the boundary part on which natural boundary conditions are imposed. Another advantage is that the interface is only supposed to satisfy a *porosity condition*. We have already mentioned this concept before: The interface is porous if and only if it is *not* full dimensional in the sense of Assouad. The upshot is that one can work again with bullet spaces inside the (flattened) boundary. Albeit this complex interpolation result is not employed in other chapters of this thesis, the success of Gröger regular sets suggests that also this result will prove useful in many real-world applications for it is much easier to check.

If we leave interpolation aside, the framework of Brewster, Mitrea, Mitrea, and Mitrea [26] could be considered to be the state of the art in the treatment of mixed boundary conditions. They use the (ε, δ)-domains introduced by Jones in a clever way as "charts" around the Neumann part to localize the mixed boundary constellation. This way, they can, for example, craft an extension operator for their geometric framework. In Chapter 3, we also use Jones' ideas to build an extension operator in the case of mixed boundary conditions. However, this operator is not based on localization but modifies the original construction of Jones. To get good estimates for his extension operator, Jones uses *connecting chains of cubes* between so-called *interior cubes*. In our construction, there are interior cubes which are not connected to other interior cubes, but "escape" the underlying set through the Dirichlet boundary part. The whole construction is highly technical, but it allows to consider constellations which are irregular arbitrarily close to the interface between Dirichlet and Neumann part, and hence are not feasible by localization methods.

The geometric setting of Chapter 3 is implied by the geometric setting of Brewster, Mitrea, Mitrea, and Mitrea. We also introduce another geometric framework in Chapter 5 which is intermediate between these two: This geometry also incorporates a "security area" around the Neumann part as is the case in [26], but it nevertheless does not involve the usage of charts in any sense, which is in accordance with the geometry from Chapter 3. We will come back to this configuration later.

Differential operators with non-smooth coefficients and the Kato problem

We leave the geometric aspects for a moment to have a look at differential operators with rough coefficients. In a series of articles [67, 68], Kato more-or-less asked the question when for an (at least maximal accretive) operator L on a Hilbert space one has the identity $\mathrm{D}(\sqrt{L}) = \mathrm{D}(\sqrt{L^*})$. After a series of examples and, in particular, counterexamples [72, 74], the question was refined to only consider the case where L is a second-order elliptic operator in divergence form with bounded, measurable, complex coefficients – in the first place as an operator on $\mathbb{R}^d$. In this case, L should be defined using a sesquilinear form $a\colon V \times V \to \mathbb{C}$, in which case the question can be reformulated as whether or not the identity $\mathrm{D}(\sqrt{L}) = V$ holds. This identity is called the *Kato square root property*. In the case of smooth coefficients, the operator L itself has optimal elliptic regularity and the square root property is an easy application of complex interpolation. Also, if the operator is self-adjoint, the square root property follows readily from Kato's so-called *second representation theorem* [69]. In the rough and non-selfadjoint situation, the square root property means that at least $\sqrt{L}$ has optimal elliptic regularity, even though this might not be the case for L itself. It turned out that the fractional exponent $1/2$ is the critical exponent for optimal elliptic regularity: For exponents strictly below $1/2$, optimal elliptic regularity follows from abstract arguments and was already known to Kato [67]. On the other hand, it is easy to construct counterexamples against optimal regularity for exponents above $1/2$ in dimension one [4].

Kato's motivation for this question came from applications to elliptic and hyperbolic equations, see [75] for more information. These ideas are nowadays successfully used in what is called the *first-order approach* [6, 7]. This underlines the relevance of Kato's question, and in particular the deviation from his original question, which was ruled out by the counterexample of Lions and McIntosh.

From the viewpoint of harmonic analysis, Kato's square root problem asks to bound certain singular integrals. Besides the square root problem, there were other challenging problems of the same kind, like the boundedness of the Cauchy operator on Lipschitz curves, which were summarized under the name *Calderón program* [16, p. 463]. Armed with many very novel techniques, Kato's square root problem was eventually solved in 2002 by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in their seminal paper [13]. For more information we refer the reader to the excellent surveys of McIntosh in [77, 78] and to the introduction of [13].

To close the loop to rough geometry and mixed boundary conditions, we take a look at a quote by Lions taken from a remark in [72], where he says the following.

> [...] par exemple, pour un opérateur elliptique A du 2ème ordre, non auto-adjoint, avec condition aux limites de Dirichlet sur une partie de la frontière et condition aux limites de Neumann sur le reste de la frontière, on ignore si $D(A^{1/2}) = D(A^{*1/2})$. Même chose d'ailleurs avec le problème de Dirichlet et une frontière irrégulière.[1]

Phrased differently, Lions suggests to combine the challenges in rough geometry and mixed boundary conditions with those in harmonic analysis coming from rough coefficients of a differential operator.

The seminal work by Axelsson, Keith, and McIntosh [16] opened the door to this problem. In that article, the authors provide *quadratic estimates* for perturbed Dirac operators. This framework allows to solve several problems from the Calderón program at once, including the Kato square root problem. Thereby, it is flexible enough to also treat systems of equations, and it could be adapted by the same authors to give a first answer [15] to the problem posed by Lions. However, the class of admissible geometries in there is not easily accessible.

This is why Egert, Haller-Dintelmann, and Tolksdorf refined the ideas from [16]. They observed that it is possible to prove quadratic estimates

[1] for example, if A is a second-order elliptic operator that is not self-adjoint and that is subject to Dirichlet boundary conditions on a part of the boundary and Neumann boundary conditions on the rest, then it is unknown if the identity $D(A^{1/2}) = D(A^{*1/2})$ holds. The same is true if the operator is subject to pure Dirichlet boundary conditions, but the boundary is irregular.

in a way that decouples harmonic analysis from geometry [39]. In a second paper, they used their modified framework to prove a very general result concerning Lions' problem [38]. Compared to the application to Kato's problem in [16], the application here is far more involved: It requires hard work to check the assumptions for the perturbed Dirac operator framework, including interpolation theory for Sobolev spaces incorporating boundary conditions, extrapolated optimal regularity for the Laplacian, or the construction of extension operators for Sobolev spaces with boundary conditions.

Extending the result of Egert, Haller-Dintelmann, and Tolksdorf is probably the deepest contribution of this thesis to the field. Their result is already fairly general, but there is some margin for improvement. For instance, they only treat bounded domains which satisfy the interior thickness condition. Moreover, they assume that the whole boundary and not only the Dirichlet boundary part are Ahlfors–David regular. In their setup, the latter is no restriction because they require Lipschitz charts around the Neumann boundary part, which in turn implies that the whole boundary is regular. Our improvement in Chapter 5 is as follows: We allow the underlying set to be disconnected and unbounded. Only the Dirichlet boundary part is supposed to be Ahlfors–David regular, and regularity around the Neumann boundary part is given by a local (ε, δ)-condition, which does not involve the usage of charts in any sense. This is the intermediate geometry that we already mentioned when comparing the setup from Brewster, Mitrea, Mitrea, and Mitrea with our setup from Chapter 3.

In our discussion concerning Chapter 3 we already mentioned that this condition must be satisfied in a neighborhood of the Neumann boundary part. This is used to show porosity of the full boundary, which is our substitute for Ahlfors–David regularity for the full boundary in [39]. Besides Ahlfors–David regularity, Lipschitz charts around the Neumann part were needed for the existence of a Sobolev extension operator in their setting. This is not an issue for us since we can rely on the results from Chapter 3. The connectedness assumption was in fact not needed in [39] and could hence be easily eliminated.

The most severe challenge is to eliminate the interior thickness condition. In other words, this condition means that the underlying set is a space of homogeneous type. For this type of spaces, Christ managed to craft "dyadic grids" [28, 80], which can be used as a substitute for dyadic cubes in $\mathbb{R}^d$. The existence of such a cube structure is essential for the harmonic analysis in the proof of quadratic estimates and cannot be circumvented. Instead, we employ an *a posteriori* argument in Section 5.5. This works in two steps:

First, we fatten the underlying set near the Dirichlet boundary part, thereby ensuring the interior thickness condition without losing any other geometric quality. Note that even if one starts with a connected set, this fattened set will be disconnected, which shows that the deviation from domains opens the road for a richer toolbox, even if one is only interested to apply the result to domains in the end. On this auxiliary set we solve the Kato problem for an "extension" of the elliptic operator. Second, we decompose the functional calculus of this extended elliptic operator to transfer regularity of the square root to the original elliptic operator. The reader is advised to also consult the roadmap in Chapter 5 to get a better understanding of the strategy of proof.

Besides this, we also have to redo the arguments from [38] in the more complex geometric constellation. A lot of this is already done in the chapter on interpolation theory, but we also need some more involved potential theory due to the lack of Ahlfors–David regularity for the full boundary.

Beyond Calderón-Zygmund theory

Another consequence of the rough nature of the coefficients is that extrapolation to L^p-spaces is much harder compared to classical Calderón-Zygmund theory. In particular, it is in general not possible to show $\mathrm{W}^{1,p} \to \mathrm{L}^p$ estimates for $\sqrt{L}$ for *all* $1 < p < \infty$. Nevertheless, extrapolation to $p \neq 2$ is possible and was pioneered, for example, by Blunck and Kunstmann [24]. In the situation of the classical Kato problem on $\mathbb{R}^d$, the L^p-extrapolation theory is well-understood [5]. These techniques go under the name "beyond Calderón-Zygmund theory". Extensions to the situation of mixed boundary conditions were first obtained in [8] in the situation of real equations and later extended in [36] to complex systems in a natural range of $p < 2 + \varepsilon$.

The geometric assumptions in [8, 36] were essentially dictated by the L^2-theory in [38, 39]. Hence, it is only natural to generalize these results to the situation from Chapter 5. This is performed in Chapter 6. Most arguments work similarly, but the unbounded underlying set demands for some extra care. The most innovative contribution of this chapter is the case $p > 2$. In the reference works, only exponents up to $2 + \varepsilon$ were considered, where ε is an abstract parameter. A quantifiable interval was only used in the work of Auscher on $\mathbb{R}^d$ in [5]. Essential tools in this case are a *conservation property* for the semigroup associated with the elliptic operator, and local Poincaré inequalities. The conservation property restricts the usage of certain lower-order terms, which is in accordance with Auscher's work, and forces one to work with pure Neumann boundary conditions. To show local Poincaré inequalites, we rely on homogeneous estimates shown in Section 3.9 for the extension operator from Chapter 3. Now the crucial point is: Homogeneous estimates hold in general only in a strip around the underlying set. But for the arguments in Chapter 6 we need *scale-invariant* local Poincaré inequalities... However, there is a situation in which global homogeneous estimates, and hence scale invariant local Poincaré inequalities, are available: When the underlying set is unbounded. This is a strong argument that it is not only often unnecessary to restrict oneself to bounded domains but might even bring certain results completely out of reach!

Acknowledgement

First of all, I want to thank my supervisors Robert Haller-Dintelmann and Moritz Egert for everything they have done for me! Robert even started to shape my mathematical career before it actually began: When I was a first-year computer science student, I visited his mathematics course for CS students, which was a major factor for me to switch subjects. I already wrote my Bachelor's and Master's thesis with him, we participated in many Internet Seminars together, he always trusted in my abilities in the transition area to PhD studies, supported me to get funding using a PhD scholarship, and supported me in the daily business during my PhD studies in Darmstadt.

Moritz was always a mentor, friend, and pace maker on the road bike to me. He shaped my style to do mathematics, was the coauthor of my first paper, supported me in everything, and spend a lot of time on my progress during both my graduate and undergraduate studies. Also, I am grateful to the Laboratoire de Mathématiques d'Orsay for hospitality during two stays in 2018 and 2019 which I used to work with Moritz on several interesting projects. Our work was similarly exhausting but fulfilling as the road biking activities to Chartres that we incorporated into these stays.

Also, I would like to thank Joachim Rehberg who suggested quite a few topics to me and who invited me two times to come to Berlin to discuss these ideas. I always enjoyed my time at the WIAS.

I thank my fellow PhD students, in particular Björn Augner (though he always had a PhD), Aday Celik, Thomas Eiter, Klaus Kress, Henning Möller, Andreas Schmidt, David Wegmann, and Marc Wrona. Special thanks go to Björn Augner, Tim Binz, Aday Celik, Thomas Eiter, Klaus Kress, Michelle Luckas, Henning Möller, and Andreas Schmidt for proofreading my thesis.

Besides that, I thank all other people from the Analysis group who made my time as a PhD student more joyful and easier. In particular, I would like to name our secretaries Renate Driessler and Verena Soga, as well as Karoline Disser and Patrick Tolksdorf. Karoline was my office neighbor and she helped me all the time, in particular when I wrote my proposal for a PhD scholarship at the very beginning of my PhD career, and Patrick was always supportive and we worked together successfully on two projects.

Some chapters of this thesis are based on publications which were jointly written with several coauthors and it is my pleasure to thank them. To be more precise, I thank Moritz Egert for his involvement in Chapter 2, Russell Brown, Robert Haller-Dintelmann, and Patrick Tolksdorf in Chapter 3, and

Moritz Egert and Robert Haller-Dintelmann in Chapter 5.

My work was supported by the Studienstiftung des deutschen Volkes. Besides the financial support, which certainly was very helpful, the academic support also meant a lot to me. In particular, I had the chance to participate in two language courses, participated in several seminars, and got to know a lot of interesting people in my local group. Especially, I enjoyed the time with my friends from the so-called "Zwischenstammtisch" group, with whom I also did quite some research in fluid dynamics.

A major factor in my work-life balance was the DGD racing team. Often (but certainly not always!) I enjoyed our time on the road bikes, be it in the Odenwald, on the Gampenpass, or at Sa Calobra.

Finally, I would like to thank my family for their support throughout my studies!

Thank you!

Zusammenfassung in deutscher Sprache

Die vorliegende Arbeit befasst sich mit Fragen der unglatten Geometrie und den gemischten Randbedingungen, sowie der Anwendung jenes Frameworks auf die Regularitätstheorie von elliptischen Differentialoperatoren in Divergenzform.

Den Start macht hierbei Kapitel 2, in dem es um die Interpolationstheorie von Sobolevräumen auf irregulären Gebieten mit gemischten Randbedingungen geht. In jenem Kapitel werden zwei Hauptresultate bewiesen: Zum einen stellt Theorem 2.1.4 ein Interpolationsresultat für die komplexe sowie für die reelle Interpolationsmethode zur Verfügung. Außerdem ist die Interpolation von Sobolevräumen mit unterschiedlichen Integrabilitätsparametern möglich. Zum Anderen wird ein Resultat zur Interpolation mit der reellen Methode unter Ausnutzen einer Spurmethode von Grisvard gezeigt, welches andere geometrische Annahmen voraus setzt, die zumindest im Fall beschränkter Gebiete strikt schwächer sind. Dieses Resultat findet sich in Theorem 2.1.7

Der Beweis des ersten Hauptresultats basiert auf zwei Hauptzutaten: Zum einem steht ein punktweises Multiplikatorresultat auf Bessel Potentialräumen von Sickel im Fokus, mit dem Interpolation auf irregulären Mengen mit reinen Dirichlet Randbedingungen gezeigt werden kann. Zum anderen spielt das Konzept von porösen Mengen eine Schlüsselrolle. Die Interpolation mit gemischten Randbedingungen in der Modellgeometrie des Halbraums kann nun behandelt werden, indem man unter Ausnutzung der Porösität in Räumen mit negativer Regularitätsordnung arbeitet.

Das zweite Hauptresultat von Kapitel 2 ist eine Verfeinerung des Beweises aus [38], wobei hier vor allem ein verbessertes Verständnis von fraktionalen

Hardyungleichungen eine wichtige Rolle spielt. Dieses Resultat ist eine wichtige Zutat für Kapitel 5.

Ein weiterer Meilenstein in der Theorie von Sobolev-Räumen mit gemischten Randbedingungen ist Kapitel 3. Hier wird ein semiuniverseller Fortsetzungsoperator für solche Räume konstruiert, der nicht auf Lokalisierung basiert. Die Konstruktion im Fall von natürlichen Randbedingungen geht auf Jones zurück [65]. Es werden auch lokale und homogene Abschätzungen gezeigt, die für die harmonische Analysis in Kapitel 6 essentiell sind, sowie eine umfangreiche *a priori* Dichtheitstheorie.

Highlight der Dissertation ist Kapitel 5. Hier wird ein sehr allgemeines Setting für elliptische Systeme in Divergenzform auf irregulären Mengen und mit gemischten Randbedingungen präsentiert, in dem die Kato'sche Wurzeleigenschaft gilt. Besonders hervorzuheben ist, dass die Regularität der zugrundeliegenden Menge ohne Verwendung jeglicher Karten formuliert wird, dass jene unterliegende Menge nicht die *interior thickness* Bedingung erfüllen muss, und dass sie unbeschränkt sein kann. Relevante Schritte im Beweis sind die Lokalisierung und Zerlegung des Funktionalkalküls des elliptischen Systems auf "Zusammenhangskomponenten", sowie die Andickung einer Menge um den Dirichletrand, um die interior thickness Bedingung sicherzustellen, ohne dabei weitere geometrische Qualität zu verlieren.

Die Idee der Andickung einer Menge wurde in Kapitel 4 nochmals aufgegriffen, um einen Fortsetzungsoperator für fraktionale Sobolevräume der Ordnung $s \in (0, 1)$ zu konstruieren, der ebenfalls nicht auf Lokalisierung beruht, sondern über eine Hilfsmenge auf den Fall von reinen Neumann Randbedingungen reduziert.

Schließlich werden in Kapitel 6 noch L^p-Abschätzungen für die Wurzel von elliptischen Systemen wie in Kapitel 5 hergeleitet. Besonders interessant ist hier der Fall $p > 2$, da solche Aussagen bisher nicht für irreguläre Gebiete gezeigt wurden. Insbesondere die lokalen und homogenen Abschätzungen für den Fortsetzungsoperator aus Kapitel 3 sind hier essentiell.

CHAPTER 1

Preliminaries

This chapter gives a brief background on interpolation theory, function spaces on the whole space, certain geometric concepts, and functional calculus. Some background on these topics is certainly helpful and each part contains references to textbooks which the reader may conduct.

We have decided to keep the preliminaries short and to put some material that could have been presented here into the individual chapters. Certainly, this leads to some redundancy. On the other hand, the individual chapters become pretty much self-contained this way. Also, we would like to use the opportunity to mention that the same symbol might be defined in different ways in different chapters. This applies in particular to the definition of function spaces. Whenever we need to use results from another chapter, we will comment on why definitions coincide under the current geometric situation.

1.1. Brief background on interpolation theory

In the following, all Banach spaces are over the complex numbers. Though some familiarity with real and complex interpolation of Banach spaces might be helpful, and we refer to the textbooks [23, 93] for further background, understanding this thesis does not require the precise construction of interpolation spaces. We shall only need the general methodology, their fundamental properties, and standard results on interpolation of function spaces on $\mathbb{R}^d$

measuring smoothness to be recalled further below in Section 1.2.

Let (X_0, X_1) be an *interpolation couple*, that is, a pair of Banach spaces that are included in a common linear Hausdorff space. Then the following Banach spaces can be defined between $X_0 \cap X_1$ and $X_0 + X_1$ with respect to continuous inclusion: For $\theta \in [0,1]$ the *complex interpolation spaces* $[X_0, X_1]_\theta$ of Calderón–Lions [23, Sec. 4.1] and for $\theta \in (0,1)$ and $p \in [1,\infty)$ the *real interpolation spaces* $(X_0, X_1)_{\theta,p}$ obtained from Peetre's K-method [23, Sec. 3.1]. In any of these spaces $X_0 \cap X_1$ is dense [23, Thm. 3.4.2 & 4.4.2]. In particular, the endpoints $[X_0, X_1]_j$, $j \in \{0,1\}$, coincide with X_j only if $X_0 \cap X_1$ is dense in X_j.

One of the most important aides in interpolation theory is the retraction-coretraction principle. Given two Banach spaces X and Y, a bounded linear operator $\mathcal{R} : X \to Y$ is called *retraction* if it has a bounded left-inverse $\mathcal{E} : Y \to X$ such that $\mathcal{R}\mathcal{E} = 1$ is the identity on Y. In this case $\mathcal{E}$ is called the associated *coretraction*. It is instructive to think of $\mathcal{R}$ as a restriction and $\mathcal{E}$ a compatible extension operator. The following is a modification of [93, Sec. 1.2.4].

Proposition 1.1.1 (Retraction-Coretraction Theorem)**.** *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples and $\mathcal{R} : X_0 + X_1 \to Y_0 + Y_1$, $\mathcal{E} : Y_0 + Y_1 \to X_0 + X_1$ be linear operators such that $\mathcal{R} : X_j \to Y_j$ is a retraction with associated coretraction $\mathcal{E} : Y_j \to X_j$ for $j = 0, 1$. Let $\langle \cdot\,, \cdot \rangle$ denote either a complex or a real interpolation bracket and put $X = \langle X_1, X_2 \rangle$ and $Y = \langle Y_1, Y_2 \rangle$. Then $\mathcal{R}(X) = Y$ holds with equivalence of norms, where $\mathcal{R}(X)$ is equipped with the quotient norm inherited from $X/\operatorname{N}(\mathcal{R})$. The implicit constants in the equivalence of norms do only depend on θ in the choice of $\langle \cdot\,, \cdot \rangle$ and the operator norms of $\mathcal{R}$ and $\mathcal{E}$ on the interpolation couples.*

Proof. Though the proof is not too long, we proceed in three steps to make this fairly abstract reasoning easier to follow.

Step 1: $\mathcal{P} = \mathcal{E}\mathcal{R}$ restricts to a projection on X. Let $x \in X_0 \cap X_1$. On using that $\mathcal{R}\mathcal{E} = 1$ on $Y_0 \cap Y_1$, we deduce $\mathcal{P}\mathcal{P}x = \mathcal{E}(\mathcal{R}\mathcal{E})\mathcal{R}x = \mathcal{P}x$. Since $X_0 \cap X_1$ is dense in X because X is an interpolation space of the couple (X_0, X_1), and $\mathcal{P}$ and $\mathcal{P}^2$ are bounded on X by interpolation, the claim follows by continuity.

Step 2: The set equality $\mathcal{P}(X) = \mathcal{E}(Y)$ holds. First, let $z \in \mathcal{P}(X)$. Since a projection acts identically on its range, $z = \mathcal{P}z = \mathcal{E}\mathcal{R}z$ implies $z \in \mathcal{E}(Y)$ owing to the fact that $\mathcal{R}z \in \mathcal{R}(X) \subseteq Y$ by interpolation. Conversely, let $z \in Y$ and note that $\mathcal{R}\mathcal{E} = 1$ extends to Y with the argument from Step 1. Consequently, $\mathcal{E}z = \mathcal{E}(\mathcal{R}\mathcal{E}z) = \mathcal{P}(\mathcal{E}z)$ yields $\mathcal{E}z \in \mathcal{P}(X)$.

Step 3: One has $\mathcal{R}(X) = Y$ topologically. The respective equality of sets follows from Step 2 on applying $\mathcal{R}$ and using the cancellation of $\mathcal{R}\mathcal{E}$. For the equivalence of norms pick $y \in Y$. Recall that $\mathcal{R}(X)$ is equipped with the quotient norm coming from $X/\operatorname{N}(\mathcal{R})$. For the continuity of the inclusion "$\supseteq$" we estimate, using that $y = \mathcal{R}\mathcal{E}y$ and that $\langle \cdot\,, \cdot \rangle$ is an exact interpolation method of exponent θ according to [93, Thm. 1.3.3 a) & Thm. 1.9.3 a)], that

$$\|y\|_{\mathcal{R}(X)} \leq \|\mathcal{E}y\|_X \leq \|\mathcal{E}\|_{Y_0 \to X_0}^{1-\theta} \|\mathcal{E}\|_{Y_1 \to X_1}^{\theta} \|y\|_Y.$$

Conversely, let $x \in X$ be *some* (arbitrary) element with $y = \mathcal{R}x$. Then

$$\|y\|_Y = \|\mathcal{R}x\|_Y \leq \|\mathcal{R}\|_{X_0 \to Y_0}^{1-\theta} \|\mathcal{R}\|_{X_1 \to Y_1}^{\theta} \|x\|_X.$$

Taking the infimum over all admissible x yields $\|y\|_Y \lesssim \|y\|_{\mathcal{R}(X)}$ with implicit constant as claimed. □

Remark 1.1.2. Many proofs of this result appeal to the closed graph theorem to show topological equality. Our direct calculation gives a better control on the implicit constants.

An important special case arises when $\mathcal{R} = \mathcal{P}$ is a projection and $\mathcal{E} = 1$ is the identity, compare with [93, Sec. 1.17.1].

Corollary 1.1.3. *Let (X_0, X_1) be an interpolation couple and $\mathcal{P}$ a bounded projection in $X_0 + X_1$ with range Z. Then $(Z \cap X_0, Z \cap X_1)$ is an interpolation couple and if $\langle \cdot\,, \cdot \rangle$ denotes either a complex or a real interpolation bracket, then up to equivalent norms*

$$\langle Z \cap X_0, Z \cap X_1 \rangle = Z \cap \langle X_0, X_1 \rangle.$$

1.2. Function spaces on the whole space

In this section, we introduce several function spaces on the whole space $\mathbb{R}^d$. All spaces will be realized within the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. With this ambient structure, tools like Fourier transforms and derivatives are *a priori* well-defined on all our function spaces, and of course we can also use these tools when defining norms. Another benefit of this approach is that we automatically have a universe in which pairs of function spaces can form interpolation couples, compare for Section 1.1.

Convention 1.2.1 (Lebesgue spaces within the tempered distributions)**.** Let $1 \leq p \leq \infty$. We say that a tempered distribution $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathrm{L}^p(\mathbb{R}^d)$ if there is a locally integrable function f which induces φ in the sense of a *regular* distribution and satisfies $f \in \mathrm{L}^p(\mathbb{R}^d)$. From now on, we won't distinguish φ and f.

Remark 1.2.2. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathrm{L}^{p'}(\mathbb{R}^d)$, we could also have defined $\mathrm{L}^p(\mathbb{R}^d)$ within $\mathcal{S}'(\mathbb{R}^d)$ by duality with $\mathrm{L}^{p'}(\mathbb{R}^d)$.

Definition 1.2.3. Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. The *Sobolev space* $\mathrm{W}^{k,p}(\mathbb{R}^d)$ consists of all tempered distributions such that

$$\|f\|_{\mathrm{W}^{k,p}} \coloneqq \Bigl(\|f\|_{\mathrm{L}^p}^p + \sum_{j=1}^d \|\partial_j^k f\|_{\mathrm{L}^p}^p\Bigr)^{1/p} < \infty,$$

with the usual modification if $p = \infty$.

We could equivalently have taken all derivatives up to order k into account, see [93, Sec. 2.3.3 Rem. 2]. This definition is complemented by introducing the scales of Bessel potential spaces and fractional Sobolev spaces.

Definition 1.2.4. Let $s \in \mathbb{R}$ and $1 < p < \infty$. The *Bessel potential space* $\mathrm{H}^{s,p}(\mathbb{R}^d)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{\mathrm{H}^{s,p}} \coloneqq \|\mathcal{F}^{-1}(1 + |\cdot|^2)^{s/2}\mathcal{F}f\|_{\mathrm{L}^p}$$

is finite. Here, $\mathcal{F}$ denotes the Fourier transform.

Remark 1.2.5.
(i) With $1/p' \coloneqq 1 - 1/p$ the spaces $\mathrm{H}^{-s,p}(\mathbb{R}^d)$ and $\mathrm{H}^{s,p'}(\mathbb{R}^d)$ are in a sesquilinear duality extending the L^2 inner product [93, Sec. 2.6.1 Thm.].

(ii) If $k \geq 0$ is an integer, then $\mathrm{H}^{k,p}(\mathbb{R}^d)$ coincides up to equivalent norms with $\mathrm{W}^{k,p}(\mathbb{R}^d)$, see [93, Sec. 2.3.3 Thm.]. Note that we have $\mathrm{H}^{0,p}(\mathbb{R}^d) = \mathrm{W}^{0,p}(\mathbb{R}^d) = \mathrm{L}^p(\mathbb{R}^d)$.

Definition 1.2.6. Let $1 < p < \infty$ and let $s = k + \sigma$ with $k \geq 0$ an integer and $\sigma \in (0, 1)$. The *fractional Sobolev space* $\mathrm{W}^{s,p}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{\mathrm{W}^{s,p}(\mathbb{R}^d)} \coloneqq \|f\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} + \Bigl(\sum_{j=1}^d \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial_j^k f(x) - \partial_j^k f(y)|^p}{|x - y|^{d+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y\Bigr)^{1/p}$$

is finite, and the space $\mathrm{W}^{s,p}(\mathbb{R}^d)$ is equipped with that norm.

Remark 1.2.7. We could also have restricted integration to $|x-y|<1$. Indeed, fix $1 \leq j \leq d$, using Fubini's theorem and polar coordinates we obtain

$$\begin{aligned} &\iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x-y|>1}} \frac{|\partial_j^k f(x) - \partial_j^k f(y)|^p}{|x-y|^{d+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^d} |\partial_j^k f(x)|^p \int_1^\infty r^{-sp}\frac{\mathrm{d}r}{r}\,\mathrm{d}x + \int_{\mathbb{R}^d} |\partial_j^k f(y)|^p \int_1^\infty r^{-sp}\frac{\mathrm{d}r}{r}\,\mathrm{d}y = \frac{2}{sp}\|\partial_j^k f\|_{\mathrm{L}^p(\mathbb{R}^d)}^p. \end{aligned}$$

Definition 1.2.8. Let $s>0$ and $1<p<\infty$. The space $\mathrm{W}^{-s,p}(\mathbb{R}^d)$ consists of the conjugate-linear functionals on $\mathrm{W}^{s,p'}(\mathbb{R}^d)$ and is equipped with the usual norm for (anti-)dual spaces,

$$\|f\|_{\mathrm{W}^{-s,p}(\mathbb{R}^d)} \coloneqq \sup_{\substack{\varphi\in\mathrm{W}^{s,p'}\\ \|\varphi\|_{\mathrm{W}^{s,p'}}=1}} |\langle f,\varphi\rangle|,$$

where we have omitted to mention $\mathbb{R}^d$ in the supremum and p' is the Hölder-conjugate exponent to p.

This is in accordance with what we have seen for Bessel potential spaces. We could have also given an equivalent intrinsic definition using the scale of Besov spaces [93, Sec. 2.3.2/2.6.1] but the viewpoint of dual spaces is better suited to our circumstances.

We collect all interpolation properties proved in [93, Sec. 2.4.2] that shall be used "off-the-shelf" in the further course of this thesis. In [93] the nomenclature is $\mathrm{H}^{s,p} = \mathrm{F}^s_{p,2}$ and $\mathrm{W}^{s,p} = \mathrm{F}^s_{p,p}$ for non-integer s.

Proposition 1.2.9. *Let $p_0,p_1 \in (1,\infty)$, $s_0,s_1\in\mathbb{R}$, and $\theta\in(0,1)$. Let* X *denote either* H *or* W*. Up to equivalent norms one has*

(i) $$[\mathrm{X}^{s_0,p_0}(\mathbb{R}^d),\mathrm{X}^{s_1,p_1}(\mathbb{R}^d)]_\theta = \mathrm{X}^{s,p}(\mathbb{R}^d),$$

(ii) $$(\mathrm{X}^{s_0,p_0}(\mathbb{R}^d),\mathrm{X}^{s_1,p_1}(\mathbb{R}^d))_{\theta,p} = \mathrm{W}^{s,p}(\mathbb{R}^d),$$

with the two exceptions that in (i) *for* $\mathrm{X}=\mathrm{W}$ *either all or none of* s_0, s_1, s *have to be integers and that in* (ii) *integer s is only permitted when $s_0 = s_1 (= s)$.*

We conclude this section with a closer look at the case $p=\infty$, at least in the case when $k=1$

Definition 1.2.10. The *Lipschitz space* is denoted by $\mathrm{Lip}(\mathbb{R}^d)$ and consists of all bounded functions on $\mathbb{R}^d$ which satisfy

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^d)} := \|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)} + \sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \left|\frac{f(x)-f(y)}{x-y}\right| < \infty.$$

We rely on the following fact from the theory of Sobolev spaces, see [58, Thm. 6.12].

Proposition 1.2.11 (Rademacher's theorem)**.** *The spaces* $\mathrm{W}^{1,\infty}(\mathbb{R}^d)$ *and* $\mathrm{Lip}(\mathbb{R}^d)$ *coincide.*

Remark 1.2.12. In [58], the notion of locally L-Lipschitz functions for some $L > 0$ is used. A bounded function on a set $E \subseteq \mathbb{R}^d$ is *locally L-Lipschitz* if for every $x \in E$ there exists a ball $B = \mathrm{B}(x, r)$ contained in E such that $|f(x)-f(y)| \leq L|x-y|$ for all $y \in B$. In the case $E = \mathbb{R}^d$ this condition implies Lipschitz continuity. Indeed, cover the connecting line segment between to arbitrary points in $\mathbb{R}^d$ by finitely many balls centered in that line segment. Then the Lipschitz seminorm can be estimated by a bootstrapping argument inside these balls. Note that this uses that the Lipschitz constant is uniform across these balls. More generally, this correspondence holds if nearby points can be connected by a comparably longer connecting path. Sets satisfying this condition are called *quasi-convex* in the literature.

1.3. Geometry

This section is first concerned with measure theoretic thickness conditions. These are closely related with the question of Sobolev extendability and the existence of bounded traces. Afterwards, we will turn our focus to metric concepts of fractional dimension, including the concept of porosity, which characterizes sets which are not full-dimensional in a certain sense.

1.3.1. Measure theoretic thickness conditions

We start with a thickness condition for full-dimensional subsets of the Euclidean space. This notion is crucial in the theory of Sobolev extendability of functions and we will revisit it several times throughout this thesis.

Definition 1.3.1. A measurable set $E \subseteq \mathbb{R}^d$ satisfies the *interior thickness condition* if it fulfills

$$\forall x \in E, r \in (0,1] : \quad |\mathrm{B}(x,r) \cap E| \gtrsim |\mathrm{B}(x,r)|.$$

Example 1.3.2. An open set whose boundary satisfies a Lipschitz condition is interior thick. This is of course not necessary, consider for example a punctured disc. On the contrary, cusps do not satisfy the interior thickness condition.

To provide a similar notion on lower dimensional subsets, let us recall the concept of Hausdorff measures. Fix an arbitrary set $E \subseteq \mathbb{R}^d$. Given $s \in [0,d]$ and $\varepsilon > 0$, put

$$\mathcal{H}^s_\varepsilon(E) \coloneqq \inf\Big\{\sum_i \mathrm{r}(B_i)^s : \bigcup_i B_i \supseteq E, \mathrm{r}(B_i) \leq \varepsilon\Big\}.$$

Here, $(B_i)_i$ is a family of balls centered in E. The case $\varepsilon = \infty$ plays a central role in potential theory and will be used in Appendix A.2, so we record it here before coming to the Hausdorff measure.

Definition 1.3.3 (Hausdorff content)**.** Let $s \geq 0$ and $E \subseteq \mathbb{R}^d$. Call the quantity $\mathcal{H}^s_\infty(E)$ the s-dimensional *Hausdorff content* of E.

Remark 1.3.4. Note that in the definition of the Hausdorff content it suffices to consider balls up to radius $\mathrm{diam}(E)$, since in a covering with larger balls we can replace these balls by concentric balls with radius $\mathrm{diam}(E)$, thereby getting a smaller competing value for the infimum.

The quantity $\mathcal{H}^s_\varepsilon(E)$ is decreasing when ε tends to zero. This makes the following definition well-defined, compare also for [2, Sec. 5.1].

Definition 1.3.5 (Hausdorff measure)**.** Let $s \geq 0$ and $E \subseteq \mathbb{R}^d$. Call the quantity $\mathcal{H}^s(E) \coloneqq \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(E) \in [0,\infty]$ the s-dimensional *Hausdorff measure* of E.

Example 1.3.6. Let $1 \leq k \leq d$. Then the outer measure $E \mapsto \mathcal{H}^k(E \times \{0\})$ on $\mathbb{R}^k$ is a translation invariant Borel measure that assigns finite measure to the unit cube [97, §27]. Here, $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^d$ and 0 the null vector of matching length. Hence, $\mathcal{H}^k$ coincides with the Lebesgue measure on $\mathbb{R}^k$ embedded into $\mathbb{R}^d$ up to a norming constant.

Example 1.3.7. The measure $\mathcal{H}^0$ coincides with the counting measure. Indeed, if E is finite then we can cover E by $\#E$ many balls of arbitrary radius. Conversely, let $\ell \geq 0$ and E an infinite set, then there is $\varepsilon > 0$ such that ℓ points in E have pairwise distance of at least ε, hence $\mathcal{H}^0_\varepsilon(E) \geq \ell$ and $\mathcal{H}^0(E) = \infty$ follows.

Definition 1.3.8. Let $E \subseteq \mathbb{R}^d$. Call E an *Ahlfors–David regular* set if

$$\forall x \in E, r \in (0, \operatorname{diam}(E)] \colon \quad \mathcal{H}^{d-1}(\mathrm{B}(x,r) \cap E) \approx r^{d-1}. \tag{1.1}$$

If condition (1.1) only holds with the restriction $r \in (0,1]$, call E a $(d-1)$-*regular* set.

Example 1.3.9. If the boundary of some open set satisfies the Lipschitz condition then it is $(d-1)$-regular. If the boundary is given by only one chart then it is moreover Ahlfors–David regular.

To stress the difference between Ahlfors–David regularity and $(d-1)$-regularity, consider logarithmically distributed line segments in the plane, that is to say, consider the set $E \coloneqq \bigcup_{j \geq 0}((2^j, 0) + S)$ for some finite line segment S. This set is 1-regular because each line segment is, but a ball around the origin with radius 2^ℓ hits only ℓ line segments, so $\mathcal{H}^1(\mathrm{B}(x, 2^\ell) \cap E) \approx \ell$.

The notions of interior thickness condition, Ahlfors–David regularity and $(d-1)$-regularity are unified in the following definition. We decided to start out with these special cases since they are the most important for this thesis and their distinguished nomenclature is commonly used in the literature.

Definition 1.3.10. A set $E \subseteq \mathbb{R}^d$ is called *ℓ-Ahlfors regular* or simply *ℓ-regular* for $0 < \ell \leq d$, if there is comparability

$$\mathcal{H}^\ell(B \cap E) \approx \mathrm{r}(B)^\ell$$

uniformly for all open balls B of radius $\mathrm{r}(B) \leq 1$ centered in E. If comparability holds for $\mathrm{r}(B) \leq \operatorname{diam}(E)$, then E is called *uniformly ℓ-regular*.

Remark 1.3.11. Our ℓ-regular sets are precisely the ℓ-sets in the sense of Jonsson–Wallin [66, Thm. II. 1]. Uniformly ℓ-regular sets are ℓ-regular and the converse holds for bounded sets, see Lemma A.1.4. Many authors consider only *closed* (uniformly) regular sets, but most considerations adapt *verbatim* since in the situation of Definition 1.3.10 the closure $\overline{E}$ is still (uniformly) ℓ-regular and $\overline{E} \setminus E$ is an $\mathcal{H}^\ell$ null set [66, Prop. VIII.1]. We shall frequently use this result without further reference.

Lemma 1.3.12. *Let $0 < \ell \leq d$ and $E_1, E_2 \subseteq \mathbb{R}^d$ be ℓ-regular. Then $E \coloneqq E_1 \cup E_2$ is also ℓ-regular. That is to say, the class of ℓ-sets is closed under finite unions.*

Proof. Let $B = \mathrm{B}(x, r)$ be a ball centered in E of radius at most 1. Without loss of generality we may assume that $x \in E_1$. The lower bound for E is for free by monotonicity. For the upper bound, first consider the case that B does not intersect E_2. Then the upper bound follows from ℓ-regularity of E_1. Otherwise, let $z \in B \cap E_2$. Then $B \subseteq \mathrm{B}(z, 2r)$ and we conclude by the calculation

$$\mathcal{H}^\ell(B \cap E) \leq \mathcal{H}^\ell(\mathrm{B} \cap E_1) + \mathcal{H}^\ell(\mathrm{B}(z, 2r) \cap E_2) \lesssim r^\ell,$$

which uses that E_1 and E_2 are ℓ-regular in the final step. $\square$

1.3.2. Dimensions and porosity

We investigate the notions of Assouad and Hausdorff dimension and explore their relation with the concept of porosity. Whenever necessary, the reader can refer to Appendix A.1 for further background on porous sets. It is instructive to think of porous sets as lower dimensional compared to the ambient space. This is made precise in Proposition A.1.9.

There is no ambiguity with uniformly ℓ-regular sets since their dimension is ℓ for any of these concepts, see Proposition A.1.6.

Definition 1.3.13. Let $E \subseteq \mathbb{R}^d$ and let $\overline{\mathcal{AS}}(E)$ denote the set of $\lambda > 0$ for which there exists $C \geq 0$ such that, if $0 < r < R < 2\operatorname{diam}(E)$ and $x \in E$, then at most $C(R/r)^\lambda$ balls of radius r centered in E are needed to cover $E \cap \mathrm{B}(x, R)$. The number $\overline{\dim}_{\mathcal{AS}}(E) \coloneqq \inf \overline{\mathcal{AS}}(E)$ is called *upper Assouad dimension* of E. The corresponding *lower Assouad dimension* is defined as $\underline{\dim}_{\mathcal{AS}}(E) \coloneqq \sup \underline{\mathcal{AS}}(E)$ with $\underline{\mathcal{AS}}(E)$ the set of $\lambda > 0$ for which there exists $C \geq 0$ such that in the former situation at least $C(R/r)^\lambda$ balls are needed. In the case where $\overline{\dim}_{\mathcal{AS}}(E) = \underline{\dim}_{\mathcal{AS}}(E)$, we put $\dim_{\mathcal{AS}}(E) \coloneqq \overline{\dim}_{\mathcal{AS}}(E)$ and call it simply the *Assouad dimension*.

Example 1.3.14. Let $E \subseteq \mathbb{R}^d$. Then $\overline{\dim}_{\mathcal{AS}}(E) \leq d$. Indeed, let B be a ball of radius R centered in E and consider the covering $\{\mathrm{B}(x, r/5)\}_{x \in B \cap E}$ of $B \cap E$. Using Vitali's covering lemma (which the reader can recall in the Appendix, see Lemma A.1.2), we find a disjoint subfamily $\{B_i\}_{i \in I}$ such that $B \cap E \subseteq \cup_{i \in I} 5B_i$. We denote by $\#_i$ the cardinality of I and calculate

$$\#_i c_d (r/5)^d = |\cup_{i \in I} B_i| \leq |2B| = c_d 2^d R^d,$$

where c_d is the measure of the unit ball. This shows $\#_i \leq 10^d(R/r)^d$ and consequently $\overline{\dim}_{\mathcal{AS}}(E) \leq d$.

Example 1.3.15. Let $E \subseteq \mathbb{R}^d$ be open. Then $\underline{\dim}_{\mathcal{AS}}(E) \geq d$. To see this, note first that we can replace balls by dyadic cubes (not necessarily centered in E) in the definition of the (lower) Assouad dimension. Let Q be a dyadic cube. Then $\underline{\dim}_{\mathcal{AS}}(Q) \geq d$ can be derived from the grid structure of dyadic subcubes. Finally, since E is open, there is some dyadic cube Q such that $Q \cap E = Q$, which allows us to conclude using that special case.

Lemma 1.3.16. *Let $E \subseteq \mathbb{R}^d$, then $\underline{\dim}_{\mathcal{AS}}(E) \leq \overline{\dim}_{\mathcal{AS}}(E)$.*

Proof. Given $\lambda \in \underline{\mathcal{AS}}(E)$ and $\mu \in \overline{\mathcal{AS}}(E)$ we have $(R/r)^\lambda \lesssim (R/r)^\mu$ for all $0 < r < R < \operatorname{diam}(E)$ and hence $\lambda \leq \mu$. □

Corollary 1.3.17. *Let $E \subseteq \mathbb{R}^d$ be open, then $\underline{\dim}_{\mathcal{AS}}(E) = \overline{\dim}_{\mathcal{AS}}(E) = d$.*

We continue with the Hausdorff dimension. The Hausdorff measure and related concepts were recalled in Section 1.3.1.

Definition 1.3.18. Let $E \subseteq \mathbb{R}^d$. Call the number

$$\dim_{\mathcal{H}}(E) \coloneqq \inf\{s > 0 \colon \mathcal{H}^s(E) = 0\}$$

the *Hausdorff dimension* of E.

The following example shows that the Hausdorff dimension is not stable under taking closures, in contrast to the Assouad dimension or the notion of being ℓ-regular (see Remark 1.3.11).

Example 1.3.19. Consider $E \coloneqq [0,1] \cap \mathbb{Q}$. Then $\mathcal{H}^s(E) = 0$ for any $s > 0$ by countability of E, hence $\dim_{\mathcal{H}}(E) = 0$. However, $\overline{E} = [0,1]$ and hence $\dim_{\mathcal{H}}(E) = 1$.

Lemma 1.3.20. *Let $E \subseteq \mathbb{R}^d$, then $\dim_{\mathcal{H}}(E) \leq \overline{\dim}_{\mathcal{AS}}(E)$.*

Proof. Let $t > \overline{\dim}_{\mathcal{AS}}(E)$ and let B be some ball centered in E with $\mathrm{r}(B) = 1$. Then we can pick a constant $C \geq 1$ such that for any $0 < r < 1$ we find a covering $(B_i)_i$ of $E \cap B$ consisting of at most Cr^{-t} balls of radius r. Now let $\ell \in (t, d)$, we estimate

$$\mathcal{H}^\ell_r(E \cap B) \leq \sum_i \mathrm{r}(B_i)^\ell = \#_i \, r^\ell \leq Cr^{\ell - t}.$$

Taking the limit as $r \to 0$, we arrive at $\mathcal{H}^\ell(E \cap B) = 0$. Finally, a countable covering of E by such balls yields $\mathcal{H}^\ell(E) = 0$, so by definition we have $\dim_{\mathcal{H}}(E) \leq t$. Finally, letting $t \to \overline{\dim}_{\mathcal{AS}}(E)$ gives the claim. □

Consequently, Example 1.3.14 yields also the following upper bound for the Hausdorff dimension of sets in Euclidean space.

Corollary 1.3.21. *Let* $E \subseteq \mathbb{R}^d$*, then* $\dim_{\mathcal{H}}(E) \in [0, d]$.

This makes the definition of the Hausdorff co-dimension meaningful.

Definition 1.3.22. Let $E \subseteq \mathbb{R}^d$. Call $\operatorname{codim}_{\mathcal{H}}(E) := d - \dim_{\mathcal{H}}(E) \in [0, d]$ the *Hausdorff co-dimension* of E.

We recall with slight modification the notion of porous sets introduced by Vaisälä [95].

Definition 1.3.23. Let $E \subseteq F \subseteq \mathbb{R}^d$. Then E is *porous in* F if there exists a constant $\kappa \in (0, 1]$ with the following property:

$$\forall x \in E, r \leq 1 \quad \exists y \in \mathrm{B}(x, r) \cap F \;:\; \mathrm{B}(y, \kappa r) \cap E = \emptyset. \tag{1.2}$$

If this holds for all $r \leq \operatorname{diam}(E)$, then E is called *uniformly porous in* F. If $F = \mathbb{R}^d$, then E is simply called *(uniformly) porous*.

Remark 1.3.24. If E is uniformly porous with constant κ, then it is porous with constant $\min\{\kappa, \kappa \operatorname{diam}(E)\}$. Condition (1.2) implies the seemingly stronger statement

$$\forall x \in F, r \leq 1 \quad \exists y \in \mathrm{B}(x, r) \cap F \;:\; \mathrm{B}(y, \kappa r/4) \subseteq \mathrm{B}(x, r) \setminus E.$$

This is seen by distinguishing whether or not $B(x, r/2)$ intersects E. An analogous remark applies to uniformly porous sets.

Example 1.3.25. If $E \subseteq \mathbb{R}^d$ is Ahlfors-David regular, then due to Propositions A.1.6 we have that $\dim_{\mathcal{AS}}(E) = d - 1$. Consequently, by Proposition A.1.9, it follows that E is uniformly porous.

1.4. Functional calculus for (bi)sectorial operators

We provide the essentials on functional calculi for (bi)sectorial operators in Hilbert spaces that are needed for understanding this thesis. Functional calculus theory is heavily and freely used in Chapters 5 and 6. The reader who is not familiar with this theory can consult [53, 76] and also [35, 62] for the bisectorial case for further guidance.

For $\omega \in (0, \pi)$ let $\mathrm{S}_\omega^+ := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}$ be the *sector* of opening angle 2ω symmetric about the positive real axis. It will be convenient to set $\mathrm{S}_0^+ := (0, \infty)$. The *bisector* of angle $\omega \in [0, \pi/2)$ is defined by $\mathrm{S}_\omega := \mathrm{S}_\omega^+ \cup (-\mathrm{S}_\omega^+)$.

Definition 1.4.1. A linear operator T in a Hilbert space H is *sectorial* of angle $\omega \in [0, \pi)$ if its spectrum $\sigma(T)$ is contained in $\overline{\mathrm{S}^+_\omega}$ and if

$$\mathbb{C} \setminus \overline{\mathrm{S}^+_\varphi} \to \mathcal{L}(H), \qquad z \mapsto z(z-T)^{-1}$$

is uniformly bounded for every $\varphi \in (\omega, \pi)$. Abbreviate the bound for $\varphi \in (\omega, \pi)$ by

$$\mathrm{M}(T, \varphi) \coloneqq \sup\Big\{ \|z(z-T)^{-1}\|_{H \to H} : z \in \mathbb{C} \setminus \overline{\mathrm{S}^+_\varphi} \Big\}.$$

Bisectorial operators of angle $\omega \in [0, \pi/2)$ are defined similarly upon replacing sectors by bisectors.

Remark 1.4.2. In this thesis, bisectorial operators do only appear in the Dirac operator framework described in Section 5.4. This part could also be used as a blackbox, but we give a brief exposition to make things more transparent to the reader. Therefore, we have decided to exclude the bisectorial case in the sequel whenever the corresponding statements are not needed and the notation becomes easier this way.

Sectorial and bisectorial operators are automatically closed and densely defined [53, Prop. 2.1.1].

Example 1.4.3. Self-adjoint operators are bisectorial of angle 0, see [53, Prop. C.4.2]. An operator T is called *maximal accretive* if its resolvent exists on the right half-plane along with the bound $\|(z+T)^{-1}\| \leq (\operatorname{Re} z)^{-1}$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. This implies sectoriality of angle $\pi/2$.

Throughout, we denote by $\mathcal{M}(U)$ and $\mathrm{H}^\infty(U)$ the meromorphic and bounded holomorphic functions on an open set $U \subseteq \mathbb{C}$, respectively.

Let T be sectorial of angle ω and let $\varphi \in (\omega, \pi)$. The construction of its functional calculus starts from the subalgebra $\mathrm{H}^\infty_0(\mathrm{S}^+_\varphi)$ of functions $f \in \mathrm{H}^\infty(\mathrm{S}^+_\varphi)$ satisfying $|f(z)| \leq C \min(|z|^\alpha, |z|^{-\alpha})$ for some $C, \alpha > 0$ and all $z \in \mathrm{S}^+_\varphi$. In this case fix $\nu \in (\omega, \varphi)$ and define

$$f(T) \coloneqq \frac{1}{2\pi i} \int_\gamma f(z)(z-T)^{-1} \,\mathrm{d}z,$$

where γ is a positively oriented parametrization of $\partial \mathrm{S}^+_\nu$. The definition is consistent in all admissible ν due to Cauchy's theorem. If the contour is clear we will occasionally just write $\int_\langle$ instead of $\int_\gamma$. The mapping $f \mapsto f(T)$ yields an algebra homomorphism $\mathrm{H}^\infty_0(\mathrm{S}^+_\varphi) \to \mathcal{L}(H)$. The canonical extension to the subalgebra

$$\mathcal{E}(\mathrm{S}^+_\varphi) \coloneqq \mathrm{H}^\infty_0(\mathrm{S}^+_\varphi) \oplus \langle (1+\mathbf{z})^{-1} \rangle \oplus \langle \mathbf{1} \rangle$$

of $\mathcal{M}(\mathrm{S}_\varphi^+)$ is well-defined and again an algebra homomorphism. Here, the bracket $\langle \cdot \rangle$ denotes the linear hull. By regularization this algebra homomorphism extends to an unbounded functional calculus within $\mathcal{M}(\mathrm{S}_\varphi^+)$ as follows. Introduce the algebra

$$\mathcal{M}(\mathrm{S}_\varphi^+)_T \coloneqq \Big\{ f \in \mathcal{M}(\mathrm{S}_\varphi^+) : \text{there exists } \ e \in \mathcal{E}(\mathrm{S}_\varphi^+) \text{ such that} \\ e(T) \text{ is injective and } ef \in \mathcal{E}(\mathrm{S}_\varphi^+) \Big\}$$

and define the closed operator $f(T)$ for $f \in \mathcal{M}(\mathrm{S}_\varphi^+)_T$ by $f(T) \coloneqq e(T)^{-1}(ef)(T)$. This definition does not depend on the choice of the *regularizer* e. The reasonable generalization of the notion of algebra homomorphisms in this context is to have $f(T) + g(T) \subseteq (f+g)(T)$ and $f(T)g(T) \subseteq (fg)(T)$ with equality if $f(T)$ is bounded. Indeed, this is the case for our construction [53, Thm. 1.3.2].

Example 1.4.4. The *fractional powers* T^α for $\operatorname{Re}\alpha > 0$ are defined via the regularizer $(1+z)^{-n}$, where n is an integer larger than $\operatorname{Re}\alpha$. They satisfy the law of exponents $T^\alpha T^\beta = T^{\alpha+\beta}$ and if T is invertible, then so is T^α. See [53, Prop. 3.1.1] for these properties.

Example 1.4.5. Let $f \in \mathrm{H}^\infty(\mathrm{S}_\varphi^+)$. If T is injective, then $z(1+z)^{-2}$ regularizes f. In general, there is a topological splitting $H = \mathrm{N}(T) \oplus \overline{\mathrm{R}(T)}$ and the part of T in $\overline{\mathrm{R}(T)}$ is an injective sectorial operator of the same angle [53, p. 24]. Hence, $f(T)$ is always defined as a closed operator in $\overline{\mathrm{R}(T)}$ via $f(T) \coloneqq f(T|_{\overline{\mathrm{R}(T)}})$.

The calculus for bisectorial operators is constructed in the same way upon systematically replacing sectors by bisectors and $(1+\mathbf{z})^{-1}$ by $(\mathrm{i}+\mathbf{z})^{-1}$. It shares the same properties except that instead of fractional powers one rather considers $(T^2)^\alpha$ for $\operatorname{Re}\alpha > 0$. No ambiguity can occur when writing down such expressions. Indeed, if T is bisectorial, then T^2 is sectorial – and if $f(T^2)$ is defined by the sectorial calculus, then $f(T^2) = [f(\mathbf{z}^2)](T)$, see [35, Thm. 3.2.20].

1.4.1. Transformed functional calculi

We investigate the behavior of functional calculi under common operations like duality or similarity. Most of the statements are well-known in the literature, but we include proofs for convenience of the reader.

Proposition 1.4.6 (Duality)**.** *Let T be a densely defined sectorial operator of angle $\omega \in [0, \pi)$ in a Hilbert space H. Then T^* is again sectorial of angle ω in H, and if $\varphi \in (\omega, \pi)$ then*

$$f(T)^* = f^*(T^*) \qquad (f \in \mathcal{E}(\mathrm{S}_\varphi^+),) \tag{1.3}$$

where $f^* := \overline{f(\overline{\mathbf{z}})} \in \mathcal{E}(\mathrm{S}_\varphi^+)$.

Proof. For the sectoriality of T^* use that one has $\rho(T^*) = \{\overline{\lambda} \colon \lambda \in \rho(T)\}$ and that the identity

$$[(\lambda - T)^{-1}]^* = (\overline{\lambda} - T^*)^{-1} \qquad (\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega^+}) \tag{1.4}$$

holds [53, Cor. C.2.2]. Since conjugation leaves sectors invariant, we conclude $\mathrm{M}(T^*, \varphi) = \mathrm{M}(T, \varphi)$ for $\varphi \in (\omega, \pi)$, which shows in particular that T^* is again sectorial of angle ω.

Next, let us show (1.3). By linearity it suffices to consider the function $\mathbf{1}$, the resolvent function $(1 + \mathbf{z})^{-1}$ and $\mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$-functions separately. For the function $\mathbf{1}$ the claim is immediate, whereas the case of the resolvent function was already treated above (note here that $[(1 + \mathbf{z})^{-1}]^* = (1 + \mathbf{z})^{-1}$).

Finally, let $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. An expansion of f into a power series readily reveals that f^* is again holomorphic and the decay is clearly preserved. Let $x, y \in H$ and compute using (1.4) and the orientation reversing transformation $w = \overline{z}$ that

$$\begin{aligned} &\int_{\langle} f^*(z)\Big((z + T^*)^{-1} x \,|\, y\Big) \,\mathrm{d}z = \int_{\langle} \overline{f(\overline{z})}\Big(x \,|\, (\overline{z} + T)^{-1} y\Big) \,\mathrm{d}z \\ &= -\int_{\langle} \overline{f(w)}\Big(x \,|\, (w + T)^{-1} y\Big) \,\mathrm{d}w = \Big(x \,\Big|\, -\int_{\langle} f(w)(z + T)^{-1} y \,\mathrm{d}w\Big). \end{aligned}$$

Divide this identity by $2\pi\mathrm{i}$ to deduce $(f^*(T^*)x \,|\, y) = (x \,|\, f(T)y)$. Therefore, $f(T)^* = f^*(T^*)$ as claimed. $\square$

Proposition 1.4.7 (Scaling)**.** *Let T be a sectorial operator of angle $\omega \in [0, \pi)$ in a Hilbert space H. For all $s > 0$ one has that sT is again a sectorial operator of angle ω in H with $\mathrm{M}(sT, \varphi) = \mathrm{M}(T, \varphi)$ for all $\varphi \in (\omega, \pi)$. Moreover,*

$$f_s(T) = f(sT) \qquad (s > 0, f \in \mathcal{E}(\mathrm{S}_\varphi^+)) \tag{1.5}$$

holds, where $f_s := f(s\mathbf{z})$ is again in $\mathcal{E}(\mathrm{S}_\varphi^+)$. Also, one has the bound

$$\sup_{s>0} \|f(sT)\|_{H \to H} \lesssim_f 1 \qquad (f \in \mathcal{E}(\mathrm{S}_\varphi^+)). \tag{1.6}$$

Proof. Let $\varphi \in (\omega, \pi)$ and $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}^+_\varphi}$. From $\lambda - sT = s(s^{-1}\lambda - T)$ and $s^{-1}\lambda \notin \overline{\mathrm{S}^+_\varphi}$ follows $\lambda \in \rho(sT)$ with $(\lambda - sT)^{-1} = s^{-1}(s^{-1}\lambda - T)^{-1}$. Multiplication by λ yields $\mathrm{M}(sT, \varphi) = \mathrm{M}(T, \varphi)$. This completes the proof of the first assertion.

Let us come to identity (1.5). Fix again $\varphi \in (\omega, \pi)$. As usual, we consider constants, resolvent functions and $\mathrm{H}^\infty_0(\mathrm{S}^+_\varphi)$-functions separately. For a constant function c we have $c = c_s$, which shows $c_s \in \mathcal{E}(\mathrm{S}^+_\varphi)$ and (1.5) is then trivial. Next, let $f = (\lambda - \mathbf{z})^{-1}$ with $\lambda \notin \overline{\mathrm{S}^+_\varphi}$ be a resolvent function. Then $f_s = (\lambda - s\mathbf{z})^{-1} = s^{-1}(s^{-1}\lambda - \mathbf{z})^{-1}$. From $s^{-1}\lambda \in \mathrm{S}^+_\varphi$, follows $(s^{-1}\lambda - \mathbf{z})^{-1} \in \mathcal{E}(\mathrm{S}^+_\varphi)$, so $f_s \in \mathcal{E}(\mathrm{S}^+_\varphi)$ as a multiple. Hence, using the resolvent identity from above

$$f_s(T) = s^{-1}\Big[(s^{-1}\lambda - \mathbf{z})^{-1}\Big](T) = s^{-1}(s^{-1}\lambda - T)^{-1} = (\lambda - sT)^{-1}.$$

The right-hand side coincides with $f(sT)$ by construction of the functional calculus, which completes this case. Finally, if $f \in \mathrm{H}^\infty_0(\mathrm{S}^+_\varphi)$ with implied constants $C, t \in (0, \infty)$, then

$$|f_s(z)| \leq C \max(s^t, s^{-t}) \min(|z|^t, |z|^{-t}) \qquad (z \in \mathrm{S}^+_\varphi),$$

hence $f_s \in \mathcal{E}(\mathrm{S}^+_\varphi)$. To see (1.5), calculate using the substitution $w = s^{-1}z$ that

$$\begin{aligned} 2\pi\mathrm{i} f(sT) &= \int_{\langle} f(z)(z - sT)^{-1} \, \mathrm{d}s = \int_{\langle} f(z) s^{-1}(s^{-1}z - T) \, \mathrm{d}z \\ &= \int_{\langle} f(sw)(w - T)^{-1} \, \mathrm{d}w = 2\pi\mathrm{i} f_s(T). \end{aligned}$$

Finally, we come to the bound (1.6). As before, we consider the building blocks of elementary functions separately. For constant functions, nothing has so be shown. Also, the resolvent functions are clear since we have shown at the very beginning of this proof that sT is a sectorial operator with same resolvent bounds as T. Lastly, for $f \in \mathcal{E}(\mathrm{S}^+_\varphi)$, the bound follows immediately from the respective bound for resolvent functions. $\square$

The functional calculus for parts was already investigated in [53, Sec. 2.6.2]. However, the precise statement of (i) in the following Proposition seems to be slightly stronger, and only this stronger version will allow us to perform certain decompositions of the functional calculus later on in Section 5.5.1, which is a central argument in this thesis.

Proposition 1.4.8 (Projections & Similarity)**.** *Let T be a (bi)sectorial operator in a Hilbert space H of angle $\omega \in [0, \pi)$.*

(i) *Let $\mathcal{P}$ be an orthogonal projection in H and let $\mathcal{P}^* : \mathcal{P}H \to H$ the inclusion map. Suppose that $\mathcal{P}T \subseteq T\mathcal{P}$. Then $T\mathcal{P}^*$ is a (bi)sectorial operator in $\mathcal{P}H$ of angle ω and one has*

$$f(T\mathcal{P}^*) = f(T)\mathcal{P}^* \quad \textit{and} \quad \mathcal{P}f(T) \subseteq f(T)\mathcal{P}$$

for every f in the functional calculus for T.

(ii) *Let $S : H \to K$ be an isomorphism onto another Hilbert space. Then $S^{-1}TS$ is again (bi)sectorial in K of angle ω. It has the same algebra of admissible functions f as T and $f(S^{-1}TS) = S^{-1}f(T)S$ with $\mathrm{D}(f(S^{-1}TS)) = S^{-1}\,\mathrm{D}(f(T))$ holds.*

Proof. We begin with the proof of (i) and consider the sectorial case first. Throughout, let $\varphi \in (\omega, \pi)$ be fixed.

Step 1: $f = (\lambda - \mathbf{z})^{-1}$ with $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\varphi^+}$. The assumption implies $\mathcal{P}(\lambda - T) \subseteq (\lambda - T)\mathcal{P}$ and hence $\mathcal{P}(\lambda - T)^{-1} = (\lambda - T)^{-1}\mathcal{P}$. On $\mathcal{P}\,\mathrm{D}(T)$ we have $\lambda - T = \lambda - T\mathcal{P}^*$ and on $\mathcal{P}H$ we have $(\lambda - T)^{-1} = (\lambda - T)^{-1}\mathcal{P}^*$. With this at hand, a direct calculation shows that $(\lambda - T)^{-1}\mathcal{P}^*$ is an operator on $\mathcal{P}H$ that acts as a two-sided inverse for $\lambda - T\mathcal{P}^*$, that is to say, $(\lambda - T\mathcal{P}^*)^{-1} = (\lambda - T)^{-1}\mathcal{P}^*$. In particular, $T\mathcal{P}^*$ is a sectorial operator in $\mathcal{P}H$ of angle ω.

Step 2: $f \in \mathcal{E}(\mathrm{S}_\varphi^+)$. As usual, we consider $\mathbf{1}$, $(1 - \mathbf{z})^{-1}$ and $g \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ separately, then the claim follows by linearity. For $\mathbf{1}$, both assertions are trivial and $(1 - \mathbf{z})^{-1}$ was already treated in Step 1. For g we compute using Step 1 that

$$g(T\mathcal{P}^*) = \frac{1}{2\pi\mathrm{i}} \int_{\langle} g(z)(z - T\mathcal{P}^*)^{-1}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_{\langle} g(z)(z - T)^{-1}\mathcal{P}^*\,\mathrm{d}z = g(T)\mathcal{P}^*.$$

The second claim follows similarly.

Step 3: $f \in \mathcal{M}(\mathrm{S}_\varphi^+)_T$. Let e be a regularizer for f. By Step 2 we have $e(T\mathcal{P}^*) = e(T)\mathcal{P}^*$ and this operator is injective by composition of injective maps. Then

$$f(T\mathcal{P}^*) = e(T\mathcal{P}^*)^{-1}(ef)(T\mathcal{P}^*) = (e(T)\mathcal{P}^*)^{-1}(ef)(T)\mathcal{P}^*.$$

But this reduces directly to $e(T)^{-1}(ef)(T)\mathcal{P}^* = f(T)\mathcal{P}^*$ since $e(T)$ and $(ef)(T)$ preserve $\mathcal{P}H$. Moreover, we obtain $\mathcal{P}f(T) \subseteq f(T)\mathcal{P}$ by the respective inclusions for $e(T)^{-1}$ and $(ef)(T)$ from Step 2.

In the proof of (ii) we directly have $(\lambda - S^{-1}TS)^{-1} = S^{-1}(\lambda - T)S$ for $\lambda \in \rho(T)$ and the rest of the proof follows the same pattern as above. $\square$

1.4.2. Bounded H^∞-calculus

In Example 1.4.5 we have seen the inclusion $\mathrm{H}^\infty(\mathrm{S}_\varphi^+) \subseteq \mathcal{M}(\mathrm{S}_\varphi^+)_T$ for a sectorial operator T of angle ω and $\varphi \in (\omega, \pi)$. The same is true for bisectorial operators with the usual modification. In fact, we will focus on the bisectorial case in the sequel due to its importance in Section 5.4. The functional calculus provides "only" algebraic relationships in the first place. From the viewpoint of analysis the question of good bounds seems natural. This is what we investigate for the functional calculus on $\mathrm{H}^\infty(\mathrm{S}_\varphi)$ in the following.

Definition 1.4.9. Let T be a bisectorial operator of angle ω in a Hilbert space H and let $\varphi \in (\omega, \pi/2)$. If $f(T)$ is a bounded operator on $\overline{\mathrm{R}(T)}$ for all $f \in \mathrm{H}^\infty(\mathrm{S}_\varphi)$ and there is a constant $C > 0$ such that the operator norm estimate

$$\|f(T)\|_{\overline{\mathrm{R}(T)} \to \overline{\mathrm{R}(T)}} \leq C\|f\|_\infty \qquad (f \in \mathrm{H}^\infty(\mathrm{S}_\varphi))$$

holds, then T is said to have a *bounded* H^∞*-calculus* of angle φ with bound C on $\overline{\mathrm{R}(T)}$. An analogous version for sectorial operators can be obtained by replacing bisectors with sectors.

The following fundamental theorem of McIntosh [76] characterizes this property through *quadratic estimates*, see also [35, Thm. 3.4.11 & Cor. 3.4.14].

Theorem 1.4.10 (M^c^Intosh's theorem)**.** *Let T be a bisectorial operator of angle ω in a Hilbert space H and let $\varphi \in (\omega, \pi/2)$. Then T has a bounded H^∞-calculus of angle φ if and only if*

$$\int_0^\infty \|tT(1+t^2T^2)^{-1}u\|_H^2 \,\frac{\mathrm{d}t}{t} \approx \|u\|_H^2 \qquad (u \in \overline{\mathrm{R}(T)}). \tag{1.7}$$

The bound for the $\mathrm{H}^\infty(\mathrm{S}_\varphi)$-calculus depends on φ, the implicit constants in (1.7) *and* $\mathrm{M}(T,\varphi)$.

Example 1.4.11. Self-adjoint operators have a bounded H^∞-calculus by the spectral theorem. One can also check quadratic estimates in an elementary manner [35, Ex. 3.4.15].

Bisectorial operators with a bounded H^∞-calculus satisfy the following abstract square root estimate [35, Prop. 3.3.15]. This essentially follows from considering the bounded operator $(z/\sqrt{z^2})(T)$ and its inverse on $\overline{\mathrm{R}(T)}$. Hence, the norm bounds are explicit. Note that the operator $\sqrt{T^2}$ is also often denoted by $|T|$ in the literature.

Example 1.4.12. Let T be a bisectorial operator in a Hilbert space H with a bounded H^∞-calculus of angle φ and bound C. It follows that $\mathrm{D}(\sqrt{T^2}) = \mathrm{D}(T)$ with comparability

$$C^{-1}\|Tu\|_H \leq \|\sqrt{T^2}u\|_H \leq C\|Tu\|_H \qquad (u \in \mathrm{D}(T)).$$

CHAPTER 2

Interpolation Theory

The following question is the driving force of this chapter. Given an open set $O \subseteq \mathbb{R}^d$ and a piece $D \subseteq \partial O$ of its boundary, define the Sobolev space $\mathrm{W}^{1,p}_D(O)$ as the $\mathrm{W}^{1,p}(O)$-closure of smooth functions whose support stays away from D. Under which geometric assumptions can one determine explicitly the interpolation spaces

$$[\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O)]_s \quad \text{and} \quad (\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O))_{s,p}$$

defined through Calderón–Lions' complex method and Peetre's real method? The space $\mathrm{W}^{1,p}_D(O)$ should be thought of the collection of $\mathrm{W}^{1,p}(O)$-functions with homogeneous Dirichlet boundary condition on D.

Interpolation theory related to the spaces $\mathrm{W}^{1,p}_D(O)$ has recently been studied in [8,15,26,38,49,56,86], but mostly with a focus on interpolating with respect to integrability. Interpolation in differentiability appears only in [15, 38] for $p = 2$ and in [49] for general p on certain model sets. The main difficulty comes from the fact that taking the boundary trace on D cannot be defined in a meaningful way on the Lebesgue space $\mathrm{L}^p(O)$. This forbids to treat the question via purely functorial techniques.

We close this gap by establishing a full interpolation theory under geometric assumptions in the spirit of what has become standard for treating mixed boundary value problems. In particular, we confirm the formula for the complex interpolation spaces that was conjectured in connection with fractional powers of divergence form operators in [8, Rem. 10.5] and listed as an open problem in [33, Sec. 5.3]. We also treat interpolation simultaneously in differentiability and integrability. Some of our results appear to be new even on much more regular domains as we do not require that the interface of D with the complementary boundary part $\partial O \setminus D$ can be parametrized by coordinate charts in any sense.

The geometric setting in a nutshell

We shall work on open sets $O \subseteq \mathbb{R}^d$, not necessarily connected or bounded, satisfying the thickness condition

$$c \leq \frac{|B \cap O|}{|B|} \leq C \tag{2.1}$$

for some constants $0 < c \leq C < 1$ and all balls B of radius $\mathrm{r}(B) \leq 1$ centered at the boundary ∂O. This excludes that O has interior or exterior cusps. We assume that the Dirichlet part $D \subseteq \partial O$ is a $(d-1)$-regular, not necessarily closed set. Only around the complementary boundary part $\overline{\partial O \setminus D}$ we demand Lipschitz coordinate charts with uniformly controlled bi-Lipschitz constants, which on domains with compact boundary reduces to the usual weak Lipschitz condition. Finally, the interface ∂D between the two boundary parts should be a *porous* subset of the full boundary. This means that there should exist some $\kappa \in (0,1)$ with the property that every ball B of radius $\mathrm{r}(B) \leq 1$ centered in ∂D contains a ball of radius κr centered in ∂O that avoids ∂D.

Porosity plays a fundamental role in our considerations. The necessary background for this concept was recalled in Section 1.3.2. We often take advantage of it in form of equivalent but less transparent conditions related to Aikawa- and Assouad dimension. In particular, all our results hold if ∂D is $(d-2)$-regular as in Figure 1. We believe that this setting is rather common in applications, for it includes for instance the Gröger regular sets [52]. Our interpolation results are new even in this context since compared to earlier work [49] we remove the requirement that the Lipschitz coordinate charts should be measure preserving.

2.1. Main results and the precise geometric constellation

Before we start, let us agree on a notational convention. In most interpolation results we shall have possibly different Lebesgue exponents $p_0, p_1 \in (1, \infty)$ and

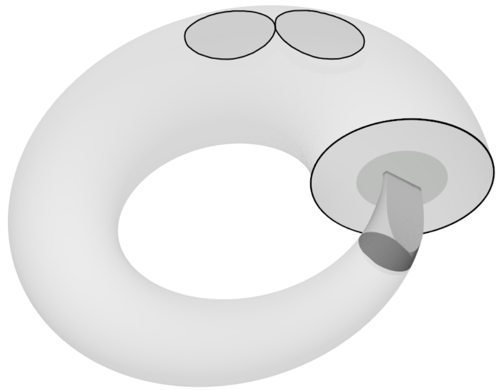


Figure 1.: The domain $O \subseteq \mathbb{R}^3$ is obtained by transforming a cylinder such that one lateral boundary part degenerates to a line segment touching the opposed side from outside. The dark-shaded boundary parts carry the Dirichlet condition.

smoothness parameters $s_0, s_1 \in \mathbb{R}$ and we interpolate in both scales simultaneously. In order to straighten the presentation, we introduce here, given $\theta \in (0,1)$, the interpolating parameters $p \in (1,\infty)$ and $s \in \mathbb{R}$ through

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad s := (1-\theta)s_0 + \theta s_1. \tag{2.2}$$

In the presence of p_i, s_i and θ as above we shall *exclusively* use the symbols p and s in that very sense, sometimes without further mentioning.

To state the central result of this chapter (Theorem 2.1.4), we introduce a set of assumptions below. The proof of this theorem is given in Section 2.4, including an informal outline in Section 2.4.1.

Assumption 2.1.1. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$.*

(O) *The sets O and ${}^cO$ are d-regular,*

(∂O) *the boundary ∂O is $(d-1)$-regular,*

(D) *the Dirichlet part D is $(d-1)$-regular,*

(∂D) *the interface ∂D between D and $\partial O \setminus D$ is porous in ∂O,*

(N) *the set O satisfies a* uniform Lipschitz condition *around $\overline{\partial O \setminus D}$, that is, for every $x \in \overline{\partial O \setminus D}$ there is an open neighborhood $U_x \ni x$ and a* *bi-Lipschitz transformation* $\Phi_x : U_x \to (-1,1)^d$ *such that* $\Phi_x(x) = 0$ *and*

$$\Phi_x(U_x \cap O) = (0,1) \times (-1,1)^{d-1}, \ \Phi_x(U_x \cap \partial O) = \{0\} \times (-1,1)^{d-1},$$

and there exists a number L *that bounds the bi-Lipschitz constants of all* Φ_x*, where* bi-Lipschitz constant *refers to the maximum of the Lipschitz constants of* Φ_x *and* Φ_x^{-1}.

The following two examples demonstrate that Assumption 2.1.1 is a reformulation of the geometric situation described in the introduction to this chapter.

Example 2.1.2. A set $O \subseteq \mathbb{R}^d$ satisfies the thickness condition (2.1) precisely if O and ${}^cO$ are both d-regular. It suffices to check d-regularity of O and ${}^cO$ for balls centered in $\partial O = \partial({}^cO)$, see Lemma 4.1.2. Assume that (2.1) holds, then the lower bound for O is clear. For ${}^cO$ we start with the calculation

$$|B \cap O| \leq C|B| = C|B \cap O| + C|B \cap {}^cO|.$$

The first term can be absorbed as $C < 1$, so we obtain $(1-C)|B \cap O| \leq C|B \cap {}^cO|$. Now, d-regularity of ${}^cO$ can be concluded from the d-regularity of O. Conversely, the lower bound for (2.1) is clear by d-regularity of O and the upper bound follows from

$$|B \cap O| = |B| - |B \cap {}^cO| \leq (1-c)|B|,$$

where $c \in (0,1]$ is the d-regularity constant of ${}^cO$.

Example 2.1.3. If D is a $(d-1)$-regular portion of the boundary of an open set $O \subseteq \mathbb{R}^d$, and O satisfies the uniform Lipschitz condition around $\overline{\partial O \setminus D}$, then the full boundary ∂O is also $(d-1)$-regular. Indeed, since the bi-Lipschitz constants are uniformly bounded, we can use that bi-Lipschitz images have comparable $\mathcal{H}^{d-1}$-measure [97, Thm. 28.10 a)] to show that $\overline{\partial O \setminus D}$ is $(d-1)$-regular. We conclude by the observation that the class of $(d-1)$-regular sets is closed under finite unions, see Lemma 1.3.12.

Theorem 2.1.4. *Let* $O \subseteq \mathbb{R}^d$ *and* $D \subseteq \partial O$ *satisfy Assumption 2.1.1 and let* $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, *and for* $\theta \in (0,1)$ *define* p *and* s *as in* (2.2). *If* X *denotes either* H *or* W, *then the complex interpolation identity*

(a) $$[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O)]_\theta = \begin{cases} \mathrm{X}^{s,p}_D(O) & (\textit{if } s > 1/p) \\ \mathrm{X}^{s,p}(O) & (\textit{if } s < 1/p) \end{cases}$$

holds up to equivalent norms as well as the real interpolation identity

$$\text{(b)} \qquad (\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O))_{\theta,p} = \begin{cases} \mathrm{W}^{s,p}_D(O) & (\text{if } s > 1/p) \\ \mathrm{W}^{s,p}(O) & (\text{if } s < 1/p) \end{cases}$$

with the exception that $s_0 \neq 0$ *and* $s_1 \neq 1$ *are required in* (a) *for* $\mathrm{X} = \mathrm{W}$.

Interpolation theory for the spaces $\mathrm{X}^{s,p}(O)$ without boundary conditions becomes apparent from the extension result of Rychkov [85] presented in Proposition 2.2.9. Abstract techniques then lead us in Section 2.3.1 to the following interpolation results for two function spaces with Dirichlet condition. In contrast to Theorem 2.1.4, this only requires O and D to be regular.

Theorem 2.1.5. *Let* $O \subseteq \mathbb{R}^d$ *be an open, d-regular set, and let* $D \subseteq \overline{O}$ *be* $(d-1)$*-regular. Let* $p_0, p_1 \in (1,\infty)$, $s_0 \in (1/p_0, 1+1/p_0)$, $s_1 \in (1/p_1, 1+1/p_1)$, *and for* $\theta \in (0,1)$ *define* p *and* s *as in* (2.2)*. Let* X *denote either* H *or* W*. Up to equivalent norms it follows that*

$$\text{(c)} \qquad [\mathrm{X}^{s_0,p_0}_D(O), \mathrm{X}^{s_1,p_1}_D(O)]_\theta = \mathrm{X}^{s,p}_D(O),$$

$$\text{(d)} \qquad (\mathrm{X}^{s_0,p_0}_D(O), \mathrm{X}^{s_1,p_1}_D(O))_{\theta,p} = \mathrm{W}^{s,p}_D(O),$$

with the two exceptions that in (c) *for* $\mathrm{X} = \mathrm{W}$ *either all or none of* s_0, s_1, s *have to be* 1 *and that in* (d) *the value* $s = 1$ *is only permitted when* $s_0 = s_1 = 1$.

As a cautionary tale, let us remark that *a priori* all function spaces are defined by restrictions. In particular, $\mathrm{W}^{1,p}(O) = \mathrm{H}^{1,p}(O)$ might be smaller than the collection of $\mathrm{L}^p(O)$-functions whose first-order distributional derivatives are in $\mathrm{L}^p(O)$ under the assumptions of Theorem 2.1.5. Under the full set of geometric assumptions in Theorem 2.1.4, however, there is no such ambiguity.

Extensions and generalizations

Abstract reiteration and duality theorems [23, 63, 93] imply numerous further interpolation results that invoke our Theorems 2.1.4 and 2.1.5 "off-the-shelf". We leave the care of writing them down to the interested readers. Here, we only present one such result that turned out useful in the $\mathrm{W}^{-1,p}$-theory of divergence form operators and previously was available only in the restrictive setup of [49, Lemma 3.4]. The proof of this result will be given in Section 2.5. We write $\mathrm{W}^{-1,p}_D(O)$ for the space of conjugate linear functionals on $\mathrm{W}^{1,p'}_D(O)$, where $1/p + 1/p' = 1$.

Theorem 2.1.6. *Let $O \subseteq \mathbb{R}^d$ and $D \subseteq \partial O$ satisfy Assumption 2.1.1 and let $p \in (1, \infty)$. Up to equivalent norms it follows that*

$$[\mathrm{W}_D^{-1,p}(O), \mathrm{W}_D^{1,p}(O)]_{1/2} = \mathrm{L}^p(O). \tag{e}$$

In Section 2.6 we present a method tailored for real interpolation of fractional Sobolev spaces with the same integrability. Though the geometric setting of Theorem 2.1.7 is only strictly weaker in the case of bounded D, it proves to be less restrictive in certain applications and we rely on it in Chapter 5.

Theorem 2.1.7. *Let $O \subseteq \mathbb{R}^d$ be an open, d-regular set with porous boundary, and let $D \subseteq \overline{O}$ be Ahlfors–David regular. Let $p \in (1, \infty)$, $s_0 \in [0, 1/p)$, $s_1 \in (1/p, 1]$, and $\theta \in (0, 1)$. Up to equivalent norms it follows that*

$$(\mathrm{W}^{s_0,p}(O), \mathrm{W}_D^{s_1,p}(O))_{\theta,p} = \begin{cases} \mathrm{W}_D^{s,p}(O) & (\textit{if } s > 1/p) \\ \mathrm{W}^{s,p}(O) & (\textit{if } s < 1/p) \end{cases}, \tag{f}$$

where $s := (1-\theta)s_0 + \theta s_1$.

Our proof simplifies [38, Sec. 7], where the case $p = 2$ was treated on bounded domains with a Lipschitz assumption around $\overline{\partial O \setminus D}$.

The results of this chapter were published in a joint paper with Moritz Egert [20].

2.2. Function spaces with a partially vanishing trace condition

Let us start with precise definitions for the spaces used in the formulation of our main results. We start with spaces on $\mathbb{R}^d$ incorporating a vanishing trace condition before turning to spaces defined on O. Important presented tools include the trace theory of Jonsson–Wallin, the extension operator of Rychkov, and the pointwise multiplier result of Sickel.

2.2.1. Function spaces on the whole space with vanishing trace condition

With the notion of $(d-1)$-regular sets $E \subseteq \mathbb{R}^d$ we define spaces of functions with positive smoothness on $\mathbb{R}^d$ which vanish on E. All this is based on celebrated results of Jonsson–Wallin [66].

We need the notion of fractional Sobolev spaces on E. They are denoted $\mathrm{B}^{p,p}_s(E)$ in [66] but to keep the analogy with Section 1.2 we shall write $\mathrm{W}^{s,p}(E)$ instead. Having equipped E with the $(d-1)$-dimensional Hausdorff measure, we define for $s \in (0,1)$ and $p \in (1,\infty)$ this space as the Banach space of those $f \in \mathrm{L}^p(E)$ for which

$$\|f\|_{\mathrm{W}^{s,p}(E)} := \bigg(\int_E |f(x)|^p \; \mathcal{H}^{d-1}(\mathrm{d}x) \bigg)^{1/p} + \bigg(\iint_{\substack{x,y \in E \\ |x-y|<1}} \frac{|f(x)-f(y)|^p}{|x-y|^{d-1+sp}} \; \mathcal{H}^{d-1}(\mathrm{d}x) \; \mathcal{H}^{d-1}(\mathrm{d}y) \bigg)^{1/p} < \infty.$$

If E is closed and $\mathrm{X} = \mathrm{H}$, the following is proved in [66, Thm. VI.1 & VII.1]. The general case follows from the discussion in Remark 1.3.11 and real interpolation.

Proposition 2.2.1 (Jonsson–Wallin)**.** *Suppose $E \subseteq \mathbb{R}^d$ is $(d-1)$-regular. Let $p \in (1,\infty)$, $s \in (1/p, 1+1/p)$, and let X denote either H or W.*

(i) *If $f \in \mathrm{X}^{s,p}(\mathbb{R}^d)$, then for $\mathcal{H}^{d-1}$-almost every $x \in E$ the limit*

$$(\mathcal{R}_E f)(x) := \lim_{r \to 0} \frac{1}{|\mathrm{B}(x,r)|} \int_{\mathrm{B}(x,r)} f(y) \; \mathrm{d}y$$

exists. The restriction operator $\mathcal{R}_E$ maps $\mathrm{X}^{s,p}(\mathbb{R}^d)$ boundedly into the trace space $\mathrm{W}^{s-1/p,p}(E)$.

(ii) *Conversely, there exists an extension operator $\mathcal{E}_E$ which is bounded from $\mathrm{W}^{s-1/p,p}(E)$ into $\mathrm{X}^{s,p}(\mathbb{R}^d)$ and that serves as a right inverse for $\mathcal{R}_E$. It does not depend on p or s.*

We often refer to $\mathcal{R}_E$ and $\mathcal{E}_E$ as the *Jonsson–Wallin operators for E*.

Definition 2.2.2. Let $E \subseteq \mathbb{R}^d$ be $(d-1)$-regular. Given $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$, define

$$\mathrm{X}^{s,p}_E(\mathbb{R}^d) := \{ f \in \mathrm{X}^{s,p}(\mathbb{R}^d) : \mathcal{R}_E f = 0 \},$$

where X denotes either H or W, and equip it with the norm inherited from $\mathrm{X}^{s,p}(\mathbb{R}^d)$.

Lemma 2.2.3. *Let $E \subseteq \mathbb{R}^d$ be $(d-1)$-regular and let $\mathcal{R}$ and $\mathcal{E}$ be the corresponding Jonsson–Wallin operators. Then $\mathcal{P} := 1 - \mathcal{E}\mathcal{R}$ is a bounded projection from $\mathrm{X}^{s,p}(\mathbb{R}^d)$ onto $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$ for any $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$. That is to say, $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$ is a closed complemented subspace of $\mathrm{X}^{s,p}(\mathbb{R}^d)$.*

Proof. The operator $\mathcal{E}\mathcal{R}$ is bounded on $\mathrm{X}^{s,p}(\mathbb{R}^d)$ by Proposition 2.2.1. Since $\mathcal{E}$ is a right inverse for $\mathcal{R}$, we have $(\mathcal{E}\mathcal{R})^2 = \mathcal{E}\mathcal{R}$, that is to say, $\mathcal{E}\mathcal{R}$ is a projection with the same nullspace as $\mathcal{R}$. Now, on the one hand, the nullspace of $\mathcal{R}$ is $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$ and on the other hand, the nullspace of $\mathcal{E}\mathcal{R}$ equals the range of $\mathcal{P}$. The conclusion follows. $\square$

Next, we turn our focus to the density of test functions in these spaces. Using Netrusov's theorem, far more general results can be derived, compare with Proposition 5.3.3, but the presented density result suffices for our needs and illustrates a neat application of the closure of first-order Sobolev spaces under truncation.

Definition 2.2.4. Given $E \subseteq \mathbb{R}^d$, define

$$\mathrm{C}^\infty_E(\mathbb{R}^d) \coloneqq \Big\{ f \in \mathrm{C}^\infty_0(\mathbb{R}^d) : \mathrm{d}(\operatorname{supp}(f), E) > 0 \Big\}.$$

Lemma 2.2.5. *Let $E \subseteq \mathbb{R}^d$ be $(d-1)$-regular. For $p \in (1,\infty)$ and $s \in (1/p, 1]$ the set $\mathrm{C}^\infty_E(\mathbb{R}^d)$ is dense in $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$.*

Proof. We shall reduce the claim to the fact that any continuous function $f \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ that vanishes everywhere on a closed set $F \subseteq \mathbb{R}^d$ can be approximated by $\mathrm{C}^\infty_F(\mathbb{R}^d)$-functions in $\mathrm{W}^{1,p}(\mathbb{R}^d)$-norm. This is easily proved by using that $\mathrm{W}^{1,p}(\mathbb{R}^d)$ is closed under truncation, see [2, Sec. 9.2].

Let $\mathcal{P} : \mathrm{X}^{s,p}(\mathbb{R}^d) \to \mathrm{X}^{s,p}_E(\mathbb{R}^d)$ be the bounded projection provided by Lemma 2.2.3. Since $\mathrm{C}^\infty_0(\mathbb{R}^d)$ is dense in $\mathrm{X}^{s,p}(\mathbb{R}^d)$, it suffices to approximate elements in $\mathcal{P}(\mathrm{C}^\infty_0(\mathbb{R}^d))$ by test functions from $\mathrm{C}^\infty_E(\mathbb{R}^d)$. Moreover, it suffices to achieve this for the $\mathrm{W}^{1,p}(\mathbb{R}^d)$-norm, which is stronger than the $\mathrm{X}^{s,p}(\mathbb{R}^d)$-norm for we have $s \le 1$. Since the projection $\mathcal{P}$ in Lemma 2.2.3 is the same for all admissible values of s and p, we have in particular

$$\mathcal{P}(\mathrm{C}^\infty_0(\mathbb{R}^d)) \subseteq \mathcal{P}((\mathrm{W}^{1,d+1} \cap \mathrm{W}^{1,p})(\mathbb{R}^d)) \subseteq (\mathrm{W}^{1,d+1} \cap \mathrm{W}^{1,p}_E)(\mathbb{R}^d).$$

Sobolev embeddings yield for every function in the right-hand space a continuous representative f that vanishes $\mathcal{H}^{d-1}$-almost everywhere on E. By $(d-1)$-regularity the intersection of E with arbitrarily small balls centered in E still has positive $\mathcal{H}^{d-1}$-measure. Thus every point on $F \coloneqq \overline{E}$ is an accumulation point of zeros of f. It follows that f vanishes everywhere on F and the above-mentioned approximation result kicks in. $\square$

2.2.2. Function spaces on open sets with and without partially vanishing trace

Throughout, X denotes either H or W. Since for $s \geq 0$ we have $\mathrm{X}^{s,p}(\mathbb{R}^d) \subseteq \mathrm{L}^p(\mathbb{R}^d)$, the *pointwise restriction* $|_O$ of functions to O is defined on $\mathrm{X}^{s,p}(\mathbb{R}^d)$.

Definition 2.2.6. Let $O \subseteq \mathbb{R}^d$ be an open set and let $s \geq 0$, $p \in (1, \infty)$. Define $\mathrm{X}^{s,p}(O) := \{f|_O : f \in \mathrm{X}^{s,p}(\mathbb{R}^d)\}$ with quotient norm

$$\|f\|_{\mathrm{X}^{s,p}(O)} := \inf \Big\{ \|F\|_{\mathrm{X}^{s,p}} : F \in \mathrm{X}^{s,p}(\mathbb{R}^d) \text{ and } F|_O = f \Big\}.$$

If in addition $E \subseteq \overline{O}$ is $(d-1)$-regular, define $\mathrm{X}^{s,p}_E(O) := \{f|_O : f \in \mathrm{X}^{s,p}_E(\mathbb{R}^d)\}$ for $s \in (1/p, 1 + 1/p)$ with quotient norm

$$\|f\|_{\mathrm{X}^{s,p}_E(O)} := \inf \Big\{ \|F\|_{\mathrm{X}^{s,p}} : F \in \mathrm{X}^{s,p}_E(\mathbb{R}^d) \text{ and } F|_O = f \Big\}.$$

Introduce a set of test functions on O similar to Definition 2.2.4.

Definition 2.2.7. Given $E \subseteq \mathbb{R}^d$ and if $O \subseteq \mathbb{R}^d$ is any open set, define

$$\mathrm{C}^\infty_E(O) := \Big\{ f|_O : f \in \mathrm{C}^\infty_E(\mathbb{R}^d) \Big\}.$$

By construction $|_O : \mathrm{X}^{s,p}_E(\mathbb{R}^d) \to \mathrm{X}^{s,p}_E(O)$ is bounded and onto. Thus, density of $\mathrm{C}^\infty_E(O)$ in $\mathrm{X}^{s,p}_E(O)$ follows readily from Lemma 2.2.5.

Lemma 2.2.8. *Let $O \subseteq \mathbb{R}^d$ be open and $E \subseteq \overline{O}$ be $(d-1)$-regular. For $p \in (1, \infty)$ and $s \in (1/p, 1]$ the set $\mathrm{C}^\infty_E(O)$ is dense in $\mathrm{X}^{s,p}_E(O)$.*

To let $\mathrm{X}^{s,p}(O)$ inherit non-trivial properties of its whole space analogue, a bounded linear right inverse is needed. If O is d-regular, this has been constructed in a beautiful paper of Rychkov [85, Thm. 5.1]. Note that a similar result was also achieved by Shvartsman [87].

Proposition 2.2.9 (Rychkov). *Let $O \subseteq \mathbb{R}^d$ be an open set satisfying the interior thickness condition. Let* X *denote either* H *or* W. *For any $s > 0$ and $p \in (1, \infty)$ there exists a bounded linear extension operator $\mathcal{E} : \mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$ that serves as a right inverse for $|_O$. Moreover, if $m \geq 1$ is an integer, then $\mathcal{E}$ can be taken the same for all $p \in (1, \infty)$ and all $s \in (0, m)$.*

Remark 2.2.10. Though not stated explicitly in [85], the consistency of the extension operator becomes apparent from an inspection of the proof. We give a guide on how to see this in that paper. In Step 1, Theorem 3.1 imports no restrictions and since $s > 0$, the embedding $A^s_{pq} \subseteq B^0_{p1}$ gives well-definedness of L according to Remark 4.5, so the restriction $s > m$ stated in [85, p.156 l.7] becomes void. In Step 2, Lemma 4.6 is used with $\tilde{L}$ instead of L (compare with Step 4), so m can be set to 0 in (5.17) and so the restriction $s > m$ in (5.18) also becomes void. Finally, in Step 3 the $\mathrm{L}^1_{\mathrm{loc}}$ argument can be performed with $m = 0$ as mentioned at the end of Step 4.

By an argument similar to that in Lemma 2.2.5, we prove the surprising feature that Rychkov's extension operator automatically preserves Dirichlet conditions on $(d-1)$-regular sets. Once again, this comes as a byproduct of consistency of the extension operator and Sobolev embeddings and has nothing to do with the particular construction.

Lemma 2.2.11. *Let $O \subseteq \mathbb{R}^d$ be an open, d-regular set, and let $E \subseteq \overline{O}$ be $(d-1)$-regular. Suppose $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$. If $\mathcal{E} : \mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$ is the extension operator of Proposition 2.2.9 constructed with $m \geq 2$, then*

$$\mathcal{E} : \mathrm{X}^{s,p}_E(O) \to \mathrm{X}^{s,p}_E(\mathbb{R}^d)$$

is bounded for the $\mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$-norm. In particular, $\mathrm{X}^{s,p}_E(O)$ is a closed subspace of $\mathrm{X}^{s,p}(O)$.

Proof. By definition of the quotient norm we obtain $\mathrm{X}^{s,p}_E(O) \subseteq \mathrm{X}^{s,p}(O)$ with continuous inclusion of Banach spaces from the fact that $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$ is a closed subspace of $\mathrm{X}^{s,p}(\mathbb{R}^d)$.

We begin with the case $s \leq 1$. Since $\mathcal{E} : \mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$ is bounded, it suffices to check that $\mathcal{E}$ maps a dense subset of $\mathrm{X}^{s,p}_E(O)$ into $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$. Owing to Lemma 2.2.8 we can take this subset to be $\mathrm{C}^\infty_E(O) = \mathrm{C}^\infty_E(\mathbb{R}^d)|_O$. So, let $f \in \mathrm{C}^\infty_E(\mathbb{R}^d)$. Since $\mathcal{E}$ acts consistently, we obtain $\mathcal{E}(f|_O) \in (\mathrm{W}^{1,d+1} \cap \mathrm{X}^{s,p})(\mathbb{R}^d)$. Due to Sobolev embeddings $\mathcal{E}(f|_O)$ admits a continuous representative and we need to check that it vanishes everywhere on E. To this end, we let $B \subseteq \mathbb{R}^d$ be an arbitrary open ball centered in $E \subseteq \overline{O}$ with radius $\mathrm{r}(B) < \mathrm{d}(\mathrm{supp}(f), E)$. Since O is d-regular, $B \cap O$ has positive Lebesgue measure but on this set we have $\mathcal{E}(f|_O) = f = 0$ almost everywhere. The conclusion follows.

If $s \in (1, 1+1/p)$ and $f \in \mathrm{X}^{s,p}_E(O)$, then we can use Proposition 2.2.9 to get $\mathcal{E}f \in \mathrm{X}^{s,p}(\mathbb{R}^d)$ and from the inclusion $\mathrm{X}^{s,p}_E(O) \subseteq \mathrm{X}^{1,p}_E(O)$ and the first part of the proof we get $\mathcal{E}f \in \mathrm{X}^{1,p}_E(\mathbb{R}^d)$. According to Definition 2.2.2 this implies $\mathcal{E}f \in \mathrm{X}^{s,p}_E(\mathbb{R}^d)$.

For the final statement, given $f \in \mathrm{X}^{s,p}_E(O)$ we have already seen $\|f\|_{\mathrm{X}^{s,p}_E(O)} \geq \|f\|_{\mathrm{X}^{s,p}(O)}$ and we have just proved $\|f\|_{\mathrm{X}^{s,p}_E(O)} \leq \|\mathcal{E}f\|_{\mathrm{X}^{s,p}_E(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{X}^{s,p}(O)}$. $\square$

Let us stress that Rychkov's operator is not defined on $\mathrm{L}^p(O)$. But in the low-regularity regime $s < 1/p$ we can simply extend $\mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$ by zero as we shall see soon. The following definition goes back to Sickel [88] and Jawerth–Frazier [64].

Definition 2.2.12. Let $t \in (0,1)$. An open set $O \subseteq \mathbb{R}^d$ belongs to the class $\mathcal{D}^t$ if

$$\sup_{x\in\partial O} \sup_{0<r\leq 1} r^{t-d} \int_{\mathrm{B}(x,r)\setminus\partial O} \mathrm{d}(y,\partial O)^{-t} \; \mathrm{d}y < \infty.$$

The relevant examples for us are as follows. For a proof we refer to Proposition A.1.10 in the appendix.

Example 2.2.13. An open set with $(d-1)$-regular boundary is of class $\mathcal{D}^t$ for any $t \in (0,1)$. An open set with porous boundary is of class $\mathcal{D}^t$ for some $t \in (0,1)$.

We cite the following multiplier theorem for characteristic functions [88, Thm. 4.4].

Proposition 2.2.14 (Sickel)**.** *Let $O \subseteq \mathbb{R}^d$ be of class $\mathcal{D}^t$ for some $t \in (0,1)$. Let $p \in (1,\infty)$ and $0 \leq s < t/p$. If X denotes either H or W, then pointwise multiplication by $\mathbb{1}_O$ is a bounded operator on $\mathrm{X}^{s,p}(\mathbb{R}^d)$. For $t(1/p-1) < s < 0$ the dual operator $\mathbb{1}_O\varphi := \varphi \circ \mathbb{1}_O$ is also bounded on $\mathrm{X}^{s,p}(\mathbb{R}^d)$.*

Corollary 2.2.15. *Let $O \subseteq \mathbb{R}^d$ be an open set with porous boundary. Let X denote either H or W. Then there exists $t \in (0,1)$ such that the zero extension operator*

$$\mathcal{E}_0 : \mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d), \quad \mathcal{E}_0 f(x) := \begin{cases} f(x) & (\text{if } x \in O) \\ 0 & (\text{if } x \in {}^cO) \end{cases}$$

is bounded provided $p \in (1,\infty)$ and $s \in [0, t/p)$.

2.3. First interpolation properties

We establish first interpolation properties which follow mostly from a purely functorial reasoning. Also, we present the technique of gluing interpolation scales together, which is often used in the course of this chapter.

2.3.1. Symmetric interpolation results

We establish symmetric interpolation results for the spaces $\mathrm{X}^{s,p}(O)$ and $\mathrm{X}^{s,p}_E(O)$. *Symmetric* means that either both or none of the spaces are with vanishing trace on E. In particular, we prove Theorem 2.1.5.

First, we obtain a result similar to Proposition 1.2.9 for spaces on d-regular sets. The argument is well-known, but we repeat it in detail since it will be re-used several times.

Proposition 2.3.1. *Let $O \subseteq \mathbb{R}^d$ be open and d-regular. Let $p_0, p_1 \in (1, \infty)$, $s_0, s_1 \in (0, \infty)$, and $\theta \in (0, 1)$. Let* X *denote either* H *or* W*. Up to equivalent norms it follows that*

(i)
$$[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}(O)]_\theta = \mathrm{X}^{s,p}(O),$$

(ii)
$$(\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}(O))_{\theta,p} = \mathrm{W}^{s,p}(O),$$

with the two exceptions that in (i) *for* $\mathrm{X} = \mathrm{W}$ *either all or none of s_0, s_1, s have to be integers and in* (ii) *integer s is only permitted when $s_0 = s_1 (= s)$.*

Proof. We apply Proposition 1.1.1 with $X_j := \mathrm{X}^{s_j,p_j}(\mathbb{R}^d)$, $Y_j := \mathrm{X}^{s_j,p_j}(O)$, $\mathcal{R} := |_O$ the pointwise restriction, and $\mathcal{E}$ Rychkov's extension operator from Proposition 2.2.9 constructed with an integer $m > \max\{s_0, s_1\}$.

Let us prove (i). According to Proposition 1.2.9 we have $X := [X_0, X_1]_\theta = \mathrm{X}^{s,p}(\mathbb{R}^d)$. By definition we have $\mathcal{R}(X) = \mathrm{X}^{s,p}(O)$, where $\mathcal{R}(X)$ carries the quotient norm inherited from $X/\,\mathrm{N}(\mathcal{R})$. Hence, $[Y_0, Y_1]_\theta =: Y = \mathrm{X}^{s,p}(O)$ with equivalent norms. The proof of (ii) follows *verbatim* from the identity $(X_0, X_1)_{\theta,p} = \mathrm{W}^{s,p}(\mathbb{R}^d)$ also provided by Proposition 1.2.9. □

Remark 2.3.2. Suppose that in addition O has a porous boundary. In the proof above we could then replace Rychkov's extension operator $\mathcal{E}$ with the zero extension operator $\mathcal{E}_0$ discussed in Corollary 2.2.15. Consequently, Proposition 2.3.1 remains valid for parameters $s_j \in [0, t/p_j)$, which includes the case of Lebesgue spaces.

The same technique yields the

Proof of Theorem 2.1.5. First, we assume $O = \mathbb{R}^d$. Proposition 1.2.9 provides the identities analogous to (c) and (d) for the spaces without Dirichlet conditions. Hence, the claim follows from Corollary 1.1.3 applied to the projection $\mathcal{P}$ provided by Lemma 2.2.3.

Having established the interpolation identities on $\mathbb{R}^d$, we can now, as in the proof of Proposition 2.3.1, pass to the spaces on O via Proposition 1.1.1. Indeed, if we take $X_j := \mathrm{X}_E^{s_j,p_j}(\mathbb{R}^d)$, $Y_j := \mathrm{X}_E^{s_j,p_j}(O) = (X_j)|_O$, $\mathcal{R} := |_O$, and $\mathcal{E}$ as Rychkov's extension operator, then the only property that needs to be checked is that $\mathcal{E}$ maps Y_j boundedly into X_j. But the latter is precisely the statement of Lemma 2.2.11. $\square$

2.3.2. Gluing interpolation scales

We recall a general interpolation technique due to Wolff [96]. Here, we cite (with adapted notation) the refined version proved in [63, Thm. 1&2]. The statement is visualized in Figure 2 for complex interpolation.

Proposition 2.3.3 (Wolff). *Let $X_0, X_\theta, X_\eta, X_1$ be Banach spaces included in a common linear Hausdorff space. Suppose $\theta, \eta, \lambda, \mu \in (0,1)$ satisfy $\theta = \lambda\eta$ and $\eta = (1-\mu)\theta + \mu$, and let $p_\theta, p_\eta \in [1,\infty]$.*

(i) *If $X_\theta = [X_0, X_\eta]_\lambda$ and $X_\eta = [X_\theta, X_1]_\mu$, then also $X_\theta = [X_0, X_1]_\theta$ and $X_\eta = [X_0, X_1]_\eta$.*

(ii) *If $X_\theta = (X_0, X_\eta)_{\lambda,p_\theta}$ and $X_\eta = (X_\theta, X_1)_{\mu,p_\eta}$, then also $X_\theta = (X_0, X_1)_{\theta,p_\theta}$ and $X_\eta = (X_0, X_1)_{\eta,p_\eta}$.*

All equalities above are in the sense of equal sets with equivalent norms.

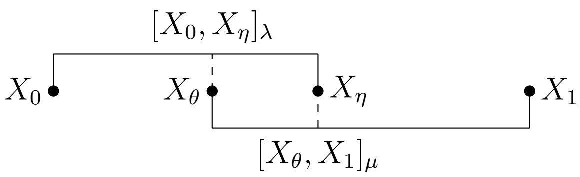


Figure 2.: Assuming the interpolation identities indicated by dashed lines, Wolff's result recovers X_θ and X_η as interpolation spaces associated with the couple (X_0, X_1) for the correct convex combination parameters θ and η, respectively.

For further reference we demonstrate once in detail how the results of Proposition 2.3.1 and Remark 2.3.2 can be patched together using Wolff's result.

Proposition 2.3.4. *If in the setting of Proposition 2.3.1 the boundary ∂O is porous, then the conclusion remains valid for $s_0, s_1 \in [0,\infty)$.*

Proof. In view of Proposition 2.3.1, Remark 2.3.2, and symmetry of the assumption, we only have to treat the case $s_0 = 0$ and $s_1 > 0$. For any $\eta \in (0,1)$ we abbreviate the relevant convex combinations by $s_\eta \coloneqq \eta s_1$ and $1/p_\eta \coloneqq (1-\eta)/p_0 + \eta/p_1$.

We begin with (i). Since s_0 is an integer, we are only claiming something new in the case $\mathrm{X} = \mathrm{H}$. We have to prove for all $\eta \in (0,1)$ the equality

$$[\mathrm{H}^{s_0,p_0}(O), \mathrm{H}^{s_1,p_1}(O)]_\eta = \mathrm{H}^{s_\eta,p_\eta}(O). \tag{2.3}$$

Throughout, the reader should keep in mind Figure 2. Let us first suppose $s_\eta < t/p_\eta$ so that $\mathrm{H}^{s_\eta,p_\eta}(\mathbb{R}^d)$ belongs to the regime covered by Remark 2.3.2. We pick $\theta \in (0,\eta)$ and $\lambda, \mu \in (0,1)$ such that $\theta = \lambda\eta$ and $\eta = (1-\mu)\theta + \mu$. The quadruple of spaces $(X_i)_i \coloneqq (\mathrm{H}^{s_i,p_i}(O))_i$ satisfies the assumption in part (i) of Wolff's result owing to Remark 2.3.2 and Proposition 2.3.1. Hence, we obtain (2.3). Now, suppose $s_\eta \geq t/p_\eta$. Due to $s_0 = 0$ we can pick $\theta \in (0,\eta)$ to arrange $s_\theta < 1/p_\theta$. The first part of the proof with s_η in place of s_1 and η replaced by λ yields $[X_0, X_\eta]_\lambda = X_\theta$. Consequently, we can apply Wolff's result with the same numerology as before to obtain (2.3).

As for (ii), the claim for W-spaces follows *verbatim* on using part (ii) of Wolff's result with p_θ, p_η corresponding to θ, η as above and systematically replacing H by W.

Real interpolation of H-spaces requires a different argument since the result will be a W-space. We rely on the one-sided reiteration theorem in Proposition 2.3.5 below. Indeed, given $\theta \in (0,1)$ we pick $\eta \in (0,\theta)$ and write $\theta = (1-\lambda)\eta + \lambda$ with $\lambda \in (0,1)$. Then we use in succession one-sided reiteration, complex interpolation of H-spaces established above, and Proposition 2.3.1, to give

$$\begin{aligned}
\Big(\mathrm{H}^{s_0,p_0}(O), \mathrm{H}^{s_1,p_1}(O)\Big)_{\theta,p_\theta} &= \Big([\mathrm{H}^{s_0,p_0}(O), \mathrm{H}^{s_1,p_1}(O)]_\eta, \mathrm{H}^{s_1,p_1}(O)\Big)_{\lambda,p_\theta} \\
&= \Big(\mathrm{H}^{s_\eta,p_\eta}(O), \mathrm{H}^{s_1,p_1}(O)\Big)_{\lambda,p_\theta} \\
&= \mathrm{W}^{s_\theta,p_\theta}(O).
\end{aligned}$$

Concerning the last line we remark that $(1-\lambda)s_\eta + \lambda s_1 = s_\theta$ and $(1-\lambda)/p_\eta + \lambda/p_1 = 1/p_\theta$ hold by construction. $\square$

The reiteration result that we have invoked above is as follows. We refer to [93, Sec. 1.10.3, Thm. 2] for real interpolation and to [30] for complex interpolation, noting that in the latter case the density of $X_0 \cap X_1$ in X_1 guarantees $[X_0, X_1]_1 = X_1$. In our application above, density is provided by Lemma 2.2.8.

Proposition 2.3.5. *Let (X_0, X_1) be an interpolation couple. Let $\eta, \lambda \in (0,1)$ and put $\theta = (1-\lambda)\eta + \lambda$. The interpolation identity*

$$\langle [X_0, X_1]_\eta, X_1 \rangle_\lambda = \langle X_0, X_1 \rangle_\theta$$

holds up to equivalent norms in the following cases. If $\langle \cdot\,, \cdot \rangle$ is a $(\cdot\,, p)$-real interpolation bracket with $p \in [1,\infty]$ fixed or if $\langle \cdot\,, \cdot \rangle$ is the complex interpolation bracket and $X_0 \cap X_1$ is dense in X_1.

2.3.3. Non-symmetric interpolation: The easy inclusion

The main difficulty in Theorem 2.1.4 lies in proving the inclusion "$\supseteq$". Indeed, here we can already prove

Proposition 2.3.6. *Let $O \subseteq \mathbb{R}^d$ be an open, d-regular set with porous boundary, and let $E \subseteq \overline{O}$ be $(d-1)$-regular. Let $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, and $\theta \in (0,1)$. Define p and s as in (2.2). Then there are continuous inclusions*

$$\text{(i)} \qquad [\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O)]_\theta \subseteq \begin{cases} \mathrm{X}^{s,p}_E(O) & (\textit{if } s > 1/p) \\ \mathrm{X}^{s,p}(O) & (\textit{if } s < 1/p) \end{cases}$$

and

$$\text{(ii)} \qquad (\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O))_{\theta,p} \subseteq \begin{cases} \mathrm{W}^{s,p}_E(O) & (\textit{if } s > 1/p) \\ \mathrm{W}^{s,p}(O) & (\textit{if } s < 1/p) \end{cases}$$

with the exception that $s_0 \neq 0$ and $s_1 \neq 1$ are required in (i) *for $\mathrm{X} = \mathrm{W}$. If $p_0 = p_1$, then the result remains true for all $s_1 \in (1/p_1, 1 + 1/p_1)$ with the additional exception that only in* (i) *for $\mathrm{X} = \mathrm{H}$ the value $s = 1$ is permitted.*

Proof. First, we check that Proposition 2.3.5 applies in its real and its complex version to the couple $(\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}(O))$. If $s_1 \leq 1$ then $\mathrm{X}^{s_0,p_0}(O) \cap \mathrm{X}^{s_1,p_1}_E(O) \supseteq \mathrm{C}^\infty_E(O)$ is dense in $\mathrm{X}^{s_1,p_1}_E(O)$ by Lemma 2.2.8 and if $p_0 = p_1$ then $\mathrm{X}^{s_0,p_0}(O) \cap \mathrm{X}^{s_1,p_1}_E(O) = \mathrm{X}^{s_1,p_1}_E(O)$ for all $s_1 \in (1/p_1, 1 + 1/p_1)$. This being said, we denote by $\langle \cdot\,, \cdot \rangle$ either the $(\cdot\,, p)$-real or the complex interpolation bracket and treat all assertions simultaneously.

By definition we have $\mathrm{X}^{s_1,p_1}_E(O) \subseteq \mathrm{X}^{s_1,p_1}(O)$ and hence we get

$$\langle \mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O) \rangle_\theta \subseteq \langle \mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}(O) \rangle_\theta$$

with continuous inclusion. The interpolation space on the right has been determined in Proposition 2.3.4. It coincides (up to equivalent norms) with $\mathrm{W}^{s,p}(O)$ in case of real interpolation and with $\mathrm{X}^{s,p}(O)$ in case of complex interpolation. In the case $s < 1/p$ this already is the desired conclusion.

Let now $s > 1/p$. We fix $\eta \in (0,\theta)$ sufficiently close to θ, so to arrange $1/p_\eta := (1-\eta)/p_0 + \eta/p_1$ and $s_\eta := (1-\eta)s_0 + \eta s_1$ satisfying $s_\eta > 1/p_\eta$. We write $\theta = (1-\lambda)\eta + \lambda$ with $\lambda \in (0,1)$. From Proposition 2.3.5 and the reasoning in the first case we obtain

$$\begin{aligned}\langle \mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O)\rangle_\theta &= \langle [\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O)]_\eta, \mathrm{X}^{s_1,p_1}_E(O)\rangle_\lambda \\ &\subseteq \langle \mathrm{X}^{s_\eta,p_\eta}(O), \mathrm{X}^{s_1,p_1}_E(O)\rangle_\lambda\end{aligned}$$

with continuous inclusion. Let $\mathcal{E}$ be Rychkov's extension operator for O. From Lemma 2.2.11 and the above we can infer by interpolation that

$$\mathcal{E} : \langle \mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O)\rangle_\theta \to \langle \mathrm{X}^{s_\eta,p_\eta}(\mathbb{R}^d), \mathrm{X}^{s_1,p_1}_E(\mathbb{R}^d)\rangle_\lambda =: Y \tag{2.4}$$

is bounded. As before, we see that Y is continuously included into $\mathrm{W}^{s,p}(\mathbb{R}^d)$ in case of real interpolation and into $\mathrm{X}^{s,p}(\mathbb{R}^d)$ in case of complex interpolation.

Consider the Jonsson–Wallin restriction operator to E, see Proposition 2.2.1. It maps $\mathrm{X}^{s_\eta,p_\eta}(\mathbb{R}^d)$ boundedly into $\mathrm{W}^{s_\eta-1/p_\eta,p_\eta}(E)$ since we have $s_\eta > 1/p_\eta$ and it maps $\mathrm{X}^{s_1,p_1}_E(\mathbb{R}^d)$ into $\{0\}$ by definition. By interpolation it maps Y into $\langle \mathrm{W}^{s_\eta-1/p_\eta,p_\eta}(E), \{0\}\rangle_\lambda$. This interpolation space equals $\{0\}$ since it contains $\{0\}$ as a dense subspace. Hence, we have continuous inclusion of Y into $\mathrm{W}^{s,p}_E(\mathbb{R}^d)$ in case of real interpolation and into $\mathrm{X}^{s,p}_E(\mathbb{R}^d)$ in case of complex interpolation. By (2.4) every function in $\langle \mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_E(O)\rangle_\theta$ has an extension in Y in virtue of a bounded extension operator. The required continuous inclusion follows. □

2.4. Proof of Theorem 2.1.4

Throughout the whole section let X denote either H or W. We are given $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, and $\theta \in (0,1)$. When concerned with complex interpolation for X = W, we implictly restrict ourselves to $s_0 \neq 0$ and $s_1 \neq 1$. Our goal is to establish set inclusions

$$[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O)]_\theta \supseteq \begin{cases} \mathrm{X}^{s,p}_D(O) & (\text{if } s > 1/p) \\ \mathrm{X}^{s,p}(O) & (\text{if } s < 1/p)\end{cases} \tag{2.5}$$

and

$$(\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O))_{\theta,p} \supseteq \begin{cases} \mathrm{W}^{s,p}_D(O) & (\text{if } s > 1/p) \\ \mathrm{W}^{s,p}(O) & (\text{if } s < 1/p) \end{cases}. \tag{2.6}$$

This will complete the proof of Theorem 2.1.4 since under Assumption 2.1.1 the converse inclusions are continuous due to Proposition 2.3.6 and hence become equalities with equivalent norms thanks to the bounded inverse theorem.

2.4.1. Road map to the proof

We give the outline for complex interpolation. The real case will be treated in the same way up to replacing the complex interpolation bracket with the $(\cdot\,, p)$-real interpolation bracket and keeping in mind that real interpolation spaces of X-spaces are always W-spaces.

First, we show in Section 2.4.3 that (2.5) and (2.6) hold in the case $D = \partial O$ of pure Dirichlet boundary condition. Then the inclusion with general D and $s \in (0, 1/p)$ follows readily:

$$\mathrm{X}^{s,p}(O) \subseteq \Big[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_{\partial O}(O)\Big]_\theta \subseteq \Big[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O)\Big]_\theta.$$

In the case $s \in (1/p, 1)$ we localize in order to reduce the problem to pure Dirichlet interpolation and interpolation with mixed boundary conditions, but for a simpler geometry. Precisely, we will have $O = \mathbb{R}^d_+$ the upper half-space and E_i a transformed version of a portion of D with a security area for good measure that is still $(d-1)$-regular and has porous boundary in $\partial\mathbb{R}^d_+ \cong \mathbb{R}^{d-1}$. Then we have to show that

$$\mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+) \subseteq \Big[\mathrm{X}^{s_0,p_0}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+)\Big]_\theta. \tag{2.7}$$

The details of this localization procedure are presented in Section 2.4.4.

The heart of the matter lies in showing (2.7) in Section 2.4.6. To do so, we decompose $f \in \mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+)$ as $f = (f - \mathcal{E}\mathcal{R}f) + \mathcal{E}\mathcal{R}f$, where $\mathcal{R}$ is the restriction to $\partial\mathbb{R}^d_+$ and $\mathcal{E}$ is a corresponding extension operator. The term $f - \mathcal{E}\mathcal{R}f$ is in $[\mathrm{X}^{s_0,p_0}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+)]_\theta$ because it satisfies pure Dirichlet boundary conditions on $\partial\mathbb{R}^d_+$. The argument for $\mathcal{E}\mathcal{R}f$ happens completely at the boundary and is displayed in Figure 3.

Here, $\mathrm{W}^{s,p}_\bullet({}^cE_i)$ is a subspace of $\mathrm{W}^{s,p}(\mathbb{R}^{d-1})$ with zero condition on the full dimensional set $E_i \subseteq \mathbb{R}^{d-1}$ and q, ε and η are parameters yet to be determined. We need to establish

$$
\begin{tikzcd}
\mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+) \arrow[dd, "\mathcal{R}"] & \Big[\mathrm{X}^{s_0,p_0}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+)\Big]_\theta \arrow[d, equal, "\text{reiteration}"'] \\
 & \Big[\mathrm{X}^{1/q-\varepsilon,q}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+)\Big]_\eta \\
\mathrm{W}^{s-1/p,p}_{\bullet}({}^cE_i) \arrow[r, equal, "(\heartsuit)"] & \Big[\mathrm{W}^{-\varepsilon,q}_{\bullet}({}^cE_i), \mathrm{W}^{s_1-1/p_1,p_1}_{\bullet}({}^cE_i)\Big]_\eta \arrow[u, "\mathcal{E}"]
\end{tikzcd}
$$

Figure 3.: Schematic presentation of the main argument to prove the inclusion "$\supseteq$" in part (a) of Theorem 2.1.4.

- the construction of an extension operator $\mathcal{E}$ from $\partial\mathbb{R}^d_+$ to $\mathbb{R}^d_+$ which is consistent in $s \in \mathbb{R} \setminus \mathbb{Z}$ and $p \in (1, \infty)$ and
- the precise definition of the spaces $\mathrm{W}^{s,p}_{\bullet}({}^cE_i)$ for a suitable range of s including verification of the interpolation identity ($\heartsuit$).

The passage through spaces of negative order in ($\heartsuit$) is inevitable and can be implemented in virtue of Proposition 2.2.14 only because ∂D is porous in ∂O.

2.4.2. Spaces of functions vanishing on a full-dimensional subset

For this part we work with a d-regular set $U \subseteq \mathbb{R}^d$ whose boundary is a Lebesgue null set and whose interior $\mathring{U}$ is of class $\mathcal{D}^t$ for some $t \in (0, 1)$, compare with Definition 2.2.12.

We remark that most results stated in Section 2.2.2 for open sets still apply in this context. Pointwise multiplication by the characteristic functions of U and $\mathring{U}$ coincide on $\mathrm{L}^p(\mathbb{R}^d)$, for $|\partial U| = 0$. Moreover, $\mathring{U}$ is d-regular by the same argument and the corresponding Rychkov's extension operators can also be regarded as extension operators for functions defined on U.

Let $\mathcal{R}$ denote the pointwise restriction operator $|_U$ and let $\mathcal{E}$ denote *some* extension operator $\mathrm{X}^{s,p}(U) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$. We will specify consistency requirements later on. For $p \in (1, \infty)$ and $s \in (0, \infty)$ we define the *bullet space*

$$\mathrm{X}^{s,p}_{\bullet}({}^cU) := \{f \in \mathrm{X}^{s,p}(\mathbb{R}^d) : \mathcal{R}f = 0\}$$

with subspace topology. This subspace is complemented in virtue of the projection $1 - \mathcal{E}\mathcal{R}$.

The pointwise multiplier $\mathbb{1}_U$ is bounded on $\mathrm{X}^{s,p}(\mathbb{R}^d)$ for $t(1/p-1) < s < t/p$ due to Proposition 2.2.14. This allows us to extend the definition of $\mathrm{X}^{s,p}_{\bullet}({}^cU)$ to such s by

$$\mathrm{X}^{s,p}_{\bullet}({}^cU) := \{f - \mathbb{1}_U f : f \in \mathrm{X}^{s,p}(\mathbb{R}^d)\},$$

where the topology is again the subspace topology. Note that for $s \in (0, t/p)$ this gives the same space as above and that now $1 - \mathbb{1}_U$ becomes the complementing projection.

The following lemma captures the interpolation behavior of these spaces.

Lemma 2.4.1. *Let $p_0, p_1 \in (1, \infty)$, $s_0 \in (t(1/p_0 - 1), \infty)$, $s_1 \in (t(1/p_1 - 1), \infty)$, and $\theta \in (0, 1)$. Up to equivalent norms it follows that*

(i) $$[\mathrm{X}^{s_0,p_0}_{\bullet}({}^cU), \mathrm{X}^{s_1,p_1}_{\bullet}({}^cU)]_\theta = \mathrm{X}^{s,p}_{\bullet}({}^cU),$$

(ii) $$(\mathrm{X}^{s_0,p_0}_{\bullet}({}^cU), \mathrm{X}^{s_1,p_1}_{\bullet}({}^cU))_{\theta,p} = \mathrm{W}^{s,p}_{\bullet}({}^cU),$$

with the two exceptions that in (i) *for* $\mathrm{X} = \mathrm{W}$ *either all or none of s_0, s_1, s have to be integers and that in* (ii) *integer s is only permitted when $s_0 = s_1 (= s)$.*

Proof. By symmetry we may assume $s_0 \leq s_1$. In virtue of Corollary 1.1.3 we shall transfer the interpolation identities of Proposition 1.2.9 for the $\mathrm{X}^{s,p}(\mathbb{R}^d)$-spaces to the $\mathrm{X}^{s,p}_{\bullet}({}^cU)$-spaces. We only have to identify suitable projections $\mathcal{P}$.

If $s_0 > 0$, then we pick Rychkov's extension operator $\mathcal{E}$ that is consistent up to a positive integer greater s_1 and use $\mathcal{P} := 1 - \mathcal{E}\mathcal{R}$.

Now assume $s_0 \leq 0$. If $s_1 < t/p_1$, then we can directly use $\mathcal{P} := 1 - \mathbb{1}_U$. Otherwise, there are $p \in (1, \infty)$ and $s \in (0, t/p)$ such that $(s, 1/p)^\top$ lies on the segment connecting $(s_0, 1/p_0)^\top$ and $(s_1, 1/p_1)^\top$ in the $(s, 1/p)$-plane. If necessary, we can arrange that s is not an integer. We have just obtained interpolation for the spaces on the segment connecting $(s_0, 1/p_0)^\top$ and $(s, 1/p)^\top$ and in order to conclude, we patch this interpolation scale together with the one for positive differentiability by the technique illustrated in the proof of Proposition 2.3.4. $\square$

2.4.3. The case of pure Dirichlet conditions

For this part we strengthen our previous requirements on $U \subseteq \mathbb{R}^d$ to the effect that it should be a closed d-regular set with $(d-1)$-regular boundary.

It follows that ∂U is a Lebesgue null set and we claim that $\mathring{U}$ is of class $\mathcal{D}^t$ for all $t \in (0,1)$. Indeed, by Example 2.2.13 the open set ${}^cU$ has this property and since we have $\partial \mathring{U} \subseteq \partial U = \partial\,({}^cU)$ with set difference of zero Lebesgue measure, we see by the very definition that if ${}^cU$ is of class $\mathcal{D}^t$, then so is $\mathring{U}$.

We start out with a reformulation of Corollary 2.2.15.

Lemma 2.4.2. *If $p \in (1,\infty)$ and $s \in [0,1/p)$, then $\mathrm{X}^{s,p}({}^cU) = \mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$ with equivalent norms.*

Proof. The inclusion $\mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU} \subseteq \mathrm{X}^{s,p}({}^cU)$ is clear. For the converse let $f \in \mathrm{X}^{s,p}({}^cU)$ and F an extension of f in $\mathrm{X}^{s,p}(\mathbb{R}^d)$. We get $\mathbb{1}_{{}^cU}\, F \in \mathrm{X}^{s,p}(\mathbb{R}^d)$ owing to Corollary 2.2.15. Hence, we have $\mathbb{1}_{{}^cU}\, F \in \mathrm{X}^{s,p}_{\bullet}({}^cU)$ and $f = (\mathbb{1}_{{}^cU}\, F)|_{{}^cU} \in \mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$ follows. For the boundedness, we calculate

$$\|f\|_{\mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}} \leq \|\mathbb{1}_{{}^cU}\, F\|_{\mathrm{X}^{s,p}(\mathbb{R}^d)} \lesssim \|F\|_{\mathrm{X}^{s,p}(\mathbb{R}^d)}$$

and take the infimum over all such extensions F. □

In order to proceed, we need a generic re-norming lemma and its consequence for the pointwise multiplication by $\mathbb{1}_{{}^cU}$. To fix ideas for the following, we include a proof even though the result is known in the literature.

Lemma 2.4.3. *If $p \in (1,\infty)$ and $s \in \mathbb{R}$, then*

$$\|f\|_{\mathrm{X}^{s,p}(\mathbb{R}^d)} \approx \|f\|_{\mathrm{X}^{s-1,p}(\mathbb{R}^d)} + \|\nabla f\|_{\mathrm{X}^{s-1,p}(\mathbb{R}^d)} \qquad (f \in \mathcal{S}'(\mathbb{R}^d)). \tag{2.8}$$

Proof. In the following all function spaces are on $\mathbb{R}^d$ and we omit the dependence. The operator $\mathcal{I}_{-1} f \coloneqq \mathcal{F}^{-1}(1+|\xi|^2)^{1/2}\mathcal{F} f$ is invertible from $\mathcal{S}'$ into itself. By definition it restricts to an isomorphism $\mathcal{I}_{-1} : \mathrm{H}^{s,p} \to \mathrm{H}^{s-1,p}$. By interpolation the same holds for $\mathcal{I}_{-1} : \mathrm{W}^{s,p} \to \mathrm{W}^{s-1,p}$, see Proposition 1.2.9. Hence, we find for all $f \in \mathcal{S}'$,

$$\|f\|_{\mathrm{X}^{s,p}} \approx \|\mathcal{F}^{-1}(1+|\xi|^2)^{1/2}\mathcal{F} f\|_{\mathrm{X}^{s-1,p}}.$$

Comparing with (2.8), we see that it remains to prove

$$\|f\|_{\mathrm{X}^{s-1,p}} + \sum_{j=1}^{d} \|\mathcal{F}^{-1}\xi_j \mathcal{F} f\|_{\mathrm{X}^{s-1,p}} \approx \|\mathcal{F}^{-1}(1+|\xi|^2)^{1/2}\mathcal{F} f\|_{\mathrm{X}^{s-1,p}}. \tag{2.9}$$

To this end we consider Fourier multipliers $f \mapsto \mathcal{F}^{-1} m \mathcal{F} f$, defined on $\mathcal{S}'$ via a smooth and bounded function $m : \mathbb{R}^d \to \mathbb{C}$, to pass from one side to the other. If such a multiplier is also bounded on L^p, then it is bounded on $\mathrm{H}^{k,p}$ for all integers k since it commutes with $\mathcal{I}_{-1}$ and its inverse. Hence, it is bounded on $\mathrm{X}^{s,p}$ for all $s \in \mathbb{R}$ by interpolation. This being said, we obtain "$\lesssim$" in (2.9) by considering the Fourier multipliers associated with $(1+|\xi|^2)^{-1/2}$ and $\xi_j(1+|\xi|^2)^{-1/2}$. Their L^p boundedness follows easily from the Mihlin multiplier theorem [23, Thm. 6.1.6]. Next, we pick a smooth function $\chi : \mathbb{R} \to [0,1]$ that vanishes on $(-1,1)$ and is identically 1 outside of $[-2,2]$ in order to write

$$(1+|\xi|^2)^{1/2} = \left(\frac{(1+|\xi|^2)^{1/2}}{1+\sum_{j=1}^d \chi(\xi_j)|\xi_j|} \right) + \sum_{j=1}^d \left(\frac{(1+|\xi|^2)^{1/2}}{1+\sum_{j=1}^d \chi(\xi_j)|\xi_j|} \right) \left(\frac{\chi(\xi_j)|\xi_j|}{\xi_j} \right) \xi_j .$$

Again by Mihlin's theorem each bracket corresponds to an L^p-bounded Fourier multiplier. This yields the converse estimate "$\gtrsim$". □

Lemma 2.4.4. *For $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$, pointwise multiplication by $\mathbb{1}_{{}^cU}$ is $\mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d) \to \mathrm{X}^{s,p}_{\bullet}({}^cU)$ bounded.*

Proof. For $f \in \mathrm{C}^\infty_{\partial U}(\mathbb{R}^d)$ we have that $\nabla(\mathbb{1}_{{}^cU} f) = \mathbb{1}_{{}^cU} \nabla f$. Hence, we can combine Lemma 2.4.3 and Proposition 2.2.14 to the effect that

$$\begin{aligned} \|\mathbb{1}_{{}^cU} f\|_{\mathrm{X}^{s,p}} &\approx \|\mathbb{1}_{{}^cU} f\|_{\mathrm{X}^{s-1,p}} + \|\mathbb{1}_{{}^cU} \nabla f\|_{\mathrm{X}^{s-1,p}} \\ &\lesssim \|f\|_{\mathrm{X}^{s-1,p}} + \|\nabla f\|_{\mathrm{X}^{s-1,p}} \\ &\approx \|f\|_{\mathrm{X}^{s,p}}. \end{aligned} \tag{2.10}$$

For $s \in (1/p, 1]$ we can use that $\mathrm{C}^\infty_{\partial U}(\mathbb{R}^d)$ is dense in $\mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d)$ by Lemma 2.2.5 to conclude that $\mathbb{1}_{{}^cU} : \mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$ is bounded. That it actually maps into the closed subspace $\mathrm{X}^{s,p}_{\bullet}({}^cU)$ follows by construction. Suppose now $s \in (1, 1+1/p)$. The commutation $\nabla(\mathbb{1}_{{}^cU} \cdot) = \mathbb{1}_{{}^cU} \nabla(\cdot)$ extends by density to all $f \in \mathrm{X}^{1,p}_{\partial U}(\mathbb{R}^d)$. Hence, it holds in particular on $\mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d)$ and the calculation (2.10) re-applies. □

We get the analogue of Lemma 2.4.2 in the case $s \in (1/p, 1+1/p)$.

Lemma 2.4.5. *If $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$, then $\mathrm{X}^{s,p}_{\partial U}({}^cU) = \mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$ with equivalent norms.*

Proof. The inclusion $\mathrm{X}^{s,p}_{\partial U}({}^cU) \subseteq \mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$ works exactly as in the proof of Lemma 2.4.2 when using Lemma 2.4.4 instead of Corollary 2.2.15.

Conversely, let $f \in \mathrm{X}^{s,p}_{\bullet}({}^cU)$. Since f is in particular a member of $\mathrm{X}^{s,p}(\mathbb{R}^d)$, we find a sequence $(f_n)_n \subseteq \mathcal{S}(\mathbb{R}^d)$ that approximates f in the topology of $\mathrm{X}^{s,p}(\mathbb{R}^d)$. Let $\mathcal{E}$ be Rychkov's extension operator for U, which, as we recall, acts consistently on $\mathrm{W}^{1,d+1}(\mathbb{R}^d)$. We apply the projection $\mathcal{P} = 1 - \mathcal{E}\mathcal{R}$ to that sequence. Since $\mathcal{P}$ projects onto $\mathrm{X}^{s,p}_{\bullet}({}^cU)$, we get $\mathcal{P}f_n = 0$ almost everywhere on U on the one hand and $\mathcal{P}f_n \in \mathrm{C}(\mathbb{R}^d)$ in virtue of Sobolev embeddings on the other hand. By d-regularity, the intersection of U with balls centered in U has positive Lebesgue measure. Hence, the $\mathcal{P}f_n$ vanish everywhere on U. In particular they vanish on ∂U, which means $\mathcal{P}f_n \in \mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d)$. Now, since $f_n \to f$ in $\mathrm{X}^{s,p}(\mathbb{R}^d)$, also $\mathcal{P}f_n \to \mathcal{P}f = f$ in $\mathrm{X}^{s,p}(\mathbb{R}^d)$. Since $\mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d)$ is a closed subspace of $\mathrm{X}^{s,p}(\mathbb{R}^d)$, this gives $f \in \mathrm{X}^{s,p}_{\partial U}(\mathbb{R}^d)$. □

Eventually, we can transfer the interpolation settled in Lemma 2.4.1 to the spaces incorporating pure Dirichlet boundary conditions. Since we can take $U = {}^cO$, this gives the full claim of Theorem 2.1.4 for pure Dirichlet conditions.

Proposition 2.4.6. *Let $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, and $\theta \in (0,1)$. There are continuous inclusions*

(i) $$[\mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_{\partial U}({}^cU)]_\theta \supseteq \begin{cases} \mathrm{X}^{s,p}_{\partial U}({}^cU) & (\textit{if } s > 1/p) \\ \mathrm{X}^{s,p}({}^cU) & (\textit{if } s < 1/p) \end{cases},$$

(ii) $$(\mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_{\partial U}({}^cU))_{\theta,p} \supseteq \begin{cases} \mathrm{W}^{s,p}_{\partial U}({}^cU) & (\textit{if } s > 1/p) \\ \mathrm{W}^{s,p}({}^cU) & (\textit{if } s < 1/p) \end{cases},$$

with the exception that $s_0 \neq 0$ and $s_1 \neq 1$ are required in (i) *for* $\mathrm{X} = \mathrm{W}$. *If in addition ${}^cU$ is d-regular, then both inclusions become equalities with equivalent norms.*

Proof. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote either the θ-complex or (θ,p)-real interpolation bracket. Using Lemma 2.4.2 and Lemma 2.4.5, we get

$$\begin{aligned} \langle \mathrm{X}^{s_0,p_0}_{\bullet}({}^cU), \mathrm{X}^{s_1,p_1}_{\bullet}({}^cU)\rangle|_{{}^cU} &\subseteq \langle \mathrm{X}^{s_0,p_0}_{\bullet}({}^cU)|_{{}^cU}, \mathrm{X}^{s_1,p_1}_{\bullet}({}^cU)|_{{}^cU}\rangle \\ &= \langle \mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_{\partial U}({}^cU)\rangle. \end{aligned}$$

Lemma 2.4.1 identifies the space on the left-hand side with either $\mathrm{X}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$ or $\mathrm{W}^{s,p}_{\bullet}({}^cU)|_{{}^cU}$, depending on the choice of the interpolation bracket above. The claim then follows from Lemma 2.4.2 in the case $s < 1/p$ and from Lemma 2.4.5 in the case $s > 1/p$.

The final statement on equalities in these inclusions follows from Proposition 2.3.6. □

2.4.4. Localization

We recall that O satisfies a uniform Lipschitz condition around $N \coloneqq \overline{\partial O \setminus D}$ with bi-Lipschitz constant L as in Assumption 2.1.1 (N). We claim that we can select countably many points $x_i \in \overline{\partial O \setminus D}$, $i \in I \subseteq \mathbb{N} \setminus \{0\}$, with corresponding coordinate charts $(U_{x_i}, \Phi_{x_i}) =: (U_i, \Phi_i)$, and an open set U_0 that does not intersect N, with the following properties. With $J \coloneqq \{0\} \cup I$, the covering

$$\overline{O} \subseteq \bigcup_{j \in J} U_j \tag{2.11}$$

admits a smooth partition of unity by functions $\eta_j \in \mathrm{C}^\infty(\mathbb{R}^d)$ satisfying

(i) $\operatorname{supp}(\eta_j) \subseteq U_j$, (ii) $\displaystyle\sum_{j \in J} \eta_j = 1$ on $\mathbb{R}^d$,

(iii) $\displaystyle\sum_{j \in J} \mathbb{1}_{U_j} \leq C$ on $\mathbb{R}^d$, (iv) $\|\eta_j\|_{\mathrm{L}^\infty} + \|\nabla \eta_j\|_{\mathrm{L}^\infty} \leq C'$,

and there are auxiliary functions $\chi_i \in \mathrm{C}^\infty(\mathbb{R}^d)$ with $\|\chi_i\|_{\mathrm{L}^\infty} + \|\nabla \chi_i\|_{\mathrm{L}^\infty} \leq C'$ such that χ_i is 1 on $\Phi_i(\operatorname{supp} \eta_i)$ and supported in $(-1,1)^d$, whereas χ_0 is 1 on $\operatorname{supp} \eta_0$ and supported in U_0. Here, C and C' are constants that depend only on L and d.

The construction is as follows. For any $x \in N$ we extend Φ_x to a bi-Lipschitz map $\overline{U_x} \to [-1,1]^d$ with the same Lipschitz constant not larger than L. From $\Phi_x(x) = 0$ we conclude that $\Phi_x(\overline{U_x} \cap \mathrm{B}(x, \frac{1}{2L}))$ is contained in $\mathrm{B}(0, \frac{1}{2})$ and hence does not intersect the boundary of the unit cube. The inclusion $B_x \coloneqq \mathrm{B}(x, \frac{1}{2L}) \subseteq U_x$ then follows from the fact that bi-Lipschitz mappings between closed sets preserve the boundaries. Starting from $\bigcup_{x \in N} \frac{1}{8} B_x \supseteq N$, we use the Vitali covering lemma (Lemma A.1.2) to extract a countable collection $(x_i)_{i \in I} \subseteq N$ such that $\bigcup_{i \in I} \frac{5}{8} B_i \supseteq N$ with the $\frac{1}{8} B_i$ mutually disjoint. We have abbreviated as usual $B_i \coloneqq B_{x_i}$.

If $x \in \mathbb{R}^d$ is contained in U_i, then $U_i \subseteq \mathrm{B}(x_i, L\sqrt{d}) \subseteq \mathrm{B}(x, 2L\sqrt{d})$ by the Lipschitz property and size of the unit cube. Due to $B(x_i, \frac{1}{16L}) \subseteq U_i$ and mutual disjointness there are at most $(32L^2\sqrt{d})^d$ such i. Bounded overlap guarantees that $U_0 \coloneqq \mathbb{R}^d \setminus \bigcup_{i \in I} \frac{5}{8} \overline{B}_i$ is an open set that pays for (2.11) and we can take $C \coloneqq 1 + (32L^2\sqrt{d})^d$ in (iii).

For $i \in I$ we pick $\varphi_i \in \mathrm{C}_0^\infty(B_i)$ with range in $[0,1]$, equal to 1 on $\frac{7}{8} B_i$, and $\|\nabla \varphi_i\|_\infty \leq cL$ for a dimensional constant c. We also pick a smooth φ_0 with range in $[0,1]$, support in $\mathbb{R}^d \setminus \bigcup_{i \in I} \frac{6}{8} B_i$, and equal to 1 on $\mathbb{R}^d \setminus \bigcup_{i \in I} \frac{7}{8} B_i$. For any $x \in \mathbb{R}^d$ the sum $\sum_{j \in J} \varphi_j(x)$ contains at most C non-zero terms, one of which is equal to 1. Hence, functions η_j with the properties specified in (i), (ii), (iv) are given by $\eta_j \coloneqq \varphi_j / \sum_{j \in J} \varphi_j$. Observe that η_j is smooth since the family $(B_i)_i$ is locally finite by the usual counting argument. For $i \in I$ we can take the χ_i all the same since $\Phi_i(\operatorname{supp}(\eta_i))$ is contained in $B(0, \frac{1}{2})$. We pick $\chi_0 \in \mathrm{C}^\infty(U_0)$ equal to 1 on $\mathbb{R}^d \setminus \bigcup_{i \in I} \frac{6}{8} B_i$ to complete the construction.

With this formalism at hand, we define the retraction-coretraction pair

$$\mathcal{E} : f \longmapsto \Big(\chi_0 f, (\chi_i(f \circ \Phi_i^{-1}))_{i \in I}\Big), \tag{2.12}$$

$$\mathcal{R} : (g_j)_{j \in J} \longmapsto \eta_0 g_0 + \sum_{i \in I} \eta_i (g_i \circ \Phi_i). \tag{2.13}$$

Indeed, we find $\mathcal{R}\mathcal{E} f = f$ for $f \in \mathrm{L}^p(O)$. It is implicitly understood that functions with compact support are extended by zero and domains of definitions are appropriately restricted to make these definitions meaningful. We introduce natural function spaces for these mappings.

Definition 2.4.7. For $p \in (1, \infty)$ and $s \in [0, 1]$ define the Banach space

$$\mathbb{X}^{s,p}(O) \coloneqq \mathrm{X}^{s,p}(O) \times \ell^p(I; \mathrm{X}^{s,p}(\mathbb{R}^d_+)), \quad \|g\|_{\mathbb{X}^{s,p}(O)} \coloneqq \Big(\sum_{j \in J} \|g_j\|^p_{\mathrm{X}^{s,p}}\Big)^{1/p}.$$

Remark 2.4.8. The space $\mathbb{X}^{s,p}(O)$ is constructed by ℓ^p-superposition from $\mathrm{X}^{s,p}(O)$ and $\mathrm{X}^{s,p}(\mathbb{R}^d_+)$. Real and complex interpolation behaves in the best possible (componentwise) way under this operation [93, Sec. 1.18.1]. Precisely, the spaces $\mathbb{X}^{s,p}(O)$ interpolate according to the same rules as do $\mathrm{X}^{s,p}(O)$ and $\mathrm{X}^{s,p}(\mathbb{R}^d_+)$ according to Proposition 2.3.4.

Lemma 2.4.9. *For $p \in (1, \infty)$ and $s \in [0, 1]$ the maps $\mathcal{E} : \mathrm{X}^{s,p}(O) \to \mathbb{X}^{s,p}(O)$ and $\mathcal{R} : \mathbb{X}^{s,p}(O) \to \mathrm{X}^{s,p}(O)$ are bounded.*

Proof. In view of Remark 2.4.8 we only have to treat the extremal cases $s = 0$ and $s = 1$. For convenience we write $\mathbb{L}^p(O)$ and $\mathbb{W}^{1,p}(O)$ instead of $\mathbb{X}^{s,p}(O)$, respectively.

Given $f \in \mathrm{L}^p(O)$, we use the uniformity and support properties of the partition of unity along with the uniform bi-Lipschitz property of the Φ_i when applying the transformation formula [81, Sec. 2.3.1], to give

$$\begin{aligned} \|\mathcal{E} f\|^p_{\mathbb{L}^p(O)} &= \int_O |\chi_0 f|^p \;\mathrm{d}x + \sum_{i \in I} \int_{\mathbb{R}^d_+} |\chi_i (f \circ \Phi_i^{-1})|^p \;\mathrm{d}x \\ &\lesssim \int_O \sum_{j \in J} \mathbb{1}_{U_j} |f|^p \;\mathrm{d}x. \end{aligned} \tag{2.14}$$

The right-hand side is bounded by $C\|f\|_{\mathrm{L}^p}^p$ due to the finite overlap property (iii). Similarly, given $g \in \mathbb{L}^p(O)$, we can estimate

(2.15)

$$\begin{aligned}\|\mathcal{R}g\|_{\mathrm{L}^p(O)}^p &\leq \int_O \Big(|\eta_0 g_0| + \sum_{i\in I} |\eta_i(g_i \circ \Phi_i)|\Big)^p \approx \int_O \Big(|\eta_0 g_0|^p + \sum_{i\in I} |\eta_i(g_i\circ\Phi_i)|^p\Big) \\ &\lesssim \int_O |g_0|^p + \sum_{i\in I}\int_{\mathbb{R}^d_+} |g_i|^p = \|g\|_{\mathbb{L}^p(O)}^p,\end{aligned}$$

where in the second step we have used again that for fixed x the sum contains at most C non-zero terms and hence the ℓ^1-norm can be replaced by an ℓ^p-norm at the expense of a constant depending on C. The previous two estimates yield the claim in case $s = 0$.

We turn to the case $s = 1$ and recall that $\mathrm{W}^{1,p}$-spaces are defined by restriction. Let $f \in \mathrm{W}^{1,p}(O)$ and let $F \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ be any extension. Calculating $\nabla(\mathcal{E}F)$ by the product and chain rules [81, Sec. 2.3.1], we can use the same argument as in (2.14) to get

$$\sum_{j\in J} \|(\mathcal{E}F)_j\|_{\mathrm{L}^p(\mathbb{R}^d)}^p + \|\nabla(\mathcal{E}F)_j\|_{\mathrm{L}^p(\mathbb{R}^d)}^p \lesssim \|F\|_{\mathrm{L}^p(\mathbb{R}^d)}^p + \|\nabla F\|_{\mathrm{L}^p(\mathbb{R}^d)}^p.$$

Since each $(\mathcal{E}F)_j$ extends $(\mathcal{E}f)_j$, the left-hand side controls $\|\mathcal{E}f\|_{\mathbb{W}^{1,p}(O)}$ from above and we can pass to the infimum over F to obtain the required boundedness of $\mathcal{E}$. Likewise, given $G \in \mathrm{X}^{s,p}(\mathbb{R}^d) \times \ell^p(I; \mathrm{X}^{s,p}(\mathbb{R}^d))$ we can recycle (2.15) to the effect that

$$\|\mathcal{R}G\|_{\mathrm{L}^p(\mathbb{R}^d)}^p + \|\nabla\mathcal{R}G\|_{\mathrm{L}^p(\mathbb{R}^d)}^p \lesssim \sum_{j\in J} \|G_j\|_{\mathrm{L}^p(\mathbb{R}^d)}^p + \|\nabla G_j\|_{\mathrm{L}^p(\mathbb{R}^d)}^p$$

and we conclude as before. □

To bring the boundary conditions into play, we introduce a modified version of $\mathbb{X}^{s,p}(O)$. We set

$$E_i := \Phi_i(D) \cup \Big(\mathbb{R}^{d-1} \setminus (-1,1)^{d-1}\Big) \qquad (i \in I) \tag{2.16}$$

and define

$$\mathbb{X}^{s,p}_E(O) := \mathrm{X}^{s,p}_{\partial O}(O) \times \bigtimes_{i\in I} \mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+) \qquad (s > 1/p), \tag{2.17}$$

which we consider as a closed subspace of $\mathbb{X}^{s,p}(O)$ in virtue of Lemma 2.2.11. Let us make sure that these transformed Dirichlet parts are of the same geometric quality as D.

Lemma 2.4.10. *The set E_i defined in* (2.16) *is* $(d-1)$*-regular in* $\mathbb{R}^{d-1}$ *and has porous boundary.*

Proof. Let $B \subseteq \mathbb{R}^{d-1}$ be a ball of radius $\mathrm{r}(B) \leq 1$ centered in $\overline{E_i}$. There are two cases. The first one is that $\frac{1}{2}B$ intersects the complement of $(-1,1)^{d-1}$. Then there is a ball B' of radius $\mathrm{r}(B)/4$ contained in $B\backslash[-1,1]^{d-1}$. The second one is that $\frac{1}{2}B$ is properly contained in the domain of Φ_i^{-1} and thus there is a ball $B'' \subseteq \mathbb{R}^d$ centered in ∂O such that $\mathrm{r}(B) \approx \mathrm{r}(B'')$ and $\Phi_i^{-1}(B) \supseteq B'' \cap \partial O$.

(i) We show that E_i is $(d-1)$-regular. In the first case we have $|B \cap E_i| \gtrsim (\mathrm{r}(B)/4)^{d-1}$. In the second case we use that bi-Lipschitz images have comparable Hausdorff measure [97, Thm. 28.10 a)] and that D is $(d-1)$-regular to conclude

$$|B \cap E_i| \approx \mathcal{H}^{d-1}(\Phi_i^{-1}(B \cap E_i)) \geq \mathcal{H}^{d-1}(B'' \cap D) \gtrsim \mathrm{r}(B)^{d-1}.$$

(ii) We show that ∂E_i is porous. Again, in the first case, already B' does not intersect ∂E_i. Otherwise, we use porosity of ∂D in ∂O, taking Remark 1.3.24 into account, to find a ball centered in ∂O and contained in B'' which avoids ∂D. Transforming this ball back using Φ_i, we find a ball centered in B with comparably smaller radius that does not intersect ∂E_i. □

The next lemma shows that $\mathcal{E}$ and $\mathcal{R}$ defined in (2.12) and (2.13) are well-behaved with respect to the Dirichlet conditions defined in (2.16) and (2.17).

Lemma 2.4.11. *For* $p \in (1,\infty)$ *and* $s \in (1/p, 1]$*, the operators* $\mathcal{E} : \mathrm{X}^{s,p}_D(O) \to \mathbb{X}^{s,p}_E(O)$ *and* $\mathcal{R} : \mathbb{X}^{s,p}_E(O) \to \mathrm{X}^{s,p}_D(O)$ *are bounded.*

Proof. For $\mathcal{E}$, it suffices to consider $f \in \mathrm{C}^\infty_D(O)$ since the general case follows by density, see Lemma 2.2.5. Since χ_0 is smooth with support away from $\overline{\partial O \setminus D}$, we get that $\chi_0 f$ is smooth with compact support away from ∂O. In particular, we have $\chi_0\, f \in \mathrm{X}^{s,p}_{\partial O}(O)$. We conclude from the bi-Lipschitz property of Φ_i that

$$\mathrm{d}(E_i, \operatorname{supp}(f \circ \Phi_i^{-1})) = \mathrm{d}(\Phi_i(D), \Phi_i(\operatorname{supp} f)) \approx \mathrm{d}(D, \operatorname{supp} f) > 0.$$

Hence, $\chi_i(f \circ \Phi_i^{-1})$ is a Lipschitz continuous function on $\mathbb{R}^d_+$ whose compact support has positive distance to E_i. Thus, it is contained in $\mathrm{W}^{1,p}_{E_i}(\mathbb{R}^d_+) \subseteq \mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+)$.

As for $\mathcal{R}$, we take $g = (g_j)_{j \in J}$ from $\mathrm{C}^\infty_{\partial O}(O) \times \bigtimes_{i \in I} \mathrm{C}^\infty_{E_i}(\mathbb{R}^d_+)$, which is dense in $\mathbb{X}^{s,p}_E(O)$ due to Lemmas 2.2.5 and 2.4.10. As before, we only have to show that the support of $\mathcal{R}g$ has positive distance to D. But $\operatorname{supp}(\eta_0\, g_0) \subseteq \operatorname{supp}(g_0)$ has positive distance to D by construction and for $\operatorname{supp}(\eta_i(g_i \circ \Phi_i))$ we can argue as above. □

Remark 2.4.12. Observe that $(\mathcal{R}, \mathcal{E})$ is a retraction-coretraction pair for the spaces $\mathrm{X}^{s,p}_D(O)$ and $\mathbb{X}^{s,p}_E(O)$ since this was the case without boundary conditions and $\mathcal{R}$ and $\mathcal{E}$ preserve the boundary conditions by the foregoing lemma. In particular, $\mathcal{R}$ is onto in the situation with boundary conditions.

We formulate a reduction result based on this localization.

Proposition 2.4.13. *The set inclusions* (2.5) *and* (2.6) *follow from the set inclusions*

$$[\mathrm{X}^{s_0,p_0}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+)]_\theta \supseteq \begin{cases} \mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+) & (\textit{if } s > 1/p) \\ \mathrm{X}^{s,p}(\mathbb{R}^d_+) & (\textit{if } s < 1/p) \end{cases} \tag{2.18}$$

and

$$(\mathrm{X}^{s_0,p_0}(\mathbb{R}^d_+), \mathrm{X}^{s_1,p_1}_{E_i}(\mathbb{R}^d_+))_{\theta,p} \supseteq \begin{cases} \mathrm{W}^{s,p}_{E_i}(\mathbb{R}^d_+) & (\textit{if } s > 1/p) \\ \mathrm{W}^{s,p}(\mathbb{R}^d_+) & (\textit{if } s < 1/p). \end{cases} \tag{2.19}$$

Proof. We apply Proposition 1.1.1 with the pair $(\mathcal{E}, \mathcal{R})$ defined in (2.12) and (2.13). Owing to the mapping properties derived in Lemmas 2.4.9 and 2.4.11, we get equal sets

$$[\mathrm{X}^{s_0,p_0}(O), \mathrm{X}^{s_1,p_1}_D(O)]_\theta = \mathcal{R}[\mathbb{X}^{s_0,p_0}(O), \mathbb{X}^{s_1,p_1}_E(O)]_\theta.$$

Remark 2.4.12 yields that the inclusion (2.5) holds provided that we can prove

$$[\mathbb{X}^{s_0,p_0}(O), \mathbb{X}^{s_1,p_1}_E(O)]_\theta \supseteq \begin{cases} \mathbb{X}^{s,p}_E(O) & (\text{if } s > 1/p) \\ \mathbb{X}^{s,p}(O) & (\text{if } s < 1/p). \end{cases} \tag{2.20}$$

The ℓ^p-superpositions of spaces $\mathrm{X}^{s,p}(O)$ and $\mathrm{X}^{s,p}_{E_i}(\mathbb{R}^d_+)$ on the left interpolate componentwise, see Remark 2.4.8. This being said, the above follows from the assumption (2.18) for the components on $\mathbb{R}^d_+$ and Proposition 2.4.6 for the component on O.

The real case is the same upon using $\mathbb{W}$-spaces on the right of (2.20) and appealing to assumption (2.19) instead. □

Remark 2.4.14. It stems from the interpolation on the left-hand side of (2.20) that at least at this stage of the proof we prefer talking about set inclusions only. Continuity of (2.20) would require continuity of (2.18) (which we could obtain) – but with uniform bounds in I (which we believe to be rather painful).

2.4.5. Extension and restriction operators for the half-space

We introduce the extension and restriction operators appearing in Figure 3. As usual, X denotes either H or W.

The restriction operator $\mathcal{R}$

Let $F \subseteq \partial\mathbb{R}^d_+$ be $(d-1)$-regular. We identify $\partial\mathbb{R}^d_+$ with $\mathbb{R}^{d-1}$ whenever convenient. Proposition 2.2.1 yields a restriction operator $\mathcal{R}_F : \mathrm{X}^{s,p}(\mathbb{R}^d) \to \mathrm{W}^{s-1/p,p}(F)$ for $p \in (1,\infty)$ and $s \in (1/p, 1+1/p)$. By construction, we have for $u \in \mathrm{X}^{s,p}(\mathbb{R}^d) \cap \mathrm{C}(\mathbb{R}^d)$,

$$\mathcal{R}_F u(x') = u(x',0) \qquad (\text{a.e. } x' \in F). \tag{2.21}$$

In virtue of this formula $\mathcal{R}_F$ is well-defined on the quotient space $\mathrm{X}^{s,p}(\mathbb{R}^d_+) \cap \mathrm{C}(\overline{\mathbb{R}^d_+})$. The inclusion chain

$$\mathrm{C}^\infty_0(\mathbb{R}^d)|_{\mathbb{R}^d_+} \subseteq \mathrm{X}^{s,p}(\mathbb{R}^d_+) \cap \mathrm{C}(\overline{\mathbb{R}^d_+}) \subseteq \mathrm{X}^{s,p}(\mathbb{R}^d_+) = \mathrm{X}^{s,p}(\mathbb{R}^d)|_{\mathbb{R}^d_+}$$

and the density of the first space in the last space shows that we can extend $\mathcal{R}_F$ to $\mathrm{X}^{s,p}(\mathbb{R}^d_+)$ by continuity. We abbreviate $\mathcal{R} := \mathcal{R}_{\partial\mathbb{R}^d_+}$.

The extension operator $\mathcal{E}$

For the extension operator we also need to consider spaces of negative smoothness. They have been defined on the whole space in Section 1.2. We set $\mathrm{X}^{s,p}(\mathbb{R}^d_+) := \mathrm{X}^{s,p}(\mathbb{R}^d)|_{\mathbb{R}^d_+}$, where the restriction of distributions $|_{\mathbb{R}^d_+} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d_+)$ coincides with the pointwise restriction when s is non-negative.

We construct $\mathcal{E}$ via the *bounded analytic C_0-semigroup* $(\mathrm{e}^{-\Lambda t})_{t\geq 0}$ generated by $\Lambda := -(1-\Delta_{x'})^{1/2}$ in $\mathrm{L}^p(\mathbb{R}^{d-1})$. Here, $\Delta_{x'}$ denotes the *Laplacian* in $\mathbb{R}^{d-1}$. A reader who is not familiar with these notions may consult the textbook [3], in particular Example 3.7.6 and Theorem 3.8.3. By means of the Fourier transform $\mathcal{F}$ in $\mathbb{R}^{d-1}$ the operators $\mathrm{e}^{-\Lambda t}$ are unambiguously defined on all of $\mathcal{S}'(\mathbb{R}^{d-1})$ through

$$\mathrm{e}^{-\Lambda t} : \mathcal{S}'(\mathbb{R}^{d-1}) \to \mathcal{D}'(\mathbb{R}^d_+), \quad u \mapsto \mathcal{F}^{-1}\Big(\mathrm{e}^{-t\sqrt{1+|\xi'|^2}}\mathcal{F}u(\xi')\Big).$$

We write $\mathrm{D}_p(\Lambda^k)$ for the maximal domain of Λ^k in $\mathrm{L}^p(\mathbb{R}^d)$ and equip it with the graph norm $\|\cdot\|_{\mathrm{L}^p} + \|\Lambda^k \cdot\|_{\mathrm{L}^p}$. By definition of Bessel potential spaces, we have for $k \in \mathbb{N}$ up to equivalent norms

$$\mathrm{D}_p(\Lambda^k) = \mathrm{H}^{k,p}(\mathbb{R}^{d-1}) = \mathrm{W}^{k,p}(\mathbb{R}^{d-1}). \tag{2.22}$$

Abstract semigroup theory [93, Thm. 1.14.5] provides an equivalent norm on the real interpolation space $\mathrm{W}^{k-1/p,p} = (\mathrm{L}^p, \mathrm{H}^{k,p})_{1-\frac{1}{kp},p}$:

$$\|\partial_t^k \mathrm{e}^{-\Lambda t} u\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{L}^p(\mathbb{R}^{d-1}))} \approx \|u\|_{\mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1})} \quad (u \in \mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1})). \tag{2.23}$$

With this at hand, we fix $\chi \in \mathrm{C}_0^\infty(\overline{\mathbb{R}_+})$ with $\chi(0) = 1$ and define the operator

$$\mathcal{E}u(x', x_d) \coloneqq \chi(x_d)\mathrm{e}^{-\Lambda x_d} u(x') \qquad (x' \in \mathbb{R}^{d-1}, x_d \geq 0). \tag{2.24}$$

Proposition 2.4.15. *The operator $\mathcal{E}$ defined in* (2.24) *is $\mathrm{W}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+)$ bounded for all $p \in (1,\infty)$ and $s \in \mathbb{R} \setminus \mathbb{Z}$.*

Proof. Our argument is an adaption of [93, Sec. 2.9.3] and divides into six steps.

Step 1: $\mathcal{E} : \mathrm{W}^{1-1/p,p}(\mathbb{R}^{d-1}) \to \mathrm{W}^{1,p}(\mathbb{R}^d_+)$ is bounded. First, we note that for $\varphi \in \mathrm{C}_0^\infty(\overline{\mathbb{R}_+})$ the multiplication operator

$$\varphi(x_d) : \mathrm{L}^p(\mathbb{R}_+; \mathrm{L}^p(\mathbb{R}^{d-1})) \to \mathrm{L}^p(\mathbb{R}_+; \mathrm{L}^p(\mathbb{R}^{d-1})) \tag{2.25}$$

is bounded. By boundedness of the semigroup the same is true for

$$\varphi(x_d)\mathrm{e}^{-\Lambda x_d} : \mathrm{L}^p(\mathbb{R}^{d-1}) \to \mathrm{L}^p(\mathbb{R}_+; \mathrm{L}^p(\mathbb{R}^{d-1})). \tag{2.26}$$

In particular, we get $\mathcal{E} : \mathrm{L}^p(\mathbb{R}^{d-1}) \to \mathrm{L}^p(\mathbb{R}^d_+)$ if we choose $\varphi = \chi$ in (2.26). Using the product rule and (2.25), we deduce from (2.23) that for $k \in \mathbb{N}$ we have

$$\|\partial_d^k \mathcal{E}u\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim \|u\|_{\mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1})}. \tag{2.27}$$

Using (2.22), (2.25), the identity $\Lambda\mathrm{e}^{-\Lambda x_d} = -\partial_d \mathrm{e}^{-\Lambda x_d}$ and (2.27), we obtain

$$\begin{aligned} \|\mathcal{E}u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d_+)} &\approx \|\mathcal{E}u\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{W}^{1,p}(\mathbb{R}^{d-1}))} + \|\mathcal{E}u\|_{\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{L}^p(\mathbb{R}^{d-1}))} \\ &\approx \|\mathcal{E}u\|_{\mathrm{L}^p(\mathbb{R}^d_+)} + \|\chi(x_d)\Lambda\mathrm{e}^{-\Lambda x_d} u(x')\|_{\mathrm{L}^p(\mathbb{R}^d_+)} + \|\partial_d \mathcal{E}u\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \\ &\lesssim \|u\|_{\mathrm{W}^{1-1/p,p}(\mathbb{R}^{d-1})}. \end{aligned}$$

Step 2: $\mathcal{E} : \mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1}) \to \mathrm{W}^{k,p}(\mathbb{R}^d_+)$ is bounded for $k \in \mathbb{N}$. We argue by induction. The case $k = 1$ was treated in Step 1. Recall from Definition 1.2.3 that we do not have to consider mixed derivatives. Moreover, the derivatives in x_d-direction are under control owing to (2.27). We fix $1 \leq j \leq d-1$. As $\mathrm{e}^{-\Lambda x_d}$ and ∂_j both are Fourier multipliers on $\mathcal{S}'(\mathbb{R}^{d-1})$, they commute. Assume the claimed boundedness holds for $k \in \mathbb{N}$. Then $\partial_j^k \mathcal{E} : \mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1}) \to \mathrm{L}^p(\mathbb{R}^d_+)$ is bounded and we conclude from Lemma 2.4.3 that

$$\|\partial_j^{k+1} \mathcal{E} u\|_{\mathrm{L}^p(\mathbb{R}^d_+)} = \|\partial_j^k \mathcal{E} \partial_j u\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim \|\partial_j u\|_{\mathrm{W}^{k-1/p,p}(\mathbb{R}^{d-1})} \lesssim \|u\|_{\mathrm{W}^{k+1-1/p,p}(\mathbb{R}^{d-1})}.$$

Step 3: Lifting property. To bring negative orders of differentiability into play, we introduce for $m \in \mathbb{N}$ the lift operator $\mathcal{I}_{2m} := \mathcal{F}^{-1}(1 + |\xi'|^2)^{-m}\mathcal{F}$ defined on $\mathcal{S}'(\mathbb{R}^{d-1})$. It is invertible and we write $\mathcal{I}_{-2m} := \mathcal{I}_{2m}^{-1}$. For $s \in \mathbb{R}$ the operator $\mathcal{I}_{2m}$ is an isomorphism $\mathrm{H}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{H}^{s+2m,p}(\mathbb{R}^{d-1})$ by definition of the norms on Bessel potential spaces. Since the Fourier multipliers $\mathcal{E}$ and $\mathcal{I}_{-2m}$ commute, we can decompose

$$\mathcal{E} = \mathcal{I}_{-2m} \circ \mathcal{E} \circ \mathcal{I}_{2m}, \tag{2.28}$$

in order to lift the argument of $\mathcal{E}$ into a space with positive order of differentiability, where $\mathcal{I}_{-2m}$ is the analogous operator in $\mathbb{R}^d$.

Step 4: $\mathcal{I}_{-2m}$ in d-dimensional space. Since $\mathcal{I}_{-2m} = (1 - \Delta_{x'})^m$ is a differential operator of order $2m$ acting only in $d-1$ coordinates, we have $\mathcal{I}_{-2m} : \mathrm{H}^{s+2m,p}(\mathbb{R}^d) \to \mathrm{H}^{s,p}(\mathbb{R}^d)$ for integer s. Interpolation by means of Proposition 1.2.9 yields $\mathcal{I}_{-2m} : \mathrm{X}^{s+2m,p}(\mathbb{R}^d) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$. The differential operator $\mathcal{I}_{-2m}$ is local in the sense that it commutes with the distributional restriction. Hence, its restriction to the upper half-space is well-defined and we get

$$\mathcal{I}_{-2m} : \mathrm{X}^{s+2m,p}(\mathbb{R}^d_+) \to \mathrm{X}^{s,p}(\mathbb{R}^d_+). \tag{2.29}$$

Step 5: Interpolation of $\mathcal{I}_{2m}$ and $\mathcal{E}$. As before, we interpolate $\mathcal{I}_{2m} : \mathrm{H}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{H}^{s+2m,p}(\mathbb{R}^{d-1})$ from Step 3 to obtain for all $s \in \mathbb{R}$ boundedness of

$$\mathcal{I}_{2m} : \mathrm{W}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{W}^{s+2m,p}(\mathbb{R}^{d-1}). \tag{2.30}$$

Similarly, real and complex interpolation of the outcome of Step 2 with the aid of Proposition 2.3.1 yields

$$\mathcal{E} : \mathrm{W}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+) \tag{2.31}$$

if $s \geq 1 - 1/p$ is not an integer.

Step 6: Patching everything together. Let $s \in \mathbb{R} \setminus \mathbb{Z}$. If $s \geq 1 - 1/p$, then $\mathcal{E} : \mathrm{W}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+)$ follows by (2.31). Otherwise, we choose $m \in \mathbb{N}$ such that $2m + s \geq 1 - 1/p$. We use the decomposition (2.28) to conclude $\mathcal{E} : \mathrm{W}^{s,p}(\mathbb{R}^{d-1}) \to \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+)$ from (2.30), (2.31) and (2.29). □

The next lemma justifies calling $\mathcal{E}$ an extension operator.

Lemma 2.4.16. *Let $F \subseteq \partial\mathbb{R}^d_+$ be $(d-1)$-regular and $\mathcal{R}_F$ the corresponding restriction operator. Let $p \in (1,\infty)$ and suppose that $s > 0$ is not an integer. If $u \in \mathrm{W}^{s,p}(\mathbb{R}^{d-1})$, then $\mathcal{R}_F\mathcal{E}u = u$ holds almost everywhere on F*

Proof. By density it suffices to prove the claim for $u \in \mathrm{C}_0^\infty(\mathbb{R}^{d-1})$. Due to (2.22) we have $u \in \mathrm{D}_p(\Lambda^k)$ for all $k \in \mathbb{N}$ and $p \in (1,\infty)$. We pick k and p such that $\mathrm{D}_p(\Lambda^k)$ is continuously included into $\mathrm{C}(\mathbb{R}^{d-1})$ in virtue of Sobolev embeddings. Since we have $\Lambda^k \mathrm{e}^{-t\Lambda}u = \mathrm{e}^{-t\Lambda}\Lambda^k u$ for $t \geq 0$, the strong continuity of the semigroup on $\mathrm{L}^p(\mathbb{R}^{d-1})$ implies $\mathcal{E}u \in \mathrm{C}(\overline{\mathbb{R}^+}; \mathrm{C}(\mathbb{R}^{d-1})) = \mathrm{C}(\overline{\mathbb{R}^d_+})$ and $\mathcal{E}u(x',0) = u(x')$ for almost every $x' \in \mathbb{R}^{d-1}$. Proposition 2.4.15 guarantees $\mathcal{E}u \in \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+)$ and we conclude from (2.21) that $\mathcal{R}_F\mathcal{E}u = u$ holds almost everywhere on F. $\square$

2.4.6. Conclusion of the proof

Here, we will verify the set inclusions (2.18) and (2.19). Thereby we complete the proof of Theorem 2.1.4.

We start out with the interpolation in the case $s \in (0, 1/p)$, which we treat slightly more generally for a later use.

Proposition 2.4.17. *Let $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, and for $\theta \in (0,1)$ define p and s as in (2.2). Suppose $s < 1/p$. Assume that $U \subseteq \mathbb{R}^d$ is a closed d-regular set with porous boundary. Moreover, assume that ${}^cU$ is also d-regular and that $F \subseteq \partial U$ is $(d-1)$-regular. Then it follows up to equivalent norms that*

(i) $$[\mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_F({}^cU)]_\theta = \mathrm{X}^{s,p}({}^cU),$$

(ii) $$(\mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_F({}^cU))_{\theta,p} = \mathrm{W}^{s,p}({}^cU),$$

with the exception that $s_0 \neq 0$ and $s_1 \neq 1$ are required in (i) *for* $\mathrm{X} = \mathrm{W}$.

Proof. The "$\subseteq$"-inclusions follow from Proposition 2.3.6. For the converse let $\langle \cdot\,,\cdot \rangle$ denote either the θ-complex or (θ,p)-real interpolation bracket. Using the inclusion $\mathrm{X}^{s_1,p_1}_{\partial U}({}^cU) \subseteq \mathrm{X}^{s_1,p_1}_F({}^cU)$, we get

$$\langle \mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_{\partial U}({}^cU) \rangle \subseteq \langle \mathrm{X}^{s_0,p_0}({}^cU), \mathrm{X}^{s_1,p_1}_F({}^cU) \rangle.$$

We identify the space on the left-hand side according to Proposition 2.4.6 to conclude. $\square$

Since this proposition can be applied to $U := \overline{\mathbb{R}^d_-}$ and $F := E_i$, we get (2.18) and (2.19) in case $s < 1/p$.

In a next step we establish the rest of Figure 3. To this end, we shall appeal to the theory of Section 2.4.2 with $U = E_i$ in $\mathbb{R}^{d-1}$. This requires E_i to be $(d-1)$-regular in $\mathbb{R}^{d-1}$, its boundary to be a Lebesgue null set, and its interior to be of some class $\mathcal{D}^t$. The first requirement is met by Lemma 2.4.10, which also guarantees that ∂E_i is porous. Hence, so is its subset ∂E_i°. In view of Example 2.2.13 the interior of E_i is of class $\mathcal{D}^t$ for some $t \in (0,1)$. Finally, the boundary of a porous set is a null set by Lemma A.1.1.

Due to Lemma 2.4.1 the spaces $\mathrm{W}^{s,p}_\bullet({}^cE_i)$ interpolate as expected. Next, we check that the extension operator constructed in the previous section preserves the zero condition when restricted to $\mathrm{W}^{s,p}_\bullet({}^cE_i)$.

Lemma 2.4.18. *Let $p \in (1,\infty)$ and let $s > t(1/p-1)$ not be an integer. If* X *denotes either* H *or* W*, then* $\mathcal{E} : \mathrm{W}^{s,p}_\bullet({}^cE_i) \to \mathrm{X}^{s+1/p,p}_{E_i}(\mathbb{R}^d_+)$.

Proof. Let $u \in \mathrm{W}^{s,p}_\bullet({}^cE_i)$. Due to Proposition 2.4.15 we have $\mathcal{E}u \in \mathrm{X}^{s+1/p,p}(\mathbb{R}^d_+)$. By Lemma 2.4.16 and the definition of $\mathrm{W}^{s,p}_\bullet({}^cE_i)$ we know that $\mathcal{R}_{E_i}\mathcal{E}u = u = 0$ holds. This means $\mathcal{E}u \in \mathrm{X}^{s+1/p,p}_{E_i}(\mathbb{R}^d_+)$. □

Let now $p_0, p_1 \in (1,\infty)$, $s_0 \in [0, 1/p_0)$, $s_1 \in (1/p_1, 1]$, and $\theta \in (0,1)$. Let us recall

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1$$

and that we assume $s > 1/p$. By these restrictions on the parameters there are $q \in (1,\infty)$ and $\varepsilon \in (0, \min\{1/q, t(1-1/q)\})$ such that the point $(1/q, -\varepsilon)^\top$ lies on the segment connecting $(1/p_1, s_1 - 1/p_1)^\top$ and $(1/p_0, s_0 - 1/p_0)^\top$ in the $(1/p, s)$-plane. Since we have by construction

$$\begin{pmatrix} 1/p \\ s - 1/p \end{pmatrix} = (1-\theta)\begin{pmatrix} 1/p_0 \\ s_0 - 1/p_0 \end{pmatrix} + \theta \begin{pmatrix} 1/p_1 \\ s_1 - 1/p_1 \end{pmatrix},$$

we can fix $\eta \in (0,\theta)$ such that

$$\begin{pmatrix} 1/p \\ s - 1/p \end{pmatrix} = (1-\eta)\begin{pmatrix} 1/q \\ -\varepsilon \end{pmatrix} + \eta \begin{pmatrix} 1/p_1 \\ s_1 - 1/p_1 \end{pmatrix}.$$

This yields identity (♡) in Figure 3. Adding both lines of the previous equation gives

$$s = (1-\eta)(1/q-\varepsilon) + \eta s_1.$$

We deduce

$$\Big(1 - \frac{\theta-\eta}{1-\eta}\Big)s_0 + \frac{\theta-\eta}{1-\eta}s_1 = 1/q - \varepsilon.$$

In the following all function spaces are on $\mathbb{R}^d_+$ and we omit the dependence. Let $\langle\cdot\,,\cdot\rangle$ denote either the complex or the $(\cdot\,,p)$-real interpolation bracket. From Proposition 2.4.17 and Proposition 2.3.5 we deduce

$$\langle \mathrm{X}^{1/q-\varepsilon,q}, \mathrm{X}^{s_1,p_1}_{E_i}\rangle_\eta = \langle [\mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{E_i}]_{\frac{\theta-\eta}{1-\eta}}, \mathrm{X}^{s_1,p_1}_{E_i}\rangle_\eta = \langle \mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{E_i}\rangle_\theta,$$

where $s_0 \neq 0$ and $s_1 \neq 1$ are required in case $\mathrm{X} = \mathrm{W}$. This establishes Figure 3 in case of complex interpolation. It also establishes the analogue that corresponds to real interpolation of H-spaces. As for real interpolation of W-spaces, we invoke the following reiteration theorem [23, Thm. 3.5.3].

Proposition 2.4.19. *Let (X_0, X_1) be an interpolation couple. Let $p \in [1,\infty]$, $\theta_0, \theta_1 \in [0,1]$ with $\theta_0 \neq \theta_1$, and $\lambda \in (0,1)$. With $\theta \coloneqq (1-\lambda)\theta_0 + \lambda\theta_1$ it follows that up to equivalent norms*

$$((X_0,X_1)_{\theta_0,p}, (X_0,X_1)_{\theta_1,p})_{\lambda,p} = (X_0,X_1)_{\theta,p},$$

subject to the interpretation $(X_0,X_1)_{j,p} \coloneqq X_j$ in the endpoint cases $j \in \{0,1\}$.

Indeed, in combination with Proposition 2.4.17 we can give

$$(\mathrm{W}^{1/q-\varepsilon,q}, \mathrm{W}^{s_1,p_1}_{E_i})_{\eta,p} = ((\mathrm{W}^{s_0,p_0}, \mathrm{W}^{s_1,p_1}_{E_i})_{\frac{\theta-\eta}{1-\eta},p}, \mathrm{W}^{s_1,p_1}_{E_i})_{\eta,p} = (\mathrm{W}^{s_0,p_0}, \mathrm{W}^{s_1,p_1}_{E_i})_{\theta,p}$$

without requiring $s_0 \neq 0$ or $s_1 \neq 1$. This completes Figure 3 in the remaining case.

With this at hand, we complete the proof of Theorem 2.1.4. Let $\langle\cdot\,,\cdot\rangle$ denote either the complex or the $(\cdot\,,p)$-real interpolation bracket. With Lemma 2.4.16 we derive $\mathcal{R}(f - \mathcal{E}\mathcal{R}f) = 0$ for $f \in \mathrm{X}^{s,p}$, which means $f - \mathcal{E}\mathcal{R}f \in \mathrm{X}^{s,p}_{\partial\mathbb{R}^d_+}$. We have

$$\langle \mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{\partial\mathbb{R}^d_+}\rangle_\theta \subseteq \langle \mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{E_i}\rangle_\theta,$$

where Proposition 2.4.6 identifies the left-hand space as $\mathrm{X}^{s,p}_{\partial\mathbb{R}^d_+}$ for complex interpolation and as $\mathrm{W}^{s,p}_{\partial\mathbb{R}^d_+}$ for real interpolation. From the decomposition

$$f = (f - \mathcal{E}\mathcal{R}f) + \mathcal{E}\mathcal{R}f$$

we conclude $f \in [\mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{E_i}]_\theta$ for $f \in \mathrm{X}^{s,p}_{E_i}$ in case of complex interpolation, which completes the proof of (2.5), and $f \in (\mathrm{X}^{s_0,p_0}, \mathrm{X}^{s_1,p_1}_{E_i})_{\theta,p}$ for $f \in \mathrm{W}^{s,p}_{E_i}$, which shows (2.6).

2.5. A complex $(\mathrm{W}^{-1,p}_D, \mathrm{W}^{1,p}_D)$-interpolation formula

In this section we prove Theorem 2.1.6. We begin by defining spaces of negative smoothness with boundary conditions on an open set.

Definition 2.5.1. Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \overline{O}$ be $(d-1)$-regular. Let $p \in (1,\infty)$ and $s \in [0,1]$. For X either H or W define

$$\mathrm{X}^{-s,p}(O) \coloneqq (\mathrm{X}^{s,p'}(O))^*$$

and if $s > 1 - 1/p$ define

$$\mathrm{X}^{-s,p}_D(O) \coloneqq (\mathrm{X}^{s,p'}_D(O))^*.$$

In the case $O = \mathbb{R}^d$, this is consistent with previous definitions.

We are concerned with interpolation spaces between $\mathrm{W}^{-1,p}_D(O)$ and $\mathrm{W}^{1,p}_D(O)$. These two spaces form an interpolation couple since we can naturally view $\mathrm{W}^{1,p}_D(O)$ as a subspace of $\mathrm{W}^{-1,p}_D(O)$ by extending the $\mathrm{L}^p(O) - \mathrm{L}^{p'}(O)$ duality. We also recall that as a consequence of Lemma 2.2.5 the inclusion $\mathrm{W}^{1,p}_D(O) \subseteq \mathrm{L}^p(O)$ is dense.

As for interpolation of dual spaces, we have the following principle [23, Cor. 4.5.2], see also [17, Cor. 2.15] for a proper treatment of the spaces of conjugate-linear functionals indicated by a superscript asterisk.

Proposition 2.5.2. *Let (X_0, X_1) be an interpolation couple such that $X_0 \cap X_1$ is dense in both X_0 and X_1 and assume that X_0 is reflexive. For $\theta \in (0,1)$ it follows that with equal norms*

$$[X_1^*, X_0^*]_{1-\theta} = ([X_0, X_1]_\theta)^*.$$

Now, the idea of proof is to patch together the interpolation scale provided by Theorem 2.1.4 with its dual scale. This requires some overlap of interpolation scales. The following lemmas use some notions introduced in Section 2.3.1.

Lemma 2.5.3. *Let $O \subseteq \mathbb{R}^d$ be an open set with porous boundary. Let $p \in (1, \infty)$, $s \in (1/p - 1, 1/p)$, and let* X *denote either* H *or* W*. There is a retraction $\mathcal{R} : \mathrm{X}^{s,p}(\mathbb{R}^d) \to \mathrm{X}^{s,p}(O)$ with corresponding coretraction $\mathcal{E} : \mathrm{X}^{s,p}(O) \to \mathrm{X}^{s,p}(\mathbb{R}^d)$. These operators are the same for all p and s.*

Proof. If $s \in [0, 1/p)$, then due to Corollary 2.2.15 we can take $\mathcal{R} \coloneqq |_O$ and $\mathcal{E} \coloneqq \mathcal{E}_0$ the extension by 0. By the usual identification of functions with distributions, these operators consistently act on $\mathrm{X}^{s,p}$ also when $s \in (1/p - 1, 0]$. Indeed, if $f \in \mathrm{X}^{s,p}(\mathbb{R}^d)$ then for all $\varphi \in \mathrm{X}^{-s,p'}(O)$ we set

$$\langle f|_O, \varphi \rangle_{\mathrm{X}^{s,p}(O), \mathrm{X}^{-s,p'}(O)} \coloneqq \langle f, \mathcal{E}_0 \varphi \rangle_{\mathrm{X}^{s,p}(\mathbb{R}^d), \mathrm{X}^{-s,p'}(\mathbb{R}^d)},$$

where $\langle \cdot\,, \cdot \rangle$ denotes the respective duality pairing. Well-definedness and boundedness of $|_O : \mathrm{X}^{s,p}(\mathbb{R}^d) \to \mathrm{X}^{s,p}(O)$ follows again from Corollary 2.2.15. Conversely, given $g \in \mathrm{X}^{s,p}(O)$, we let the zero extension $\mathcal{E}_0 g$ act on $\psi \in \mathrm{X}^{-s,p'}(\mathbb{R}^d)$ via

$$\langle \mathcal{E}_0 g, \psi \rangle_{\mathrm{X}^{s,p}(\mathbb{R}^d), \mathrm{X}^{-s,p'}(\mathbb{R}^d)} \coloneqq \langle g, \psi|_O \rangle_{\mathrm{X}^{s,p}(O), \mathrm{X}^{-s,p'}(O)}.$$

It is bounded since $|_O : \mathrm{X}^{-s,p'}(\mathbb{R}^d) \to \mathrm{X}^{-s,p'}(O)$ is bounded by definition of the quotient norm. Finally, $(\mathcal{E}_0 g)|_O = g$ follows by concatenating the two identities above. □

Lemma 2.5.4. *Let $O \subseteq \mathbb{R}^d$ be an open set with $(d-1)$-regular boundary. Let $p \in (1, \infty)$, $s_0, s_1 \in (1/p - 1, 1/p)$, $\theta \in (0, 1)$, and set $s \coloneqq (1-\theta)s_0 + \theta s_1$. If* X *denotes either* H *or* W*, then up to equivalent norms*

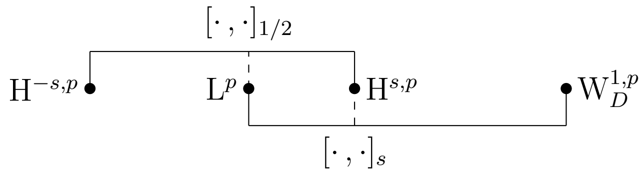
$$[\mathrm{X}^{s_0,p}(O), \mathrm{X}^{s_1,p}(O)]_\theta = \mathrm{X}^{s,p}(O)$$

with the exception that $s = 0$ is only allowed if $\mathrm{X} = \mathrm{H}$.

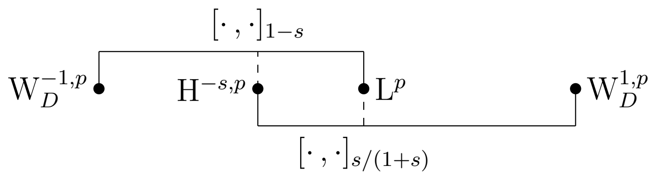
Proof. The corresponding identities on $O = \mathbb{R}^d$ are due to Proposition 1.2.9. The conclusion follows from Proposition 1.1.1 applied with the retraction-coretraction pair from Lemma 2.5.3. □

With these tools at hand, we can give the

Proof of Theorem 2.1.6. We appeal to Wolff's result, Proposition 2.3.3. All function spaces will be on O and we omit the dependence. We fix some $s \in (0, \min\{1/p, 1 - 1/p\})$ and consider the following diagram.



The 1/2-interpolation is due to Lemma 2.5.4 and s-interpolation is due to Theorem 2.1.4. Proposition 2.3.3 yields $\mathrm{L}^p = [\mathrm{H}^{-s,p}, \mathrm{W}^{1,p}_D]_{s/(1+s)}$. Therefore we can consider the diagram



where the $(1-s)$-interpolation follows from Theorem 2.1.4 by means of the duality principle of Proposition 2.5.2. Another application of Proposition 2.3.3 completes the proof. □

2.6. Real interpolation via the trace method

Here, we present the proof of Theorem 2.1.7.

2.6.1. Road map

The main new ingredient is Grisvard's trace characterization of real interpolation spaces [50, Thm. 5.12] stated in Proposition 2.6.1 below.

For a Banach space X we need the usual Bochner–Lebesgue space $\mathrm{L}^p(\mathbb{R}; X)$ of X-valued p-integrable functions on the real line and for $s > 0$ the respective (fractional) Sobolev spaces $\mathrm{W}^{s,p}(\mathbb{R}; X)$ that are defined as in the scalar case upon replacing absolute values by norms on X. For $s > 1/p$ such functions have a continuous representative and in that sense $\mathrm{W}^{s,p}(\mathbb{R}; X) \subseteq \mathrm{C}(\mathbb{R}; X)$ holds with continuous inclusion [90, Cor. 26]. In particular, the pointwise evaluation $|_{t=0} : \mathrm{W}^{s,p}(\mathbb{R}; X) \to X$ is well-defined and bounded. All this was already used in [50] and was known at the time by different proofs.

Proposition 2.6.1 (Grisvard)**.** *Let X_0, X_1 be Banach spaces such that $X_1 \subseteq X_0$ with dense and continuous inclusion. Let $p \in (1,\infty)$ and $s > 1/p$. Then*

$$\Big(\mathrm{L}^p(\mathbb{R}; X_1) \cap \mathrm{W}^{s,p}(\mathbb{R}; X_0)\Big)|_{t=0} \subseteq \Big(X_0, X_1\Big)_{1-\frac{1}{sp},p}.$$

The strategy to obtain Theorem 2.1.7 is schematically displayed in Figure 4. Owing to Proposition 2.3.6 and the bounded inverse theorem, we only need to prove the set inclusion "$\supseteq$" in (f). As usual, it suffices to only consider the case $s_0 = 0$ and $s_1 = 1$.

$$\begin{tikzcd}
\mathrm{W}^{s+1/p,p}_{\mathbb{R}\times D}(\mathbb{R}^{d+1}) \arrow[r, hook, "\text{Fubini}"] & \mathrm{W}^{s+1/p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p}_D(\mathbb{R}^d)) \arrow[d, "|_{x\in O}"] \\
\mathrm{W}^{s,p}(O \perp D) \arrow[u, "\mathcal{E}_{O\perp D}"] & \mathrm{W}^{s+1/p}(\mathbb{R}; \mathrm{L}^p(O)) \cap \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p}_D(O)) \arrow[d, "|_{t=0}"] \\
\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \arrow[u, "\mathcal{E}_0"] & \Big(\mathrm{L}^p(O), \mathrm{W}^{s+1/p,p}_D(O)\Big)_{\vartheta,p} \arrow[d, "\text{reiteration}"] \\
\mathrm{W}^{s,p}_D(O) \arrow[u, "\text{Hardy}"] & \Big(\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O)\Big)_{s,p}
\end{tikzcd}$$

Figure 4.: Schematic presentation of the argument for obtaining the inclusion "$\supseteq$" in Theorem 2.1.7 for $s > 1/p$. For $s < 1/p$ the diagram would start with $\mathrm{W}^{s,p}(O)$ instead.

The key observation is that functions in the second space of Figure 4 can be extended by zero to the set

$$O \perp D := \Big(O \times \{0\}\Big) \cup \Big(D \times \mathbb{R}\Big),$$

without losing Sobolev regularity. We shall see that $O \perp D$ is, as expected, a d-regular subset of $\mathbb{R}^{d+1}$. By means of the Jonsson–Wallin operator $\mathcal{E}_{O\perp D}$ we can then extend to all of $\mathbb{R}^{d+1}$ and via a Fubini property we end up in a space suitable for Grisvard's result. Taking the trace yields the desired inclusion, up to applying reiteration techniques from Proposition 2.4.19 in the final step.

Unless otherwise stated, we make the following

Assumption 2.6.2. *The set $O \subseteq \mathbb{R}^d$ is open and d-regular. The Dirichlet part $D \subseteq \overline{O}$ is Ahlfors–David regular.*

Only the final step will use the $(d-1)$-regularity of the full boundary ∂O additionally assumed in Theorem 2.1.7.

2.6.2. Hardy's inequality

In order to obtain the first inclusion in Figure 4, we establish a fractional Hardy inequality adapted to mixed boundary conditions that might be of independent interest. In contrast to related inequalities in [38] we completely avoid the use of capacities.

Definition 2.6.3. A set $U \subseteq \mathbb{R}^d$ is *plump* if there exists $\kappa \in (0,1)$ with the property:

$$\forall x \in \overline{U}, r \leq \operatorname{diam}(U) \quad \exists y \in \mathrm{B}(x,r) \ : \ \mathrm{B}(y, \kappa r) \subseteq U.$$

Remark 2.6.4. A comparison with Definition 1.3.23 yields first examples of plump sets. Namely, if $E \subseteq \mathbb{R}^d$ is uniformly porous, then ${}^c E$ is plump. This example can be modified to the effect that E is bounded and (uniformly) porous and $Q \subseteq \mathbb{R}^d$ is an open cube containing $\overline{E}$: Still we have that $Q \setminus E$ is plump.

We cite a result of Dyda–Vähäkangas [34, Thm. 1]. Upper and lower Assouad dimension have been introduced in Definition 1.3.13, and for uniformly ℓ-regular sets they coincide and equal ℓ.

Proposition 2.6.5 (Dyda–Vähäkangas)**.** *Let $p \in (1,\infty)$ and $s \in (0,1)$. Suppose that $U \subseteq \mathbb{R}^d$ is a proper, plump, open set in $\mathbb{R}^d$. Assume one of the following conditions:*

(i) $\overline{\dim}_{\mathcal{AS}}(\partial U) < d - sp$ *and U is unbounded.*

(ii) $\underline{\dim}_{\mathcal{AS}}(\partial U) > d - sp$ *and either U is bounded or ∂U is unbounded.*

Then there exists a constant c such that the inequality

$$\int_U \frac{|f(x)|^p}{\mathrm{d}(x, \partial U)^{sp}} \, \mathrm{d}x \leq c \int_U \int_U \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}} \, \mathrm{d}x \, \mathrm{d}y$$

holds for all measurable functions f for which the left-hand side is finite.

Remark 2.6.6. In the proof of the previous proposition, a clever absorption into the term on the left-hand side is used. This is how the finiteness condition enters the game.

With this at hand, we can state and prove the Hardy inequality alluded to above.

Proposition 2.6.7. *Let $p \in (1,\infty)$ and $s \in (0,1)$, $s \neq 1/p$. Under Assumption 2.6.2 there is a constant $C > 0$ such that the fractional Hardy inequality*

$$\int_O \frac{|f(x)|^p}{\mathrm{d}(x,D)^{sp}} \, \mathrm{d}x \leq C \|f\|^p_{\mathrm{W}^{s,p}(O)} \tag{2.32}$$

holds for all $f \in \mathrm{W}^{s,p}(O)$ if $s < 1/p$ and for all $f \in \mathrm{W}^{s,p}_D(O)$ if $s > 1/p$.

Proof. In both cases we shall reduce the claim to Proposition 2.6.5 on some auxiliary set.

Case 1: $s < 1/p$. Since D is Ahlfors–David regular, so is $\overline{D}$ by Remark 1.3.11. According to Example 1.3.25, $\overline{D}$ is uniformly porous. Hence, $U := \mathbb{R}^d \setminus \overline{D}$ is plump. Since $(d-1)$-regular sets have empty interior, we conclude $\partial U = \overline{D}$, which has upper and lower Assouad dimension $d-1$. Part (i) of Proposition 2.6.5 yields for all measurable f for which the left-hand side is finite

$$\int_{\mathbb{R}^d} \frac{|f(x)|^p}{\mathrm{d}_D(x)^{sp}} \, \mathrm{d}x \leq c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}} \, \mathrm{d}x \, \mathrm{d}y \leq c \|f\|^p_{\mathrm{W}^{s,p}(\mathbb{R}^d)}. \tag{2.33}$$

Example 2.2.13 guarantees that $x \mapsto \mathrm{d}_D(x)^{-sp}$ is locally integrable. Thus, (2.33) applies to every $f \in \mathrm{C}_0^\infty(\mathbb{R}^d)$, a dense subspace of $\mathrm{W}^{s,p}(\mathbb{R}^d)$, and we can use Fatou's lemma to extend (2.33) to all $f \in \mathrm{W}^{s,p}(\mathbb{R}^d)$. Restriction to O yields (2.32) for $f \in \mathrm{W}^{s,p}(O)$.

Case 2: $s > 1/p$ and D is unbounded. Part (ii) of Proposition 2.6.5 applies to $U := \mathbb{R}^d \setminus \overline{D}$ and we can argue as before, except that now we have (2.33) *a priori* for $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)$, a dense class of $f \in \mathrm{W}^{s,p}_D(\mathbb{R}^d)$ in view of Lemma 2.2.5. Hence we get (2.32) for $f \in \mathrm{W}^{s,p}_D(O)$.

Case 3: $s > 1/p$ and D is bounded. Let Q be an open cube that contains $\overline{D}$. As before we obtain that $U := 2Q \setminus \overline{D}$ is plump, where $2Q$ denotes the concentric cube with twice the sidelength. Moreover, $\partial U = \partial(2Q) \cup \overline{D}$ is uniformly $(d-1)$-regular as a finite union of sets with that property. Hence, it has lower Assouad dimension $d-1$. Part (ii) of Proposition 2.6.5 yields for all measurable g for which the left-hand side is finite

$$\int_U \frac{|g(x)|^p}{\mathrm{d}(x,\partial U)^{sp}} \, \mathrm{d}x \leq c \int_U \int_U \frac{|g(x)-g(y)|^p}{|x-y|^{sp+d}} \, \mathrm{d}x \, \mathrm{d}y \leq c \|g\|^p_{\mathrm{W}^{s,p}(U)}. \tag{2.34}$$

This applies to $g \in \mathrm{C}^\infty_{\partial U}(U)$ and extends to $g \in \mathrm{W}^{s,p}_{\partial U}(U)$ as in Case 2.

Let now $f \in \mathrm{W}^{s,p}_D(O)$. Let us fix $\eta \in \mathrm{C}^\infty_0(2Q)$ equal to 1 on Q and let $\mathcal{E}$ be an extension operator for O as in Proposition 2.2.9. We can bound

$$\begin{aligned}\int_O \frac{|f(x)|^p}{\mathrm{d}_D(x)^{sp}} \,\mathrm{d}x &\leq \int_{O\cap Q} \frac{|\eta(x)\mathcal{E}f(x)|^p}{\mathrm{d}_D(x)^{sp}} \,\mathrm{d}x + \int_{O\setminus Q} \frac{|f(x)|^p}{\mathrm{d}_D(x)^{sp}} \,\mathrm{d}x \\ &\leq \int_U \frac{|\eta(x)\mathcal{E}f(x)|^p}{\mathrm{d}(x,\partial U)^{sp}} \,\mathrm{d}x + \int_{O\setminus Q} \frac{|f(x)|^p}{\mathrm{d}_D(x)^{sp}} \,\mathrm{d}x =: I_1 + I_2,\end{aligned}$$

where we have used $O \cap Q \subseteq U$ and $D \subseteq \partial U$ to obtain I_1. Since on ${}^cQ$ we have $\mathrm{d}(\cdot\,, D) \geq \mathrm{d}(D, \partial Q) > 0$, we control I_2 by $\|f\|^p_{\mathrm{L}^p(O)}$. Next, $\mathcal{E}f \in \mathrm{W}^{s,p}_D(\mathbb{R}^d)$ follows from Lemma 2.2.11. Pointwise multiplication by η is bounded on $\mathrm{W}^{s,p}(\mathbb{R}^d)$ and maps $\mathrm{W}^{s,p}_D(\mathbb{R}^d) \to \mathrm{W}^{s,p}_{\partial U}(\mathbb{R}^d)$ since this is true for the respective dense subsets provided by Lemma 2.2.5. In conclusion, we have $\eta\mathcal{E}f \in \mathrm{W}^{s,p}_{\partial U}(\mathbb{R}^d)$. Hence, the extension of (2.34) gives control on I_1 by $\|\eta\mathcal{E}f\|^p_{\mathrm{W}^{s,p}(\mathbb{R}^d)}$. The boundedness of $\mathcal{E}$ leads us to a desirable bound for I_1. $\square$

2.6.3. Details of the proof

We are in the position to give a precise meaning to Figure 4. The first inclusion was established in Section 2.6.2. We continue with the zero extension part on the left.

The following lemma is a straightforward consequence of a product formula for the Hausdorff measure [42, Thm. 2.10.45]. Full details are written out in [38, Cor. 7.6].

Lemma 2.6.8. *If $O \subseteq \mathbb{R}^d$ is d-regular and $D \subseteq \mathbb{R}^d$ is $(d-1)$-regular, then $D \times \mathbb{R}$, $O \times \{0\}$, and $O \perp D \subseteq \mathbb{R}^d$ are d-regular.*

Since $O \perp D$ is a d-regular subset of $\mathbb{R}^{d+1}$, the fractional Sobolev spaces $\mathrm{W}^{s,p}(O \perp D)$ are defined as in Section 2.2.1 and there is a corresponding Jonsson–Wallin theory in $\mathbb{R}^{d+1}$. In the following, we systematically use bold face to distinguish geometric objects such as points, balls, and Hausdorff measures in $\mathbb{R}^{d+1}$ from their counterparts in $\mathbb{R}^d$.

Proposition 2.6.9. *Let $p \in (1,\infty)$ and $s \in (0,1)$. Under Assumption 2.6.2 the zero extension operator*

$$(\mathcal{E}_0 f)(x,t) := \begin{cases} f(x) & (\textit{if } x \in O,\ t = 0) \\ 0 & (\textit{if } x \in D,\ t \in \mathbb{R}) \end{cases}$$

is $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}(O \perp D)$ bounded.

Proof. Since the outer measure $E \mapsto \boldsymbol{\mathcal{H}}^d(E \times \{0\})$ on $\mathbb{R}^d$ is a translation invariant Borel measure that assigns finite measure to the unit cube [97, §27], the induced measure coincides up to a norming constant $c_d > 0$ with the d-dimensional Lebesgue measure. Thus, $\mathcal{E}_0 f \in \mathrm{L}^p(O \perp D)$ is a consequence of $f \in \mathrm{L}^p(O)$.

We use Tonelli's theorem to bound the remaining part of the $\mathrm{W}^{s,p}(O \perp D)$-norm by

$$
\begin{aligned}
&\iint_{\substack{\boldsymbol{x},\boldsymbol{y} \in O \perp D \\ |\boldsymbol{x}-\boldsymbol{y}|<1}} \frac{|\mathcal{E}_0 f(\boldsymbol{x}) - \mathcal{E}_0 f(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{d+sp}} \, \boldsymbol{\mathcal{H}}^d(\mathrm{d}\boldsymbol{x}) \, \boldsymbol{\mathcal{H}}^d(\mathrm{d}\boldsymbol{y}) \\
&\leq c_d \iint_{\substack{x,y \in O \\ |x-y|<1}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + 2 \int_O \int_{\substack{\boldsymbol{y} \in D \times \mathbb{R} \\ |\boldsymbol{y}-(x,0)|<1}} \frac{|f(x)|^p}{|\boldsymbol{y}-(x,0)|^{d+sp}} \, \boldsymbol{\mathcal{H}}^d(\mathrm{d}\boldsymbol{y}) \, \mathrm{d}x.
\end{aligned}
\tag{2.35}
$$

The first integral on the right is bounded by $\|f\|^p_{\mathrm{W}^{s,p}(O)}$. If the inner domain of integration in the second integral is non-empty, then there exists an integer $n_0 \geq 0$ such that $2^{-n_0-1} < \mathrm{d}_D(x) \leq 2^{-n_0}$. We then split the integral into dyadic annuli

$$
\boldsymbol{C}_n := \Big(D \times \mathbb{R}\Big) \cap \Big((\mathbf{B}((x,0),2^{-n}) \setminus \mathbf{B}((x,0),2^{-n-1})\Big),
$$

each of which satisfies $\boldsymbol{\mathcal{H}}^d(\boldsymbol{C}_n) \lesssim 2^{-dn}$ since $D \times \mathbb{R}$ is d-regular, to give

$$
\begin{aligned}
&\int_{\substack{\boldsymbol{y} \in D \times \mathbb{R} \\ |\boldsymbol{y}-(x,0)|<1}} \frac{1}{|\boldsymbol{y}-(x,0)|^{d+sp}} \, \boldsymbol{\mathcal{H}}^d(\mathrm{d}\boldsymbol{y}) \\
&\lesssim \sum_{n=0}^{n_0} 2^{(n+1)(d+sp)} 2^{-dn} = \frac{2^{d+sp}}{2^{sp}-1}(2^{sp(n_0+1)} - 1).
\end{aligned}
$$

By choice of n_0, the right-hand side is controlled by $\mathrm{d}_D(x)^{-sp}$. In conclusion, the second integral on the right of (2.35) is bounded by

$$
\int_O \int_{\substack{\boldsymbol{y} \in D \times \mathbb{R} \\ |\boldsymbol{y}-(x,0)|<1}} \frac{|f(x)|^p}{|\boldsymbol{y}-(x,0)|^{d+sp}} \, \boldsymbol{\mathcal{H}}^d(\mathrm{d}\boldsymbol{y}) \, \mathrm{d}x \lesssim \int_O \frac{|f(x)|^p}{\mathrm{d}_D(x)^{sp}} \, \mathrm{d}x. \qquad \square
$$

The Fubini property appearing in Figure 4 is as follows. Throughout, we canonically identify $\mathrm{L}^p(\mathbb{R}^{d+1})$ with $\mathrm{L}^p(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d))$ by means of Fubini's theorem.

Lemma 2.6.10. *If $p \in (1,\infty)$ and $s \geq 0$, then up to equivalent norms*

$$
\mathrm{W}^{s,p}(\mathbb{R}^{d+1}) = \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s,p}(\mathbb{R}^d)) \cap \mathrm{W}^{s,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d)).
\tag{2.36}
$$

Proof. For $s \geq 0$ an integer, the claim follows directly from Fubini's theorem. Let now $s = k + \sigma$, where $k \geq 0$ is an integer and $\sigma \in (0,1)$. According to [93, Sec. 2.5.1] we can equivalently norm $\mathrm{W}^{s,p}(\mathbb{R}^d)$ by

$$\|f\| \coloneqq \|f\|_{\mathrm{L}^p} + \sum_{j=1}^{d} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \frac{\partial^k f}{\partial x_j^k}(x + he_j) - \frac{\partial^k f}{\partial x_j^k}(x) \right|^p \mathrm{d}x \, \frac{\mathrm{d}h}{|h|^{1+sp}} \right)^{1/p},$$

where $(e_j)_j$ denotes the standard unit vectors in $\mathbb{R}^d$. This equivalent norm only takes into account differences of f along the coordinate axes. Therefore we obtain (2.36) from Fubini's theorem if we equivalently norm all appearing spaces as described before. □

The next lemma makes Figure 4 precise, except for the final step.

Lemma 2.6.11. *Let $p \in (1,\infty)$ and $s \in (0,1)$. Under Assumption 2.6.2 the set inclusion*

$$(\mathrm{L}^p(O), \mathrm{W}^{s+1/p,p}_D(O))_{\vartheta,p} \supseteq \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}),$$

holds for $\vartheta \in (0,1)$ satisfying $\vartheta(s + 1/p) = s$.

Proof. We fix a function f in $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$.

The inclusion $\mathrm{W}^{s+1/p,p}_D(O) \subseteq \mathrm{L}^p(O)$ is continuous and it is dense since already $\mathrm{C}^\infty_{\partial O}(O)$ is dense in $\mathrm{L}^p(O)$. In view of Proposition 2.6.1 it suffices to construct a function

$$(2.37)\quad F \in \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p,p}_D(O)) \cap \mathrm{W}^{s+1/p,p}(\mathbb{R}; \mathrm{L}^p(O)) \quad \text{such that} \quad F|_{t=0} = f.$$

For the construction we start by extending f to $O \perp D$ by zero. This extension $\mathcal{E}_0 f$ is in $\mathrm{W}^{s,p}(O \perp D)$ due to Proposition 2.6.9. Since $O \perp D$ is a d-regular subset of $\mathbb{R}^{d+1}$ according to Lemma 2.6.8, we can use Proposition 2.2.1 to extend $\mathcal{E}_0 f$ to a function $G \in \mathrm{W}^{s+1/p,p}(\mathbb{R}^{d+1})$ in virtue of the corresponding Jonsson–Wallin operator. In view of Lemma 2.6.10 we have by canonical identification

$$G \in \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p,p}(\mathbb{R}^d)) \cap \mathrm{W}^{s+1/p,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d)).$$

A closer inspection reveals the following.

(i) Let $\mathcal{R}$ be the Jonsson–Wallin restriction to the d-set $D \times \mathbb{R}$ in $\mathbb{R}^{d+1}$. We have $\mathcal{R}G = 0$ by construction and therefore $G \in \mathrm{W}^{s+1/p,p}_{D\times\mathbb{R}}(\mathbb{R}^{d+1})$. Owing to Lemma 2.2.5, we approximate $G \in \mathrm{W}^{s+1/p,p}_{D\times\mathbb{R}}(\mathbb{R}^{d+1})$ in that space by test functions $G_n \in \mathrm{C}^\infty_{D\times\mathbb{R}}(\mathbb{R}^{d+1})$. By slicing

$$G_n \in \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p,p}_D(\mathbb{R}^d)) \cap \mathrm{W}^{s+1/p,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d))$$

and due to the Fubini property of Lemma 2.6.10 the limit G is contained in the same space, which is to say,

$$G \in \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p,p}_D(\mathbb{R}^d)) \cap \mathrm{W}^{s+1/p,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d)).$$

(ii) Let $\mathcal{R}$ be the Jonsson–Wallin restriction to the d-set $\mathbb{R}^d \times \{0\}$ in $\mathbb{R}^{d+1}$. This operator is bounded from $\mathrm{W}^{s+1/p}(\mathbb{R}^{d+1})$ into $\mathrm{L}^p(\mathbb{R}^d \times \{0\})$ by Proposition 2.2.1. On the other hand, we can look at the restriction $|_{t=0}$ defined on $\mathrm{L}^p(\mathbb{R}; \mathrm{W}^{s+1/p,p}(\mathbb{R}^d)) \cap \mathrm{W}^{s+1/p,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d))$ and bounded into $\mathrm{L}^p(\mathbb{R}^d)$. Identifying corresponding objects via Fubini's theorem as before, it turns out that these two restrictions are the same since they obviously agree on a dense class of continuous functions. Since $\mathcal{R}G$ and f coincide $\boldsymbol{\mathcal{H}}^d$-almost everywhere on $O \times \{0\}$ by construction, we can record

$$G|_{t=0} = f \quad \text{almost everywhere on } O.$$

The outcome of observations (i) and (ii) shows that $F := G|_{O\times\mathbb{R}}$ verifies (2.37). $\square$

Together with Proposition 2.3.6 we obtain

Corollary 2.6.12. *If in addition to Assumption 2.6.2 the set O has $(d-1)$-regular boundary, then the set inclusion in Lemma 2.6.11 is an equality with equivalent norms.*

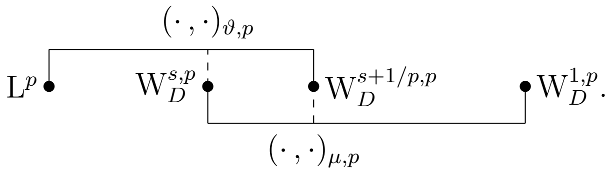
Eventually, we can complete the

Proof of Theorem 2.1.7. In the following all function spaces will be on O and we omit the dependence on O for clarity. In view of the reiteration theorem above it suffices to treat the case $s_0 = 0$ and $s_1 = 1$ and prove for $s \in (0,1)$ that up to equivalent norms it follows that

$$(\mathrm{L}^p, \mathrm{W}^{1,p}_D)_{s,p} = \begin{cases} \mathrm{W}^{s,p}_D & (\text{if } s > 1/p) \\ \mathrm{W}^{s,p} & (\text{if } s < 1/p) \end{cases}. \tag{2.38}$$

If $s + 1/p = 1$, then the claim follows from Corollary 2.6.12. The proof for $s + 1/p \neq 1$ divides into four cases.

Case 1: $s > 1/p$ and $s + 1/p < 1$. We have a diagram suitable for Wolff interpolation:



Indeed, the (ϑ, p)-interpolation is due to Corollary 2.6.12 and the (μ, p)-interpolation with suitable $\mu \in (0, 1)$ is due to Theorem 2.1.5. The claim follows by Proposition 2.3.3.

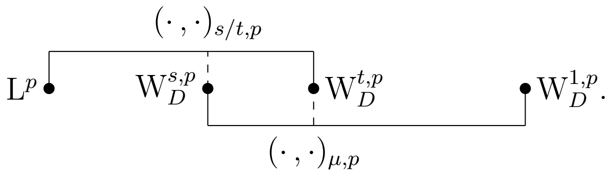
Case 2: $s > 1/p$ and $s + 1/p > 1$. We fix any $t \in (s, 1)$ and let $\lambda \in (0, 1)$ satisfy $(1-\lambda)s + \lambda(s+1/p) = t$. Applying one after the other Theorem 2.1.5, Corollary 2.6.12, and Proposition 2.4.19, we obtain

$$\mathrm{W}^{t,p}_D = (\mathrm{W}^{s,p}_D, \mathrm{W}^{s+1/p,p}_D)_{\lambda,p} = ((\mathrm{L}^p, \mathrm{W}^{s+1/p,p}_D)_{\vartheta,p}, \mathrm{W}^{s+1/p,p}_D)_{\lambda,p} = (\mathrm{L}^p, \mathrm{W}^{s+1/p,p})_{\theta,p},$$

with $\theta = t/(s + 1/p)$. Once again by Proposition 2.4.19 and Corollary 2.6.12 we find

$$(\mathrm{L}^p, \mathrm{W}^{t,p}_D)_{s/t,p} = (\mathrm{L}^p, (\mathrm{L}^p, \mathrm{W}^{s+1/p,p}_D)_{\theta,p})_{s/t,p} = (\mathrm{L}^p, \mathrm{W}^{s+1/p,p}_D)_{\vartheta,p} = \mathrm{W}^{s,p}_D.$$

Thus we obtain the desired result (2.38) from Proposition 2.3.3 applied as follows:



Indeed, we have obtained the $(s/t, p)$-interpolation above and the (μ, p)-interpolation for appropriately chosen μ is due to Theorem 2.1.5. Note that because of the exceptional case for real interpolation of Sobolev spaces we cannot pick $t = 1$ right away.

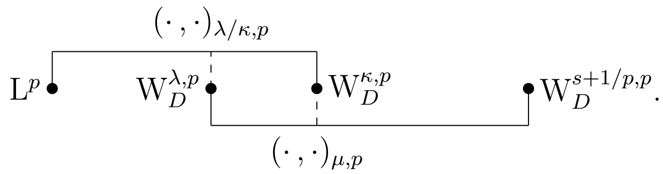
Case 3: $s < 1/p$ and $s + 1/p < 1$. We can apply one of the previous two cases with $s+1/p$ in place of s to obtain $(\mathrm{L}^p, \mathrm{W}^{1,p}_D)_{s+1/p,p} = \mathrm{W}^{s+1/p,p}_D$. Together with Corollary 2.6.12 in the first and reiteration in the third step, we are led to the desired result

$$\mathrm{W}^{s,p} = (\mathrm{L}^p, \mathrm{W}^{s+1/p,p}_D)_{\vartheta,p} = (\mathrm{L}^p, (\mathrm{L}^p, \mathrm{W}^{1,p}_D)_{s+1/p,p})_{\vartheta,p} = (\mathrm{L}^p, \mathrm{W}^{1,p}_D)_{s,p}.$$

Case 4: $s < 1/p$ and $s + 1/p > 1$. We pick $1/p < \lambda < \kappa < 1$. By one of the first two cases along with Proposition 2.4.19, we find

$$(\mathrm{L}^p, \mathrm{W}_D^{\kappa,p})_{\lambda/\kappa,p} = (\mathrm{L}^p, (\mathrm{L}^p, \mathrm{W}_D^{1,p})_{\kappa,p})_{\lambda/\kappa,p} = (\mathrm{L}^p, \mathrm{W}_D^{1,p})_{\lambda,p} = \mathrm{W}_D^{\lambda,p}.$$

Together with Theorem 2.1.5 this establishes for suitable μ the diagram



Proposition 2.3.3 yields $\mathrm{W}_D^{\kappa,p} = (\mathrm{L}^p, \mathrm{W}_D^{s+1/p,p})_{\theta,p}$ with $\theta = \kappa/(s + 1/p)$. We conclude by using one after the other Corollary 2.6.12, reiteration, one of the first two cases, and again reiteration:

$$\begin{aligned}\mathrm{W}^{s,p} &= (\mathrm{L}^p, \mathrm{W}_D^{s+1/p,p})_{\vartheta,p} = (\mathrm{L}^p, (\mathrm{L}^p, \mathrm{W}_D^{s+1/p,p})_{\theta,p})_{s/\kappa,p} = (\mathrm{L}^p, \mathrm{W}_D^{\kappa,p})_{s/\kappa,p} \\ &= (\mathrm{L}^p, (\mathrm{L}^p, \mathrm{W}_D^{1,p})_{\kappa,p})_{s/\kappa,p} = (\mathrm{L}^p, \mathrm{W}_D^{1,p})_{s,p}. \qquad \square\end{aligned}$$

CHAPTER 3

Extension operators for Sobolev spaces with boundary conditions

Sobolev spaces $\mathrm{W}^{k,p}_D(O)$ that contain functions that only vanish on a portion D of the boundary of some given open set $O \subseteq \mathbb{R}^d$ play an eminent role in the study of the mixed problem for elliptic operators. In the study of these spaces, an extension operator is a crucial tool.

Early contributions to the history of Sobolev extension operators include the works of Stein [92, pp. 180–192] and Calderón [27] on Lipschitz domains as well as the seminal paper of Jones [65] on (ε, δ)-domains. The latter work was later refined by Chua [29] and Rogers [84]. Though all these constructions aim at the full Sobolev space $\mathrm{W}^{1,p}(O)$, they restrict to bounded extension operators on the space with vanishing trace on D and the extensions preserve the trace condition on D if a mild regularity assumption is imposed, see Lemma 2.2.11 (the lemma is formulated for Rychkov's extension operator, but the proof applies verbatim if $s = 1$).

All these constructions rely on regularity assumptions for the full boundary of the underlying set O. However, if we consider a (relatively) interior point of D, then it is possible to extend the function by zero around that point, so that a relaxation on the boundary regularity is feasible. This effect was exploited using localization techniques by several authors, see Brewster, Mitrea, Mitrea, and Mitrea [26] for a very mature incarnation of this idea using local (ε, δ)-charts, and [56] for a version using Lipschitz manifolds. We will present both frameworks in detail in Section 3.2 and show that they are included in our setup.

One drawback of this method is that the regularity assumption for the Neumann boundary part $\partial O \setminus D$ has to hold not merely on this boundary portion but in a neighbourhood of it, which in particular contains interior points of D. This forbids all kinds of cusps that are arbitrarily close to the interface between the Dirichlet and the Neumann boundary parts.

In this work, we will introduce an (ε, δ)-condition that is adapted to the Dirichlet condition on D. To be more precise, we also connect nearby points in O by ε-cigars, but these are with respect to the Neumann boundary part $\partial O \setminus D$ and *not* the full boundary ∂O, which means that ε-cigars may "leave" the domain across the Dirichlet part D to some extent that is measured by a *quasi-hyperbolic distance* condition. This allows to have certain inward and outward cusps arbitrarily close to the interface between the Dirichlet and Neumann parts, see Example 3.2.5 for an illustrating example. However, there are types of cusps that are particularly nasty and which are excluded from our setting by the aforementioned quasihyperbolic distance condition. In Example 3.2.7 we show that in these kinds of configurations there cannot exist a bounded extension operator, which emphasizes that it is indeed necessary that we have incorporated some further restriction in our setup. A detailed description of our geometric framework will be given in Assumption 3.1.1.

Next, we give a precise definition of what we mean by the term *extension operator*, followed by our main result.

Definition 3.0.1. Call a linear mapping $\mathcal{E}$ defined on $\mathrm{L}^1_{\mathrm{loc}}(O)$ into the measurable functions on $\mathbb{R}^d$ an *extension operator* if it satisfies $\mathcal{E}f(x) = f(x)$ for almost every $x \in O$ and for all $f \in \mathrm{L}^1_{\mathrm{loc}}(O)$.

Theorem 3.0.2. *Let $O \subseteq \mathbb{R}^d$ be open and let $D \subseteq \partial O$ be closed. Assume that O and D are subject to Assumption 3.1.1. Moreover, fix an integer $k \geq 0$. Then there exists an extension operator $\mathcal{E}$ such that for all $1 \leq p < \infty$ and $0 \leq \ell \leq k$ one has that $\mathcal{E}$ restricts to a bounded mapping from $\mathrm{W}^{\ell,p}_D(O)$ to $\mathrm{W}^{\ell,p}_D(\mathbb{R}^d)$. The operator norm of $\mathcal{E}$ depends on geometry only via the implicit constants and parameters in Assumption 3.1.1.*

In addition, we will present a further improvement for the first-order case in Theorem 3.8.3. We also show *local* and *homogeneous* estimates in Theorem 3.9.2.

This chapter is based in a joint publication with Russell Brown, Robert Haller-Dintelmann, and Patrick Tolksdorf, see [19].

Outline of the chapter

First of all, we introduce our geometric setting in Section 3.1 and provide examples of admissible geometries. We also give counterexamples for $\mathrm{W}^{1,p}_D(O)$ extension domains.

Right after that, we dive into the construction of the extension operator. Sections 3.3 and 3.4 are all about cubes. In there, we will define collections of exterior and interior cubes coming from two different Whitney decompositions, and will explain how an exterior cube can be reflected "at the Neumann boundary" to obtain an associated interior cube. In contrast to Jones, not all small cubes in the Whitney decomposition of $\overline{O}$ are exterior cubes. The treatment of Whitney cubes which are "almost" exterior cubes are the central deviation from Jones' construction and are thus the heart of the matter in this construction. These two sections are highly technical.

Eventually, we come to the actual crafting of the extension operator for Theorem 3.0.2 in Section 3.5. This section also contains results on (adapted) polynomials which are needed to define the extension operator via "reflection". The proof of Theorem 3.0.2 will be completed in Section 3.7. Before that, we introduce an approximation scheme that yields more regular test functions for $\mathrm{W}^{k,p}_D(O)$ in Section 3.6. This additional regularity is crucial for Proposition 3.7.1.

Finally, we present some additional first-order theory in Section 3.8, followed by some short observations on locallity and homogeneity in Section 3.9 which build on an observation made in Remark 3.5.12.

3.1. Geometry and Function Spaces

3.1.1. Geometry

Let $\Xi \subseteq \mathbb{R}^d$ be open. For two points $x, y \in \Xi$ their *quasihyperbolic distance*, first introduced by Gehring and Palka [46], is given by

$$\mathrm{k}_\Xi(x,y) := \inf_\gamma \int_\gamma \frac{1}{\mathrm{d}(z, \partial\Xi)} \, |\,\mathrm{d}z|,$$

where the infimum is taken over all rectifiable curves γ in Ξ joining x with y. Notice that its value might be ∞. This is the case if there is no path connecting x with y in Ξ. The function k_Ξ is called the *quasihyperbolic metric*. If $\Xi' \subseteq \Xi$

define

$$\mathrm{k}_\Xi(x, \Xi') := \inf\{\mathrm{k}_\Xi(x,y) : y \in \Xi'\} \qquad (x \in \Xi).$$

To construct the Sobolev extension operator in Theorem 3.0.2, we will rely on the following geometric assumption.

Assumption 3.1.1. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$ be closed, and define $N := \partial O \setminus D$. We assume that there exist $\varepsilon \in (0,1]$, $\delta \in (0,\infty]$ and $K > 0$ such that for all points $x, y \in O$ with $|x-y| < \delta$ there exists a rectifiable curve γ that joins x and y and takes values in $\Xi := \mathbb{R}^d \setminus \overline{N}$ and satisfies*

(LC) $$\mathrm{length}(\gamma) \leq \varepsilon^{-1}|x-y|,$$

(CC) $$\mathrm{d}(z, N) \geq \varepsilon \frac{|x-z||y-z|}{|x-y|} \qquad (z \in \gamma),$$

(QHD) $$\mathrm{k}_\Xi(z, O) \leq K \qquad (z \in \gamma).$$

Furthermore, assume that there exists $\lambda > 0$ such that for each connected component O_m of O holds

(DC) $$\partial O_m \cap N \neq \emptyset \qquad \Longrightarrow \qquad \mathrm{diam}(O_m) \geq \lambda\delta.$$

Remark 3.1.2. Let $(\Xi_m)_m$ denote the connected components of Ξ. From $\mathrm{d}(z, \partial\Xi_m) = \mathrm{d}(z, \partial\Xi)$ for $z \in \Xi_m$ follows directly that $\mathrm{k}_\Xi(x,y) = \mathrm{k}_{\Xi_m}(x,y)$ holds for all $x, y \in \Xi_m$. Note that $\partial\Xi = \overline{N}$ since $\overline{N} \subseteq \partial O$ contains no interior points. Moreover, for $x \in \Xi_m$ and $y \in \Xi_n$ with $m \neq n$ one has $\mathrm{k}_\Xi(x,y) = \infty$ since there is no connecting path between those points. Finally, $\mathrm{k}_{\mathbb{R}^d}(x,y) = 0$ holds for all $x, y \in \mathbb{R}^d$ by the convention $1/\infty = 0$.

Remark 3.1.3. (i) Consider the pure Dirichlet case $D = \partial O$. Then the ε-cigars are allowed to take values in all of $\mathbb{R}^d$. In particular, we may connect points by a straight line, so that (LC) is clearly satisfied. Condition (CC) is void and also (QHD) is trivially fulfilled, see Remark 3.1.2. Moreover, the diameter condition is always fulfilled since there are no connected components that intersect $N = \emptyset$. Consequently, if $D = \partial O$, then Assumption 3.1.1 is fulfilled for any open set O.

(ii) Consider the pure Neumann case $D = \emptyset$ and fix ε, δ. The curve γ can only connect points in the same connected component of O. Thus, O is the union of at most countably many (ε, δ)-domains, whose pairwise distance is at least δ and whose diameters stay uniformly away from zero. In particular, if $\delta = \infty$, then O is connected and unbounded.

(iii) A similar condition on the diameter of connected components was introduced in [26, Sec. 2] in order to transfer Jones' construction of the Sobolev extension operator in [65] to disconnected sets. In the situation of Assumption 3.1.1 the positivity of the radius *only* ensures that the connected components of O whose boundaries have a common point with N do not become arbitrarily small. This is because our construction is global and not using a localization procedure. We will present a thorough comparison with the geometry from [26] in Section 3.2.

3.1.2. Function spaces

Write $\mathrm{W}^{k,p}(O)$ for the collection of all $\mathrm{L}^p(O)$ functions that have weak derivatives up to the integer order k and which are again in $\mathrm{L}^p(O)$. Equip $\mathrm{W}^{k,p}(O)$ with the usual norm. Recall that by Proposition 1.2.11 the space $\mathrm{W}^{1,\infty}(\mathbb{R}^d)$ coincides with the space $\mathrm{Lip}(\mathbb{R}^d)$ of Lipschitz continuous functions. A particular consequence is that (locally) Lipschitz continuous functions are weakly differentiable. We will exploit this fact in Section 3.7. Note that on domains a mild geometric assumption is needed to ensure that $\mathrm{W}^{1,\infty}(O)$ coincides with $\mathrm{Lip}(O)$, see also Remark 1.2.12. This can be observed by considering $O = \mathrm{B}(0,1) \setminus [0,1)$ as a counterexample.

Definition 3.1.4. Let $O \subseteq \mathbb{R}^d$ be open and let $D \subseteq \overline{O}$ be closed. Define the space of smooth functions on O which vanish in a neighborhood of D by

$$\mathrm{C}_D^\infty(O) \coloneqq \Big\{ f \in \mathrm{C}^\infty(O) \colon \ \mathrm{d}(\mathrm{supp}(f), D) > 0 \Big\}.$$

Using this space of test functions, we define Sobolev functions vanishing on D. Note that we exclude the endpoint case $p = \infty$ in that definition. However, in the case $k = 1$, we will work with a related space in Section 3.8.

Definition 3.1.5. Let $O \subseteq \mathbb{R}^d$ be open and let $D \subseteq \overline{O}$ be closed. For an integer k and $p \in [1, \infty)$ define the Sobolev space $\mathrm{W}_D^{k,p}(O)$ as the closure of $\mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$ in $\mathrm{W}^{k,p}(O)$.

In Section 3.6 we will see that even the space $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,p}(O)$ is dense in $\mathrm{W}_D^{k,p}(O)$ as long as we assume the geometry from Assumption 3.1.1; In fact, we will approximate by compactly supported $\mathrm{C}_D^\infty(\mathbb{R}^d)$ functions, which are therefore in particular in $\mathrm{W}^{k,p}(O)$.

3.2. Comparison with other results and examples

The most general geometric setup to construct a Sobolev extension operator for the spaces $\mathrm{W}^{1,p}_D(O)$ was considered in the work of Brewster, Mitrea, Mitrea, and Mitrea [26, Thm. 1.3, Def. 3.4] and reads as follows.

Assumption 3.2.1. *Let $O \subseteq \mathbb{R}^d$ be an open, non-empty, and proper subset of $\mathbb{R}^d$, $D \subseteq \partial O$ be closed, and let $N \coloneqq \partial O \setminus D$. Let $\varepsilon, \delta > 0$ be fixed. Assume there exist an at most countable family $\{U_i\}_i$ of open subsets of $\mathbb{R}^d$ satisfying*

(i) *$\{U_i\}_i$ is locally finite and has bounded overlap,*

(ii) *there exists $r_0 > 0$ such that for all i there exists an (ε, δ)-domain $O_i \subseteq \mathbb{R}^d$ whose connected components are all of diameter at least r_0 and $O \cap U_i = O_i \cap U_i$,*

(iii) *there exists $r \in (0, \infty]$ such that for all $x \in N$ there exists i for which $B(x, r) \subseteq U_i$.*

Here, an open set O_i is called an (ε, δ)-domain if there exist $\varepsilon, \delta > 0$ such that for all $x, y \in O_i$ there exists a rectifiable curve γ that joins x and y, takes its values in O_i, and satisfies (LC) and (CC) with respect to ∂O_i instead of N. The standard example of a fractal (ε, δ)-domain is the von Koch snowflake [94, Fig. 3.5]. The following proposition is a special case of Proposition 5.1.4, which also takes into account the geometry used in Chapter 5.

Proposition 3.2.2. *Assumption 3.2.1 implies Assumption 3.1.1.*

A common geometric setup which is used in many works dealing with mixed Dirichlet/Neumann boundary conditions requires Lipschitz charts around points on the closure of N and is presented in the following assumption.

Assumption 3.2.3. *Let $O \subseteq \mathbb{R}^d$ be a bounded open set and $D \subseteq \partial O$ be closed. Put $N \coloneqq \partial O \setminus D$. Assume that around each point $x \in \overline{N}$ there exists a neighborhood U_x of x and a bi-Lipschitz homeomorphism $\Phi_x : U_x \to (-1,1)^d$ such that $\Phi_x(x) = 0$, $\Phi_x(U_x \cap O) = (-1,1)^{d-1} \times (0,1)$, and $\Phi_x(U_x \cap \partial O) = (-1,1)^{d-1} \times \{0\}$.*

Proposition 3.2.4. *Assumption 3.2.3 implies Assumption 3.2.1.*

Proof. By [35, Lem. 2.2.20], for any $x \in \overline{N}$ the set $O_x \coloneqq U_x \cap O$ is an (ε, δ)-domain. Here, ε and δ do only depend on d and the Lipschitz constant. The compactness of $\overline{N}$ implies that there exist finitely many $x_1, \dots, x_m \in \overline{N}$ such that $\overline{N} \subseteq \bigcup_{j=1}^m U_{x_j}$. Define $U_i \coloneqq U_{x_i}$ and $O_j \coloneqq O_{x_j}$ for $j = 1, \dots, m$. Due to the finiteness of the family $\{O_j\}_{j=1}^m$, the constants ε and δ can be chosen to be uniform in j. Finally, if $r > 0$ is the Lebesgue number of the covering $\{U_{x_i}\}_{i=1}^m$, then for all $x_0 \in \overline{N}$ there exists $1 \leq i \leq m$ such that $B(x_0, r) \subseteq U_i$. Thus, all requirements in Assumption 3.2.1 are fulfilled. □

Next, we give an example of a two-dimensional domain that satisfies Assumption 3.1.1 but not Assumption 3.2.1. We further show that, within this configuration, the geometry described in Assumption 3.1.1 is in some sense optimal.

Example 3.2.5. Let $\theta \in (0, \pi)$ and let $\mathrm{S}_\theta^+ \subseteq \mathbb{R}^2$ denote the open sector symmetric around the positive x-axis with opening angle 2θ. Let $O \subseteq \mathbb{R}^2$ be any domain satisfying

$$O \cap \mathrm{S}_\theta^+ = \{(x, y) \in \mathrm{S}_\theta^+ : y < 0\},$$

and define

$$D \coloneqq \partial O \cap \Big[\mathbb{R}^2 \setminus \mathrm{S}_\theta^+\Big] \qquad \text{and} \qquad N \coloneqq \partial O \setminus D = (0, \infty) \times \{0\}.$$

Essentially, this means that inside the sector S_θ^+ the domain O looks like the lower half-space and the half-space boundary that lies inside S_θ^+ is N. In the complement of the sector S_θ^+, O could be any open set and the boundary of O in the complement of S_θ^+ is defined to be D. See FIGURE 5 for an example of such a configuration.

To verify that such a domain fulfills the geomteric setup described in Assumption 3.1.1, consider first the set

$$\Delta_\theta \coloneqq (\mathbb{R}^2 \setminus \overline{\mathrm{S}_\theta^+}) \cup \{(x, y) \in \mathbb{R}^2 : y < 0\},$$

which is an (ε, δ)-domain for some values $\varepsilon, \delta > 0$. Since $O \subseteq \Delta_\theta$ and $\overline{N} \subseteq \partial \Delta_\theta$, the (ε, δ)-paths with respect to Δ_θ for points in O satisfy (LC) and (CC). Hence, to conclude the example, we only have to show that there exists $K > 0$ such that for all $z \in \Delta_\theta$ and with $\Xi = \mathbb{R}^2 \setminus \overline{N}$ it holds

$$\mathrm{k}_\Xi(O, z) \leq K. \tag{3.1}$$

Since the paths obtained above take their values only in Δ_θ this will establish the remaining condition (QHD). Notice that since

$$\mathrm{S}_\theta^+ \cap \{(x, y) \in \mathbb{R}^2 : y < 0\} \subseteq O$$

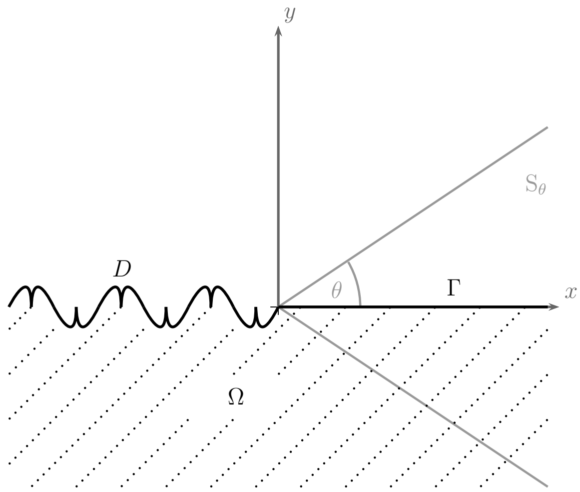


Figure 5.: A generic picture of a domain described in Example 3.2.5.

it suffices to show that there exists $K > 0$ such that for all $z \in \Delta_\theta$ it holds

$$\mathrm{k}_\Xi(\mathrm{S}_\theta^+ \cap \{(x,y) \in \mathbb{R}^2 : y < 0\}, z) \leq K.$$

We only describe one particular case in detail, the remaining cases are similar and left to the reader. Assume that $\theta < \pi/2$ and pick $z = (v,w) \in \Delta_\theta$ with $v \geq 0$ and $w > 0$. Choose $(x,y) \in \partial\mathrm{S}_\theta^+$ such that $y \coloneqq -w$ and let $\gamma \coloneqq \gamma_1 + \gamma_2 + \gamma_3$ with

$$\begin{aligned}
\gamma_1 &: [0,1] \to \mathbb{R}^2, \quad t \mapsto (x,y) + t(y-x,0),\\
\gamma_2 &: [0,1] \to \mathbb{R}^2, \quad t \mapsto (y,y) + t(0,w-y),\\
\gamma_3 &: [0,1] \to \mathbb{R}^2, \quad t \mapsto (y,w) + t(v-y,0).
\end{aligned}$$

This construction is depicted in Figure 6. The path γ then connects (x,y) to (v,w) and

$$\begin{aligned}
\mathrm{k}_\Xi((x,y),(v,w)) &\leq \int_0^1 \frac{|y-x|}{|y|}\,\mathrm{d}t + \int_0^1 \frac{|w-y|}{|y|}\,\mathrm{d}t + \int_0^1 \frac{|v-y|}{w}\,\mathrm{d}t\\
&= 4 + \frac{x+v}{w}.
\end{aligned}$$

Notice that $x = w/\tan(\theta)$ and that $v \leq w/\tan(\theta)$, so that

$$\mathrm{k}_\Xi((x,y),(v,w)) \leq 2\Big(2 + \frac{1}{\tan(\theta)}\Big).$$

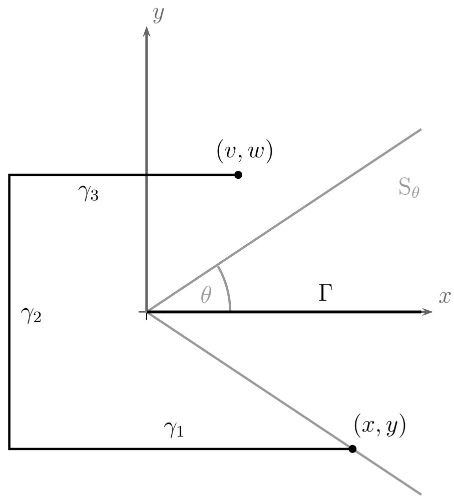


Figure 6.: A path connecting (v, w) and (x, y) that is 'short' with respect to the quasihyperbolic distance.

In the remaining cases $v < 0$ and $w \geq 0$, $v < 0$ and $w < 0$, or $v \geq 0$ and $w < 0$ the quasihyperbolic distance to O will even be smaller. This proves the validity of (3.1) and thus, since O is connected and hence (DC) is void, that O fulfills Assumption 3.1.1.

Remark 3.2.6. Notice that the geometric setup in Assumption 3.2.1 imposes boundary regularity in a neighborhood of $\overline{N}$, while in the situation described in Example 3.2.5 the portion D of ∂O can be arbitrarily irregular as long as it stays outside of S_θ^+.

We conclude this section by giving examples of domains where the boundary portion D fails to remain outside of a sector S_θ^+ and show that the $\mathrm{W}_D^{1,p}$-extension property fails for these types of domains. These examples show that interior cusps that lie directly on the interface separating D and N destroy the $\mathrm{W}_D^{1,p}$-extension property. The same happens with "interior cusps at infinity", that is to say, if D and N approach each other at infinity at a certain rate.

Example 3.2.7 (Interior boundary cusp in zero)**.** Let $\alpha \in (1, \infty)$ and consider

$$O \coloneqq \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \colon x \geq 0 \text{ and } -x^\alpha \leq y \leq 0\}.$$

Define D and N via

$$D \coloneqq \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } -x^\alpha = y\} \qquad \text{and} \qquad N \coloneqq (0, \infty) \times \{0\}.$$

To prove that the $\mathrm{W}^{1,p}_D$-extension property fails, let $1 < p < \infty$ and $0 < r < \infty$. Let f_r be a smooth function, that is supported in

$$Q_r \coloneqq \{(x,y) \in \mathbb{R}^2 : r/2 \leq x \leq 2r \text{ and } 0 \leq y \leq r\},$$

satisfies $0 \leq f_r \leq 1$, and is identically 1 on

$$R_r \coloneqq \{(x,y) \in \mathbb{R}^2 : 3r/4 \leq x \leq 3r/2 \text{ and } 0 \leq y \leq r/2\}.$$

Moreover, let f_r be such that $\|\nabla f_r\|_{\mathrm{L}^\infty} \lesssim r^{-1}$. In this case

$$\|f_r\|^p_{\mathrm{W}^{1,p}(O)} \lesssim (r^2 + r^{2-p}). \tag{3.2}$$

Next, employ the fundamental theorem of calculus and a density argument to conclude that for all $F \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ it holds

$$\int_{3r/4}^{3r/2} F(x,0)\ \mathrm{d}x - \int_{3r/4}^{3r/2} F(x,-x^\alpha)\ \mathrm{d}x = \int_{3r/4}^{3r/2}\int_{-x^\alpha}^{0} \partial_y F(x,y)\ \mathrm{d}y\ \mathrm{d}x.$$

If there exists a bounded extension operator $\mathcal{E} : \mathrm{W}^{1,p}_D(O) \to \mathrm{W}^{1,p}_D(\mathbb{R}^2)$, put $F \coloneqq \mathcal{E} f_r$ and conclude that the second integral on the left-hand side vanishes since $\mathcal{E} f_r \in \mathrm{W}^{1,p}_D(\mathbb{R}^2)$. Using further that by construction the trace of $\mathcal{E} f_r$ onto the set $(3r/4, 3r/2) \times \{0\}$ is identically 1, one concludes

$$\frac{3r}{4} \leq \int_{\frac{3r}{4}}^{\frac{3r}{2}} \int_{-x^\alpha}^{0} |\partial_y \mathcal{E} f_r(x,y)|\ \mathrm{d}y\ \mathrm{d}x \lesssim r^{(\alpha+1)/p'} \|\mathcal{E} f_r\|_{\mathrm{W}^{1,p}(\mathbb{R}^2)}.$$

Dividing by r and using that $\mathcal{E}$ is bounded delivers together with (3.2) the relation

$$1 \lesssim r^{(\alpha+1)/p'-1}(r^{2/p} + r^{2/p-1}),$$

which results for $r \to 0$ in the condition

$$\frac{\alpha+1}{p'} + \frac{2}{p} - 2 \leq 0 \qquad \Longleftrightarrow \qquad \alpha \leq 1.$$

This is a contradiction since α is assumed to be in $(1,\infty)$. Thus, there cannot be a bounded extension operator $\mathcal{E} : \mathrm{W}^{1,p}_D(O) \to \mathrm{W}^{1,p}_D(\mathbb{R}^2)$.

Example 3.2.8 (Interior boundary cusp at infinity)**.** Let $\alpha \in (0,\infty)$ and consider

$$O \coloneqq \{(x,y) \in \mathbb{R}^2 : \text{either } y > 0 \text{ or } x > 0 \text{ and } y < -x^{-\alpha}\}.$$

Define D and N via

$$D \coloneqq \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } -x^{-\alpha} = y\} \qquad \text{and} \qquad N \coloneqq \mathbb{R} \times \{0\}.$$

The proof that in this situation there does not exists a bounded extension operator $\mathcal{E}$ from $\mathrm{W}^{1,p}_D(O)$ to $\mathrm{W}^{1,p}_D(\mathbb{R}^2)$ for any $p \in (1,\infty)$ is similar to Example 3.2.7 and we omit the details.

3.3. Whitney decompositions and the quasihyperbolic distance

In this section, we introduce the Whitney decomposition of an open subset of $\mathbb{R}^d$ and show how condition (QHD) relates to properties of Whitney cubes. A cube $Q \subseteq \mathbb{R}^d$ is always closed and is said to be *dyadic* if there exists $k \in \mathbb{Z}$ such that Q coincides with a cube of the mesh determined by the lattice $2^{-k}\mathbb{Z}^d$. Two cubes are said to *touch*, if a face of one cube lies in a face of the other cube, and they are said to *intersect* if their intersection is non-empty. The sidelength of a cube is denoted by $l(Q)$. For a number $\alpha > 0$ the dilation of Q about its center by the factor α is denoted by αQ.

Let $F \subseteq \mathbb{R}^d$ be a non-empty closed set. Then, by [92, Thm. VI.1] there exists a collection of cubes $\{Q_j\}_{j\in\mathbb{N}}$ with pairwise disjoint interiors such that

(i) $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^d \setminus F$,

(ii) $\operatorname{diam}(Q_j) \leq \mathrm{d}(Q_j, F) \leq 4\operatorname{diam}(Q_j)$ for all $j \in \mathbb{N}$,

(iii) the cubes $\{Q_j\}_{j\in\mathbb{N}}$ are dyadic,

(iv) $\frac{1}{4}\operatorname{diam}(Q_j) \leq \operatorname{diam}(Q_k) \leq 4\operatorname{diam}(Q_j)$ if $Q_j \cap Q_k \neq \emptyset$,

(v) each cube has at most 12^d intersecting cubes.

The collection $\{Q_j\}_{j\in\mathbb{N}}$ is called Whitney cubes and will be referred to as $\mathcal{W}(F)$. In connection with Whitney cubes, the letters (i)-(v) refer always to the above properties. We say that a collection of cubes $Q_1, \dots, Q_m \in \mathcal{W}(F)$ is a *touching chain* if Q_j and Q_{j+1} are touching cubes and that it is an *intersecting chain* if $Q_j \cap Q_{j+1} \neq \emptyset$ for all $j = 1, \dots, m-1$. The *length* of a chain is the number m.

Let us mention that for a cube $Q \in \mathcal{W}(F)$ and $x \in Q$ we have $\operatorname{diam}(Q) \geq \frac{1}{5}\mathrm{d}(x, F)$. This follows from

$$4\operatorname{diam}(Q) \geq \mathrm{d}(Q, F) \geq \mathrm{d}(x, F) - \operatorname{diam}(Q),$$

and will be used freely in the rest of this chapter.

The following lemma translates (QHD) to the existence of intersecting chains of uniformly bounded length. Notice that if $(\Xi_m)_{m\in\mathcal{I}}$ denotes the connected components of the set $\Xi = \mathbb{R}^d \setminus \overline{N}$, Gehring and Osgood [45, Lem. 1] proved that for any two points $x, y \in \Xi_m$ there exists a quasihyperbolic geodesic $\gamma_{x,y}$ with endpoints x and y satisfying

$$\mathrm{k}_\Xi(x,y) = \int_{\gamma_{x,y}} \frac{1}{\mathrm{d}(z, \partial\Xi)} \, |\,\mathrm{d}z|.$$

Trivially, if $\Xi = \mathbb{R}^d$, then any path connecting x and y is a quasihyperbolic geodesic.

Lemma 3.3.1. *Fix $k > 0$. There exists a constant $M = M(d,k) \in \mathbb{N}$ such that for all $x, y \in \Xi$ with $\mathrm{k}_\Xi(x,y) \leq k$ there exists an intersecting chain $Q_1, \dots, Q_m \in \mathcal{W}(\overline{N})$ with $x \in Q_1$ and $y \in Q_m$ and $m \leq M$.*

Conversely, if for $x, y \in \Xi$ there exists an intersecting chain connecting x and y of length less than $M \in \mathbb{N}$, then $\mathrm{k}_\Xi(x,y) \leq M$.

Proof. Notice that $\mathrm{k}_\Xi(x,y) < \infty$ implies that x and y lie in the same connected component of Ξ. Assume first that

$$|x - y| \leq \frac{1}{10\sqrt{d}} \min\{\mathrm{d}(x, N), \mathrm{d}(y, N)\}. \tag{3.3}$$

Let $Q_x, Q_y \in \mathcal{W}(\overline{N})$ with $x \in Q_x$ and $y \in Q_y$, and let $\widetilde{Q}_x$ denote the region occupied by Q_x and all its intersecting Whitney cubes and similarly let $\widetilde{Q}_y$ denote its counterpart for Q_y. Then by (iv)

$$\mathrm{d}(x, {}^c\widetilde{Q}_x) \geq \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_x) \qquad \text{and} \qquad \mathrm{d}(y, {}^c\widetilde{Q}_y) \geq \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_y).$$

This combined with (3.3) yields

$$\mathrm{d}(x, {}^c\widetilde{Q}_x) \geq \frac{1}{4\sqrt{d}} \operatorname{diam}(Q_x) \geq \frac{1}{2}|x - y|.$$

By symmetry, the same is valid for y instead of x. Consequently, $\widetilde{Q}_x$ and $\widetilde{Q}_y$ have a common point and thus, x and y can be connected by an intersecting chain of length at most 4.

Now, let

$$|x - y| > \frac{1}{10\sqrt{d}} \min\{\mathrm{d}(x, N), \mathrm{d}(y, N)\}.$$

Assume without loss of generality that $\mathrm{d}(x, N) \leq \mathrm{d}(y, N)$. Fix a quasihyperbolic geodesic $\gamma_{x,y}$ that connects x with y (see the discussion before this proof). Then Herron and Koskela [59, Prop. 2.2] ensures the existence of points $y_0 \coloneqq x, y_1, \dots, y_\ell \in \mathbb{R}^d \setminus \overline{N}$ such that $\gamma_{x,y}$ is contained in the closure of $\bigcup_{i=0}^{\ell} B_i$, where $B_i \coloneqq B(y_i, r_i)$ with $r_i \coloneqq \mathrm{d}(y_i, N)/(10\sqrt{d})$, and such that

$$\ell \leq 20\sqrt{d}\, \mathrm{k}_\Xi(x, y). \tag{3.4}$$

Next, we estimate the number of Whitney cubes that cover each of these balls. Denote the number of Whitney cubes that cover $\overline{B}_i$ by W_i. Let $Q \in \mathcal{W}(\overline{N})$ be such that $Q \cap \overline{B}_i \neq \emptyset$. Then,

$$\operatorname{diam}(Q) \geq \frac{1}{4}\, \mathrm{d}(Q, N) \geq \frac{1}{4}[\mathrm{d}(y_i, N) - r_i - \operatorname{diam}(Q)],$$

so that by definition of r_i

$$\operatorname{diam}(Q) \geq \frac{(10\sqrt{d} - 1)\, \mathrm{d}(y_i, N)}{50\sqrt{d}}.$$

Moreover,

$$\operatorname{diam}(Q) \leq \mathrm{d}(Q, N) \leq \mathrm{d}(B_i \cap Q, N) \leq \mathrm{d}(y_i, N) + r_i = \left[1 + \frac{1}{10\sqrt{d}}\right] \mathrm{d}(y_i, N).$$

Consequently,

$$W_i \left[\frac{(10\sqrt{d} - 1)\, \mathrm{d}(y_i, N)}{50d}\right]^d \leq \sum_{\substack{Q \in \mathcal{W}(\overline{N}) \\ Q \cap \overline{B}_i \neq \emptyset}} |Q| \leq \left|B\Big(y_i, \left[1 + \frac{1}{5\sqrt{d}}\right] \mathrm{d}(y_i, N)\Big)\right|,$$

which proves that W_i is controlled by a constant depending only on d. We conclude by (3.4) and the bound on each W_i that there exists an intersecting chain connecting x and y of length bounded by a constant depending only on d and k.

For the other direction, let $Q_1, \dots, Q_m$ be an intersecting chain that connects x with y and with $m \leq M$. Thus, by definition $Q_j \cap Q_{j+1} \neq \emptyset$. Let γ be a path connecting x and y which is constructed by linearly connecting a point in $Q_{j-1} \cap Q_j$ with a point in $Q_j \cap Q_{j+1}$. Thus, employing (ii) delivers

$$\mathrm{k}_\Xi(x, y) \leq \sum_{j=1}^{m} \int_{\gamma \cap Q_j} \frac{1}{\mathrm{d}(Q_j, N)}\, |\,\mathrm{d}z| \leq \sum_{j=1}^{m} \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q_j)} = m. \qquad \square$$

3.4. Cubes and chains

In this section, we describe how to “reflect” cubes at $\overline{N}$ if O is subject to Assumption 3.1.1, and establish some natural properties of theses “reflections”. This is an adaptation of an argument of Jones presented in [65]. Throughout, assume in Sections 3.4 and 3.5 that O is an open set subject to Assumption 3.1.1 which satisfies $\overline{O} \neq \mathbb{R}^d$. (When O is dense in $\mathbb{R}^d$, Theorem 3.0.2 follows in a trivial way. The details will be presented separately in the proof of the theorem.) Recall that we assume $\operatorname{diam}(O_m) \geq \lambda\delta$, where $(O_m)_m$ are the connected components of O whose boundary hits N. This is in contrast to [65] where Jones assumes without loss of generality (by scaling) that the domain has connected components of diameter at least 1 and that δ is at most 1. This has the disadvantage that homogeneous estimates are only achievable on small scales even if $\delta = \infty$ and the domain is unbounded. We will comment on this topic later on in Remark 3.5.12.

Lemma 3.4.1. *We have $|N| = 0$.*

Proof. Fix $x_0 \in N$ and $y \in O$ with $|x_0 - y| < \frac{\delta}{2}$. Let Q be *any* cube in $\mathbb{R}^d$ centered in x_0 with $l(Q) \leq \frac{1}{2}|x_0 - y|$. We will show that $[\mathbb{R}^d \setminus N] \cap Q$ has Lebesgue measure comparable to that of Q. Let $x \in O$ with $|x - x_0| \leq \frac{1}{8} l(Q)$. Then, we have

$$|x - y| \geq \frac{15}{8} l(Q) \qquad \text{and} \qquad |x - y| \leq \frac{17}{16}|x_0 - y|. \tag{3.5}$$

Let γ be a path connecting x and y subject to Assumption 3.1.1 (note that $|x - y| < \delta$ is either void if $\delta = \infty$ or otherwise it follows from the second inequality in (3.5)). By virtue of (3.5), the intermediate value theorem implies that there exists $z \in \gamma$ with $|x - z| = \frac{1}{8} l(Q)$. This point lies in $\frac{1}{2}Q$ by construction. Moreover, (CC) together with $|y - z| \geq |x - y| - |x - z|$ implies

$$\mathrm{d}(z, N) \geq \frac{\varepsilon l(Q)}{8} \frac{|x - y| - |x - z|}{|x - y|} \geq \frac{\varepsilon l(Q)}{8}\Big(1 - \frac{l(Q)}{8|x - y|}\Big) \geq \frac{7\varepsilon}{60} l(Q).$$

Thus, $\limsup_{l(Q) \to 0} \frac{|[\mathbb{R}^d \setminus N] \cap Q|}{|Q|} > 0$, where the lim sup is taken over all cubes centered at x_0. Since $\mathbf{1}_{\mathbb{R}^d \setminus N}(x_0) = 0$ and $\mathbf{1}_{\mathbb{R}^d \setminus N} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, Lebesgue's differentiation theorem implies $|N| = 0$. □

To proceed, we define two families of cubes. The family of *interior cubes* is given by

$$\mathcal{W}_i \coloneqq \{Q \in \mathcal{W}(\overline{N}) : Q \cap O \neq \emptyset\}.$$

These interior cubes will be the reflections of *exterior cubes* $\mathcal{W}_e$. To define $\mathcal{W}_e$ we use numbers $A > 0$ and $B > 2$ whose values are to be fixed during this section and define

$$\mathcal{W}_e \coloneqq \{Q \in \mathcal{W}(\overline{O}) : \operatorname{diam}(Q) \leq A\delta \text{ and } \mathrm{d}(Q, N) < B\, \mathrm{d}(Q, D)\}.$$

Remark 3.4.2. First, the collection $\mathcal{W}_e$ is empty if and only if $D = \partial O$. Indeed, if $D = \partial O$ then the second condition in the definition $\mathcal{W}_e$ can never be fulfilled. To the contrary, if N is non-empty, then, using the relative openness of N, one can fix a ball centered in N that does not intersect D, and small cubes inside this ball will satisfy both conditions. Second, if $D \neq \partial O$, then for a cube $Q \in \mathcal{W}_e$ we have

$$\mathrm{d}(Q, O) = \min\{\mathrm{d}(Q, N), \mathrm{d}(Q, \partial O \setminus N)\} \geq B^{-1}\, \mathrm{d}(Q, N)$$

what implies that for all $Q \in \mathcal{W}_e$ it holds

$$\mathrm{d}(Q, O) \leq \mathrm{d}(Q, N) \leq B\, \mathrm{d}(Q, O). \tag{3.6}$$

Thus, the diameter of Q is comparable to its distance to N.

For the rest of this section, we assume that $N \neq \emptyset$. Before we present how to “reflect” cubes, we prove a technical lemma that, given an exterior cube $Q \in \mathcal{W}_e$, allows us to find a connected component of O whose boundary intersects N and which is not too far away from Q.

Lemma 3.4.3. *Let $Q \in \mathcal{W}_e$. Then there exists a connected component O_m of O with $N \cap \partial O_m \neq \emptyset$ and $x \in O_m$ with*

$$\mathrm{d}(x, Q) \leq 5B \operatorname{diam}(Q).$$

Proof. By (ii) and Remark 3.4.2, there exists $x' \in \overline{N}$ such that $\mathrm{d}(x', Q) \leq 4B \operatorname{diam}(Q)$. Since $x' \in \overline{N}$ there is $x'' \in N$ with $\mathrm{d}(x'', Q) \leq \frac{9}{2} B \operatorname{diam}(Q)$.

Denote the at most countable family of connected components of O whose boundary has a non-empty intersection with N by $\{O_m\}_m$ and the connected components whose boundary has an empty intersection with N by $\{\Upsilon_m\}_m$.

If there is O_m with $x'' \in \partial O_m$, then the proof is finished. If not, pick a sequence $(x_n)_n$ in O that converges to $x'' \in N \subseteq \partial O$. If all but finitely many x_n are contained in the union of the O_m, this concludes the proof as well. Otherwise, choose a subsequence (again denoted by x_n) for which there are indices m_n such that $x_n \in \Upsilon_{m_n}$. Furthermore, $x'' \in {}^c\overline{\Upsilon}_m$ for all m since $N \cap$

$\partial\Upsilon_m = \emptyset$. Now, by connecting x'' and x_n by a straight line, the intermediate value theorem implies the existence of a point $x_n' \in \partial\Upsilon_{m_n}$ with

$$|x'' - x_n'| \leq |x'' - x_n|.$$

We have $x_n' \in D$. Passing to the limit $n \to \infty$ yields $x'' \in D$ by the closedness of D and thus a contradiction. $\square$

The following lemma assigns to every cube in $\mathcal{W}_e$ a "reflected" cube in $\mathcal{W}_i$. For the rest of Sections 3.4 and 3.5 we will reserve the letter M to denote the constant M appearing in Lemma 3.3.1 applied with $k = 2K$, where K is the number from Assumption 3.1.1. Notice that M solely depends on d and K. For the rest of this section we make the following agreement.

Agreement 1. If X and Y are two quantities and if there exists a constant C depending only on d, p, K, λ, and ε such that $X \leq CY$ holds, then we will write $X \lesssim Y$ or $Y \gtrsim X$. If both $\frac{Y}{C} \leq X \leq CY$ holds, then we will write $X \approx Y$.

Lemma 3.4.4. *There exist constants $C_1 = C_1(M, \varepsilon) > 0$ and $C_2 = C_2(\lambda) > 0$ such that if $AB \leq C_1$ and $B \geq C_2$, then for every $Q \in \mathcal{W}_e$ there exists a cube $R \in \mathcal{W}_i$ satisfying*

$$\operatorname{diam}(Q) \leq \operatorname{diam}(R) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q) \tag{3.7}$$

and

$$\mathrm{d}(R, Q) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q). \tag{3.8}$$

Proof. Fix $Q \in \mathcal{W}_e$ and recall that $\operatorname{diam}(Q) \leq A\delta$ by definition of $\mathcal{W}_e$ and that $B > 2$. By Lemma 3.4.3 there exists a connected component O_m of O with $N \cap \partial O_m \neq \emptyset$ and $x \in O_m$ with $\mathrm{d}(x, Q) \leq 5B \operatorname{diam}(Q)$. We introduce the additional lower bound $B \geq 3/\lambda$, which is only needed in the case $\delta < \infty$ but we choose B always that large for good measure.

So, in the case $\delta < \infty$, since $(AB)^{-1} \operatorname{diam}(Q) \leq \delta B^{-1} < \min(\delta, \lambda\delta/2) \leq \min(\delta, \operatorname{diam}(O_m)/2)$ owing to (DC), we find $y \in O_m$ satisfying

$$|x - y| = (AB)^{-1} \operatorname{diam}(Q) \qquad \text{and} \qquad |x - y| < \delta. \tag{3.9}$$

If $\delta = \infty$, then O is unbounded by (DC) and $N \neq \emptyset$, so we again find $y \in O$ satisfying the first condition whereas the second becomes void for Assumption 3.1.1.

Hence, let γ be a path provided by Assumption 3.1.1 connecting x and y, and let $z \in \gamma$ with $|x-z| = \frac{1}{2}|x-y|$. Estimate by virtue of (CC) and (3.9)

$$\mathrm{d}(z,N) \geq \frac{\varepsilon}{2}|y-z| \geq \frac{\varepsilon}{2}\Big(|x-y| - |x-z|\Big) = \frac{\varepsilon}{4}(AB)^{-1}\operatorname{diam}(Q). \tag{3.10}$$

By Assumption 3.1.1 we have $\mathrm{k}_\Xi(z,O) \leq K$, hence there exists $z' \in O$ with $\mathrm{k}_\Xi(z,z') \leq 2K$. Thus, by Lemma 3.3.1 there exists an intersecting chain $Q_1,\dots,Q_m \in \mathcal{W}(\overline{N})$ with $Q_m \cap O \neq \emptyset$, $z \in Q_1$, and $m \leq M$. Choose the reflected cube as $R \coloneqq Q_m$. Using (ii) and (iv), one gets

$$4\operatorname{diam}(R) \geq \mathrm{d}(R,N) \geq \mathrm{d}(z,N) - \sum_{j=1}^m \operatorname{diam}(Q_j) \geq \mathrm{d}(z,N) - \sum_{j=1}^m 4^{m-j}\operatorname{diam}(R).$$

Thus, by (3.10) and $m \leq M$

$$\frac{11+4^M}{3}\operatorname{diam}(R) \geq \frac{\varepsilon}{4}(AB)^{-1}\operatorname{diam}(Q).$$

Consequently, there exists $C = C(M,\varepsilon) > 0$ such that $AB \leq C$ implies $\operatorname{diam}(Q) \leq \operatorname{diam}(R)$.

In order to control $\operatorname{diam}(R)$ by $\operatorname{diam}(Q)$, employ (ii) and (iv) and the triangle inequality to deduce

$$4^{1-m}\operatorname{diam}(R) \leq \operatorname{diam}(Q_1) \leq \mathrm{d}(z,N) \leq \mathrm{d}(z,Q) + \operatorname{diam}(Q) + \mathrm{d}(Q,N).$$

The right-hand side is estimated by the triangle inequality, followed by (3.6) and (ii), the choice $|x-z| = \frac{1}{2}|x-y|$ combined with (3.9), and $\mathrm{d}(x,Q) \leq 5B\operatorname{diam}(Q)$, yielding

$$\begin{aligned}\mathrm{d}(z,Q) + \operatorname{diam}(Q) + \mathrm{d}(Q,N) &\leq |z-x| + \mathrm{d}(x,Q) + \operatorname{diam}(Q) + B\,\mathrm{d}(Q,O)\\ &\leq ((2AB)^{-1} + 1 + 9B)\operatorname{diam}(Q).\end{aligned}$$

Taking into account that $\mathrm{d}(z,R) \leq \operatorname{diam}(R)(4^m-1)/3$ (estimate the sizes of the cubes in the connecting chain using a geometric sum), the distance from R to Q is estimated similarly, yielding

$$\begin{aligned}\mathrm{d}(R,Q) &\leq \operatorname{diam}(R) + \operatorname{diam}(Q) + |x-z| + \mathrm{d}(z,R) + \mathrm{d}(x,Q)\\ &\leq (1 + (2AB)^{-1} + 5B)\operatorname{diam}(Q) + \frac{4^m+2}{3}\operatorname{diam}(R).\end{aligned}$$

Together with the previous estimate, this concludes the proof. □

For the rest of this chapter, we fix the notation that if $Q \in \mathcal{W}_e$ and $R \in \mathcal{W}_i$ is the cube constructed in Lemma 3.4.4, then R is denoted by $R = Q^*$ and Q^* is called the *reflected* cube of Q. The next lemma gives a bound on the distance of reflected cubes of two intersecting cubes.

Lemma 3.4.5. *If $Q_1, Q_2 \in \mathcal{W}_e$ with $Q_1 \cap Q_2 \neq \emptyset$, then*

$$\mathrm{d}(Q_1^*, Q_2^*) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_1).$$

Proof. Fix $z \in Q_1 \cap Q_2$ and let $x_1 \in Q_1$, $x_2 \in Q_2$, $x_1^* \in Q_1^*$, and $x_2^* \in Q_2^*$. One gets by the triangle inequality

$$|x_1^* - x_2^*| \leq |x_1^* - x_1| + |x_1 - z| + |z - x_2| - |x_2 - x_2^*|,$$

and taking the infimum then reveals

$$\mathrm{d}(Q_1^*, Q_2^*) \leq \mathrm{d}(Q_1, Q_1^*) + \operatorname{diam}(Q_1) + \operatorname{diam}(Q_2) + \mathrm{d}(Q_2, Q_2^*).$$

The desired bound for the first term is just Lemma 3.4.4. For the third term, employ (iv). Finally, the fourth term can be handled by combining the previous two arguments. □

In the proof of the boundedness of the extension operator, one needs to connect Whitney cubes by appropriate touching chains. The following lemma presents a basic principle of how to build a chain out of a path γ and how the quantities $\operatorname{length}(\gamma)$ and $\mathrm{d}(\gamma, N)$ translate into the length of the chain and the distance of the cubes of the chain to N.

Lemma 3.4.6. *Let $R_1, R_2 \in \mathcal{W}(\overline{N})$ with $R_1 \neq R_2$ and let $x \in R_1$, $y \in R_2$, and γ be a rectifiable path in $\mathbb{R}^d \setminus \overline{N}$ connecting x and y. Assume that there exist constants $C_1, C_2 > 0$ such that $\operatorname{length}(\gamma) \leq C_1 \operatorname{diam}(R_1)$ and $\mathrm{d}(z, N) \geq C_2 \operatorname{diam}(R_1)$ for all $z \in \gamma$, then there exists a touching chain of cubes $R_1 = S_1, \ldots, S_m = R_2$ in $\mathcal{W}(\overline{N})$, where m is bounded by a number depending only on d, C_1, and C_2. Moreover,*

$$\frac{C_2}{5} \operatorname{diam}(R_1) \leq \operatorname{diam}(S_i) \leq (5 + C_1) \operatorname{diam}(R_1) \qquad (i = 1, \ldots, m).$$

Proof. Let $\mathcal{S}$ be the *finite* set of cubes in $\mathcal{W}(\overline{N})$ intersecting γ. For $S \in \mathcal{S}$ one finds by (ii) and by assumption that $\operatorname{diam}(S) \geq \frac{C_2}{5} \operatorname{diam}(R_1)$. Fix $z \in S \cap \gamma$, then

$$\mathrm{d}(z, N) \leq \mathrm{d}(x, N) + |x - z| \leq 5 \operatorname{diam}(R_1) + \operatorname{length}(\gamma) \leq (5 + C_1) \operatorname{diam}(R_1),$$

so that $\operatorname{diam}(S) \leq (5+C_1)\operatorname{diam}(R_1)$ by (ii). This, together with $\operatorname{length}(\gamma) \leq C_1 \operatorname{diam}(R_1)$ implies that $S \subseteq B(x,(5+2C_1)\operatorname{diam}(R_1))$. Because all elements of $\mathcal{S}$ have mutually disjoint interiors, one finds

$$\sharp(\mathcal{S}) \leq \frac{|B(x,(5+2C_1)\operatorname{diam}(R_1))|}{(\frac{C_2}{5\sqrt{d}}\operatorname{diam}(R_1))^d} = c_d \biggl(\frac{5\sqrt{d}(5+2C_1)}{C_2}\biggr)^d,$$

where $\sharp(\mathcal{S})$ denotes the cardinality of $\mathcal{S}$ and $c_d := |B(0,1)|$. By (iii), the elements of $\mathcal{S}$ are dyadic and thus one finds a subset of $\mathcal{S}$ which is a touching chain starting at R_1 and ending at R_2. □

Lemma 3.4.7. *There exist constants $C_1, C_2 > 0$ depending only on ε, d, λ, and K such that if $A \leq C_1$ and $B \geq C_2$ and if $Q_j, Q_k \in \mathcal{W}_e$ with $Q_j \cap Q_k \neq \emptyset$, then there exists a touching chain $F_{j,k} = \{Q_j^* = S_1, \dots, S_m = Q_k^*\}$ of cubes in $\mathcal{W}(\overline{N})$ connecting Q_j^* and Q_k^*, where m can be bounded uniformly by a constant depending only on ε, d, K, A, and B. Moreover, there exist $K_1, K_2 > 0$ depending only on ε, d, K, A, and B such that*

$$K_1 \operatorname{diam}(Q_j) \leq \operatorname{diam}(S_i) \leq K_2 \operatorname{diam}(Q_j) \qquad (i = 1, \dots, m).$$

Proof. If $Q_j^* = Q_k^*$ there is nothing to show. Thus, assume $Q_j^* \neq Q_k^*$ and let $x \in Q_j^*$ and $y \in Q_k^*$. We show in the following that the assumptions of Lemma 3.4.6 are satisfied.

Using Lemmas 3.4.4 and 3.4.5 in conjunction with (iv) gives

$$\begin{aligned} |x-y| &\leq \operatorname{d}(Q_j^*, Q_k^*) + \operatorname{diam}(Q_j^*) + \operatorname{diam}(Q_k^*) \\ &\lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j). \end{aligned} \tag{3.11}$$

If δ is finite we get from $Q_j \in \mathcal{W}_e$ that

$$|x-y| \lesssim (A + AB + B^{-1})\delta,$$

so we obtain $|x-y| < \delta$ when we first choose B large enough and afterwards A sufficiently small. Let γ be a path connecting x and y according to Assumption 3.1.1. By (LC), (3.11), and (3.7) one finds

$$\operatorname{length}(\gamma) \lesssim (1 + B + (AB)^{-1}) \operatorname{diam}(Q_j^*).$$

To estimate the distance between $z \in \gamma$ and N, notice that if $|x-z| \leq \frac{1}{2}\operatorname{diam}(Q_j^*)$, then $\operatorname{d}(z,N) \geq \frac{1}{2}\operatorname{diam}(Q_j^*)$. Analogously, but by employing additionally (3.7) twice and (iv), if $|y-z| \leq \frac{1}{2}\operatorname{diam}(Q_k^*)$, then

$$\begin{aligned} \operatorname{d}(z,N) &\geq \frac{1}{2}\operatorname{diam}(Q_k^*) \geq \frac{1}{2}\operatorname{diam}(Q_k) \geq \frac{1}{8}\operatorname{diam}(Q_j) \\ &\gtrsim \Bigl(1 + B + (AB)^{-1}\Bigr)^{-1} \operatorname{diam}(Q_j^*). \end{aligned} \tag{3.12}$$

In the remaining case, one estimates by (CC), the calculation performed in (3.12), and (3.11) that

$$\mathrm{d}(z, N) \gtrsim \frac{\mathrm{diam}(Q_j^*)^2}{(1 + B + (AB)^{-1})|x - y|} \gtrsim \frac{\mathrm{diam}(Q_j^*)}{(1 + B + (AB)^{-1})^2}. \qquad \square$$

The following lemma provides the existence of chains that "escape O" for reflections of cubes $Q \in \mathcal{W}(\overline{O})$ that are close to a relatively open portion of D. These chains will be important to obtain a Poincaré inequality with a quantitative control of the constants.

Lemma 3.4.8. *There exist constants $C_1, C_2 > 0$ depending only on ε, d, λ, and K such that if $A \leq C_1$ and $B \geq C_2$ and if $Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e$ satisfies* $\mathrm{diam}(Q) \leq A\delta$ *and has a non-empty intersection with a cube in $\mathcal{W}_e$, then for each intersecting cube $Q_j \in \mathcal{W}_e$ of Q there exists a touching chain $F_{P,j} = \{Q_j^* = S_1, \dots, S_m\}$ of cubes in $\mathcal{W}(\overline{N})$, where m is bounded by a constant depending only on ε, d, K, A, and B, and $S_m \cap Q_j$ is a dyadic cube that satisfies*

$$|S_m \cap Q_j| \gtrsim \mathrm{diam}(Q_j)^d.$$

Furthermore, all $S_i \in F_{P,j}$ satisfy

$$K_1 \,\mathrm{diam}(Q_j) \leq \mathrm{diam}(S_i) \leq K_2 \,\mathrm{diam}(Q_j) \qquad (i = 1, \dots, m).$$

The constants $K_1, K_2 > 0$ depend only on ε, d, K, A, and B.

Proof. Let $Q_j \in \mathcal{W}_e$ be an intersecting cube of Q. Then, using properties of the Whitney decomposition and $Q \notin \mathcal{W}_e$, one estimates

$$B\,\mathrm{d}(Q_j, \partial O \setminus N) \leq 6B\,\mathrm{d}(Q, \partial O \setminus N) \leq 6\,\mathrm{d}(Q, N) \leq 36\,\mathrm{d}(Q_j, N). \tag{3.13}$$

Let $B \geq 720$, then (3.13) and Remark 3.4.2 imply that $\mathrm{d}(Q_j, \partial O \setminus N) = \mathrm{d}(Q_j, O)$. Hence, using (3.13) again and (iv), one finds that $\mathrm{d}(Q_j, N) \geq \frac{B}{36}\,\mathrm{d}(Q_j, O) \geq \frac{B}{36}\,\mathrm{diam}(Q_j)$. Let $x_0 \in D$ be such that $\mathrm{d}(x_0, Q_j) = \mathrm{d}(Q_j, O) \leq 4\,\mathrm{diam}(Q_j)$. The properties collected above then imply

$$\mathrm{d}(x_0, N) \geq \mathrm{d}(Q_j, N) - \mathrm{d}(x_0, Q_j) - \mathrm{diam}(Q_j) \geq (36^{-1}B - 5)\,\mathrm{diam}(Q_j),$$

and if y is any point from $B(x_0, 5\,\mathrm{diam}(Q_j))$ then the previous estimate delivers

$$\mathrm{d}(y, N) \geq \mathrm{d}(x_0, N) - 5\,\mathrm{diam}(Q_j) \geq (36^{-1}B - 10)\,\mathrm{diam}(Q_j) \geq 10\,\mathrm{diam}(Q_j).$$

Fix $y \in B(x_0, 5\,\mathrm{diam}(Q_j)) \cap O$. Notice that the midpoint z of Q_j is contained in $B(x_0, 5\,\mathrm{diam}(Q_j))$. Thus, each point on the line segment γ_1 connecting y to z has at least a distance which is larger than $10\,\mathrm{diam}(Q_j)$ to N.

For $x \in Q_j^* \cap O$ Lemma 3.4.4 together with $\{y\} \cup Q_j \subseteq B(x_0, 6\,\mathrm{diam}(Q_j))$ implies

$$\begin{aligned} |x - y| &\leq \mathrm{d}(y, Q_j) + \mathrm{d}(Q_j, Q_j^*) + \mathrm{diam}(Q_j) + \mathrm{diam}(Q_j^*) \\ &\lesssim (1 + B + (AB)^{-1})\,\mathrm{diam}(Q_j). \end{aligned}$$

If δ is finite, we can ensure $|x - y| < \delta$ using exactly the same argument as in the proof of Lemma 3.4.7 and otherwise this condition is again meaningless. Let γ_2 be the path connecting x and y subject to Assumption 3.1.1 and let $Q \in \mathcal{W}(\overline{N})$ with $y \in Q$. Since $\mathrm{d}(Q, N) \geq C\,\mathrm{diam}(Q_j^*)$ for some $C > 0$ depending only on ε, K, d, A, and B, one concludes as in the proof of Lemma 3.4.7 that the path γ_2, and hence, by the consideration above, also the path $\gamma = \gamma_1 + \gamma_2$ which connects $x \in Q_j^*$ with $z \in Q_j$, satisfies the assumptions of Lemma 3.4.6, where Q_j^* fulfills the role of R_1 and R_2 is some cube in $\mathcal{W}(\overline{N})$ that contains z. Note that the constants appearing in Lemma 3.4.6 depend only on ε, K, d, A, and B.

As in the statement of the lemma we write S_m for R_2 and distinguish cases for the relation between S_m and Q_j. Since $Q_j \cap S_m \neq \emptyset$ and since Whitney cubes are dyadic, it either holds $S_m \subseteq Q_j$ or $Q_j \subseteq S_m$. If $Q_j \subseteq S_m$ the proof is finished. If $S_m \subseteq Q_j$, then

$$4\,\mathrm{diam}(S_m) \geq \mathrm{d}(S_m, N) \geq \mathrm{d}(Q_j, N) \geq 36^{-1} B\,\mathrm{diam}(Q_j),$$

so that $|S_m \cap Q_j| \gtrsim \mathrm{diam}(Q_j)^d$. □

The next lemma shows that for a fixed cube $R \in \mathcal{W}_i$ there are only finitely many cubes in $\mathcal{W}_e$ whose reflected cube is R.

Lemma 3.4.9. *There is a constant $C \in \mathbb{N}$ such that for each $R \in \mathcal{W}_i$ there are at most C cubes $Q \in \mathcal{W}_e$ such that $Q^* = R$, where C solely depends on d, K, A, B, and ε.*

Proof. Let α denote the implied constant from (3.8) and let $Q \in \mathcal{W}_e$ be a cube with reflected cube R, then it follows with (3.7) that $\mathrm{d}(R, Q) \leq \alpha\,\mathrm{diam}(R)$. So, if x_R denotes the center of R, every cube Q with $Q^* = R$ must be contained in $B(x_R, (\alpha + \frac{3}{2})\,\mathrm{diam}(R))$. Because for those cubes $\mathrm{diam}(Q)$ is controlled from below by $\mathrm{diam}(R)$ according to (3.7) and because cubes from $\mathcal{W}_e$ have disjoint interiors, the lemma follows by the usual counting argument. □

3.5. Construction of the extension operator and exterior estimates

This section is devoted to the construction of the extension operator from Theorem 3.0.2. We also already establish the respective estimates on $\mathbb{R}^d \setminus \overline{O}$. To do so, we start with a preparatory part on (adapted) polynomials, followed by some overlap considerations. We proceed with the construction of the actual operator, in which the adapted polynomials will appear, followed by the exterior estimates, for which we will need the results on overlap.

Agreement 2. If not otherwise mentioned, the symbols k and p are supposed to refer to these parameters in Theorem 3.0.2. The numbers A and B, which were introduced in Section 3.4, will be considered as fixed numbers depending only on ε, d, K and λ such that all statements in Section 3.4 are valid. From now on we will use the symbols $\lesssim$ and $\gtrsim$ in a more liberal way than described in Agreement 1.

Polynomial fitting and Poincaré-type estimates

We record some results on polynomial approximation and Poincaré type estimates. Most of them stem from [29] and were used therein for a similar purpose.

We start out with the following generic norm comparison lemma for polynomials of fixed degree, see [29, Lem. 2.3].

Lemma 3.5.1. *Let Q, R be cubes with $R \subseteq Q$ and assume that there exists a constant $\kappa > 0$ such that $|R| \geq \kappa |Q|$. Then for each polynomial P of degree at most m one has*

$$\|P\|_{\mathrm{L}^p(Q)} \lesssim \|P\|_{\mathrm{L}^p(R)},$$

where the implicit constant does only depend on d, κ, p and m. In particular, if S is another cube with $S \subseteq Q$ and $|S| \geq \kappa |Q|$, then the L^p norms over R and S are equivalent norms on $\mathcal{P}_m$ and the implicit constants do only depend on d, κ, p and m.

The following lemma provides “adapted” polynomials together with corresponding Poincaré-type estimates. A proof can be found in [29, Thm. 4.5, Thm. 4.7 & Rem. 4.8]; Note that the proof of the remark still works when replacing Whitney cubes by cubes of the same size.

Lemma 3.5.2. *Let Q be a cube, R a touching cube of Q of the same size, and $k \geq 0$ an integer. Then there exists a linear projection $P : \mathrm{L}^1(Q) \to \mathcal{P}_{k-1}$ that satisfies the estimate*

$$\|\partial^\alpha Pf\|_{\mathrm{L}^p(Q)} \lesssim \|\partial^\alpha f\|_{\mathrm{L}^p(Q)} \tag{3.14}$$

for $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)|_Q \cap \mathrm{W}^{k,p}(Q)$, $|\alpha| \leq k$, and $1 \leq p \leq \infty$. Moreover, the Poincaré-type estimate

$$\|\partial^\alpha(f - Pf)\|_{\mathrm{L}^p(Q\cup R)} \lesssim \operatorname{diam}(Q)^{\ell-|\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(Q\cup R)},$$

holds for $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)|_{Q\cup R} \cap \mathrm{W}^{k,p}(Q\cup R)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$. The implicit constants depend only on d, k, and p. Of course, the case $Q = R$ is also permitted.

Remark 3.5.3. (i) The polynomial Pf will be denoted by $(f)_Q$. The case $|\alpha| = k$ in (3.14) was not stated in [29] but follows since the degree of Pf is at most $k - 1$.

(ii) That the projection is always meaningfully defined on $\mathrm{L}^1(Q)$ becomes evident from (4.2) in [29].

(iii) In the case $\alpha = 0$ we can drop the intersection with $\mathrm{C}_D^\infty(\mathbb{R}^d)$ in both estimates in Lemma 3.5.2. This follows from a direct computation using the representation formula for the projection given in [29].

Combining these results gives a Poincaré-type estimate where the polynomial is only adapted to a subcube of the domain of integration.

Corollary 3.5.4. *Let Q and R be cubes with $R \subseteq Q$ such that there is $\kappa > 0$ with $|R| \geq \kappa|Q|$. Then with $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)|_Q \cap \mathrm{W}^{k,p}(Q)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$ we obtain*

$$\|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q)} \lesssim \operatorname{diam}(Q)^{\ell-|\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(Q)},$$

where the implicit constant does only depend on d, k, p, and κ. We may also replace Q by the union of two touching cubes of the same size where one of them contains R as a subcube.

Proof. Step 1. We start with the case that Q is a single cube. Using Lemma 3.5.2 and Lemma 3.5.1 we get from the fact that $\partial^\alpha((f)_Q - (f)_R)$ is a polynomial of degree at most $k - 1$ the estimate

$$\begin{aligned}\|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q)} &\leq \|\partial^\alpha(f - (f)_Q)\|_{\mathrm{L}^p(Q)} + \|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(Q)} \\ &\lesssim \operatorname{diam}(Q)^{\ell-|\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(Q)} + \|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(R)}.\end{aligned}$$

The first term is already fine so we focus on the second one. Using that $R \subseteq Q$ and Lemma 3.5.2 twice, we estimate further

$$\begin{aligned}\|\partial^\alpha((f)_Q - (f)_R)\|_{\mathrm{L}^p(R)} &\leq \|\partial^\alpha(f - (f)_Q)\|_{\mathrm{L}^p(R)} + \|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(R)} \\ &\lesssim \|\partial^\alpha(f - (f)_Q)\|_{\mathrm{L}^p(Q)} + \operatorname{diam}(R)^{\ell - |\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(R)} \\ &\lesssim \operatorname{diam}(Q)^{\ell-|\alpha|}\|\nabla^\ell f\|_{\mathrm{L}^p(Q)}.\end{aligned}$$

Step 2. Now assume that $Q = Q_1 \cup Q_2$ is the union of two touching cubes with $R \subseteq Q_1$ and $|R| \geq \kappa|Q_1|$. We reduce this case to the already shown case. Start with the triangle inequality to get

$$\|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q)} \leq \|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q_1)} + \|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q_2)}.$$

The first term is fine by Step 1 and for the second one we continue with

$$\|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q_2)} \leq \|\partial^\alpha(f - (f)_{Q_2})\|_{\mathrm{L}^p(Q_2)} + \|\partial^\alpha((f)_{Q_2} - (f)_R)\|_{\mathrm{L}^p(Q_2)}.$$

Again, the first term is good and for the other one we exploit $Q_1 \subseteq 3Q_2$ to derive with Lemma 3.5.1

$$\begin{aligned}\|\partial^\alpha((f)_{Q_2} - (f)_R)\|_{\mathrm{L}^p(Q_2)} &\lesssim \|\partial^\alpha((f)_{Q_2} - (f)_R)\|_{\mathrm{L}^p(Q_1)} \\ &\leq \|\partial^\alpha(f - (f)_{Q_2})\|_{\mathrm{L}^p(Q)} + \|\partial^\alpha(f - (f)_R)\|_{\mathrm{L}^p(Q_1)}.\end{aligned}$$

The first term is good by Lemma 3.5.2 and the second one by Step 1. □

Some overlap considerations

Let $Q_j \in \mathcal{W}_e$ and let $Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e$ be such that it intersects a cube in $\mathcal{W}_e$ and satisfies $\operatorname{diam}(Q) \leq A\delta$. Define

$$F(Q_j) \coloneqq \bigcup_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \bigcup_{S \in F_{j,k}} 2S \qquad \text{and} \qquad F_P(Q) \coloneqq \bigcup_{\substack{Q_k \in \mathcal{W}_e \\ Q \cap Q_k \neq \emptyset}} \bigcup_{S \in F_{P,k}} 2S.$$

We count how many of these "extended" chains $F(Q_j)$ and $F_P(Q)$ can intersect a fixed point $x \in \mathbb{R}^d$. We present the case of $F(Q_j)$ in full detail and only sketch the necessary changes when considering $F_P(Q)$.

By Lemma 3.4.7, we know that a chain $F_{j,k}$ has length less than a constant M which does only depend on d, K, λ, and ε. If $x \in F(Q_j)$, then there exist $k \in \mathbb{N}$ and $S \in F_{j,k}$ with $x \in 2S$. Assume $R \in \mathcal{W}(\overline{N})$ is any cube such that also $x \in 2R$. By (ii) and an elementary geometric consideration one infers for $z \in S$ that

$$4\operatorname{diam}(R) \geq \mathrm{d}(R,N) \geq \mathrm{d}(z,N) - |x-z| - \frac{3}{2}\operatorname{diam}(R).$$

Pick some z that satisfies $|x-z| \leq \operatorname{diam}(S)/2$, then

$$4\operatorname{diam}(R) \geq \mathrm{d}(S,N) - \frac{1}{2}\operatorname{diam}(S) - \frac{3}{2}\operatorname{diam}(R) \geq \frac{1}{2}\operatorname{diam}(S) - \frac{3}{2}\operatorname{diam}(R).$$

By symmetry (interchange S and R), this implies that

$$\frac{1}{11}\operatorname{diam}(S) \leq \operatorname{diam}(R) \leq 11\operatorname{diam}(S). \tag{3.15}$$

Now let $F_{\alpha,\beta}$ be another chain such that $x \in \cup_{S \in F_{\alpha,\beta}} 2S$. This means that there is a cube in $F_{\alpha,\beta}$ that fulfills the role of R above. Since Q_α^* and R as well as Q_j^* and S are connected by touching chains of Whitney cubes each of length at most M, we deduce from (3.15) that $\operatorname{diam}(Q_\alpha^*) \approx \operatorname{diam}(Q_j^*)$ and conclude $\mathrm{d}(Q_j^*, Q_\alpha^*) \lesssim \operatorname{diam}(Q_j^*)$. Then the usual counting argument yields a bound on such reflected cubes Q_α^*. Finally, Lemma 3.4.9 implies that there exists a constant $C > 0$ that depends only on d, K, λ, and ε such that

$$\sum_{Q_j \in \mathcal{W}_e} \mathbf{1}_{F(Q_j)}(x) \leq C. \tag{3.16}$$

Let us now sketch the case of $F_P(Q)$. Fix $x \in F_P(Q)$ and let $Q' \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e$ be another cube that intersects a cube in $\mathcal{W}_e$ and satisfies $\operatorname{diam}(Q') \leq A\delta$. Assume that also $x \in F_P(Q')$. This means that for some cubes $Q_k \in \mathcal{W}_e$ with $Q_k \cap Q \neq \emptyset$ and $Q_\ell \in \mathcal{W}_e$ with $Q_\ell \cap Q' \neq \emptyset$ we find cubes $S \in F_{P,k}$ and $R \in F_{P,\ell}$ such that $x \in 2S \cap 2R$. The lengths of these chains are again bounded by a number M owing to Lemma 3.4.8 instead of Lemma 3.4.7. Hence, the comparability argument from above yields again $\mathrm{d}(Q^*, (Q')^*) \lesssim \operatorname{diam}(Q^*)$. Finally, employing the same counting arguments as before reveals the bound

$$\sum_{\substack{Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e \\ Q \cap \mathcal{W}_e \neq \emptyset \\ \operatorname{diam}(Q) \leq A\delta}} \mathbf{1}_{F_P(Q)}(x) \leq C \tag{3.17}$$

for some constant $C > 0$ that depends only on d, K, λ, and ε.

Construction of the extension operator

Fix an enumeration $(Q_j)_j$ of $\mathcal{W}_e$ and take a partition of unity $(\varphi_j)_j$ on $\bigcup_{Q_j \in \mathcal{W}_e} Q_j$ valued in $[0,1]$ and satisfying $\operatorname{supp}(\varphi_j) \subseteq \frac{17}{16} Q_j$ as well as the bound $\|\partial^\alpha \varphi_j\|_{\mathrm{L}^\infty} \lesssim \operatorname{diam}(Q_j)^{-|\alpha|}$ for $|\alpha| \leq k$ and an implicit constant only depending on k and d.

Let f be a measurable function on O and $A \subseteq \mathbb{R}^d$ closed. Write $\mathcal{E}_A f$ for the zero extension of f to $A \cup O$. Clearly, $\mathcal{E}_A$ is isometric from $\mathrm{L}^p(A \cap O)$ to $\mathrm{L}^p(A)$ for all $1 \leq p \leq \infty$. Moreover, if $f \in \mathrm{C}^\infty_D(O) \cap \mathrm{W}^{k,p}(O)$ and $A \cap \overline{N} = \emptyset$, then $\mathcal{E}_A f$ is again in $\mathrm{C}^\infty_D(A) \cap \mathrm{W}^{k,p}(A)$. A relevant example is $A = Q \in \mathcal{W}_i$. Note that then $\|\partial^\alpha \mathcal{E}_A f\|_{\mathrm{L}^p(A)} = \|\partial^\alpha f\|_{\mathrm{L}^p(A \cap O)}$ holds for any $|\alpha| \leq k$.

Recall the notation introduced in Remark 3.5.3. Define the extension operator $\mathcal{E}$ on some locally integrable f by

$$\mathcal{E} f(x) := \begin{cases} f(x), & x \in O, \\ 0, & x \in D, \\ \sum_{Q_j \in \mathcal{W}_e} (\mathcal{E}_{Q_j^*} f)_{Q_j^*}(x) \varphi_j(x), & x \in {}^c\overline{O}. \end{cases}$$

If $\mathcal{W}_e$ is empty (which is the case if $D = \partial O$ according to Remark 3.4.2) then the sum is empty and its value is considered to be zero.

Remark 3.5.5. One has $\operatorname{supp}(\varphi_j) \cap \operatorname{supp}(\varphi_k) \neq \emptyset$ if and only if $Q_j \cap Q_k \neq \emptyset$. To see this, assume to the contrary that $\operatorname{supp}(\varphi_j) \cap \operatorname{supp}(\varphi_k) \neq \emptyset$, but Q_j and Q_k are disjoint. Moreover, assume without loss of generality that $\operatorname{diam}(Q_j) \geq \operatorname{diam}(Q_k)$. It follows from the dyadic structure of the cubes that $\operatorname{diam}(Q_k) \leq \frac{1}{16} \operatorname{diam}(Q_j)$. Finally, use (ii) in a short computation to arrive at a contradiction.

Remark 3.5.6. If f is locally integrable on O then $\mathcal{E} f$ is defined almost everywhere on $\mathbb{R}^d$ according to Lemma 3.4.1. Moreover, $\mathcal{E} f$ is smooth on $\mathbb{R}^d \setminus \overline{O}$ by construction. Note that the sets $\operatorname{supp}(\varphi_j)$ for $j \in J$ have bounded overlap according to Remark 3.5.5 and (v). Hence, due to Remark 3.5.3, $\mathcal{E}$ restricts to a bounded operator from $\mathrm{L}^p(O)$ to $\mathrm{L}^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$. If $f \in \mathrm{C}^\infty_D(O) \cap \mathrm{W}^{k,p}(O)$ then $\mathcal{E} f$ vanishes almost everywhere around D. Indeed, this follows from the support assumption on f and the fact that Q_j^* is close to D if $x \in Q_j$ is close to D, see (3.8).

Estimates for the extension operator

We show estimates for the extension operator on different types of cubes. The overlap considerations from before will permit us to sum them up in

Proposition 3.5.11 to arrive at exterior estimates for the extension operator.

Lemma 3.5.7. *Let $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$. If $S_1, \dots, S_m$ is a touching chain of Whitney cubes with respect to $\overline{N}$ whose length is bounded by a constant M, then*

$$\|\partial^\alpha((\mathcal{E}_{S_1} f)_{S_1} - (\mathcal{E}_{S_m} f)_{S_m})\|_{\mathrm{L}^p(S_1)} \lesssim \operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(\bigcup_{r=1}^m (2S_r) \cap O)},$$

where the implicit constant does only depend on d, k, p, and M. The assertion remains true if the chain consists of cubes in $\Xi = \mathbb{R}^d \setminus \overline{N}$ of fixed size (not necessarily Whitney cubes). In that case, the set $\bigcup_{r=1}^m (2S_r) \cap O$ in the L^p-norm on the right-hand side can be replaced by $\bigcup_{r=1}^m S_r \cap O$.

Proof. We focus first on the case of Whitney cubes.

Note first that the sizes of cubes from the chain are pairwise comparable due to the bound on the chain length. Using Lemma 3.5.1 repeatedly in the sequel (observe that the whole chain is contained in a comparably larger cube), we begin with

$$\|\partial^\alpha((\mathcal{E}_{S_1} f)_{S_1} - (\mathcal{E}_{S_m} f)_{S_m})\|_{\mathrm{L}^p(S_1)} \lesssim \sum_{r=1}^{m-1} \|\partial^\alpha((\mathcal{E}_{S_r} f)_{S_r} - (\mathcal{E}_{S_{r+1}} f)_{S_{r+1}})\|_{\mathrm{L}^p(S_r)}$$

Fix $r = 1, \dots, m-1$ and estimate one summand from the right-hand side just before by

$$\begin{aligned}
&\lesssim \|\partial^\alpha((\mathcal{E}_{S_r} f)_{S_r} - (\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r)} \\
&\qquad + \|\partial^\alpha((\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r}) - (\mathcal{E}_{S_{r+1}} f)_{S_{r+1}})\|_{\mathrm{L}^p(S_{r+1})} \\
&\lesssim \|\partial^\alpha((\mathcal{E}_{S_r} f)_{S_r} - \mathcal{E}_{S_r} f)\|_{\mathrm{L}^p(S_r)} + \|\partial^\alpha(\mathcal{E}_{S_r} f - (\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r)} \\
&\qquad + \|\partial^\alpha(\mathcal{E}_{S_{r+1}} f - (\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_{r+1})} \\
&\qquad + \|\partial^\alpha((\mathcal{E}_{S_{r+1}} f)_{S_{r+1}} - \mathcal{E}_{S_{r+1}} f)\|_{\mathrm{L}^p(S_{r+1})} \\
&\lesssim \|\partial^\alpha((\mathcal{E}_{S_r} f)_{S_r} - \mathcal{E}_{S_r} f)\|_{\mathrm{L}^p(S_r)} + \|\partial^\alpha(\mathcal{E}_{S_r \cup S_{r+1}} f - (\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r \cup S_{r+1})} \\
&\qquad + \|\partial^\alpha((\mathcal{E}_{S_{r+1}} f)_{S_{r+1}} - \mathcal{E}_{S_{r+1}} f)\|_{\mathrm{L}^p(S_{r+1})}.
\end{aligned}$$

By virtue of Lemma 3.5.2, the first and the last term on the right-hand side are controlled by $\operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_r \cap O)}$ and $\operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_{r+1} \cap O)}$.

If S_r and S_{r+1} are of the same size, the second term can be controlled using Corollary 3.5.4. Otherwise, assume without loss of generality that $\operatorname{diam}(S_{r+1}) < \operatorname{diam}(S_r)$. Since the cubes are dyadic, it follows that $S_r \cup S_{r+1} \subseteq 2S_r$. Moreover,

$$\mathrm{d}(2S_r, N) \geq \mathrm{d}(S_r, N) - \frac{1}{2} \operatorname{diam}(S_r) \geq \frac{1}{2} \operatorname{diam}(S_r).$$

Hence, $\mathcal{E}_{2S_r} f$ is a smooth extension of $\mathcal{E}_{S_r \cup S_{r+1}} f$ to $2S_r$, in particular we have $(\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r} = (\mathcal{E}_{2S_r} f)_{S_r}$. Invoking Corollary 3.5.4 yields

$$\begin{aligned} &\|\partial^\alpha(\mathcal{E}_{S_r \cup S_{r+1}} f - (\mathcal{E}_{S_r \cup S_{r+1}} f)_{S_r})\|_{\mathrm{L}^p(S_r \cup S_{r+1})} \\ &\leq \|\partial^\alpha(\mathcal{E}_{2S_r} f - (\mathcal{E}_{2S_r} f)_{S_r})\|_{\mathrm{L}^p(2S_r)} \lesssim \operatorname{diam}(S_1)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p((2S_r) \cap O)}. \end{aligned}$$

The case of cubes of fixed size in $\mathbb{R}^d \setminus \overline{N}$ is even simpler. First, we do not have to argue that the cubes have comparable size. Second, we can omit the argument for the case $\operatorname{diam}(S_{r+1}) < \operatorname{diam}(S_r)$, which furthermore allows us to estimate against $S_r \cap O$ instead of $(2S_r) \cap O$. Besides that, the proof applies verbatim to this case. ☐

Lemma 3.5.8. *Let $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$. If $Q_j \in \mathcal{W}_e$, then*

$$\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q_j)} \lesssim \operatorname{diam}(Q_j)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F(Q_j) \cap O)} + \|\partial^\alpha f\|_{\mathrm{L}^p(Q_j^* \cap O)}.$$

Proof. Observe that φ_k vanishes on Q_j if $Q_k \cap Q_j = \emptyset$. Hence, by definition it holds $\mathcal{E} f = \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} (\mathcal{E}_{Q_k^*} f)_{Q_k^*} \varphi_k$ and $\sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \varphi_k = 1$ on Q_j. Consequently, using the Leibniz rule we get

$$\begin{aligned} \|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q_j)} \leq & \Big\| \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \sum_{\beta \leq \alpha} c_{\alpha,\beta} \partial^{\alpha - \beta} \Big[(\mathcal{E}_{Q_k^*} f)_{Q_k^*} - (\mathcal{E}_{Q_j^*} f)_{Q_j^*} \Big] \partial^\beta \varphi_k \Big\|_{\mathrm{L}^p(Q_j)} \\ & + \|\partial^\alpha (\mathcal{E}_{Q_j^*} f)_{Q_j^*}\|_{\mathrm{L}^p(Q_j)} =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We employ the estimate for $\partial^\beta \varphi_k$ and Lemma 3.5.1 (taking Lemma 3.4.4 into account), followed by Lemma 3.5.7 and (v) to derive

$$\begin{aligned} \mathrm{I} &\lesssim \sum_{\substack{Q_k \in \mathcal{W}_e \\ Q_j \cap Q_k \neq \emptyset}} \sum_{\beta \leq \alpha} \operatorname{diam}(Q_k)^{-|\beta|} \|\partial^{\alpha - \beta} \Big[(\mathcal{E}_{Q_k^*} f)_{Q_k^*} - (\mathcal{E}_{Q_j^*} f)_{Q_j^*} \Big] \|_{\mathrm{L}^p(Q_j^*)} \\ &\lesssim \operatorname{diam}(Q_j)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F(Q_j) \cap O)}. \end{aligned}$$

Term II is controlled by $\|\partial^\alpha f\|_{\mathrm{L}^p(Q_j^* \cap O)}$ using (3.14) from Lemma 3.5.2; Note that we can switch to Q_j^* using Lemma 3.5.1 as in the estimate for term I. ☐

Lemma 3.5.9. *Let $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$. If $Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e$ intersects a cube in $\mathcal{W}_e$ and satisfies* $\operatorname{diam}(Q) \leq A\delta$, *then*

$$\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q)} \lesssim \operatorname{diam}(Q)^{\ell - |\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(F_P(Q) \cap O)}.$$

Proof. Note that Q satisfies the assumptions of Lemma 3.4.8. For $Q_j \in \mathcal{W}_e$ an intersecting cube of Q let $Q_j^* = S_1, \dots, S_{m_j}$ be the corresponding touching chain. Then

$$
\begin{aligned}
\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q)} &\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leq \alpha} \operatorname{diam}(Q_j^*)^{-|\beta|} \Big\| \partial^{\alpha-\beta} \Big[(\mathcal{E}_{Q_j^*} f)_{Q_j^*} \Big] \Big\|_{\mathrm{L}^p(Q_j^*)} \\
&\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leq \alpha} \operatorname{diam}(Q_j^*)^{-|\beta|} \Big\{ \Big\| \partial^{\alpha-\beta} \Big[(\mathcal{E}_{S_1} f)_{S_1} - (\mathcal{E}_{S_{m_j}} f)_{S_{m_j}} \Big] \Big\|_{\mathrm{L}^p(S_1)} \\
&\qquad + \|\partial^{\alpha-\beta} (\mathcal{E}_{S_{m_j}} f)_{S_{m_j}} \|_{\mathrm{L}^p(S_{m_j})} \Big\}.
\end{aligned}
$$

By virtue of Lemma 3.5.7 and (v) the first term in the sum is controlled by $\operatorname{diam}(Q)^{\ell-|\alpha|} \|\nabla^\ell f\|_{\mathrm{L}^p(\bigcup_{r=1}^{m_j} (2S_r) \cap O)}$. For the second term in the sum, note that $\mathcal{E}_{S_{m_j}} f = 0$ on the cube $S_{m_j} \cap Q_j$ and that $|S_{m_j} \cap Q_j| \gtrsim \operatorname{diam}(Q_j)^d$ by Lemma 3.4.8. By using Lemma 3.5.1 and the fact that $(\mathcal{E}_{S_{m_j}} f)_{S_{m_j} \cap Q_j}$ vanishes, estimate that

$$
\|\partial^{\alpha-\beta} (\mathcal{E}_{S_{m_j}} f)_{S_{m_j}} \Big\|_{\mathrm{L}^p(S_{m_j})} \lesssim \|\partial^{\alpha-\beta} \Big[(\mathcal{E}_{S_{m_j}} f)_{S_{m_j}} - (\mathcal{E}_{S_{m_j}} f)_{S_{m_j} \cap Q_j} \Big] \Big\|_{\mathrm{L}^p(S_{m_j} \cap Q_j)}.
$$

Using Lemma 3.5.2 and $\operatorname{diam}(S_{m_j}) \approx \operatorname{diam}(Q_j)$, we further estimate

$$
\begin{aligned}
&\leq \Big\| \partial^{\alpha-\beta} \Big[\mathcal{E}_{S_{m_j}} f - (\mathcal{E}_{S_{m_j}} f)_{S_{m_j}} \Big] \Big\|_{\mathrm{L}^p(S_{m_j})} \\
&\qquad + \Big\| \partial^{\alpha-\beta} \Big[\mathcal{E}_{S_{m_j} \cap Q_j} f - (\mathcal{E}_{S_{m_j}} f)_{S_{m_j} \cap Q_j} \Big] \Big\|_{\mathrm{L}^p(S_{m_j} \cap Q_j)} \\
&\lesssim \operatorname{diam}(S_{m_j})^{\ell-|\alpha|+|\beta|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap O)} \\
&\qquad + \operatorname{diam}(S_{m_j} \cap Q_j)^{\ell-|\alpha|+|\beta|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap Q_j \cap O)} \\
&\lesssim \operatorname{diam}(Q_j)^{\ell-|\alpha|+|\beta|} \|\nabla^\ell f\|_{\mathrm{L}^p(S_{m_j} \cap O)}.
\end{aligned}
$$

With (v) and $\operatorname{diam}(Q_j) \approx \operatorname{diam}(Q)$ this concludes the proof. □

Lemma 3.5.10. *Let $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$, $0 \leq \ell \leq k$, $|\alpha| \leq \ell$, and $1 \leq p \leq \infty$. If $Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e$ intersects a cube in $\mathcal{W}_e$ and satisfies* $\operatorname{diam}(Q) > A\delta$, *then*

$$
\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q)} \lesssim \max(1, \delta^{-\ell}) \|f\|_{\mathrm{W}^{\ell,p}(\bigcup_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} Q_j^* \cap O)}.
$$

Proof. Note that in fact $\operatorname{diam}(Q) \approx \delta$ because Q intersects $\mathcal{W}_e$. The same is true for its intersecting Whitney cubes. Hence, with a similar calculation as in Lemma 3.5.8 we derive

$$
\begin{aligned}
\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(Q)} &\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leq \alpha} \delta^{-|\beta|} \|\partial^{\alpha-\beta} (\mathcal{E}_{Q_j^*} f)_{Q_j^*}\|_{\mathrm{L}^p(Q_j^*)} \\
&\lesssim \sum_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} \sum_{\beta \leq \alpha} \delta^{-|\beta|} \|\partial^{\alpha-\beta} f\|_{\mathrm{L}^p(Q_j^* \cap O)} \\
&\lesssim \max(1, \delta^{-\ell}) \|f\|_{\mathrm{W}^{\ell,p}(\bigcup_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} Q_j^* \cap O)}.
\end{aligned}
$$ $\square$

Proposition 3.5.11. *For all $1 \leq p \leq \infty$ and $0 \leq \ell \leq k$ there exists a constant $C > 0$ depending only on d, ε, δ, k, p, λ, and K such that for all $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$ and $|\alpha| \leq \ell$ one has*

$$
\|\partial^\alpha \mathcal{E} f\|_{\mathrm{L}^p(\mathbb{R}^d \setminus \overline{O})} \leq C \|f\|_{\mathrm{W}^{\ell,p}(O)}. \tag{3.18}
$$

Proof. The estimates for the derivatives in the case $p < \infty$ are deduced by the following calculation based on Lemmas 3.5.8, 3.5.9, and 3.5.10

$$
\begin{aligned}
\|\partial^\alpha \mathcal{E} f\|^p_{\mathrm{L}^p(\mathbb{R}^d \setminus \overline{O})} &= \sum_{Q_j \in \mathcal{W}_e} \|\partial^\alpha \mathcal{E} f\|^p_{\mathrm{L}^p(Q_j)} + \sum_{\substack{Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e \\ Q \cap \mathcal{W}_e \neq \emptyset}} \|\partial^\alpha \mathcal{E} f\|^p_{\mathrm{L}^p(Q)} \\
&\lesssim \sum_{Q_j \in \mathcal{W}_e} \Big(\operatorname{diam}(Q_j)^{(\ell-|\alpha|)p} \|\nabla^\ell f\|^p_{\mathrm{L}^p(F(Q_j) \cap O)} + \|\partial^\alpha f\|^p_{\mathrm{L}^p(Q_j^* \cap O)} \Big) \\
&\quad + \sum_{\substack{Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e \\ Q \cap \mathcal{W}_e \neq \emptyset \\ \operatorname{diam}(Q) > A\delta}} \max(1, \delta^{-\ell p}) \, \|f\|^p_{\mathrm{W}^{\ell,p}(\bigcup_{\substack{Q_j \in \mathcal{W}_e \\ Q \cap Q_j \neq \emptyset}} Q_j^* \cap O)} \\
&\quad + \sum_{\substack{Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e \\ Q \cap \mathcal{W}_e \neq \emptyset \\ \operatorname{diam}(Q) \leq A\delta}} \operatorname{diam}(Q_j)^{(\ell-|\alpha|)p} \|\nabla^\ell f\|^p_{\mathrm{L}^p(F_P(Q) \cap O)}.
\end{aligned}
$$

Since $\ell - |\alpha| \geq 0$ and $\operatorname{diam}(Q_j)$ is comparably smaller than δ, we can get rid of the factors in front of the norm terms to the cost of an implicit constant depending on δ and k. The estimate then follows from Lemma 3.4.9 in conjunction with (v), (3.16), and (3.17).

The estimate in the case $p = \infty$ is even simpler because we can use the same estimates but can omit the overlap argument. $\square$

Remark 3.5.12. Assume that E and F are subsets of $\mathbb{R}^d$ such that the following versions of (3.16) and (3.17) hold true:

$$\sum_{\substack{Q_j \in \mathcal{W}_e \\ Q_j \cap E \neq \emptyset}} \mathbf{1}_{F(Q_j)} \leq C \mathbf{1}_F \quad \text{and} \quad \sum_{\substack{Q \in \mathcal{W}(\overline{O}) \setminus \mathcal{W}_e \\ Q \cap \mathcal{W}_e \neq \emptyset \\ \operatorname{diam}(Q) \leq A\delta \\ Q \cap E \neq \emptyset}} \mathbf{1}_{F_P(Q)} \leq C \mathbf{1}_F.$$

Then we may replace the $\mathrm{L}^p(\mathbb{R}^d \setminus \overline{O})$-norm on the left-hand side of (3.18) by an $\mathrm{L}^p(E \cap (\mathbb{R}^d \setminus \overline{O}))$-norm and the $\mathrm{W}^{\ell,p}(O)$-norm on the right-hand side by an $\mathrm{W}^{\ell,p}(F \cap O)$-norm. Moreover, if $|\alpha| = \ell$ and E is contained in the neighborhood $N_{A\delta}(O)$, then it suffices to estimate against $\|\nabla^\ell f\|_{\mathrm{L}^p(F \cap O)}$. Indeed, in this case the second term in the final estimate in the proof of Proposition 3.5.11 vanishes. We will benefit from these observations in Section 3.9.

3.6. Approximation with smooth functions

In this section, we show that smooth and compactly supported functions on $\mathbb{R}^d$ whose support stays away from D are dense in $\mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$. In particular, both classes of functions have the same $\mathrm{W}^{k,p}(O)$-closure. We will benefit from this fact in Section 3.7. To do so, we use an approximation scheme similar to that introduced in [65, Sec. 4]. The arguments rely on techniques similar to what we have used in the construction of the extension operator.

To begin with, let $f \in \mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$ and put $\kappa \coloneqq \mathrm{d}(\operatorname{supp}(f), D) > 0$. Furthermore, let $\eta > 0$ quantify the approximation error. We need parameters A, B, s, t, and ρ for which we will collect several constraints in the course of this section (similar to what we have done in Section 3.4). Some parameters depend on each other, but there is a non-cyclic order in which they can be picked. This will enable us to show the following proposition.

Proposition 3.6.1. *Let f, η, and κ be as above. Then there is a function g which is smooth on $\mathbb{R}^d$, satisfies $\mathrm{d}(\operatorname{supp}(g), D) > \kappa/2$, and $\|f - g\|_{\mathrm{W}^{k,p}(O)} \lesssim \eta$. In particular, smooth and compactly supported functions on $\mathbb{R}^d$ whose support have positive distance to D are dense in $\mathrm{C}_D^\infty(O) \cap \mathrm{W}^{k,p}(O)$ with respect to the $\mathrm{W}^{k,p}(O)$ topology.*

For brevity, put $\tilde{B}_t \coloneqq N_t(\partial O)$ for the tubular neighborhood of size t around ∂O and choose $s \in (0,1)$ in such a way that we have the estimate

$$\|f\|_{\mathrm{W}^{k,p}(\tilde{B}_{3s} \cap O)} \leq \eta. \tag{3.19}$$

We may assume that s is smaller than $\kappa/2$. Furthermore, we define a region near N that stays away from D and is adapted to the support of f, namely

$$B_t \coloneqq \Big\{x \in \mathbb{R}^d \colon \ \mathrm{d}(x, N) < t \text{ and } \mathrm{d}(x, D) > \frac{\kappa}{2}\Big\}.$$

Later on, we will only deal with $t \in (0, 3s)$, so that (3.19) will in particular be applicable on $B_t \cap O$.

Denote the zero extension of f to

$$O_0 \coloneqq O \cup \bigcup_{x \in D} \mathrm{B}(x, 3\kappa/4)$$

by $\mathcal{E}_0 f$. Note that this function is again smooth since $\mathrm{d}(\mathrm{B}(x, 3\kappa/4), \mathrm{supp}(f)) \geq \frac{\kappa}{4}$ for $x \in D$.

Lemma 3.6.2. *Let $x \in O \setminus B_s$, then $\mathrm{B}(x, t) \subseteq O_0$ for all $0 < t < s/2$.*

Proof. Recall $s < \kappa/2$. We distinguish two cases.
Case 1: $\mathrm{d}(x, D) \leq \kappa/2$. Let $z \in D$ with $|x - z| = \mathrm{d}(x, D)$. For $y \in \mathrm{B}(x, \kappa/4)$ we derive

$$|y - z| \leq |x - y| + |x - z| < \kappa/4 + \kappa/2 = 3\kappa/4,$$

so by choice of s we see

$$\mathrm{B}(x, t) \subseteq \mathrm{B}(x, \kappa/4) \subseteq \mathrm{B}(z, 3\kappa/4) \subseteq O_0.$$

Case 2: $\mathrm{d}(x, D) > \kappa/2$ and consequently $\mathrm{d}(x, N) \geq s$. Then $\mathrm{d}(x, \partial O) \geq \min(\kappa/2, s) = s > t$, therefore $\mathrm{B}(x, t) \subseteq O \subseteq O_0$ (keep in mind $x \in O$). $\square$

A family of interior cubes

Assume that ρ is a dyadic number and $\mathcal{G}$ is the collection of all dyadic cubes of sidelength ρ. Recall $\Xi = \mathbb{R}^d \setminus \overline{N}$. As before, write $(O_m)_m$ for the connected components of O whose boundary intersects N and $(\Upsilon_m)_m$ for the remaining ones. Write Σ' for the collection of cubes in $\mathcal{G}$ that are contained in Ξ. Moreover, we introduce the collection of cubes

$$\Sigma \coloneqq \Big\{R \in \mathcal{G} \colon \text{there exist } S \in \mathcal{W}(\overline{N}) \text{ and } m \colon \ \mathrm{diam}(S) \geq A\rho, \ R \subseteq S \\ \& \ R \cap O_m \neq \emptyset\Big\}.$$

These cubes take the role of $\mathcal{W}_i$ in the upcoming approximation construction. Note that $\Sigma \subseteq \Sigma'$. For $R \in \Sigma$ define enlarged cubes

$$\hat{R} \coloneqq B\,R \qquad \text{and} \qquad \hat{\hat{R}} \coloneqq 2B\,R.$$

We claim that if we choose $\rho \leq \frac{\kappa}{2\sqrt{d}}$, then $R \subseteq O_0$. Indeed, if $R \cap D = \emptyset$, then R is properly contained in O since it has a non-trivial intersection with O and avoids its boundary. Otherwise, let $z \in R \cap D$, then $R \subseteq \mathrm{B}(z, \operatorname{diam}(R)) \subseteq \mathrm{B}(z, 3\kappa/4)$.

Before we turn to our first lemma, let us mention that for a cube $Q \in \mathcal{W}(\overline{N})$ and $x \in Q$ we have $\operatorname{diam}(Q) \geq \frac{1}{5}\,\mathrm{d}(x, N)$. This follows from

$$4\operatorname{diam}(Q) \geq \mathrm{d}(Q, N) \geq \mathrm{d}(x, N) - \operatorname{diam}(Q)$$

and was already used in earlier sections of this chapter.

Lemma 3.6.3. *There are constants $C_1 = C_1(d)$ and $C_2 = C_2(A, s)$ such that*

$$\bigcup_m O_m \setminus N_s(N) \subseteq \bigcup_{R \in \Sigma} R,$$

provided $A \geq C_1$ and $\rho \leq C_2$.

Proof. Let $x \in O_m \setminus N_s(N)$. In particular, $x \in \Xi$ and hence there exists $S \in \mathcal{W}(\overline{N})$ that contains x. Since $\mathrm{d}(x, N) \geq s$ by choice of x we conclude $\operatorname{diam}(S) \geq \frac{1}{5}\,\mathrm{d}(x, N) \geq \frac{s}{5}$. Hence, if we choose $\rho \leq \frac{s}{5A}$, then $\operatorname{diam}(S) \geq A\rho$. Let R be some cube in $\mathcal{G}$ that contains x. If we demand $A \geq \sqrt{d}$, then $R \subseteq S$ because both are dyadic cubes, $\operatorname{diam}(S) \geq \sqrt{d}\rho = \operatorname{diam}(R)$, and they have a common point. Finally, $R \cap O_m \neq \emptyset$ since $x \in O_m$, so $R \in \Sigma$. $\square$

If we do not allow to keep some distance to N, then at least the enlarged cubes $\hat{R}$ cover the whole Neumann boundary region where f is non-zero.

Lemma 3.6.4. *There are constants $C_1 = C_1(A, \varepsilon)$ and $C_2 = C_2(A, \delta, \varepsilon, \kappa, \lambda)$ such that*

$$B_{2s} \cap \bigcup_m O_m \subseteq \bigcup_{R \in \Sigma} \hat{R},$$

provided $B \geq C_1$ and $\rho \leq C_2$.

Proof. Let $x \in B_{2s} \cap O_m$. Choose $\rho \leq \frac{\varepsilon}{80A} \min(\delta, \lambda\delta)$. Then $\frac{20A}{\varepsilon}\rho < \lambda\delta/2 \leq \operatorname{diam}(O_m)/2$ by Assumption 3.1.1, hence there exists some $y \in O_m$ satisfying

$|x-y| = \frac{20A}{\varepsilon}\rho$. Moreover, since $|x-y| < \delta$, there is an ε-cigar provided by Assumption 3.1.1 that connects x and y. Let $z \in \gamma$ with $|x-z| = \frac{1}{2}|x-y|$. Then

$$\mathrm{d}(z,N) \geq \frac{\varepsilon}{2}|y-z| \geq \frac{\varepsilon}{4}|x-y| = 5A\rho.$$

Since γ takes its values in Ξ, there exists a cube $S \in \mathcal{W}(\overline{N})$ with $z \in S$. We deduce $\operatorname{diam}(S) \geq \frac{1}{5}\mathrm{d}(z,N) \geq A\rho$. If we require $A \geq \sqrt{d}$, then as in the previous lemma there is some cube $R \in \mathcal{G}$ that contains z and consequently is a subcube of S. To conclude that $R \in \Sigma$ it suffices to ensure that γ does not escape O_m. To this end, let us assume that $z \notin O_m$. Since $x \in O_m$, there would be some $\tilde{z} \in \gamma$ with $\tilde{z} \in \partial O_m$. Since $\tilde{z} \notin N$ by definition of γ, we must have $\tilde{z} \in D$. Now recall that by definition of B_{2s} it holds $\mathrm{d}(x,D) > \frac{\kappa}{2}$. On imposing the constraint $\rho \leq \frac{\varepsilon^2\kappa}{40A}$ we then get the contradiction

$$\mathrm{d}(x,D) \leq |x-\tilde{z}| \leq \operatorname{length}(\gamma) \leq \frac{20A}{\varepsilon^2}\rho \leq \frac{\kappa}{2} < \mathrm{d}(x,D).$$

So, indeed, $z \in O_m$ and therefore $R \in \Sigma$. Denote the center of R by x_R and estimate

$$|x-x_R|_\infty \leq |x-z| + |x_R - z|_\infty \leq \left(\frac{10A}{\varepsilon} + \frac{1}{2}\right)\rho.$$

So, if we choose $B \geq \frac{20A}{\varepsilon} + 1$, then $x \in \hat{R}$. □

We have already mentioned that the collection Σ is a substitute for $\mathcal{W}_i$, so it is not surprising that we want to connect nearby cubes in Σ by a touching chain of cubes (which we even allow to be in Σ') of bounded length.

Lemma 3.6.5. *There are constants $C_1 = C_1(B,d,\varepsilon)$, $C_2 = C_2(d,\varepsilon)$, and $C_3 = C_3(B,d,\delta)$ such that any pair of cubes $R,S \in \Sigma$ with $\hat{\hat{R}} \cap \hat{\hat{S}} \neq \emptyset$ can be connected by a touching chain of cubes in Σ' whose length is controlled by C_1, provided that $A \geq C_2$ and $\rho \leq C_3$.*

Proof. By definition of Σ we can pick $x \in R \cap O_m$ and $y \in S \cap O_\ell$. By assumption we moreover fix $z \in \hat{\hat{R}} \cap \hat{S}$. Let x_R, y_S denote the centers of R and S, then

$$\begin{aligned} |x-y| &\leq \sqrt{d}\Big(|x-x_R|_\infty + |x_R - z|_\infty + |y_S - z|_\infty + |y - y_S|_\infty\Big) \\ &\leq \sqrt{d}(1+2B)\rho. \end{aligned} \tag{3.20}$$

If we choose $\rho \le \frac{\delta}{2\sqrt{d}(1+B)}$, then $|x-y| < \delta$ and we can connect x and y by an ε-cigar. Fix any $z \in \gamma$ and pick $Q \in \mathcal{G}$ such that $z \in Q$. By symmetry we assume without loss of generality that $|x-z| \le |y-z|$. This implies, in particular, that $|x-y| \le 2|y-z|$.

Case 1: $|x-z| \le \frac{4\sqrt{d}}{\varepsilon}\rho$. Then, since $R \in \Sigma$, we find $\tilde{Q} \in \mathcal{W}(\overline{N})$ with $R \subseteq \tilde{Q}$ and $\operatorname{diam}(\tilde{Q}) \ge A\rho$. Using $x \in R \subseteq \tilde{Q}$, it follows

$$\mathrm{d}(x,N) \ge \mathrm{d}(\tilde{Q},N) \ge \operatorname{diam}(\tilde{Q}) \ge A\rho,$$

consequently

$$\mathrm{d}(Q,N) \ge \mathrm{d}(x,N) - |x-z| - \operatorname{diam}(Q) \ge \Big(A - \frac{4\sqrt{d}}{\varepsilon} - \sqrt{d}\Big)\rho.$$

We choose $A \ge \sqrt{d}(4/\varepsilon + 2)$ to conclude $\mathrm{d}(Q,N) \ge \operatorname{diam}(Q)$, in particular $Q \in \Sigma'$.

Case 2: $|x-z| > \frac{4\sqrt{d}}{\varepsilon}\rho$. We calculate using (CC)

$$\mathrm{d}(Q,N) \ge \mathrm{d}(z,N) - \operatorname{diam}(Q) \ge \frac{\varepsilon}{2}|x-z| - \sqrt{d}\rho > \operatorname{diam}(Q).$$

So, as before, $Q \in \Sigma'$.

Taking (LC) and (3.20) into account, we get

$$\operatorname{length}(\gamma) + \operatorname{diam}(Q) \le \sqrt{d}\Big(\frac{2B+1}{\varepsilon} + 1\Big)\rho \;\; \& \;\; Q \subseteq \overline{\mathrm{B}}(x, \operatorname{length}(\gamma) + \operatorname{diam}(Q)).$$

By the usual counting argument that we have already used in Lemma 3.4.6, it follows that the number of such cubes Q can be bounded by a constant depending only on B, d, and ε. We select a touching chain out of that collection of cubes to conclude the proof. □

Remark 3.6.6. There is a constant $C = C(B,d,\varepsilon,s)$ such that for $R, S \in \Sigma$ as in the foregoing lemma with $R \cap B_{2s} \neq \emptyset$ we have that the connecting chain stays in $\tilde{B}_{3s}$ provided $\rho \le C$. Indeed, let $\tilde{C}$ be the constant C_1 from that lemma with dependence on B, d, and ε. If x is contained in some cube from the connecting chain between R and S and $y \in R \cap B_{2s}$, then $\mathrm{d}(x, \partial O) \le \mathrm{d}(y,N) + \tilde{C}\sqrt{d}\rho < 2s + \tilde{C}\sqrt{d}\rho$, so the claim follows if we choose $\rho \le s(\tilde{C}\sqrt{d})^{-1}$.

So far, we have seen that near N and away from D we can reasonably cover the components O_m. The next two lemmas show that we will not have to bother with the components Υ_m.

Lemma 3.6.7. *There is a constant $C = C(B, d, \delta, \varepsilon, \kappa)$ such that for any $R \in \Sigma$ with $\hat{\hat{R}} \cap B_{2s} \neq \emptyset$ it follows $\hat{\hat{R}} \cap \bigcup_m \Upsilon_m = \emptyset$ provided that $\rho \leq C$.*

Proof. Assume there exists $y \in \hat{\hat{R}} \cap \Upsilon_m$. Furthermore, let $x \in R \cap O_\ell$. It holds $|x-y| \leq 2B\sqrt{d}\rho$, so x and y can be connected by a path in Ξ if we ensure $\rho \leq (4\sqrt{d}B)^{-1}\delta$, and its length can be controlled by $\operatorname{length}(\gamma) \leq \varepsilon^{-1}|x-y|$ according to (LC). Since x and y are in different connected components by assumption, there must be a point $z \in \gamma$ which satisfies $z \in D$. By assumption we may pick some $\tilde{z} \in \hat{\hat{R}} \cap B_{2s}$. Then

$$\mathrm{d}(x, D) \geq \mathrm{d}(\hat{\hat{R}}, D) \geq \mathrm{d}(\tilde{z}, D) - \operatorname{diam}(\hat{\hat{R}}) \geq \kappa/2 - 2B\sqrt{d}\rho.$$

On the other hand,

$$|x - z| \leq \operatorname{length}(\gamma) \leq \frac{2B\sqrt{d}}{\varepsilon}\rho.$$

If we choose $\rho \leq \frac{\varepsilon\kappa}{16\sqrt{d}B}$ as well as $\rho \leq \frac{\kappa}{8B\sqrt{d}}$, then we arrive at the contradiction

$$\mathrm{d}(x, D) \leq |x - z| \leq \frac{\kappa}{8} < \frac{\kappa}{4} \leq \mathrm{d}(x, D). \qquad \square$$

Lemma 3.6.8. *Let $x \in B_{2s} \cap \bigcup_m \Upsilon_m$, then $x \notin \operatorname{supp}(f)$.*

Proof. Let $x \in B_{2s} \cap \Upsilon_m$, then there is $y \in N$ such that $|x - y| < 2s$. Since $y \notin \overline{\Upsilon}_m$, there is $z \in \partial\Upsilon_m \subseteq D$ on the connecting line between x and y. Thus,

$$\mathrm{d}(x, D) \leq |x - z| \leq |x - y| < 2s < \kappa = \mathrm{d}(\operatorname{supp}(f), D).$$

Consequently, $x \notin \operatorname{supp}(f)$. $\square$

Construction of the approximation and estimates

Let ψ be a cutoff function valued in $[0, 1]$ which is 1 in a neighborhood of $\overline{B_s}$, supported in $N_s(\overline{B_s})$, and satisfies $|\partial^\alpha \psi| \lesssim s^{-|\alpha|}$ for $|\alpha| \leq k$. Moreover, fix an enumeration $(R_j)_j$ of Σ and let φ_j be a partition of unity on $\bigcup_j \hat{R}_j$ with $\operatorname{supp}(\varphi_j) \subseteq \hat{\hat{R}}_j$ and $|\partial^\alpha \varphi_j| \lesssim \rho^{-|\alpha|}$. The implicit constants depend on α, d, and B. Note that according to Lemma 3.6.4 this partition of unity exists in particular on $B_{2s} \cap \bigcup_m O_m$.

Now we may construct the approximation g of f for Proposition 3.6.1. With Lemma 3.6.2 in mind, choose $t \in (0, s/2)$ small enough that

$$\|f - \mathcal{E}_0 f * \Phi_t\|_{\mathrm{W}^{k,p}(O \setminus \overline{B_s})} \leq \eta s^k, \tag{3.21}$$

where Φ_t is a mollifier function supported in $B(0,t)$. Recall the notation for adapted polynomials introduced in Remark 3.5.3 and put

$$g_1 := \sum_j (\mathcal{E}_0 f)_{R_j}\varphi_j, \qquad g_2 := \mathcal{E}_0 f * \Phi_t, \qquad \text{and} \qquad g := \psi g_1 + (1-\psi)g_2.$$

With a further constraint on ρ we see that g_1 vanishes near D.

Lemma 3.6.9. *There is a constant* $C = C(d,B,\kappa)$ *such that* $\mathrm{d}(\mathrm{supp}(g_1), D) \geq 3\kappa/4$ *provided* $\rho \leq C$.

Proof. Let $x \in \mathbb{R}^d$ with $\mathrm{d}(x,D) \leq \frac{3\kappa}{4}$. If $x \in \mathrm{supp}(\varphi_j)$ then fix any $y \in R_j$. We estimate (with z the center of R_j)

$$\mathrm{d}(y,D) \leq |y-z| + |x-z| + \mathrm{d}(x,D) \leq \frac{1}{2}\sqrt{d}\rho + B\sqrt{d}\rho + \frac{3\kappa}{4}.$$

Chose $\rho \leq \frac{\kappa}{4(1+2B)\sqrt{d}}$, then $\mathrm{d}(y,D) \leq \frac{7}{8}\kappa < \mathrm{d}(\mathrm{supp}(f),D)$, so $y \notin \mathrm{supp}(\mathcal{E}_0 f)$ and $(\mathcal{E}_0 f)_{R_j} = 0$ by linearity of the projection. But this means $g_1(x) = 0$. □

Finally, we get to the

Proof of Proposition 3.6.1. Assume that all constraints on the parameters collected in this section are fulfilled. We split the proof into several steps.

Step 1: *g is well-defined and smooth.* We have already noticed after the definition of Σ that we can ensure that all its cubes are contained in O_0, so the usage of polynomial approximations is justified and yields the smooth function g_1 on $\mathbb{R}^d$. By definition of the mollification, g_2 is a smooth function in $O \setminus \overline{B_s}$. If $x \in O$ with $\mathrm{d}(x,D) \leq \kappa/2$, then we get as in Lemma 3.6.2 that $\mathrm{B}(x,\kappa/4) \subseteq \mathrm{B}(z,3\kappa/4) \subseteq O_0$ for some $z \in D$ and $\mathcal{E}_0 f$ vanishes on this ball, so by definition of the mollification, g_2 vanishes in that neighborhood of D. Together with the knowledge on the support of $1-\psi$ we infer that $(1-\psi)g_2$ can be extend by zero to a smooth function on $\mathbb{R}^d$.

Step 2: $\mathrm{d}(\mathrm{supp}(g),D) \geq \kappa/2$. First, we have $\mathrm{d}(\mathrm{supp}(g_1),D) \geq 3\kappa/4$ by Lemma 3.6.9. On the other hand, we have already noticed in Step 1 that $\mathrm{d}(\mathrm{supp}(g_2),D) \geq \kappa/2$, which in total gives a distance of at least $\kappa/2$ to D.

Step 3: *Split up the terms for estimation.* Let α be some multi-index with $|\alpha| \leq k$. Then

$$\begin{aligned}\partial^\alpha(f-g) &= \partial^\alpha(\psi(f-g_1)) + \partial^\alpha((1-\psi)(f-g_2)) \\ &= \sum_{\beta\leq\alpha} c_{\alpha,\beta}\Big(\partial^{\alpha-\beta}\psi\partial^\beta(f-g_1) + \partial^{\alpha-\beta}(1-\psi)\partial^\beta(f-g_2)\Big) \\ &=: \sum_{\beta\leq\alpha} c_{\alpha,\beta}(\mathrm{I}_{\alpha,\beta} + \mathrm{II}_{\alpha,\beta}).\end{aligned}$$

Clearly, it suffices to estimate for fixed α and β the terms $\mathrm{I}_{\alpha,\beta}$ and $\mathrm{II}_{\alpha,\beta}$ in the $\mathrm{L}^p(O)$-norm. The estimate for $\mathrm{II}_{\alpha,\beta}$ is possible in a uniform manner whereas for $\mathrm{I}_{\alpha,\beta}$ we will have to carefully consider different relations between $|\alpha|$, $|\beta|$, and k.

Step 4: *Estimate for* $\mathrm{II}_{\alpha,\beta}$. Owing to (3.21), this term is under control on keeping $|\partial^{\alpha-\beta}(1-\psi)| \lesssim s^{-|\alpha-\beta|} \leq s^{-k}$ in mind (recall $s < 1$).

Step 5: *Reduction of the area of integration for* $\mathrm{I}_{\alpha,\beta}$. Since the support of ψ is contained in $N_s(\overline{B_s})$, we only have to consider this region. Assume $x \in N_s(\overline{B_s}) \setminus B_{2s}$. Then we must have $\mathrm{d}(x, D) \leq \kappa/2$. But in this region f and g_1 vanish according to the definition of κ and Step 2. So we only have to deal with B_{2s}. Furthermore, f vanishes on $B_{2s} \cap \bigcup_m \Upsilon_m$ according to Lemma 3.6.8 and the same is true for g_1 owing to Lemma 3.6.7. So in summary, we only need to estimate the term $\mathrm{I}_{\alpha,\beta}$ on $B_{2s} \cap \bigcup_m O_m$.

Step 6: *Estimate for* $\mathrm{I}_{\alpha,\beta}$ *if* $|\beta| < |\alpha|$. Since $\psi = 1$ on B_s and $|\alpha - \beta| \neq 0$, we even only have to estimate the L^p norm over $(B_{2s} \setminus B_s) \cap \bigcup_m O_m$. Write M for this set. The fact $(B_{2s} \setminus B_s) \cap N_s(N) = \emptyset$ allows us to use Lemma 3.6.3 to cover M by cubes from Σ to calculate

$$\|\partial^{\alpha-\beta}\psi\partial^\beta(f-g_1)\|^p_{\mathrm{L}^p(M)} \leq \sum_{\substack{R_j\in\Sigma\\ R_j\cap B_{2s}\neq\emptyset}} s^{p(|\beta|-|\alpha|)}\|\partial^\beta\Big(\mathcal{E}_0 f - \sum_{\substack{R_k\in\Sigma\\ \hat{\hat{R}}_k\cap R_j\neq\emptyset}}(\mathcal{E}_0 f)_{R_k}\varphi_k\Big)\|^p_{\mathrm{L}^p(R_j)}.$$

Using that $(\varphi_k)_k$ is a partition of unity on R_j, we derive using the Leibniz rule that on R_j we have

$$\partial^\beta \sum_{\substack{R_k\in\Sigma\\ \hat{\hat{R}}_k\cap R_j\neq\emptyset}}(\mathcal{E}_0 f)_{R_k}\varphi_k = \partial^\beta(\mathcal{E}_0 f)_{R_j} + \partial^\beta \sum_{\substack{R_k\in\Sigma\\ \hat{\hat{R}}_k\cap R_j\neq\emptyset}}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\varphi_k.$$

Using Lemma 3.5.2 we can estimate the norm of $\partial^\beta\Big[\mathcal{E}_0 f - (\mathcal{E}_0 f)_{R_j}\Big]$ against $\rho^{k-|\beta|}\|\nabla^k\mathcal{E}_0 f\|_{\mathrm{L}^p(R_j)}$. From $\rho \leq s \leq 1$ we obtain $s^{|\beta|-|\alpha|}\rho^{k-|\beta|} \leq 1$, so we infer with (3.19) that

$$\sum_{\substack{R_j\in\Sigma\\ R_j\cap B_{2s}\neq\emptyset}} s^{p(|\beta|-|\alpha|)}\|\partial^\beta\Big[\mathcal{E}_0 f - (\mathcal{E}_0 f)_{R_j}\Big]\|^p_{\mathrm{L}^p(R_j)} \lesssim \|\nabla^k f\|^p_{\mathrm{L}^p(\tilde{B}_{3s}\cap O)} \leq \eta^p.$$

For the second term, we first expand using the Leibniz rule to obtain

$$\begin{aligned}&\partial^\beta \sum_{\substack{R_k\in\Sigma\\ \hat{\hat{R}}_k\cap R_j\neq\emptyset}}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\varphi_k \\ &= \sum_{\substack{R_k\in\Sigma\\ \hat{\hat{R}}_k\cap R_j\neq\emptyset}}\sum_{\gamma\leq\beta} c_{\beta,\gamma}\partial^{\beta-\gamma}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\partial^\gamma\varphi_k.\end{aligned}$$

According to Lemma 3.6.5 we can apply Lemma 3.5.7 to the effect that

$$\|\partial^{\beta-\gamma}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\|_{\mathrm{L}^p(R_j)} \lesssim \rho^{k-|\beta|+|\gamma|}\|\nabla^k f\|_{\mathrm{L}^p(G_{j,k}\cap O)},$$

where $G_{j,k}$ denotes the union over the cubes of the connecting chain from Lemma 3.6.5 between R_j and R_k. The ρ factor compensates for $s^{|\beta|-|\alpha|}$ and $|\partial^\gamma \varphi_k|$ as before. The sums in k and j add up by similar (but simpler) overlap considerations as already seen in Section 3.5 for $F_{j,k}$. Finally, since $G_{j,k}$ stays in $\tilde{B}_{3s}$ by Remark 3.6.6, we get an estimate against η as was the case for the first term.

Step 7: *Estimate for* $\mathrm{I}_{\alpha,\beta}$ *if* $|\beta| = |\alpha|$. The estimate follows the same ideas as in Step 6, so we only mention which modifications are needed.

First of all, we have to estimate over the whole $B_{2s} \cap \bigcup_m O_m$. According to Lemma 3.6.4, this set can be covered by the enlarged cubes $\hat{R}_j$. As there are no derivatives on ψ, this term can be ignored. For the $\mathrm{L}^p(\hat{R}_j)$ norm of

$$\partial^\beta \sum_{\substack{R_k \in \Sigma \\ \hat{R}_k \cap R_j \neq \emptyset}} \Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\varphi_k$$

we use Lemma 3.5.1 to estimate

$$\|\partial^{\beta-\gamma}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\|_{\mathrm{L}^p(\hat{R}_j)} \lesssim \|\partial^{\beta-\gamma}\Big[(\mathcal{E}_0 f)_{R_k} - (\mathcal{E}_0 f)_{R_j}\Big]\|_{\mathrm{L}^p(R_j)},$$

where the implicit constant imports a dependence on B (which determines κ in that lemma). Then this term can be handled as in Step 6.

For the term $\partial^\beta\Big[f - (\mathcal{E}_0 f)_{R_j}\Big]$ we crudely apply the triangle inequality. Then we can estimate $\partial^\beta f$ directly with (3.19), and for $\partial^\beta(\mathcal{E}_0 f)_{R_j}$ we estimate with Lemma 3.5.1 and Lemma 3.5.2 that

$$\|\partial^\beta(\mathcal{E}_0 f)_{R_j}\|_{\mathrm{L}^p(\hat{R}_j)} \lesssim \|\partial^\beta(\mathcal{E}_0 f)_{R_j}\|_{\mathrm{L}^p(R_j)} \lesssim \|\nabla^k f\|_{\mathrm{L}^p(R_j\cap O)}.$$

Step 8: *Approximation by compactly supported functions.* As we have seen in the previous steps, g is an approximation to f that satisfies all properties but the compact support. But if we multiply g with a cutoff ψ_n from $\mathrm{B}(0,n)$ to $\mathrm{B}(0,2n)$ then this sequence does the job. □

3.7. Conclusion of the proof of Theorem 3.0.2

First, we show that the extension of a compactly supported function in $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,p}(O)$ constructed in Section 3.5 is weakly differentiable up to order k. More precisely, we show this for the larger class $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,\infty}(O)$, which makes this result also applicable for Section 3.8. Clearly, compactly supported functions in $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,p}(O)$ belong to this class, though the inclusion is not topological. Combined with the exterior estimates from Proposition 3.5.11 and the density result from Section 3.6, this will allow us to conclude Theorem 3.0.2.

Proposition 3.7.1. *Let $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,\infty}(O)$ and $|\alpha| \leq k-1$. Then $\partial^\alpha \mathcal{E} f$ exists and has a Lipschitz continuous representative g_α which satisfies* $\mathrm{d}(\mathrm{supp}(g_\alpha), D) > 0$.

Proof. Fix an extension $F \in \mathrm{C}_D^\infty(\mathbb{R}^d)$ of f. We show the claim by induction over $|\alpha|$. By Proposition 3.5.11, $\mathcal{E} f$ is well-defined and bounded. Now assume that $|\alpha| < k$ and $\partial^\alpha \mathcal{E} f$ is well-defined and bounded. It suffices to show that $\partial^\alpha \mathcal{E} f$ is given by a Lipschitz function. To this end, define g_α to equal $\partial^\alpha F$ on $\overline{O}$ and $\partial^\alpha \mathcal{E} f$ otherwise. We proceed in two steps.

Step 1: g_α is a representative of $\partial^\alpha \mathcal{E} f$. That g_α and $\partial^\alpha \mathcal{E} f$ coincide on $O \cup {}^c\overline{O}$ is by definition. It follows from Remark 3.5.6 that $\partial^\alpha \mathcal{E} f$ vanishes around D. The same is true for F by assumption. Consequently, Lemma 3.4.1 reveals that g_α is a representative of $\partial^\alpha \mathcal{E} f$.

Step 2: g_α is Lipschitz continuous. By assumption, g_α is Lipschitz on $\overline{O}$. Furthermore, g_α is smooth on $\mathbb{R}^d \setminus \overline{O}$ and its gradient is bounded according to Proposition 3.5.11. Hence, g_α is Lipschitz on any line segment contained in the exterior of O. The claim follows if we show that g_α is continuous on ∂O. This is already established around D, so it only remains to show continuity in $x \in N$ with $\mathrm{d}(x, D) > 0$.

Clearly, it suffices to consider $y \in \mathbb{R}^d \setminus \overline{O}$ close to x to show continuity. Moreover, using the positive distance of x to D, we may assume using Lemma 3.4.4 that $y \in Q_j$ for some cube $Q_j \in \mathcal{W}_e$ and that $Q_j^* \subseteq O$. Write y^j for the center of Q_j. Fix some cube R which contains Q_j and Q_j^* with size comparable to Q_j^*. Also, note that $\mathcal{E} f(z) = (\mathcal{E}_{Q_j^*} f)_{Q_j^*}(z)$ in a neighborhood of y^j by choice of the partition of unity used in the construction of $\mathcal{E}$, and that $\mathcal{E}_{Q_j^*} f = F$ on Q_j^* since Q_j^* is properly contained in O. Then

$$
\begin{aligned}
&|g_\alpha(x) - g_\alpha(y)| \\
&\leq |\partial^\alpha F(x) - \partial^\alpha F(y^j)| + |(\partial^\alpha F - \partial^\alpha (\mathcal{E}_{Q_j^*} f)_{Q_j^*})(y^j)| + |\partial^\alpha \mathcal{E} f(y^j) - \partial^\alpha \mathcal{E} f(y)| \\
&\leq \|\partial^\alpha F\|_{\mathrm{Lip}(\mathbb{R}^d)} |x - y^j| + \|\partial^\alpha \Big(F - (F)_{Q_j^*}\Big)\|_{\mathrm{L}^\infty(R)} + \|\partial^\alpha \mathcal{E} f\|_{\mathrm{Lip}(Q_j)} \operatorname{diam}(Q_j).
\end{aligned}
$$

Clearly, the first and the last term tend to zero when y approaches x. Finally, we estimate the second term using Corollary 3.5.4 to get decay of order

$\operatorname{diam}(R) \approx \operatorname{diam}(Q_j)$. Hence, g_α is indeed continuous in x. $\square$

We are now in the position to prove Theorem 3.0.2.

Proof of Theorem 3.0.2. Let $f \in \mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,p}(O)$ with compact support. First, we treat the trivial case $\overline{O} = \mathbb{R}^d$. In this situation, we extend f to D by zero. This is a representative according to Lemma 3.4.1, it is weakly differentiable of all orders by assumption on f, and the extension is isometric with respect to the norm of $\mathrm{W}^{k,p}(O)$. Hence, this case can be completed by continuity, compare with the conclusion of the general case below.

Otherwise, derive from Proposition 3.7.1 that $\mathcal{E}f$ has weak derivates up to order k and satisfies $\mathrm{d}(\operatorname{supp}(\mathcal{E}f), D) > 0$. From the latter follows in particular that $\partial^\alpha \mathcal{E}f$ vanishes in D. Proposition 3.5.11 yields the desired estimate on $\mathbb{R}^d \setminus \overline{O}$. Taking Lemma 3.4.1 into account, these estimates sum up to an estimate that holds almost everywhere on $\mathbb{R}^d \setminus O$, which completes the boundedness assertion.

Because we have the positive distance of the support of $\mathcal{E}f$ to D, a convolution argument shows that moreover $\mathcal{E}f \in \mathrm{W}_D^{k,p}(\mathbb{R}^d)$. Finally, we can extend $\mathcal{E}$ by density to $\mathrm{W}_D^{k,p}(O)$ using the definition of that space and the density of $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{k,p}(O)$ shown in Section 3.6. $\square$

3.8. Extending Lipschitz functions which vanish on D

Definition 3.8.1. Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \overline{O}$. Define

$$\operatorname{Lip}_D(O) \coloneqq \{f : \overline{O} \to \mathbb{C} \colon f \in \operatorname{Lip}(O) \quad \text{and} \quad f \text{ vanishes on } D\}.$$

By the canonical identification of $\operatorname{Lip}_D(O)$ as a subspace of $\operatorname{Lip}(O)$ a norm is inherited.

The following approximation lemma for functions in $\operatorname{Lip}_D(O)$ is a modified version of an argument of Stein [92, p. 188] and is used as a substitute for the result from Section 3.6.

Lemma 3.8.2. *Let $f \in \operatorname{Lip}_D(O)$. Then there exists a bounded sequence $(f_n)_n \subseteq \mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{1,\infty}(O)$ that converges to f in $\mathrm{L}^\infty(O)$ and satisfies the estimate $\|f_n\|_{\operatorname{Lip}(O)} \lesssim \|f\|_{\operatorname{Lip}(O)}$, where the implicit constant does only depend on d.*

Proof. By the Whitney extension theorem, it suffices to show the claim for whole space functions. For convenience, we drop $\mathbb{R}^d$ in the notation of function spaces for the rest of this proof.

Pick a family of functions $\varphi_n : [0,\infty) \to [0,1]$ satisfying for $y \geq x > 0$

(i) $\varphi_n = 0$ on $[0, 1/n)$, (ii) $\varphi_n = 1$ on $(2/n, \infty)$,

(iii) $|\varphi_n(x) - \varphi_n(y)| \lesssim \dfrac{1}{x}|x-y|$,

for an explicit construction see [56, Thm. 3.7]. Put $\psi_n(x) \coloneqq \varphi_n(\mathrm{d}_D(x))$. By construction, ψ_n vanishes around D and, by Lipschitz continuity of the distance function, (iii) yields for $x, y \in \mathbb{R}^d$ with $\mathrm{d}_D(x) \leq \mathrm{d}_D(y)$ the bound

$$|\psi_n(x) - \psi_n(y)| \lesssim \mathrm{d}_D(x)^{-1}|x-y|. \tag{3.22}$$

It suffices to show that there is a sequence of Lipschitz functions whose supports have positive distance to D which fulfill all claims but smoothness, since then we can conclude using mollification. Note that the mollified sequence still converges in L^∞ owing to the Lipschitz continuity.

In this light, define the sequence of functions $f_n \coloneqq \psi_n f$. Clearly, these functions are Lipschitz, and their supports stay away from D because ψ_n has this property. Next, we show that f_n converges to f in L^∞. To this end, let $x \in \mathbb{R}^d$ and pick $z \in D$ satisfying $|x - z| = \mathrm{d}_D(x)$. Since $f(z) = 0$, we get

$$|f(x) - f_n(x)| = (1 - \psi_n(x))|f(x) - f(z)| \leq \|f\|_{\mathrm{Lip}}(1 - \psi_n(x))\,\mathrm{d}_D(x).$$

By definition of ψ_n, $(1-\psi_n(x))\,\mathrm{d}_D(x) \leq 2/n$. Consequently, $|f(x)-f_n(x)| \to 0$ uniformly in x.

It remains to show that the Lipschitz seminorms of f_n can be estimated against $\|f\|_{\mathrm{Lip}}$. The argument uses the same trick (using an element from D) that we have just seen. So, let $x, y \in \mathbb{R}^d$. Assume without loss of generality that $\mathrm{d}_D(x) \leq \mathrm{d}_D(y)$ and let z realize the distance from x to D. Using (3.22), we obtain

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f(x) - f(y)|\psi_n(y) + |f(x)||\psi_n(x) - \psi_n(y)| \\ &\lesssim \|f\|_{\mathrm{Lip}}|x-y| + |f(x) - f(z)|\,\mathrm{d}_D(x)^{-1}|x-y|. \end{aligned}$$

The first term is fine and for the second we notice that

$$|f(x) - f(z)| \leq \|f\|_{\mathrm{Lip}}|x - z| = \|f\|_{\mathrm{Lip}}\,\mathrm{d}_D(x). \qquad \square$$

Theorem 3.8.3. *Let $O \subseteq \mathbb{R}^d$ be an open set and $D \subseteq \overline{O}$ be closed such that O and D are subject to Assumption 3.1.1. Then there exists an extension operator $\mathcal{E}$ which is bounded from* $\mathrm{Lip}_D(O)$ *to* $\mathrm{Lip}_D(\mathbb{R}^d)$.

Proof. Let $f \in \mathrm{Lip}_D(O)$ and let $(\varphi_n)_n$ be the approximation in $\mathrm{C}_D^\infty(\mathbb{R}^d)|_O \cap \mathrm{W}^{1,\infty}(O)$ constructed in Lemma 3.8.2. Write $\mathcal{E}$ for the extension operator constructed in Section 3.5 for the case $k = 1$. According to Proposition 3.5.11 we have L^∞ bounds for $\mathcal{E}$ on φ_n. In particular, this shows the $\mathrm{L}^\infty(\mathbb{R}^d)$ bound for $\mathcal{E}$ on f, where the latter is defined by approximation. Moreover, this permits us to calculate for almost every $x, y \in \mathbb{R}^d$ that

$$|\mathcal{E}f(x) - \mathcal{E}f(y)| = \lim_{n\to\infty} |\mathcal{E}\varphi_n(x) - \mathcal{E}\varphi_n(y)|.$$

By Proposition 3.7.1, $\mathcal{E}\varphi_n$ is Lipschitz and hence

$$\lim_{n\to\infty} |\mathcal{E}\varphi_n(x) - \mathcal{E}\varphi_n(y)| \leq \liminf_{n\to\infty} \|\nabla\mathcal{E}\varphi_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)}|x - y|.$$

Proceeding by Proposition 3.5.11 and Lemma 3.8.2, we obtain

$$\liminf_{n\to\infty} \|\nabla\mathcal{E}\varphi_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \liminf_{n\to\infty} \|\varphi_n\|_{\mathrm{W}^{1,\infty}(O)} \lesssim \|f\|_{\mathrm{Lip}(O)}.$$

So, $\mathcal{E}f$ satisfies a Lipschitz estimate against $\|f\|_{\mathrm{Lip}(O)}$ almost everywhere. Hence, $\mathcal{E}f$ possesses a representative which is Lipschitz on $\mathbb{R}^d$ and satisfies the boundedness estimate. □

3.9. Homogeneous estimates

We provide further estimates for the extension operator from Theorem 3.0.2 which concern homogeneous estimates and locality (see Definition 3.9.1 for a proper definition). These results build on the observations made in Remark 3.5.12.

Definition 3.9.1. An extension operator $\mathcal{E}$ on $\mathrm{W}_D^{k,p}(O)$ is called *local* if there exist constants $r_0, \kappa > 0$ such that

$$\|\nabla^\ell \mathcal{E}f\|_{\mathrm{L}^p(\mathrm{B}(x,r))} \lesssim \|f\|_{\mathrm{W}^{k,p}(O\cap\mathrm{B}(x,\kappa r))}$$

for all $x \in \partial O$, $r \in (0, r_0)$, and $\ell \leq k$. Moreover, call $\mathcal{E}$ *homogeneous* if one can replace the right-hand side of that estimate by $\|\nabla^\ell f\|_{\mathrm{L}^p(O\cap\mathrm{B}(x,\kappa r))}$.

To verify that $\mathcal{E}$ is local, we chose $E = \mathrm{B}(x,r)$ in Remark 3.5.12 and let $Q_j \in \mathcal{W}_e$ with $Q_j \cap \mathrm{B}(x,r) \neq \emptyset$. Using (3.7), (3.8), the bound on the chain length from Lemma 3.4.7 as well as the properties of Whitney cubes, we see that $F(Q_j)$ is contained in the ball $\mathrm{B}(x,\kappa r)$ for some κ depending only on ε, d, K, and λ (as before, an analogous version for $F_P(Q)$ holds on using Lemma 3.4.8 instead of Lemma 3.4.7 and a similar reasoning). So, with $F = \mathrm{B}(x,\kappa r)$ we derive locality from Remark 3.5.12 with $r_0 = \infty$. If we restrict to $r_0 = A\delta$, the same remark also yields that $\mathcal{E}$ is homogeneous. Note that in the case of $\delta = \infty$ this restriction is void. We summarize this result in the following theorem.

Theorem 3.9.2. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$ be closed such that O and D are subject to Assumption 3.1.1, and fix some integer $k \geq 0$. Then there exist $A, \kappa > 0$ and an extension operator $\mathcal{E}$ such that for all $1 \leq p < \infty$ one has that $\mathcal{E}$ restricts to a bounded mapping from $\mathrm{W}^{k,p}_D(O)$ to $\mathrm{W}^{k,p}_D(\mathbb{R}^d)$ which is homogeneous and local, that is, the estimate*

$$\|\nabla^\ell \mathcal{E} f\|_{\mathrm{L}^p(\mathrm{B}(x,r))} \lesssim \|\nabla^\ell f\|_{\mathrm{L}^p(\mathrm{B}(x,\kappa r)\cap O)}$$

holds for $f \in \mathrm{W}^{k,p}_D(O)$, $\ell \leq k$, $x \in \partial O$, and $r \in (0, A\delta)$. The implicit constant in that estimate depends on geometry only via the implied constants and parameters in Assumption 3.1.1.

CHAPTER 4

Extension operators for fractional Sobolev spaces with boundary conditions

Let $O \subseteq \mathbb{R}^d$ be open. For $s \in (0,1)$ and $p \in (1,\infty)$ the fractional Sobolev space $\mathrm{W}^{s,p}(O)$ is defined in Definition 4.1.6 using an intrinsic norm. Under the interior thickness condition

$$\text{(ITC)} \qquad \forall x \in O, r \in (0,1] : \quad |\mathrm{B}(x,r) \cap O| \gtrsim |\mathrm{B}(x,r)|$$

an extension operator for these spaces was constructed by Jonsson–Wallin [66], see Proposition 4.1.13. In fact, the interior thickness condition is equivalent for $\mathrm{W}^{s,p}(O)$ to admit whole space extensions, see [98]. However, in case there is a vanishing trace condition on ∂O in a suitable sense, zero extension is possible, so it is a priori clear that the thickness condition can be relaxed in the presence of zero boundary conditions.

In Section 4.2 we will construct an extension operator which is adapted to a vanishing trace condition on a portion of the boundary of O. More precisely, let $D \subseteq \partial O$ and put $N \coloneqq \partial O \setminus D$. We incorporate a vanishing trace condition on D into $\mathrm{W}^{s,p}(O)$ by intersection with the space $\mathrm{L}^p(O, \mathrm{d}_D^{-sp})$. On this space, the interior thickness condition in N (see Definition 4.1.1) turns out to be sufficient for the existence of an extension operator, and the constructed extensions also vanish in D in the sense of Definition 2.2.2 if D is $(d-1)$-regular. An example of a now admissible configuration is a self-touching with a cusp, see Example 4.1.3.

Theorem 4.0.1. *Let $O \subseteq \mathbb{R}^d$ be open and let $D \subseteq \partial O$ be closed, $p \in (1,\infty)$ and $s \in (0,1)$. If O satisfies the interior thickness condition in $\partial O \setminus D$ (see Definition 4.1.1), then there exists a bounded extension operator*

$$\mathcal{E} : \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}(\mathbb{R}^d).$$

If D is $(d-1)$-regular and $s \in (1/p, 1)$, then $\mathcal{E}$ maps even into the subspace $\mathrm{W}^{s,p}_D(\mathbb{R}^d)$ of $\mathrm{W}^{s,p}(\mathbb{R}^d)$.

Since the thickness condition does not hold in a neighborhood of N, localization techniques are not applicable. We will construct a superset $\boldsymbol{O}$ of O which is enlarged near D to satisfy (ITC) and permits for zero extension. Phrased differently, we reduce the case $D \neq \emptyset$ to the case $D = \emptyset$ by means of zero extension. The enlargement is carried out in Section 4.2.1. This type of construction will also be central in Chapter 5 later on. Afterwards, we will provide the aforementioned zero extension operator from O to $\boldsymbol{O}$ in Section 4.2.2, which is bounded due to the additional fractional Hardy term coming from the $\mathrm{L}^p(O, \mathrm{d}_D^{-sp})$-norm. With this in hand, we can conclude Theorem 4.0.1 in Section 4.2.3.

Using the pointwise restriction of $\mathrm{W}^{s,p}_D(\mathbb{R}^d)$-functions to O the space $\mathcal{W}^{s,p}_D(O)$ was defined in Definition 2.2.6 (although the notation varies between the chapters). We have already investigated their interpolation behavior in Chapter 2. The next theorem shows that we can identify the intrinsic space $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$ with $\mathcal{W}^{s,p}_D(O)$ provided a scale-invariant thickness condition for D holds.

Theorem 4.0.2. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$ be closed, $p \in (1,\infty)$ and $1/p \neq s \in (0,1)$. If O satisfies the interior thickness condition in $\partial O \setminus D$ and if D is Ahlfors–David regular (see Definition 1.3.8), then*

$$\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) = \mathcal{W}^{s,p}_D(O)$$

holds up to equivalent norms.

The inclusion "$\subseteq$" follows immediately from Theorem 4.0.1, whereas the converse inclusion is due to Proposition 2.6.7.

For the endpoint case $s = 1$ we work with the geometry from Chapter 3 described in Assumption 3.1.1, in which the space $\mathcal{W}^{1,p}_D(O)$ carries the usual local Sobolev norm. Hardy's inequality for these spaces is provided by the following theorem, whose proof is given in Section 4.3. It builds on the approach from [37]; Our primary improvement lies in allowing unbounded open sets instead of merely bounded domains without demanding a scale-invariant condition on D.

Theorem 4.0.3. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$ be closed and $(d-1)$-regular, $p \in (1, \infty)$ and assume that Assumption 3.1.1 is fulfilled. Then* Hardy's inequality *holds for $\mathrm{W}^{1,p}_D(O)$, that is, for all $f \in \mathrm{W}^{1,p}_D(O)$ holds*

$$\int_O \left| \frac{f}{\mathrm{d}_D} \right|^p \, \mathrm{d}x \lesssim \|f\|^p_{\mathrm{W}^{1,p}(O)}.$$

Finally, we transfer the interpolation property of the $\mathcal{W}^{s,p}_D(O)$-spaces to arrive at the following purely intrinsic interpolation formula.

Theorem 4.0.4. *Let $O \subseteq \mathbb{R}^d$ be open and let $D \subseteq \partial O$ be closed. Put $N \coloneqq \partial O \setminus D$. If O satisfies the interior thickness condition in N, D is Ahlfors–David regular and Assumption 3.1.1 is fulfilled, then*

$$\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) = (\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O))_{s,p},$$

where $p \in (1, \infty)$ and $s \in (0,1) \setminus \{\frac{1}{p}\}$. Moreover, the inclusion "$\supseteq$" holds also if we relax the Ahlfors–David regularity condition to $(d-1)$-regularity and in this case also $s = 1/p$ is admissible.

To conclude, we consider the necessity of the geometric assumption from Section 4.2 in Section 4.5. More precisely, we introduce a condition in Definition 4.5.1 that is strictly weaker than that from Definition 4.1.1 imposed in Theorem 4.0.1. Proposition 4.5.2 shows that this condition is necessary for extension operators on the space $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$. Example 4.5.5 is a geometry in which such an extension operator is available but which is not admissible in Theorem 4.0.1.

The results of this chapter were (partially) published in a journal article [18].

4.1. Geometry and function spaces

We are going to take a look on geometry and function spaces first, before we continue with the construction of the extension operator in Section 4.2 afterwards.

4.1.1. Geometry

We start with the definition of the interior thickness condition in a part of the boundary and relate it to (ITC). We will give an example of an admissible geometry that is not covered by the previous theory afterwards.

Definition 4.1.1. Let $E \subseteq \mathbb{R}^d$ and $F \subseteq \partial E$. Then E satisfies the *interior thickness condition in F* if

$$\forall x \in F, r \in (0,1] : \quad |\mathrm{B}(x,r) \cap E| \gtrsim |\mathrm{B}(x,r)|.$$

Lemma 4.1.2. *Let $E \subseteq \mathbb{R}^d$. Then E satisfies* (ITC) *if and only if E satisfies the interior thickness condition in ∂E.*

Proof. Assume (ITC) and let $x \in \partial E$, $r \in (0,1]$. Then pick some $y \in \mathrm{B}(x, r/2) \cap E$ and calculate

$$|\mathrm{B}(x,r) \cap E| \geq |\mathrm{B}(y,r/2) \cap E| \gtrsim |\mathrm{B}(y,r/2)| \approx |\mathrm{B}(x,r)|.$$

Conversely, let $x \in E$, $r \in (0,1]$ and E is interior thick in ∂E. If $\mathrm{B}(x,r/2) \subseteq E$ then the claim follows immediately. Otherwise, pick again some $y \in \mathrm{B}(x,r/2) \cap \partial E$ and argue as above. $\square$

We stress that Definition 4.1.1 provides a way to formulate a sharp condition at the interface between Dirichlet and Neumann boundary part. The following simple example shows that a set can satisfy the thickness condition in some closed subset of the boundary but fails to have it in any neighborhood. We will later see the more elaborate Example 4.4.1 which additionally satisfies Assumption 3.1.1. However, the example here is much simpler, so we include it for good measure.

Example 4.1.3. Let O be the right half-plane touched by a cusp from the left, for example this could mean

$$O = \{(x,y) \in \mathbb{R}^2 \colon |y| < x^2, x < 0\} \cup \{(x,y) \in \mathbb{R}^2 \colon x > 0\}.$$

Put D to be the boundary of the cusp and N is the y-axis except the origin. Then the (ITC) estimate holds in $\overline{N}$ since each ball hits the half-plane with half its area, but any proper neighborhood around N would contain a region around the tip of the cusp, in which thickness does not hold (consider a sequence that approximates the tip of the cusp and test with balls that do not reach N).

4.1.2. Function spaces

We start out with the classical (intrinsic) definitions of Lebesgue and (fractional) Sobolev spaces up to order 1.

Definition 4.1.4. Let μ be a measure, $E \subseteq \mathbb{R}^d$ be measurable and $p \in (1, \infty)$. Then $\mathrm{L}^p(E, \mu)$ is the space of p-integrable complex-valued functions on E with respect to μ. Write $\mathrm{L}^p(E)$ if μ is the Lebesgue measure and $\mathrm{L}^p(E, w)$ if μ is the Lebesgue measure weighted by some positive function w.

Definition 4.1.5. Let $O \subseteq \mathbb{R}^d$ be open and $p \in (1, \infty)$. The Sobolev space $\mathrm{W}^{1,p}(O)$ consists of those $f \in \mathrm{L}^p(O)$ for which their distributional gradient lies again in $\mathrm{L}^p(O)$, normed by

$$\|f\|_{\mathrm{W}^{1,p}(O)} \coloneqq \Big(\|f\|^p_{\mathrm{L}^p(O)} + \|\nabla f\|^p_{\mathrm{L}^p(O)} \Big)^{\frac{1}{p}}.$$

Definition 4.1.6. Let $O \subseteq \mathbb{R}^d$ be open, $p \in (1, \infty)$ and $s \in (0, 1)$. Then $\mathrm{W}^{s,p}(O)$ denotes the *fractional Sobolev space of order s*, which consists of those $f \in \mathrm{L}^p(O)$ for which

$$\|f\|_{\mathrm{W}^{s,p}(O)} \coloneqq \left(\|f\|^p_{\mathrm{L}^p(O)} + \iint_{\substack{x,y \in O \\ |x-y|<1}} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Remark 4.1.7. Dropping the restriction $|x - y| < 1$ leads to an equivalent norm, compare with the calculation in Remark 1.2.7.

We also define spaces with vanishing trace condition in this “intrinsic” context.

Definition 4.1.8. Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \overline{O}$. The set $\mathrm{C}^\infty_D(\mathbb{R}^d)$ consists of those smooth and compactly supported functions on $\mathbb{R}^d$ whose support has strictly positive distance to D. Then

$$\mathrm{C}^\infty_D(O) \coloneqq \Big\{ f|_O \colon f \in \mathrm{C}^\infty_D(\mathbb{R}^d) \Big\}$$

and $\mathrm{W}^{1,p}_D(O)$ denotes the closure of $\mathrm{C}^\infty_D(O)$ in $\mathrm{W}^{1,p}(O)$, where $p \in (1, \infty)$.

Remark 4.1.9. Observe that $\mathrm{W}^{1,p}_\emptyset(\mathbb{R}^d) = \mathrm{W}^{1,p}(\mathbb{R}^d)$ by the Meyers–Serrin Theorem.

Besides these intrinsic definitions we can also define spaces (with and without boundary conditions) by means of whole-space restrictions. The following proposition on traces is taken from [66, Thm. VI.1 & VII.1].

Proposition 4.1.10. *Let $D \subseteq \mathbb{R}^d$ be $(d-1)$-regular, $p \in (1,\infty)$, $s \in (\frac{1}{p}, 1]$, and $f \in \mathrm{W}^{s,p}(\mathbb{R}^d)$. For $\mathcal{H}^{d-1}$-almost every $x \in D$ the limit*

$$(\mathcal{R}_D f)(x) \coloneqq \lim_{r \to 0} \frac{1}{|\mathrm{B}(x,r)|} \int_{\mathrm{B}(x,r)} f(y)\,\mathrm{d}y$$

exists, and the restriction operator $\mathcal{R}_D$ maps $\mathrm{W}^{s,p}(\mathbb{R}^d)$ boundedly into the trace space $\mathrm{L}^p(D, \mathcal{H}^{d-1})$.

Definition 4.1.11. Let $D \subseteq \mathbb{R}^d$ be $(d-1)$-regular, $p \in (1,\infty)$ and $s \in (\frac{1}{p}, 1]$. Then $\mathcal{W}^{s,p}_D(\mathbb{R}^d)$ denotes the null space of $\mathcal{R}_D$. Moreover, if $O \subseteq \mathbb{R}^d$ is open, put

$$\mathcal{W}^{s,p}_D(O) \coloneqq \left\{ f|_O \colon f \in \mathcal{W}^{s,p}_D(\mathbb{R}^d) \right\}$$

and equip it with the quotient norm.

Remark 4.1.12. In the situation of Definition 4.1.11, the spaces $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$ and $\mathcal{W}^{1,p}_D(\mathbb{R}^d)$ coincide, see Lemma 2.2.5.

The following proposition is the full-dimensional case in [66, V.1.3]. Note that the consistency becomes apparent from the formula for the extension operator on p. 109 in [66].

Proposition 4.1.13. *Let $\Xi \subseteq \mathbb{R}^d$ be a set that satisfies* (ITC)*, $p \in (1,\infty)$ and $s \in (0,1)$. Then $\mathrm{W}^{s,p}(\Xi)$ admits a bounded extension operator $\mathbf{E}$ to $\mathrm{W}^{s,p}(\mathbb{R}^d)$ which is consistent in s and p.*

4.2. The extension operator

The purpose of this section is to prove Theorem 4.0.1. This follows the plan outlined in the introduction to this chapter. Throughout, O and D are as in Theorem 4.0.1 and we put $N \coloneqq \partial O \setminus D$ for convenience.

4.2.1. Embedding of O into an interior thick set

We construct an open set $\boldsymbol{O} \subseteq \mathbb{R}^d$ with $O \subseteq \boldsymbol{O}$, $\partial O \subseteq \partial \boldsymbol{O}$, and that satisfies (ITC). According to the assumption on N and Lemma 4.1.2 it suffices to check that $\boldsymbol{O}$ is interior thick in D and the "added" boundary. Of course we could take $\boldsymbol{O}$ as $\mathbb{R}^d \setminus \partial O$ in this step but this would make zero extension impossible in Section 4.2.2. Therefore, our construction will be in such a way that $|x-y| \gtrsim \mathrm{d}_D(x)$ whenever $x \in O$ and $y \in \boldsymbol{O} \setminus O$, which will do the trick in the second step.

Let $\{Q_j\}_j$ be a Whitney decomposition for the complement of $\overline{N}$, which means that the Q_j are disjoint dyadic open cubes such that

$$\text{(i)} \quad \bigcup_j \overline{Q_j} = \mathbb{R}^d \setminus \overline{N} \qquad \text{(ii)} \quad \operatorname{diam}(Q_j) \leq \mathrm{d}(Q_j, N) \leq 4 \operatorname{diam}(Q_j).$$

Using the Whitney decomposition we define

$$\Sigma \coloneqq \{Q_j \colon \overline{Q_j} \cap \overline{O} \neq \emptyset\} \qquad \text{and} \qquad \boldsymbol{O} \coloneqq O \cup \Big(\bigcup_{Q \in \Sigma} Q \setminus D \Big).$$

All claimed properties of $\boldsymbol{O}$ except (ITC) follow immediately by definition (keep in mind $N \cap Q = \emptyset$ when checking the inclusion of the boundaries). So, let $x \in \partial \boldsymbol{O}$ and $r \in (0,1]$. If $x \in \overline{N}$ then we are done by assumption (argue as in the proof of Lemma 4.1.2). Otherwise, either $x \in D$ or $x \in \partial Q$ for some $Q \in \Sigma$ (to see this, use that the Whitney decomposition is locally finite). But if $x \in D$ then $x \in \overline{Q}$ for some $Q \in \Sigma$ by property (i) of the Whitney decomposition and the definition of Σ. Hence, in either case $x \in \overline{Q}$ for some $Q \in \Sigma$. Now we make a case distinction on the radius size compared to the size of Q. If $r \geq 4\,\mathrm{d}(Q,N)$, pick $y \in \overline{Q}$ and $z \in \overline{N}$ with $\mathrm{d}(Q,N) = |y-z|$. Then with (ii) we get

$$|x-z| \leq |x-y| + |y-z| \leq \operatorname{diam}(Q) + \mathrm{d}(Q,N) \leq 2\,\mathrm{d}(Q,N) \leq r/2,$$

hence $\mathrm{B}(x,r)$ contains a ball of radius $r/2$ centered in $\overline{N}$ and we are done. Finally, if $r < 4\,\mathrm{d}(Q,N)$, then by (ii) we get $r < 16 \operatorname{diam}(Q)$ and the claim follows from (ITC) for Q.

4.2.2. Zero extension

Let $\boldsymbol{O}$ denote the set constructed in the previous section. We define the zero extension Operator $\mathcal{E}_0$ from O to $\boldsymbol{O} \cup D$ and claim $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{sp}) \to \mathrm{W}^{s,p}(\boldsymbol{O} \cup D)$ boundedness. We start with a preparatory lemma.

Lemma 4.2.1. *One has $2|x-y| \geq \mathrm{d}_D(x)$ whenever $x \in O$ and $y \in (\boldsymbol{O} \setminus O) \cup D$.*

Proof. The case $y \in D$ is trivial so let us consider $y \in \boldsymbol{O} \setminus O$. We distinguish whether or not x and y are far away from each other in relation to $\operatorname{diam}(Q)$, where $Q \in \Sigma$ contains y.

Case 1: $|x-y| < \operatorname{diam}(Q)$. Fix a point $z \in \partial O$ on the line segment connecting x with y. Assume for the sake of contradiction that $z \in N$. Then using (ii) we calculate

$$\mathrm{d}(Q,N) \leq |y-z| \leq |x-y| < \operatorname{diam}(Q) \leq \mathrm{d}(Q,N),$$

hence $z \in D$. Thus, $|x-y| \geq |x-z| \geq \mathrm{d}_D(x)$.

Case 2: $|x-y| \geq \operatorname{diam}(Q)$. By definition of Σ and $y \notin O$ we can pick $z \in \overline{Q} \cap D$. Then

$$|x-z| \leq |x-y| + |y-z| \leq |x-y| + \operatorname{diam}(Q) \leq 2|x-y|,$$

hence $2|x-y| \geq \mathrm{d}_D(x)$. $\square$

This enables us to estimate $\mathcal{E}_0$. Clearly, we only have to estimate the $\mathrm{W}^{s,p}(O)$-seminorm since extension by zero is always isometric on L^p. Let $f \in \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$, then

$$\begin{aligned} &\iint_{\substack{x,y \in \boldsymbol{O} \cup D \\ |x-y|<1}} \frac{|\mathcal{E}_0 f(x) - \mathcal{E}_0 f(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}y\,\mathrm{d}x \\ &\leq \iint_{\substack{x,y \in O \\ |x-y|<1}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y + 2 \iint_{\substack{x \in O, y \in (\boldsymbol{O} \setminus O) \cup D \\ |x-y|<1}} \frac{|f(x)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.1}$$

The first term is bounded by $\|f\|_{W^{s,p}(O)}^p$, so it only remains to bound the second term. On using Lemma 4.2.1 and calculating in polar coordinates we find for $x \in O$

$$\int_{\substack{y \in (\boldsymbol{O} \setminus O) \cup D \\ |x-y|<1}} |x-y|^{-d-sp} \,\mathrm{d}y \leq c_d \int_{\mathrm{d}_D(x)/2}^{1} t^{-sp-1} \,\mathrm{d}t \lesssim \mathrm{d}_D(x)^{-sp}.$$

Plugging this back into (4.1) yields that we can bound the second term therein by the Hardy term $\|f\|_{\mathrm{L}^p(O, \mathrm{d}_D^{-sp})}^p$.

4.2.3. Proof of Theorem 4.0.1

We combine the results from above with the extension operator of Jonsson–Wallin to conclude.

Proof of Theorem 4.0.1. Put $\mathcal{E} = \mathbf{E} \circ \mathcal{E}_0$, where $\mathbf{E}$ denotes the extension operator from Proposition 4.1.13. From Section 4.2.1, $\boldsymbol{O} \subseteq \boldsymbol{O} \cup D \subseteq \overline{\boldsymbol{O}}$ and Lemma 4.1.2 follows that $\boldsymbol{O} \cup D$ satisfies (ITC), so by construction $\mathcal{E} : \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}(\mathbb{R}^d)$.

It only remains to verify the vanishing trace condition if D is $(d-1)$-regular and $s > 1/p$. This amounts to showing that $\mathcal{E}$ maps into the kernel of $\mathcal{R}_D$. To this end, let $f \in \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$, $t \in (1/p, s)$, and $(f_n)_n$ be the approximation from Lemma 4.2.2 below. Fix n and let $x \in D$ for which $(\mathcal{R}_D \mathcal{E} f_n)(x)$ is defined. By assumption on the support of f_n we find an $r > 0$ such that $\mathrm{B}(x, r)$ is disjoint to $\operatorname{supp}(f_n)$. Since the Whitney decomposition is locally finite, it is moreover possible to choose r small enough that each Whitney cube that intersects $\mathrm{B}(x,r)$ contains x in its closure. Consequently, since

$$\mathrm{B}(x,r) \subseteq (\mathrm{B}(x,r) \cap O) \cup D \cup \Big(\bigcup_{\substack{Q \in \Sigma \\ x \in \overline{Q}}} \overline{Q} \Big),$$

we get $\mathcal{E} f_n = \mathcal{E}_0 f_n = 0$ almost everywhere on $\mathrm{B}(x,r)$. Therefore, it follows by the very definition of $\mathcal{R}_D$ using mean values over small balls that $\mathcal{R}_D \mathcal{E} f_n(x) = 0$. Finally, $\mathcal{R}_D \mathcal{E} f = 0$ by continuity and consistency of $\mathcal{R}_D$ and $\mathcal{E}$. □

Lemma 4.2.2. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$, $p \in (1,\infty)$, $s \in (1/p, 1)$ and $f \in \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$. Then for any $t \in (1/p, s)$ one has that f can be approximated in $\mathrm{W}^{t,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-tp})$ by a sequence $(f_n)_n$ of functions vanishing almost everywhere in a neighborhood of D.*

Proof. For $n \geq 1$ define the cutoff function $\delta_n : (0,\infty) \to [0,1]$ by

$$\delta_n(x) = \begin{cases} 0, & \text{if } x < 1/n, \\ nx - 1, & \text{if } 1/n \leq x \leq 2/n, \\ 1, & \text{if } x > 2/n. \end{cases}$$

This sequence was already used for a similar purpose in [56, Thm. 3.7] and it is known from that proof (or distinguishing cases) that

$$|\delta_n(x) - \delta_n(y)| \lesssim \frac{1}{x}|x-y| \qquad (y \geq x > 0). \tag{4.2}$$

Put $f_n \coloneqq \delta_n(\mathrm{d}_D) f$. By construction, f_n vanishes identically in a neighborhood of D. Moreover, since $\delta_n(\mathrm{d}_D)$ converges pointwise and boundedly to 1, and taking into account that $\mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \cap \mathrm{L}^p(O) \subseteq \mathrm{L}^p(O, \mathrm{d}_D^{-tp})$, we get convergence of f_n to f in both $\mathrm{L}^p(O)$ and $\mathrm{L}^p(O, \mathrm{d}_D^{-tp})$ by Lebesgue's theorem.

It remains to show convergence in the $\mathrm{W}^{t,p}(O)$-seminorm. For convenience, we put $\eta_n := \delta_n(\mathrm{d}_D)$ and obtain

$$\begin{aligned}
&\iint\limits_{\substack{x,y\in O\\|x-y|<1}} \frac{|(1-\eta_n)(x)f(x)-(1-\eta_n)(y)f(y)|^p}{|x-y|^{d+tp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \iint\limits_{\substack{x,y\in O\\|x-y|<1}} \frac{|\eta_n(x)-\eta_n(y)|^p|f(x)|^p}{|x-y|^{d+tp}}\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\substack{x,y\in O\\|x-y|<1}} \frac{(1-\eta_n(y))|f(x)-f(y)|^p}{|x-y|^{d+tp}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Again, the second term goes to zero by Lebesgue's theorem. In case of the first term, it is also evident that the integrand goes pointwise almost everywhere to zero, but we have to show that there exists an integrable bound for the sequence to apply Lebesgue's theorem once more. To this end, we calculate using $|\eta_n(x)-\eta_n(y)| \leq 1$ and with the aid of (4.2) along with Lipschitz continuity of d_D with constant 1 that

$$\begin{aligned}
|\eta_n(x)-\eta_n(y)| &\leq |\eta_n(x)-\eta_n(y)|^s \lesssim \mathrm{d}_D(x)^{-s}|\,\mathrm{d}_D(x)-\mathrm{d}_D(y)|^s\\
&\leq \mathrm{d}_D(x)^{-s}|x-y|^s.
\end{aligned}$$

Hence, the integral over y is not singular anymore and the integral over x can be estimated by $\|f\|_{\mathrm{L}^p(O,\mathrm{d}_D^{-sp})}$. ☐

Corollary 4.2.3. *Under the assumptions of Theorem 4.0.1 one has* $\mathrm{W}^{s,p}(O)\cap \mathrm{L}^p(O,\mathrm{d}_D^{-sp}) \subseteq \mathcal{W}_D^{s,p}(O)$.

4.3. Hardy's inequality

To prove Theorem 4.0.3, we show the following version on $\mathbb{R}^d \setminus D$ first. Then the theorem follows readily using boundedness of the extension operator from Chapter 3. The advantage is that by this decoupling the Hardy inequality is immediately available whenever there is an extension operator $\mathrm{W}^{1,p}_D(O) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$.

Proposition 4.3.1. *Let $D \subseteq \mathbb{R}^d$ be closed and $(d-1)$-regular, and let $p \in (1,\infty)$. Then Hardy's inequality holds for $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$, that is, for all $f \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ holds*

$$\int_{\mathbb{R}^d} \left|\frac{f}{\mathrm{d}_D}\right|^p \mathrm{d}x \lesssim \|f\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

Before we turn to the proof of this proposition, we show how it implies the theorem from the introduction.

Proof of Theorem 4.0.3. Let $\mathcal{E}$ be the extension operator from Theorem 3.0.2. Then Proposition 4.3.1 and boundedness yield

$$\int_O \left|\frac{f}{\mathrm{d}_D}\right|^p \mathrm{d}x \le \int_{\mathbb{R}^d} \left|\frac{\mathcal{E}f}{\mathrm{d}_D}\right|^p \mathrm{d}x \lesssim \|\mathcal{E}f\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \lesssim \|f\|^p_{\mathrm{W}^{1,p}(O)}. \qquad \square$$

The proof of Proposition 4.3.1 relies on the following Hardy's inequality with pure Dirichlet boundary conditions, which is essentially contained in [70], see also [54].

Proposition 4.3.2. *Let $O \subseteq \mathbb{R}^d$ with Ahlfors–David regular boundary, where either O is bounded or ∂O is unbounded. Then we get the estimate*

$$\int_O \left|\frac{f}{\mathrm{d}_{\partial O}}\right|^p \mathrm{d}x \lesssim \int_O |\nabla f|^p \,\mathrm{d}x \qquad (f \in \mathrm{C}^\infty_{\partial O}(O)).$$

The implicit constant depends on geometry only via the implied constants from Ahlfors–David regularity of ∂O. The inequality extends to $\mathrm{W}^{1,p}_{\partial O}(O)$ owing to Fatou's Lemma.

Proof of Proposition 4.3.1. Let $(Q_k)_k$ be a grid of open cubes of diameter $1/4$. We consider the sets $O_k \coloneqq 2Q_k \setminus D$. Each O_k has an Ahlfors–David regular boundary where the implicit constants depend only on the $(d-1)$-regularity constants of D and dimension.

To see this, take a ball B centered in ∂O_k with radius r at most $1/2$ (which equals the diameter of O_k). The lower bound follows from the $(d-1)$-regularity of $\partial(2Q_k)$ or the $(d-1)$-regularity of D depending on in which part the center of B lies. The upper bound follows similarly if B doesn't intersect either $\partial(2Q_k)$ or D. Otherwise, say B is centered in $\partial(2Q_k)$ and intersects D in x. Then we estimate $\mathcal{H}^{d-1}(B \cap \partial O_k) \le \mathcal{H}^{d-1}(B \cap \partial(2Q_k)) + \mathcal{H}^{d-1}(\mathrm{B}(x, 2r) \cap D)$ and the estimate follows again from the $(d-1)$-regularity of the two portions of the boundary. Note that all constants are uniform in k.

Now take a cutoff function χ_k which is supported in $2Q_k$ and equals 1 on $\overline{Q_k}$. We can essentially use the same cut-off function for each k by translation. Then we estimate for $f \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ using Proposition 4.3.2 and the bounded overlap of $(O_k)_k$ that

$$\int_{\mathbb{R}^d \setminus D} \left|\frac{f}{\mathrm{d}_D}\right|^p \mathrm{d}x \le \sum_k \int_{O_k} \left|\frac{\chi_k f}{\mathrm{d}_{\partial O_k}}\right|^p \mathrm{d}x \lesssim \sum_k \|\chi_k f\|^p_{\mathrm{W}^{1,p}(2Q_k)} \lesssim \|f\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

Note that at the first "$\lesssim$" we crucially use the dependence of the constant in the Dirichlet Hardy inequality. $\square$

4.4. Interpolation with intrinsic spaces

We combine Theorem 4.0.1, Theorem 2.1.7, Theorem 3.0.2 and Theorem 4.0.3 to conclude Theorem 4.0.4.

Proof of Theorem 4.0.4. Throughout, let $\mathcal{E}$ denote the extension operator from Theorem 4.0.1. Recall that $\mathcal{E}$ maps into $\mathcal{W}^{s,p}_D(\mathbb{R}^d)$.

We start with the inclusion "$\subseteq$". To this end, let $f \in \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$. Then we get with Theorem 2.1.7 that

$$\mathcal{E}f \in \mathcal{W}^{s,p}_{(D)}(\mathbb{R}^d) = (\mathrm{L}^p(\mathbb{R}^d), \mathcal{W}^{1,p}_D(\mathbb{R}^d))_{s,p}.$$

Here, we put the subscript D on the left-hand side whenever it is meaningful. Consequently, $f = (\mathcal{E}f)|_O \in (\mathrm{L}^p(O), \mathcal{W}^{1,p}_D(O))_{s,p}$. This completes this inclusion since $\mathcal{W}^{1,p}_D(O) = \mathrm{W}^{1,p}_D(O)$ owing to Theorem 3.0.2.

Conversely, if $f \in (\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O))_{s,p}$, then

$$\mathcal{E}f \in (\mathrm{L}^p(\mathbb{R}^d), \mathrm{W}^{1,p}_D(\mathbb{R}^d))_{s,p} \subseteq (\mathrm{L}^p(\mathbb{R}^d), \mathrm{W}^{1,p}(\mathbb{R}^d))_{s,p} = \mathrm{W}^{s,p}(\mathbb{R}^d).$$

Restriction to O gives the embedding into $\mathcal{W}^{s,p}(O) \subseteq \mathrm{W}^{s,p}(O)$. For the embedding into $\mathrm{L}^p(O, \mathrm{d}_D^{-sp})$ we argue similarly using Theorem 4.0.3 and L^p-interpolation with weights (see [93, Thm. 1.18.5]) to obtain

$$(\mathrm{L}^p(O), \mathrm{W}^{1,p}_D(O))_{s,p} \subseteq (\mathrm{L}^p(O), \mathrm{L}^p(O, \mathrm{d}_D^{-p}))_{s,p} = \mathrm{L}^p(O, \mathrm{d}_D^{-sp}).$$

Note that this inclusion did not use that $s \neq \frac{1}{p}$. □

The following example is an elaboration of Example 4.1.3. We construct an open set O and a Dirichlet part $D \subseteq \partial O$ which are admissible for Theorem 4.0.4 but which do not fulfill the interior thickness condition in *any* neighborhood of the Neumann boundary part $\partial O \setminus D$.

Example 4.4.1. To construct O, we start with the lower half-space in $\mathbb{R}^2$. We decompose the negative x-axis into dyadic chunks indexed by the integers, that is, $I_k \coloneqq [-2^{-k+1}, -2^{-k})$. For positive k we add a hat to O above I_k with height $\operatorname{len}(I_k)$ and width $2^{-k} \operatorname{len}(I_k)$. Finally, put $N \coloneqq (0, \infty) \times \{0\}$ and $D \coloneqq \partial O \setminus N$.

By Example 3.2.5, Assumption 3.1.1 is satisfied. Moreover, O satisfies the interior thickness condition in N but not in any neighborhood of N since such a neighborhood would contain arbitrarily peaked hats. Hence, it only remains to verify Ahlfors–David regularity for D.

Let Q_k be the closed dyadic cube over I_k. Note that $\mathcal{H}^1(Q_k \cap D) \approx \operatorname{len}(I_k)$ and that $D \subseteq \bigcup_k Q_k$. We verify Ahlfors–David regularity using cubes instead of balls and using dyadic side lengths only. So, let $x \in D$, $\ell = 2^m$ a dyadic side length and Q_k a cube from above that contains x. We compare ℓ with the side length of Q_k. If $\ell \leq 2^k$, the upper bound follows from the Ahlfors–David regularity of Q_k and its adjacent cubes. Otherwise, $\mathrm{Q}(x, \ell)$ intersects D at most in $\bigcup_{j \leq m} Q_j$, so by a geometric sum we find the upper bound

$$\mathcal{H}^1(\mathrm{Q}(x,\ell) \cap D) \leq \sum_{j \leq m} \mathcal{H}^1(Q_j \cap D) \lesssim \sum_{j \leq m} 2^j = 2^{m+1} \approx \ell.$$

For the lower bound we start with the case $\ell \leq 2^{k-1}$. Then the lower bound follows again from the Ahlfors–David regularity of Q_k. Otherwise, $Q(x, \ell)$ contains Q_{m-1} and we get the lower bound from this cube.

4.5. On necessary conditions for the existence of an extension operator

In this final section we consider the necessity of the geometric assumption in Theorem 4.0.1. We introduce a modified version of the interior thickness condition in $N \subseteq \partial O$ that degenerates near $\partial O \setminus N$ in Definition 4.5.1 and show that this conditions is necessary for $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$-extension domains. This condition also automatically holds whenever Assumption 3.1.1 is satisfied. Finally, we give an example of a geometry that satisfies the degenerate thickness conditions but is not covered by Theorem 4.0.1.

Definition 4.5.1. Say that O satisfies the *degenerate interior thickness condition in N* if $O \subseteq \mathbb{R}^d$ is open, $N \subseteq \partial O$ and they fulfill

$$\forall x \in N, r \leq \min(1, \mathrm{d}_{\partial O \setminus N}(x))\colon |\mathrm{B}(x,r) \cap O| \gtrsim |\mathrm{B}(x,r)|.$$

This condition is necessary for $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$-extension operators as the following proposition shows. The technique is due to Y. Zhou, see [98].

Proposition 4.5.2. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$ be closed, $p \in (1, \infty)$, $s \in (0,1)$ and put $N := \partial O \setminus D$. If there exists an extension operator E : $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}(\mathbb{R}^d)$, then O satisfies the degenerate interior thickness condition in N.*

Before we come to the proof we provide a handy lemma needed therein, see [98, Lemma 2.4].

Lemma 4.5.3. *Let $O \subseteq \mathbb{R}^d$ be open, $p \in (1,\infty)$ and $s \in (0,1)$. For $x \in O$ and $0 < t < r \leq 1$ define the cutoff function $f_{r,t}$ on O by*

$$f_{r,t}(y) := \begin{cases} 1, & \text{if } y \in \mathrm{B}(x,t) \cap O, \\ \frac{r-|y-x|}{r-t}, & \text{if } y \in (\mathrm{B}(x,r) \cap O) \setminus \mathrm{B}(x,t), \\ 0, & \text{if } y \in O \setminus \mathrm{B}(x,r). \end{cases}$$

Then one has the estimate

$$\|f_{r,t}\|_{\mathrm{W}^{s,p}(O)} \lesssim \frac{|\mathrm{B}(x,r) \cap O|^{\frac{1}{p}}}{(r-t)^s},$$

where the implicit constant does not depend on x.

Proof of Proposition 4.5.2. We only treat the case $sp < d$. The necessary modifications of the proof in [98] become already apparent from this case and we invite the interested reader to check the other cases himself.

Take $x \in N$ and a radius $r \leq \min(1, \mathrm{d}_D(x))$. We claim that whenever $0 < t < \frac{1}{2}\min(1, \mathrm{d}_D(x))$ and $b \in (0,1)$ are such that

$$|\mathrm{B}(x,bt) \cap O| = \frac{1}{2}|\mathrm{B}(x,t) \cap O|, \tag{4.3}$$

then

$$t - bt \lesssim |\mathrm{B}(x,t) \cap O|^{\frac{1}{d}}. \tag{4.4}$$

Indeed, we calculate using the fractional Sobolev inequality [31, Thm. 6.5] that

$$\begin{aligned} \|f_{t,bt}\|_{\mathrm{L}^{\frac{pd}{d-sp}}(O)} &\leq \|Ef_{t,bt}\|_{\mathrm{L}^{\frac{pd}{d-sp}}(\mathbb{R}^d)} \lesssim \|Ef_{t,bt}\|_{\mathrm{W}^{s,p}(\mathbb{R}^d)} \\ &\lesssim \|f_{t,bt}\|_{\mathrm{W}^{s,p}(O)} + \|f_{t,bt}\|_{\mathrm{L}^p(O,\mathrm{d}_D^{-sp})}. \end{aligned} \tag{4.5}$$

If y is in the support of $f_{t,bt}$ then $\mathrm{d}_D(y) \geq \mathrm{d}_D(x) - |y-x| \geq \frac{1}{2}\mathrm{d}_D(x)$ by choice of t. This is where the restriction of admissible radii enters the scene. Hence, we get the estimate

$$\|f_{t,bt}\|_{\mathrm{L}^p(O,\mathrm{d}_D^{-sp})} \lesssim \mathrm{d}_D(x)^{-s}|\mathrm{B}(x,t) \cap O|^{\frac{1}{p}} \leq \frac{|\mathrm{B}(x,t) \cap O|^{\frac{1}{p}}}{(t-bt)^s}.$$

Now, we get a lower bound for the $\mathrm{L}^{\frac{pd}{d-sp}}(O)$-norm of $f_{t,bt}$ in terms of $|\mathrm{B}(x,t) \cap O|$ by the definition of $f_{t,bt}$ and (4.3), and an upper bound if we apply

Lemma 4.5.3 to the first summand in the final estimate of (4.5) and use the previously shown bound for its second term. This gives in summary

$$|\mathrm{B}(x,t) \cap O|^{\frac{d-sp}{pd}} \lesssim \|f_{t,bt}\|_{\mathrm{L}^{\frac{pd}{d-sp}}(O)} \lesssim \frac{|\mathrm{B}(x,t) \cap O|^{\frac{1}{p}}}{(t-bt)^s}.$$

Sorting all terms gives (4.4) as claimed.

To conclude, we define a sequence of "halfing factors" as follows. Put $b_0 := 1$. Since the function $\varphi : b \mapsto |\mathrm{B}(x, bt) \cap O|$ is continuous on $[0,\infty)$ for any radius t in virtue of Lebesgue's theorem, we inductively find for $j \geq 1$ a factor b_j such that $|\mathrm{B}(x, b_j r) \cap O| = \frac{1}{2}|\mathrm{B}(x, b_{j-1}r) \cap O|$. In particular, $|\mathrm{B}(x, b_j r) \cap O| = 2^{-j}|\mathrm{B}(x,r) \cap O|$. By continuity of φ and $\varphi(0) = 0$ we deduce that b_j is a null sequence. Moreover, with $t := b_{j-1}r$ and $b := b_j/b_{j-1}$ we can employ (4.4), which leads to the calculation

$$\begin{aligned} r = \sum_{j\geq 1} b_{j-1}r - b_j r &\lesssim \sum_{j\geq 0} |\mathrm{B}(x, b_j r) \cap O|^{\frac{1}{d}} \\ &\lesssim \sum_{j\geq 0} 2^{-j/d}|\mathrm{B}(x,r) \cap O|^{\frac{1}{d}} \approx |\mathrm{B}(x,r) \cap O|^{\frac{1}{d}}. \end{aligned}$$

Raising to the power of d concludes the proof. □

We also verify Definition 4.5.1 in the situation of Assumption 3.1.1.

Proposition 4.5.4. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$ be closed such that Assumption 3.1.1 holds. Then O satisfies the degenerate interior thickness condition in $\partial O \setminus D$.*

Proof. For convenience, put $N := \partial O \setminus D$ and let $x \in N$ and r an admissible radius. First, we note that it suffices to consider radii that obey the restriction $r \leq \frac{1}{2}\min(\delta, \lambda\delta, 1)\min(1, \mathrm{d}_D(x))$ since we can put $C := \frac{1}{2}\min(\delta, \lambda\delta, 1)$ to get

$$|\mathrm{B}(x,r) \cap O| \geq |\mathrm{B}(x, Cr) \cap O| \gtrsim (Cr)^d \approx r^d.$$

So we assume the aforementioned restriction on r and pick $y \in \mathrm{B}(x, r/8) \cap O$. We claim that there exists $z \in O$ such that

$$r/2 \leq |y - z| \leq 3r/4. \tag{4.6}$$

Otherwise, let O' denote the connected component of O that contains y and let $z \in O'$. We cannot have $|y - z| > 3r/4$ since then we could connect y and z in O' by a path which would contain a point satisfying (4.6). Hence, $O' \subseteq$

$\mathrm{B}(y, r/2)$ and consequently $\operatorname{diam}(O') < r/2 \leq c$. But $\mathrm{B}(x,r) \cap O$ is connected and contains y, so $\mathrm{B}(x,r) \cap O \subseteq O'$ and $x \in \partial O'$, which contradicts (DC) in Assumption 3.1.1.

To proceed, fix some $z \in O$ satisfying (4.6). Due to $|y-z| \leq 3r/4 < \delta$ there is some ε-cigar γ that connects y with z. By continuity we find $w \in \gamma$ with $|w-y| = \frac{1}{2}|y-z|$. We calculate the distance of w to D and N. First, condition (CC) in Assumption 3.1.1 and (4.6) yield

$$\mathrm{d}_N(w) \geq \frac{\varepsilon |y-z||w-z|}{2|y-z|} = \frac{\varepsilon}{2}|w-z| \geq \frac{\varepsilon}{4}|y-z| \geq \frac{\varepsilon}{8} r.$$

Second, from $|w-y| = |y-z|/2 \leq 3r/8$ follows

$$|w-x| \leq |w-y| + |y-x| \leq \frac{4}{8} r \leq \frac{1}{2}\,\mathrm{d}_D(x),$$

with which we derive

$$\mathrm{d}_D(w) \geq \mathrm{d}_D(x) - |w-x| \geq \frac{1}{2}\,\mathrm{d}_D(x) \geq r.$$

Combining both estimates and using that $w \in O$ gives $\mathrm{B}(w, \varepsilon r/8) \subseteq O \cap \mathrm{B}(x,r)$. Since $|\mathrm{B}(w, \varepsilon r/8)| \approx r^d$, the assertion follows. $\square$

The following example shows that the condition in Definition 4.5.1 is strictly weaker compared to the interior thickness condition in N in the sense that there is a geometry that allows for extension operators (and hence satisfies the degenerate interior thickness condition), but is *not* admissible for Theorem 4.0.1 (nor Theorem 3.0.2).

Example 4.5.5. Consider the cusp $O := \{(x,y) \in \mathbb{R}^2 \colon x > 0, 0 < y < x^2\}$ and put $N := (0,\infty) \times \{0\}$ and $D := \partial O \setminus N$. To construct an extension operator on $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$, extend to the upper half-plane by zero (the calculation is the same as in Section 4.2.2, use that in the mixed case the connecting straight line intersects D) and extend to the whole space by reflection (use here that upon reflection the distance of points increases). The same construction yields a $\mathrm{W}^{1,p}_D(O)$-extension operator. On the contrary, it is easy to verify that O does not satisfy the interior thickness condition in N.

Remark 4.5.6. Note that the extension operator in Example 4.5.5 decomposed (similarly to our extension operator in Theorem 4.0.1) into a zero extension operator and an extension operator for the pure Neumann case.

CHAPTER 5

Kato's square root property: $\mathbf{L^2}$–Theory

Let L be a second order elliptic operator in divergence form on an open, possibly unbounded set $O \subseteq \mathbb{R}^d$, $d \geq 2$, with bounded measurable complex coefficients, formally given by

$$Lu = -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) - \sum_{i=1}^{d} \partial_i(b_i u) + \sum_{j=1}^{d} c_j \partial_j u + du. \tag{5.1}$$

Let D be a closed, possibly empty, subset of the boundary ∂O. We complement L with Dirichlet boundary conditions on D and Neumann boundary conditions on $\partial O \setminus D$. More generally, L can be an $(m \times m)$-system in which case u takes its values in $\mathbb{C}^m$ and each coefficient is valued in $\mathcal{L}(\mathbb{C}^m)$.

Let $V \coloneqq \mathrm{W}^{1,2}_D(O)^m$ be the $\mathrm{W}^{1,2}(O)^m$-closure of smooth functions that vanish in a neighborhood of D (Definition 5.1.11). Note that this definition of $\mathrm{W}^{1,2}_D(O)$ coincides with Definition 2.2.6 in virtue of Corollary 5.1.12 under suitable geometric assumptions, so that the results from Chapter 2 still apply. The superscript m indicates that we consider $\mathbb{C}^m$-valued functions. As usual, we interpret L as the maximal accretive operator in $\mathrm{L}^2(O)^m$ associated with the sesquilinear form $a : V \times V \to \mathbb{C}$ defined by

$$a(u,v) = \int_O \sum_{i,j=1}^{d} a_{ij}\partial_j u \cdot \overline{\partial_i v} + \sum_{i=1}^{d} b_i u \cdot \overline{\partial_i v} + \sum_{j=1}^{d} c_j \partial_j u \cdot \overline{v} + du \cdot \overline{v} \,\mathrm{d}x, \tag{5.2}$$

which we assume to satisfy for some $\lambda > 0$ the (inhomogeneous) *Gårding inequality*

$$\operatorname{Re} a(u,u) \geq \lambda(\|u\|_2^2 + \|\nabla u\|_2^2) \qquad (u \in V). \tag{5.3}$$

Then L is invertible and there is a unique maximal accretive operator $\sqrt{L}$ in $\mathrm{L}^2(O)^m$ that satisfies $(\sqrt{L})^2 = L$. We give a more detailed account in Section 5.2. More generally, operators like $\sqrt{L}$ arise from functional calculus. Though we assume some familiarity with this concept, we have supplied the necessary background for understanding this chapter in Section 1.4.

The Kato problem is to identify the domain of the square root operator as $\mathrm{D}(\sqrt{L}) = V$ with equivalent norms. If L is self-adjoint, then this essentially follows from the (formal) calculation

$$\|u\|_V^2 \approx a(u,u) = (Lu \,|\, u)_2 = (\sqrt{L}u \,|\, (\sqrt{L})^* u)_2 = \|\sqrt{L}u\|_2^2,$$

where the first step uses (5.3), the second step is the definition of L and the final step uses self-adjointness in a crucial way. Indeed, this can be turned into a complete proof, Kato's so-called second representation theorem [69, VI.§2.6]. No such abstract argument can work when L is not self-adjoint [74]. The problem becomes incomparably harder and tied to deep results in harmonic analysis. On $O = \mathbb{R}^d$ it was eventually solved by Auscher–Hofmann–Lacey–McIntosh–Tchamitchian in their 2001 breakthrough paper [13] and extended by four of them to systems [14]. For a historical account and connections to other fields of analysis the reader can refer to the introduction of [13].

On general open sets O the problem as posed above is wide open till this day. Applications on sets with "rough" geometry come from various fields including, with exemplary references, elliptic and parabolic regularity [25,57], Lions' non-autonomous maximal regularity problem [1,41] and boundary value problems [7,11]. This motivates the search for minimal geometric assumptions that allow to solve the Kato problem. The main result of this chapter improves on all available results (to be reviewed momentarily) and reads as follows.

Theorem 5.0.1. *Let $O \subseteq \mathbb{R}^d$ be an open set and $D \subseteq \partial O$ a closed subset of the boundary. Suppose that D is Ahlfors–David regular and that O is locally uniform near $\partial O \setminus D$ (Definition 5.1.1). Then $\mathrm{D}(\sqrt{L}) = \mathrm{W}^{1,2}_D(O)^m$ holds with equivalence of norms*

$$\|u\|_2 + \|\nabla u\|_2 \approx \|\sqrt{L}u\|_2 \qquad (u \in \mathrm{W}^{1,2}_D(O)^m),$$

and the implicit constants depend on the coefficients of L only through the coefficient bounds.

Here, *coefficient bounds* refers to the lower bound λ in (5.3) and a pointwise upper bound Λ for the coefficients. The geometric framework in Theorem 5.0.1 includes the one proposed by Brewster–Mitrea–Mitrea–Mitrea in their influential paper [26] for treating various aspects of mixed boundary value problems in vast generality. We have already had a look on it in Assumption 3.2.1 earlier on. It will be discussed in detail in Section 5.1. We do not require coordinate charts around $\partial O \setminus D$ in any sense and we do not assume that O satisfies the *interior thickness condition*

$$\exists c > 0 \quad \forall x \in O,\ r \leq 1\ :\ |\mathrm{B}(x,r) \cap O| \geq c|\mathrm{B}(x,r)|. \tag{5.4}$$

Those are two main points compared to all earlier results.

Indeed, for pure Neumann boundary conditions ($D = \emptyset$) the solution of the Kato problem was only known on Lipschitz domains [12, 15]. Our assumption reduces to O being an (ε, δ)-domain (with positive radius if it has infinitely many connected components, see Remark 5.1.5). For example, O could be the interior of the von Koch snowflake [94, Fig. 3.5]. Pure Dirichlet conditions ($D = \partial O$) were first treated in [12] on Lipschitz domains. The Lions problem on mixed boundary conditions was solved in [15] on a class of Lipschitz domains if D is a Lipschitz submanifold of ∂O. An elaboration on their method of proof in [38] yielded the solution on bounded interior thick sets with Ahlfors–David regular boundary that are locally Lipschitz regular around $\partial O \setminus D$.

The proof of Theorem 5.0.1 divides into three steps. They correspond to the three Sections 5.3 - 5.5. Here, we give an informal overview on the strategy of proof and to fix ideas it will be somewhat more convenient to reverse the order of Section 5.3 and Section 5.4.

Some parts of the chapter require that O satisfies (5.4) nonetheless. We avoid any ambiguity by the following convention. In a context where the underlying set is interior thick, we use bold letters for the relevant objects and write for example $\boldsymbol{O}$ instead of O.

Step 1: Dirac operator framework

As many before us, we cast the Kato problem in the abstract first order framework of perturbed Dirac operators that was introduced by Axelsson–Keith–McIntosh in their remarkable elaboration of the original proof of the Kato conjecture [16]. More precisely, we use the refinement in [39] that will allow us to work on interior thick sets with porous boundary (Definition 1.3.23).

The latter holds true under our assumptions (Corollary 5.1.10) and for the moment let us assume in addition that $\boldsymbol{O}$ is interior thick in the sense of (5.4).

Consequently, we can use the Dirac operator framework as a black box. In Section 5.4 we explain what is going on "behind the scenes" in more detail, but what basically happens is that harmonic analysis (present due to the non-smooth coefficients) is decoupled from geometry (present due to the rough nature of $\boldsymbol{O}$). The harmonic analysis is taken care of by the Axelsson–Keith–McIntosh framework. It then turns out that in order to prove our main theorem, still under the additional assumption (5.4), we only need the following *higher regularity result* for the Laplacian with boundary conditions on $\boldsymbol{O}$.

Step 2: Higher regularity for the Laplacian

In light of the growing interest in fractional Laplacians in other fields of analysis this might be of independent interest. The (componentwise) Laplacian $-\Delta_{\boldsymbol{D}} + 1$ corresponds to $\boldsymbol{a}_{ij} = \delta_{ij}$, $\boldsymbol{b}_i = 0 = \boldsymbol{c}_j$, $\boldsymbol{d} = 1$ in (5.1). Definitions of fractional Sobolev spaces $\mathrm{W}^{\alpha,2}(\boldsymbol{O})$ and $\mathrm{W}^{\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O})$ will be given in Section 5.3.

Theorem 5.0.2. *Let $\boldsymbol{O} \subseteq \mathbb{R}^d$ be open and interior thick, let $\boldsymbol{D} \subseteq \partial\boldsymbol{O}$ be closed and Ahlfors–David regular and assume that $\boldsymbol{O}$ is locally uniform near $\partial\boldsymbol{O} \setminus \boldsymbol{D}$. Then there exists $\varepsilon \in (0, \frac{1}{2})$ such that*

$$-\Delta_{\boldsymbol{D}} + 1 : \mathrm{W}^{1+s,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \to \mathrm{W}^{-1+s,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$$

is an isomorphism for all $s \in (-\varepsilon, \varepsilon)$. Its fractional power domains in $\mathrm{L}^2(\boldsymbol{O})$ are given by

$$\mathrm{D}((-\Delta_{\boldsymbol{D}} + 1)^{\frac{\alpha}{2}}) = \begin{cases} \mathrm{W}^{\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O})^m & \text{if } \alpha \in (\frac{1}{2}, 1+\varepsilon), \\ \mathrm{W}^{\alpha,2}(\boldsymbol{O})^m & \text{if } \alpha \in (0, \frac{1}{2}). \end{cases}$$

We give the proof in Section 5.3 below. We remark that the results for $s = 0$ and $\alpha = 1$ are elementary consequences of the Lax–Milgram lemma and the Kato problem for self-adjoint operators, respectively. Hence, we are concerned with a question of extrapolation. Compared to certain forerunners [15, 38, 83] there are two new ingredients that allow us to relax the required geometric quality of $\boldsymbol{O}$: Improved complex interpolation theory for (fractional) Sobolev spaces with boundary conditions, developed in Chapter 2, see also [20], and Netrusov's spectral synthesis [2, Ch. 10] replacing more naive measure theoretic considerations in [38].

Step 3: Eliminating the interior thickness condition

At this stage the proof of Theorem 5.0.1 has been completed under the additional assumption (5.4). The final step, carried out in Section 5.5, consists in eliminating this assumption by an *ad hoc* method.

First, we observe that if an open set $\boldsymbol{O}$ can be written as a countable union of disjoint open sets $\boldsymbol{O} = \bigcup_i O_i$ in such a way that the canonical identification

$$\mathrm{L}^2(\boldsymbol{O})^m \cong \bigotimes_i \mathrm{L}^2(O_i)^m$$

also gives rise to a decomposition of form domains

$$\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \cong \bigotimes_i \mathrm{W}^{1,2}_{D_i}(O_i)^m,$$

then a divergence form operator $\boldsymbol{L}$ on $\boldsymbol{O}$ with Dirichlet boundary part $\boldsymbol{D}$ inherits a decomposition of its functional calculus as

$$f(\boldsymbol{L}) \cong \bigotimes_i f(L_i),$$

where $L_i \coloneqq \boldsymbol{L}|_{O_i}$ is subject to Dirichlet conditions on $D_i \coloneqq \boldsymbol{D} \cap \partial O_i$. Obviously we are somewhat sketchy and some caution is needed to make such decomposition precise, see Section 5.5. Solving the Kato problem for the triple $(\boldsymbol{L}, \boldsymbol{O}, \boldsymbol{D})$ will therefore be equivalent to solving it for all triples (L_i, O_i, D_i) with uniform control of the constants in i (Proposition 5.5.8).

This being said, we reverse the order of reasoning. We let $(L_0, O_0, D_0) \coloneqq (L, O, D)$ be the original operator in Theorem 5.0.1 and construct an open set O_1 disjoint to O with Dirichlet part $D_1 \coloneqq \partial O_1$ such that $\boldsymbol{O} \coloneqq O_0 \cup O_1$ with Dirichlet part $\boldsymbol{D} \coloneqq D_0 \cup D_1$ is a "fattened version" of (O, D): It has the same geometric quality and additionally satisfies (5.4). Then we set $L_0 \coloneqq L$ and $L_1 \coloneqq -\Delta_{\partial O_1} + 1$. In the interior thick setting we have already solved the Kato problem. Hence, we have the solution for $(\boldsymbol{L}, \boldsymbol{O}, \boldsymbol{D})$ and obtain the solution for (L, O, D) by restriction to that triple.

The results of this chapter were published in a joint paper with Moritz Egert and Robert Haller-Dintelmann in [21].

5.1. Discussion of the geometric setup

5.1.1. Locally uniform domains

Definition 5.1.1. Let $\varepsilon \in (0, 1]$ and $\delta \in (0, \infty]$. Let $O \subseteq \mathbb{R}^d$ be open and $N \subseteq \partial O$. Set $N_\delta \coloneqq \{z \in \mathbb{R}^d : \mathrm{d}(z, N) < \delta\}$. Then O is called *locally an*

(ε, δ)-*domain near* N if the following properties hold.

(i) All points $x, y \in O \cap N_\delta$ with $|x - y| < \delta$ can be joined in O by an ε-*cigar with respect to* $\partial O \cap N_\delta$, that is to say, a rectifiable curve $\gamma \subseteq O$ of length $\ell(\gamma) \leq \varepsilon^{-1}|x - y|$ such that

$$\mathrm{d}(z, \partial O \cap N_\delta) \geq \frac{\varepsilon|z - x|\,|z - y|}{|x - y|} \qquad (z \in \gamma). \tag{5.5}$$

(ii) O has *positive radius near* N, that is, there exists $\lambda > 0$ such that all connected components O' of O with $\partial O' \cap N \neq \emptyset$ satisfy $\operatorname{diam}(O') \geq \lambda\delta$.

If the values of ε, δ, c need not be specified, then O is simply called *locally uniform near* N.

Remark 5.1.2. Definition 5.1.1 describes a quantitative local connectivity property of O near N. For an illustration of ε-cigars with respect to ∂O the reader can refer for instance to [94, Fig. 3.1]. Having positive radius is of course only a restriction if O has infinitely many connected components.

Remark 5.1.3. The positive radius condition is scale invariant in the sense that $\delta = \infty$ forces O to be connected and unbounded in the case $N \neq \emptyset$. We will *only* need this scale invariant formulation later on in Chapter 6. Hence, we assume $\delta < \infty$ for the rest of this chapter. In particular, the positive radius condition then reduces to the existence of a constant $c > 0$ such that all connected components O' of O with $\partial O' \cap N \neq \emptyset$ satisfy $\operatorname{diam}(O') \geq c$.

Condition (5.5) originates from Jones' influential paper [65]. For his (ε, δ)-domains he requires that all $x, y \in O$ with $|x - y| < \delta$ can be joined by an ε-cigar with respect to ∂O. Locally (ε, δ)-domains near a part of the boundary have been pioneered in [26] using (ε, δ)-domains as charts around N in analogy with how Lipschitz graphs give rise to the notion of sets with Lipschitz boundary. Our novel definition avoids charts and is inspired by Assumption 3.1.1. Let us give a concise comparison, which also includes the proof of Proposition 3.2.2.

Proposition 5.1.4. *Let* $O \subseteq \mathbb{R}^d$ *be an open set and let* $N \subseteq \partial O$.

(i) *If* O *is locally an* (ε, δ)*-domain near* N *in the sense of Assumption 3.2.1 (which is Definition 3.4 in [26]), then it is locally uniform near* N *in the sense of Definition 5.1.1.*

(ii) *If O is locally uniform near N in the sense of Definition 5.1.1, then (O, N) is an admissible geometry for Assumption 3.1.1 in Chapter 3.*

Proof. We start with (i). Besides further quantitative conditions, being an (ε, δ)-domain near N in the sense of Assumption 3.2.1 means that there exist (at most) countably many open sets U_i and constants $r, c > 0$ such that

(a) for each $x \in N$ there exists some i such that $B(x, 8r) \subseteq U_i$ and

(b) for each i there is an (ε, δ)-domain O_i with connected components all of diameter at least c such that $O \cap U_i = O_i \cap U_i$.

We take $\delta' := \min(\delta, \varepsilon r)$. Note that in particular $\delta' \leq r$. We show that O is locally an (ε, δ')-domain near N in the sense of Definition 5.1.1. To this end, let $x, y \in O \cap N_{\delta'}$ be such that $|x - y| < \delta'$. According to (a) there is a ball B of radius r centered in N and an index i such that $x, y \in 2B$ and $8B \subseteq U_i$. Due to (b) we have $x, y \in O_i$. Consequently, there is a rectifiable curve $\gamma \subseteq O_i$ of length $\ell(\gamma) \leq \varepsilon^{-1}|x - y|$ that joins x to y in such a way that

$$\mathrm{d}(z, \partial O_i) \geq \frac{\varepsilon |z - x|\,|z - y|}{|x - y|} \qquad (z \in \gamma). \tag{5.6}$$

From $\ell(\gamma) < r$ we obtain $\gamma \subseteq 3B$ and (b) yields $\gamma \subseteq O_i \cap U_i \subseteq O$. Given $z \in \gamma$, we let z' be a point in $\overline{\partial O \cap N_{\delta'}}$ closest to z. We have $|z - z'| \leq 3r$ since B is centered in N, which shows that $z' \in \partial O \cap 6B$. Now, (b) yields $z' \in \partial O_i$. Thus we proved $\mathrm{d}(z, \partial O_i) \leq \mathrm{d}(z, \partial O \cap N_{\delta'})$ and by (5.6) we see that γ is an ε-cigar with respect to $\partial O \cap N_{\delta'}$.

Let O' be a connected component of O with $\partial O' \cap N \neq \emptyset$. We complete the proof of (i) by demonstrating $\operatorname{diam}(O') \geq \min(2r, c)$. Suppose we have $\operatorname{diam}(O') < 2r$. As above, we find a ball B and an index i such that $O' \subseteq 2B$ and $8B \subseteq U_i$. From (b) we obtain that O contains all $x \in O_i$ with $\mathrm{d}(x, O') < 6r$ and that $O' \subseteq O_i$. In particular, O' is an open and connected subset of O_i. Since O' is a maximal connected subset of O, we also get that no continuous curve $\gamma \subseteq O_i$ can join points from $O_i \setminus O'$ and O'. Hence, O' is a connected component of O_i and $\operatorname{diam}(O') \geq c$ follows.

Let us prove (ii). Besides O having positive radius near N, for (O, N) being an admissible geometry for Assumption 3.1.1 we need that there exists $\varepsilon', \delta', K > 0$ such that

(c) all $x, y \in O$ with $|x - y| < \delta'$ can be joined by an ε'-cigar γ with respect to $\overline{N}$, *not necessarily contained in* O, such that $k(z, O) := \inf_O k(z, \cdot) \leq K$

for all $z \in \gamma$, where $k(\cdot\,,\cdot)$ denotes the *hyperbolic distance*

$$k(x',y') \coloneqq \inf_{\substack{\gamma' \subseteq \mathbb{R}^d \setminus \overline{N} \text{ rect.} \\ \text{curve from } x' \text{ to } y'}} \int_{\gamma'} \mathrm{d}(z',N)^{-1} \, |\,\mathrm{d}z'|.$$

Let ε, δ be as in Definition 5.1.1. We check (c) for $\delta' \coloneqq \frac{\delta}{2}$, $\varepsilon' \coloneqq \varepsilon$ and $K \coloneqq 1$. If $x, y \in N_\delta$, then we can use the ε-cigar provided by Definition 5.1.1, on noting that for $z \in \gamma \subseteq O$ we have $k(z,O) = 0$ and $\mathrm{d}(z, \partial O \cap N_\delta) \leq \mathrm{d}(z,N)$. Now, suppose $x \notin N_\delta$. Let γ be the straight line segment to y and take any $z \in \gamma$. First,

$$\frac{\varepsilon|z-x||z-y|}{|x-y|} \leq \varepsilon|x-y| < \frac{\delta}{2} \leq \mathrm{d}(x,N) - \frac{\delta}{2} \leq \mathrm{d}(z,N)$$

shows that γ is an ε-cigar with respect to $\overline{N}$. Second, on taking γ' as the segment from z to x in the definition of hyperbolic distance, we find $k(z,O) \leq k(z,x) \leq \ell(\gamma')\frac{2}{\delta} \leq 1$. □

Remark 5.1.5. By definition, $(O, \partial O)$ is an admissible geometry in [19] if and only if O is an (ε,δ)-domain with positive radius. Thus, all introduced notions of locally (ε,δ)-domains near the full boundary imply that O has positive radius. This observation sharpens [26, Lem. 3.7].

Remark 5.1.6. The proof moreover shows that if $\delta = \infty$ in Definition 5.1.1 then the same is true for Assumption 3.1.1.

Bounded Lipschitz domains are locally uniform, see [38, Rem. 5.11] or [94, Prop. 3.8]. In particular, the local (ε,δ)-condition near N in the sense of [26] already comprises Lipschitz regular sets near N, see Proposition 3.2.4. The standard example of a fractal locally uniform domain is the von Koch snowflake [94, Fig. 3.5].

5.1.2. Corkscrew condition and porosity

We establish the corkscrew condition in our context. As before, we write N_δ for the set of points with distance to N less than δ.

Proposition 5.1.7. *Suppose that $O \subseteq \mathbb{R}^d$ is open and locally an (ε,δ)-domain near $N \subseteq \partial O$. Then there exists a constant $\kappa \in (0,1]$ such that:*

$$\forall x \in \overline{N_{\delta/2} \cap O},\ r \leq 1 \quad \exists z \in \mathrm{B}(x,r) \ : \ \mathrm{B}(z,\kappa r) \subseteq O \cap \mathrm{B}(x,r).$$

Proof. Let $C \coloneqq \frac{1}{2}\min(\delta, c, 1)$. It suffices to obtain some κ that works for all radii $r \leq C$ and all $x \in N_{\delta/2} \cap O$. Indeed, for $r \leq 1$ we find $z \in \mathrm{B}(x, Cr) \subseteq \mathrm{B}(x, r)$ with $\mathrm{B}(z, (\kappa C)r) \subseteq O \cap \mathrm{B}(x, r)$, so we only need to use κC instead of κ. Finally, with a constant *strictly* smaller than κC and a limiting argument, we can allow all $x \in \overline{N_{\delta/2} \cap O}$.

Let $x \in N_{\delta/2} \cap O$. We claim that there is $y \in O$ satisfying $\frac{r}{2} \leq |x-y| \leq \frac{3r}{4}$. Suppose this was not true and let O' be the connected component of O that contains x. Then $O' \subseteq \mathrm{B}(x, \frac{r}{2}) \subseteq N_\delta$ and we also have $\mathrm{B}(x, \delta) \cap N_\delta \cap O \subseteq O'$ since all points in the left-hand set can be joined to x via a curve in O. The first inclusion gives $\operatorname{diam}(O') < c$, whereas the second one gives $\partial O' \cap N \neq \emptyset$ in contradiction with Definition 5.1.1.

We fix any $y \in O$ as above. Then $|x - y| \leq \frac{\delta}{2}$ and in particular $y \in N_\delta \cap O$. Let $\gamma \subseteq O$ be the joining ε-cigar with respect to $\partial O \cap N_\delta$. By continuity we pick $z \in \gamma$ with $|z - x| = \frac{1}{2}|x - y|$ and verify the required properties for $\kappa \coloneqq \frac{\varepsilon}{8}$. First, we have $\mathrm{B}(z, \kappa r) \subseteq \mathrm{B}(x, r)$ by construction. Second, $r \leq \frac{\delta}{2}$ yields $\mathrm{d}(z, \mathbb{R}^d \setminus N_\delta) \geq \frac{\delta}{2} - |x - z| \geq \frac{r}{2}$. Third, $|z-y| \geq \frac{1}{2}|x-y|$ and $|x-y| \geq \frac{r}{2}$ plugged into (5.5) give $\mathrm{d}(z, \partial O \cap N_\delta) \geq \kappa r$. The last two bounds imply $\mathrm{B}(z, \kappa r) \subseteq \mathbb{R}^d \setminus \partial O$. But as $z \in O$ we must have $\mathrm{B}(z, \kappa r) \subseteq O$. □

Remark 5.1.8. If $\delta = \infty$ and $N \neq \emptyset$, then there exists a constant $\kappa \in (0, 1]$ such that

$$\forall x \in O, r > 0 \quad \exists z \in \mathrm{B}(x, r) \ : \ \mathrm{B}(z, \kappa r) \subseteq O \cap \mathrm{B}(x, r).$$

Indeed, the positive radius condition then forces O to be a connected and unbounded open set. In particular, there is $y \in O$ satisfying $r/2 \leq |x - y| \leq 3r/4$. Connect x and y by an ε-cigar γ. By continuity, there is $z \in \gamma$ with $|z-x| = \frac{1}{2}|x-y|$. Put $\kappa \coloneqq \varepsilon/8$. By the triangle inequality, $\mathrm{B}(z, \kappa r) \subseteq \mathrm{B}(x, r)$. Also, it follows from (5.5) that $\mathrm{d}(z, \partial O) = \mathrm{d}(z, \partial O \cap N_\delta) \geq \kappa r$. Hence, $\mathrm{B}(z, \kappa r) \subseteq \mathbb{R}^d \setminus \partial O$ and $z \in O$ lets us conclude $\mathrm{B}(z, \kappa r) \subseteq O \cap \mathrm{B}(x, r)$.

The property above is closely related to porosity in the following sense.

Definition 5.1.9. A set $E \subseteq \mathbb{R}^d$ is *porous* if there exists $\kappa \in (0, 1]$ with the property that:

$$\forall x \in E,\, r \leq 1 \quad \exists z \in \mathrm{B}(x, r) \ : \ \mathrm{B}(z, \kappa r) \subseteq \mathrm{B}(x, r) \setminus E.$$

Proposition 5.1.7 entails in particular that N is porous. As a non-trivial example let us mention that Ahlfors–David regular sets are porous by Lemma A.1.7. This leads to the following important corollary of Proposition 5.1.7.

Corollary 5.1.10. *Under the assumptions of Theorem 5.0.1 the full boundary ∂O is porous.*

Proof. In view of the examples given above, the claim amounts to showing that the union of two porous sets E_0, E_1 is again porous. To this end, start without loss of generality with a ball B centered in E_0 and obtain a ball $B' \subseteq B \setminus E_0$ with comparable radius. Then either $\frac{1}{2}B' \subseteq B \setminus (E_0 \cup E_1)$ or $\frac{1}{2}B' \cap E_1 \neq \emptyset$. In the first case we are done and in the second case porosity of E_1, applied with $r = \frac{1}{2}\,\mathrm{r}(B')$ and x an intersection point, furnishes a comparably sized ball $B'' \subseteq B' \setminus E_1 \subseteq B \setminus (E_0 \cup E_1)$. $\square$

5.1.3. Sobolev extensions

The Hilbert space $\mathrm{W}^{1,2}(O)$ on an open set $O \subseteq \mathbb{R}^d$ is the collection of all $u \in \mathrm{L}^2(O)$ such that $\nabla u \in \mathrm{L}^2(O)^d$ with norm

$$\|u\|_{\mathrm{W}^{1,2}(O)} := \Big(\|u\|^2_{\mathrm{L}^2(O)} + \|\nabla u\|^2_{\mathrm{L}^2(O)^d}\Big)^{1/2}. \tag{5.7}$$

We introduce the subspace of functions that vanish on a subset of $\overline{O}$ as follows.

Definition 5.1.11. Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \overline{O}$ be closed. The Hilbert space $\mathrm{W}^{1,2}_D(O)$ is the $\mathrm{W}^{1,2}(O)$-closure of the set of test functions

$$\mathrm{C}^\infty_D(O) := \Big\{u|_O : u \in \mathrm{C}^\infty_0(\mathbb{R}^d) \text{ and } \mathrm{d}(\mathrm{supp}(u), D) > 0\Big\}.$$

For pure Dirichlet conditions we recover $\mathrm{W}^{1,2}_0(O) = \mathrm{W}^{1,2}_{\partial O}(O)$. If O is an (ε, δ)-domain with positive radius, then Jones' extension operator and density of $\mathrm{C}^\infty_0(\mathbb{R}^d)$ in $\mathrm{W}^{1,2}(\mathbb{R}^d)$ allows to obtain $\mathrm{W}^{1,2}(O) = \mathrm{W}^{1,2}_\emptyset(O)$. In view of Proposition 5.1.4.(ii) we may state a special case of Theorem 3.0.2 in the following

Corollary 5.1.12. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$ be closed. If O is locally uniform near $\partial O \setminus D$, then there is a bounded linear extension operator $\mathcal{E} : \mathrm{W}^{1,2}_D(O) \to \mathrm{W}^{1,2}_D(\mathbb{R}^d)$.*

There are further natural choices for the test function class $\mathrm{C}^\infty_D(O)$ in Definition 5.1.11 that all lead to the same $\mathrm{W}^{1,2}(O)$-closure, see Section 3.6.

5.2. Definition of the elliptic operator

Throughout, we assume that $O \subseteq \mathbb{R}^d$ is open and $D \subseteq \partial O$ is closed. Identifying $\mathrm{L}^2(O)^m$ with its anti-dual space $(\mathrm{L}^2(O)^m)^*$, we have dense embeddings

$$V = \mathrm{W}^{1,2}_D(O)^m \subseteq \mathrm{L}^2(O)^m \subseteq (\mathrm{W}^{1,2}_D(O)^m)^*.$$

We assume that the coefficients $a_{ij}, b_i, c_j, d : O \to \mathcal{L}(\mathbb{C}^m)$ in (5.1) are *bounded and measurable*. We group them as $A \coloneqq (a_{ij})_{ij}$, $b \coloneqq (b_i)_i$ and $c \coloneqq (c_j)_j$ in the coefficient matrix

$$\begin{bmatrix} d & c \\ b & A \end{bmatrix} : O \to \mathcal{L}(\mathbb{C}^m)^{(1+d)\times(1+d)}, \tag{5.8}$$

and we introduce for a $\mathbb{C}^m$-valued function u the gradient $\nabla u \coloneqq (\partial_i u)_i$ as a vector in $(\mathbb{C}^m)^d$. Here, i and j always refer to column and row notation, respectively. With this notation, the sesquilinear form in (5.2) can be rewritten as

$$a\colon V \times V \to \mathbb{C}, \qquad a(u,v) = \int_O \begin{bmatrix} d & c \\ b & A \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} \, \mathrm{d}x. \tag{5.9}$$

Our *ellipticity* assumption is the lower bound (5.3). Note also that $(\| \cdot \|_2^2 + \|\nabla \cdot \|_2^2)^{1/2}$ is the Hilbert space norm on $\mathrm{W}^{1,2}_D(O)^m$.

The Lax-Milgram lemma associates with a the bounded and invertible operator

$$\mathcal{L} : \mathrm{W}^{1,2}_D(O)^m \to (\mathrm{W}^{1,2}_D(O)^m)^*, \qquad \langle \mathcal{L}u, v\rangle = a(u,v).$$

We define L to be the maximal restriction of $\mathcal{L}$ to an operator in $\mathrm{L}^2(O)^m$. Then L is an invertible, maximal accretive, sectorial operator in $\mathrm{L}^2(O)$ of some angle $\omega \in [0, \pi/2)$, see [53, Prop. 7.3.4]. The adjoint L^* is associated in the same way with $a^*(u,v) \coloneqq \overline{a(v,u)}$, see [69, Thm. VI§2.5]. Consequently, L is self-adjoint when the matrix in (5.8) is Hermitian.

The square root $\sqrt{L}$ is defined via the sectorial functional calculus for L. It is invertible and maximal accretive [53, Cor. 7.1.13]. In particular, it coincides with Kato's original definition of the square root [69, Thm. V§3.35] as the unique maximal accretive operator in $\mathrm{L}^2(O)^m$ that satisfies $(\sqrt{L})^2 = L$.

It will be convenient to write L as a composition of differential and multiplication operators, more in the spirit of the formal definition (5.1). To this end, we introduce the closed and densely defined operator

$$\nabla_D : \mathrm{W}^{1,2}_D(O)^m \subseteq \mathrm{L}^2(O)^m \to \mathrm{L}^2(O)^{dm}, \qquad \nabla_D u = \nabla u \tag{5.10}$$

and let $-\operatorname{div}_D$ be its (unbounded) adjoint. In view of (5.9) it follows that

$$L = \begin{bmatrix} 1 & -\operatorname{div}_D \end{bmatrix} \begin{bmatrix} d & c \\ b & A \end{bmatrix} \begin{bmatrix} 1 \\ \nabla_D \end{bmatrix} \tag{5.11}$$

with maximal domain in $\mathrm{L}^2(O)^m$. Integration by parts reveals $\operatorname{div}_D (u_i)_i = \sum_{i=1}^d \partial_i u_i$ for $(u_i)_i \in (\mathrm{C}_0^\infty(O)^m)^d$ but in this generality no explicit description of $\mathrm{D}(\operatorname{div}_D)$ is available.

5.3. Higher regularity for fractional powers of the Laplacian

The goal of this section is to show Theorem 5.0.2, thereby accomplishing Step 2 from the introduction. In the whole section we fix $\boldsymbol{O}$ and $\boldsymbol{D}$ satisfying the assumptions from Theorem 5.0.2. It suffices to treat the case $m = 1$ since $-\Delta_{\boldsymbol{D}}$ in $\mathrm{L}^2(\boldsymbol{O})^m$ acts componentwise. We adopt the convention that function spaces without reference to an underlying set are understood on the whole space, e.g. we write $\mathrm{W}^{1,2}$ instead of $\mathrm{W}^{1,2}(\mathbb{R}^d)$.

5.3.1. Fractional Sobolev spaces on open sets with vanishing trace condition

To make this chapter as self-contained as possible, we recall the *(fractional) Sobolev spaces* of regularity $s \in (0, \frac{3}{2})$ in the Hilbertian case. Whereas this is a classical topic on $\mathbb{R}^d$ and was presented in Section 1.2 in the preliminaries, different definitions suiting different purposes are possible on $\boldsymbol{O}$. We follow the treatment from Chapter 2 with a focus on interpolation theory.

If $s \in (0,1)$, then $\mathrm{W}^{s,2}$ consists of all $u \in \mathrm{L}^2$ such that

$$\|u\|^2_{\mathrm{W}^{s,2}} := \|u\|^2_{\mathrm{L}^2} + \iint_{\substack{x,y \in \mathbb{R}^d \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

If $s \in (0, \frac{1}{2})$, then $\mathrm{W}^{1+s,2}$ consists of all $u \in \mathrm{L}^2$ with $\|u\|_{\mathrm{W}^{1+s,2}}^2 := \|u\|_{\mathrm{L}^2}^2 + \|\nabla u\|_{\mathrm{W}^{s,2}}^2 < \infty$. It will be convenient to set $\mathrm{W}^{0,2} := \mathrm{L}^2$. It follows from the structure of the norms that these spaces are Hilbert spaces and that ∇ maps $\mathrm{W}^{1+s,2}$ into $\mathrm{W}^{s,2}$. The *Bessel potential space*

$$\mathrm{H}^{s,2} := \Big\{u \in \mathrm{L}^2 : \|u\|_{\mathrm{H}^{s,2}} := \|(1-\Delta)^{\frac{s}{2}} u\|_{\mathrm{L}^2} < \infty\Big\} \tag{5.12}$$

coincides with $\mathrm{W}^{s,2}$ by [93, Sec. 2.4.2. Rem. 2]. Here, Δ is the Laplacian in $\mathbb{R}^d$.

Since $\boldsymbol{D}$ is a $(d-1)$-set, see Remark 1.3.11 for this terminology, a version of the Lebesgue differentiation theorem allows us to define traces on $\boldsymbol{D}$. The following is a weakened version of Proposition 2.2.1 that suffices for our purpose.

Proposition 5.3.1. *Let $s \in (\frac{1}{2}, \frac{3}{2})$ and $u \in \mathrm{W}^{s,2}$. For $\mathcal{H}^{d-1}$-almost every $x \in \boldsymbol{D}$ the limit*

$$(\mathcal{R}_{\boldsymbol{D}} u)(x) := \lim_{r \to 0} \frac{1}{|\mathrm{B}(x,r)|} \int_{\mathrm{B}(x,r)} u(y) \,\mathrm{d}y$$

exists. The restriction operator $\mathcal{R}_{\boldsymbol{D}}$ maps $\mathrm{W}^{s,2}$ boundedly into $\mathrm{L}^2(\boldsymbol{D}, \mathcal{H}^{d-1})$.

With the trace operator at hand, we introduce the closed subspace $\mathrm{W}^{s,2}_{\boldsymbol{D}}$ of $\mathrm{W}^{s,2}$ by

$$\mathrm{W}^{s,2}_{\boldsymbol{D}} := \Big\{u \in \mathrm{W}^{s,2} : \mathcal{R}_{\boldsymbol{D}} u = 0\Big\}.$$

In the case $s = 1$ this notion is consistent with Definition 5.1.11, see Lemma 2.2.5.

Finally, we denote the distributional restriction to $\boldsymbol{O}$ by $|_{\boldsymbol{O}}$ and define fractional Sobolev spaces on $\boldsymbol{O}$ by restriction. Let $s \in [0, \frac{3}{2})$ and $t \in (\frac{1}{2}, \frac{3}{2})$. Put $\mathrm{W}^{s,2}(\boldsymbol{O}) := \{u|_{\boldsymbol{O}} : u \in \mathrm{W}^{s,2}\}$ and $\mathrm{W}^{t,2}_{\boldsymbol{D}}(\boldsymbol{O}) := \{u|_{\boldsymbol{O}} : u \in \mathrm{W}^{t,2}_{\boldsymbol{D}}\}$ and equip them with quotient norms.

Remark 5.3.2. These spaces are again Hilbert spaces by construction as quotients of Hilbert spaces. Since $\boldsymbol{O}$ is interior thick, we have that $\mathrm{W}^{t,2}_{\boldsymbol{D}}(\boldsymbol{O})$ is a closed subspace of $\mathrm{W}^{t,2}(\boldsymbol{O})$ with an equivalent norm, see Lemma 2.2.11. As a cautionary tale, let us stress that in the context of this section $\mathrm{W}^{1,2}(\boldsymbol{O})$ is embedded into but possibly *not* equal to the collection of all $u \in \mathrm{L}^2(\boldsymbol{O})$ with $\nabla u \in \mathrm{L}^2(\boldsymbol{O})^d$ and norm (5.7). However, as a consequence of Corollary 5.1.12, the definition of $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ above coincides with the original one from Definition 5.1.11 up to equivalent norms.

Spaces of negative smoothness are defined by duality extending the inner product on $\mathrm{L}^2(\boldsymbol{O})$. For $s \in [0, \frac{3}{2})$ and $t \in (\frac{1}{2}, \frac{3}{2})$ let $\mathrm{W}^{-s,2}(\boldsymbol{O})$ and $\mathrm{W}_{\boldsymbol{D}}^{-t,2}(\boldsymbol{O})$ be the anti-dual spaces of $\mathrm{W}^{s,2}(\boldsymbol{O})$ and $\mathrm{W}_{\boldsymbol{D}}^{t,2}(\boldsymbol{O})$, respectively.

We turn our focus to the density of test functions in these spaces. The following proposition follows as a special case of Netrusov's Theorem [2, Thm. 10.1.1] if one replaces the appearing capacities by the Hausdorff measure using [2, Thm. 5.1.9]. To make the statement more concise, we use the concept of *Hausdorff co-dimension*, defined by

$$\operatorname{codim}_{\mathcal{H}}(E) \coloneqq d - \dim_{\mathcal{H}}(E),$$

where

$$\dim_{\mathcal{H}}(E) \coloneqq \inf\Big\{s \in (0, d] : \mathcal{H}^s(E) = 0\Big\}$$

is the *Hausdorff dimension* of $E \subseteq \mathbb{R}^d$. For more information take a look at Section 1.3.2.

Proposition 5.3.3 (A Version of Netrusov's Theorem)**.** *Let $0 < s < \frac{d}{2}$ and let $E \subseteq \mathbb{R}^d$ be closed. If $2s < \operatorname{codim}_{\mathcal{H}}(E)$, then C_E^∞ is dense in $\mathrm{W}^{s,2}$.*

In order to show that this version of Netrusov's Theorem is applicable in our setting, we use the elementary covering lemma for porous sets presented in Lemma A.1.8.

Proposition 5.3.4. *There exists $0 < s_0 \leq \frac{1}{2}$ such that $\mathrm{C}_{\partial\boldsymbol{O}}^\infty(\boldsymbol{O})$ is dense in $\mathrm{W}^{s,2}(\boldsymbol{O})$ and the zero extension operator $\mathcal{E}_0 : \mathrm{W}^{s,2}(\boldsymbol{O}) \to \mathrm{W}^{s,2}$ is bounded for $0 < s < s_0$.*

Proof. Since $\partial\boldsymbol{O}$ is porous by Corollary 5.1.10, we can pick $C \geq 1$ and $0 < t < d$ as in Lemma A.1.8. We take $0 < s < 1$ such that $2s < d - t$. Then, let B be some ball centered in $\partial\boldsymbol{O}$ with $\mathrm{r}(B) = 1$ and let $0 < r < 1$. Given $(B_i)_i$ a covering of $\partial\boldsymbol{O} \cap B$ provided by the lemma mentioned above and $\ell \in (t, d)$, we estimate

$$\mathcal{H}_r^\ell(\partial\boldsymbol{O} \cap B) \leq \sum_i \mathrm{r}(B_i)^\ell = \#_i\, r^\ell \leq C r^{\ell - t}.$$

Taking the limit as $r \to 0$, we arrive at $\mathcal{H}^\ell(\partial\boldsymbol{O} \cap B) = 0$. Finally, a countable covering of $\partial\boldsymbol{O}$ by such balls yields $\mathcal{H}^\ell(\partial\boldsymbol{O}) = 0$, so by definition we have $\dim_{\mathcal{H}}(\partial\boldsymbol{O}) \leq t$ and therefore $2s < \operatorname{codim}_{\mathcal{H}}(\partial\boldsymbol{O})$. Now, Netrusov's Theorem gives density of $\mathrm{C}_{\partial\boldsymbol{O}}^\infty$ in $\mathrm{W}^{s,2}$ and the first claim follows by restriction to $\boldsymbol{O}$.

Boundedness of the zero extension operator for s sufficiently small follows from a result of Sickel presented in Proposition 2.2.14, take Example 2.2.13 into account. Inspecting the proof of Lemma A.1.8 reveals that the same range of s as before would work but we do not need such precision. $\square$

5.3.2. Interpolation scales

Some general background on interpolation can be found in Section 1.1. The following notion of *interpolation scale* comprises a particular easy way to use interpolation theory. We will see below that particularly nice interpolation couples induce interpolation scales.

Definition 5.3.5. Let $I \subseteq \mathbb{R}$ be an interval and for each $i \in I$ let H_i be a Hilbert space. Call $(H_i)_i$ a *complex interpolation scale* if whenever $i_0 < i_1$, then $H_{i_1} \subseteq H_{i_0}$ with dense and continuous inclusion and

$$[H_{i_0}, H_{i_1}]_\theta = H_{(1-\theta)i_0+\theta i_1} \tag{5.13}$$

for all $\theta \in (0,1)$ up to equivalent norms.

We could have introduced a similar notion for $(\theta, 2)$-real interpolation, but it is also a good opportunity to note that this coincides with θ-complex interpolation when working with Hilbert spaces [61, Cor. C.4.2.]. We will freely use this fact. If H_0, H_1 are Hilbert spaces with dense inclusion $H_1 \subseteq H_0$, then $([H_0, H_1]_\theta)_{\theta\in[0,1]}$ is a complex interpolation scale, see [93, Sec. 1.9.3. Thm. (c) & (d)] and [93, Sec. 1.10.3. Thm. 2]. In this context (5.13) is called *reiteration* in the literature and we use the convention $[H_0, H_1]_j = H_j$ for $j = 0, 1$.

For the proof of Theorem 5.0.2 we need the following interpolation scales:

(a) $(\mathrm{W}^{s,2}_{\boldsymbol{D}}(\boldsymbol{O}))_{s\in(\frac{1}{2},\frac{3}{2})}$, (b) $(\mathrm{W}^{s,2}_{\boldsymbol{D}}(\boldsymbol{O}))_{s\in(-\frac{3}{2},-\frac{1}{2})}$,

(c) $(\mathrm{D}((1-\Delta_{\boldsymbol{D}})^{\frac{s}{2}}))_{s\in[0,\infty)}$.

Since $1 - \Delta_{\boldsymbol{D}}$ is self-adjoint we obtain (c) from [53, Thm. 6.6.9 & Cor. 7.1.6]. Also, (b) follows from (a) by duality [93, Sec. 1.11.2]. Parts (a) uses the standing geometric assumptions and has been obtained in Theorem 2.1.5. It will be important in the proof of Theorem 5.0.2 to identify some spaces in (c) with fractional Sobolev spaces using Theorem 2.1.7.

5.3.3. Mapping properties for $1 - \Delta_{\boldsymbol{D}}$

We start with mapping properties for the distributional gradient on spaces of fractional smoothness.

Lemma 5.3.6. *Let $s, t > 0$ satisfy $t < \frac{1}{2}$ and $s < s_0$, where s_0 was determined in Proposition 5.3.4. Then ∇ is a bounded operator $\mathrm{W}^{1+t,2}(\boldsymbol{O}) \to \mathrm{W}^{t,2}(\boldsymbol{O})^d$ and $\mathrm{W}^{1-s,2}(\boldsymbol{O}) \to \mathrm{W}^{-s,2}(\boldsymbol{O})^d$.*

Proof. For the first claim we simply note that $\nabla : \mathrm{W}^{1+t,2} \to (\mathrm{W}^{t,2})^d$ is bounded and that $\nabla Eu \in (\mathrm{W}^{t,2})^d$ is an extension of ∇u if $Eu \in \mathrm{W}^{1+t,2}$ is an extension of $u \in \mathrm{W}^{1+t,2}(\boldsymbol{O})$.

For the second claim let $u \in \mathrm{W}^{1-s,2}(\boldsymbol{O})$ and let $Eu \in \mathrm{W}^{1-s,2}$ be some extension of u. Let $i = 1, \dots, d$. Given $\varphi \in \mathrm{C}^\infty_{\partial\boldsymbol{O}}(\boldsymbol{O})$, we first rewrite the duality pairing as

$$\begin{aligned} -\langle \partial_i u, \varphi\rangle &= (u \,|\, \partial_i \varphi)_{\mathrm{L}^2(\boldsymbol{O})} \\ &= (Eu \,|\, \mathcal{E}_0 \partial_i \varphi)_{\mathrm{L}^2} \\ &= (Eu \,|\, \partial_i \mathcal{E}_0 \varphi)_{\mathrm{L}^2} \\ &= \Big((1-\Delta)^{\frac{1-s}{2}} Eu \,\Big|\, \partial_i (1-\Delta)^{-\frac{1}{2}} (1-\Delta)^{\frac{s}{2}} \mathcal{E}_0 \varphi\Big)_{\mathrm{L}^2}, \end{aligned}$$

where we used again the fractional powers of the Laplacian on $\mathbb{R}^d$ and commuted the respective Fourier multiplication operators. Since the Riesz transform $\partial_i(1-\Delta)^{-\frac{1}{2}}$ is bounded on L^2 by Plancherel's theorem and the Bessel spaces in (5.12) coincide with the fractional Sobolev spaces, we obtain

$$\begin{aligned} |\langle \partial_i u, \varphi\rangle| &\lesssim \|Eu\|_{\mathrm{W}^{1-s}} \|\mathcal{E}_0 \varphi\|_{\mathrm{W}^{s,2}} \\ &\lesssim \|Eu\|_{\mathrm{W}^{1-s}} \|\varphi\|_{\mathrm{W}^{s,2}(\boldsymbol{O})}, \end{aligned}$$

where the final step is due to Proposition 5.3.4. By passing to the infimum over all extensions Eu we arrive at $|\langle \partial_i u, \varphi\rangle| \lesssim \|u\|_{\mathrm{W}^{1-s,2}(\boldsymbol{O})} \|\varphi\|_{\mathrm{W}^{s,2}(\boldsymbol{O})}$. But since $\mathrm{C}^\infty_{\partial\boldsymbol{O}}(\boldsymbol{O})$ is dense in $\mathrm{W}^{s,2}(\boldsymbol{O})$ by Proposition 5.3.4, this shows $\partial_i u \in \mathrm{W}^{-s,2}(\boldsymbol{O})$ with $\|\partial_i u\|_{\mathrm{W}^{-s,2}(\boldsymbol{O})} \lesssim \|u\|_{\mathrm{W}^{1-s,2}(\boldsymbol{O})}$. □

Proposition 5.3.7. *Let $0 < s < s_0$ with s_0 as in Proposition 5.3.4. Then $1 - \Delta_{\boldsymbol{D}} : \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O}) \to \mathrm{W}^{-1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ restricts/extends to a bounded operator $\mathrm{W}^{1\pm s,2}_{\boldsymbol{D}}(\boldsymbol{O}) \to \mathrm{W}^{-1\pm s,2}_{\boldsymbol{D}}(\boldsymbol{O})$.*

Proof. By definition, we have

$$\langle (1-\Delta_{\boldsymbol{D}})u, v\rangle = (u \mid v)_2 + (\nabla u \mid \nabla v)_2 \qquad (u, v \in \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})).$$

If in addition $u \in \mathrm{W}^{1\pm s,2}_{\boldsymbol{D}}(\boldsymbol{O})$ and $v \in \mathrm{W}^{1\mp s,2}_{\boldsymbol{D}}(\boldsymbol{O})$, then we control the first part of the right-hand side by

$$|(u \mid v)_2| \le \|u\|_2 \|v\|_2 \le \|u\|_{\mathrm{W}^{1\pm s,2}_{\boldsymbol{D}}(\boldsymbol{O})} \|v\|_{\mathrm{W}^{1\mp s,2}_{\boldsymbol{D}}(\boldsymbol{O})},$$

whereas for the second part we first use that the $\mathrm{W}^{\pm s,2}(\boldsymbol{O})$ - $\mathrm{W}^{\mp s,2}(\boldsymbol{O})$ duality extends the inner product on $\mathrm{L}^2(\boldsymbol{O})$ and then apply Lemma 5.3.6 to give

$$|(\nabla u \mid \nabla v)_2| \le \|\nabla u\|_{\mathrm{W}^{\pm s,2}(\boldsymbol{O})} \|\nabla v\|_{\mathrm{W}^{\mp s,2}(\boldsymbol{O})} \lesssim \|u\|_{\mathrm{W}^{1\pm s,2}_{\boldsymbol{D}}(\boldsymbol{O})} \|v\|_{\mathrm{W}^{1\mp s,2}_{\boldsymbol{D}}(\boldsymbol{O})}.$$

By virtue of the scale (a) from Section 5.3.2 the fractional Sobolev spaces form a hierarchy of densely included spaces. This yields the claim. $\square$

We continue by recalling an abstract extrapolation result due to Šneĭberg [9, 91].

Proposition 5.3.8. *Let $a < b$, let $(H_i)_{i\in[a,b]}$ and $(K_i)_{i\in[a,b]}$ be complex interpolation scales and let $T : H_a \to K_a$ be a bounded linear operator that is also $H_b \to K_b$ bounded. Then the set $\{i \in (a, b) \mid T : H_i \to K_i$ is an isomorphism$\}$ is open.*

This enables us to complete the first part of Theorem 5.0.2 through the following

Proposition 5.3.9. *There exists $\varepsilon \in (0, \frac{1}{2})$ such that $1 - \Delta_{\boldsymbol{D}}$ is an isomorphism between $\mathrm{W}^{1+s,2}_{\boldsymbol{D}}(\boldsymbol{O})$ and $\mathrm{W}^{-1+s,2}_{\boldsymbol{D}}(\boldsymbol{O})$ for all $s \in (-\varepsilon, \varepsilon)$.*

Proof. We define $\beta \coloneqq \frac{1}{2}s_0$ and $I = [-\beta, \beta]$. According to Proposition 5.3.7, $1-\Delta_{\boldsymbol{D}}$ extends to a bounded operator $\mathrm{W}^{1-\beta,2}_{\boldsymbol{D}}(\boldsymbol{O}) \to \mathrm{W}^{-1-\beta,2}_{\boldsymbol{D}}(\boldsymbol{O})$. We denote this extension by T and the same proposition shows that T also restricts to a bounded operator $\mathrm{W}^{1+\beta,2}_{\boldsymbol{D}}(\boldsymbol{O}) \to \mathrm{W}^{-1+\beta,2}_{\boldsymbol{D}}(\boldsymbol{O})$. From Section 5.3.2 we know that $(\mathrm{W}^{1+s,2}_{\boldsymbol{D}}(\boldsymbol{O}))_{s\in I}$ and $(\mathrm{W}^{-1+s,2}_{\boldsymbol{D}}(\boldsymbol{O}))_{s\in I}$ are complex interpolation scales. We know by the Lax–Milgram lemma that $s = 0$ is contained in the isomorphism set in Šneĭberg's Theorem, thus the claim follows. $\square$

5.3.4. Domains of fractional powers of the Laplacian

In this section we complete the proof of Theorem 5.0.2.

Proof of Theorem 5.0.2. In view of Proposition 5.3.9 it only remains to determine the fractional power domains in $\mathrm{L}^2(\boldsymbol{O})$. The starting point is that $1-\Delta_{\boldsymbol{D}}$ is a self-adjoint operator and therefore we have $\mathrm{D}((1 - \Delta_{\boldsymbol{D}})^{\frac{1}{2}}) = \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ by the Kato property for self-adjoint operators. Combining Proposition 2.1.7 and the interpolation scale (c) in Section 5.3.2 we obtain

$$\mathrm{D}((1 - \Delta_{\boldsymbol{D}})^{\frac{\alpha}{2}}) = \begin{cases} \mathrm{W}^{\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O}) & (\text{if } \alpha > 1/2) \\ \mathrm{W}^{\alpha,2}(\boldsymbol{O}) & (\text{if } \alpha < 1/2) \end{cases}$$

for $\alpha \in [0, 1]$, all with equivalent norms. For the extrapolation we decompose fractional powers of $1 - \Delta_{\boldsymbol{D}}$ above $\frac{1}{2}$ into the inverse of the full operator and fractional powers of lower order. Since the fractional powers of $1 - \Delta_{\boldsymbol{D}}$ are invertible, see Example 1.4.4,

$$(1 - \Delta_{\boldsymbol{D}})^{\frac{1-\alpha}{2}} : \mathrm{W}^{1-\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O}) \to \mathrm{L}^2(\boldsymbol{O})$$

is an isomorphism for $\alpha \in [0, \frac{1}{2})$. Using duality and self-adjointness, we moreover get that $(1 - \Delta_{\boldsymbol{D}})^{\frac{1-\alpha}{2}}$ extends to an isomorphism between $\mathrm{L}^2(\boldsymbol{O})$ and $\mathrm{W}^{-1+\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O})$. In particular, with ε from Proposition 5.3.9 and $\alpha \in (0, \varepsilon)$, $(1-\Delta_{\boldsymbol{D}})^{\frac{1-\alpha}{2}}$ maps into the domain of the extrapolated Lax–Milgram isomorphism. On the dense subset $\mathrm{D}((1 - \Delta_{\boldsymbol{D}})^{\frac{1-\alpha}{2}})$ of $\mathrm{L}^2(\boldsymbol{O})$ we have the decomposition

$$(1 - \Delta_{\boldsymbol{D}})^{-\frac{1+\alpha}{2}} = (1 - \Delta_{\boldsymbol{D}})^{-1}(1 - \Delta_{\boldsymbol{D}})^{\frac{1-\alpha}{2}}.$$

Again by example 1.4.4, the left-hand side is an isomorphism from $\mathrm{L}^2(\boldsymbol{O})$ onto $\mathrm{D}((1 - \Delta_{\boldsymbol{D}})^{\frac{1+\alpha}{2}})$. But the right-hand side extends to an isomorphism onto $\mathrm{W}^{1+\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O})$, which reveals that indeed $\mathrm{D}((1 - \Delta_{\boldsymbol{D}})^{\frac{1+\alpha}{2}}) = \mathrm{W}^{1+\alpha,2}_{\boldsymbol{D}}(\boldsymbol{O})$. $\square$

5.4. Proof of Theorem 5.0.1 on interior thick sets

This section corresponds to Step 1 of the introduction. Throughout we assume that $\boldsymbol{O} \subseteq \mathbb{R}^d$ is an open and interior thick set, that $\boldsymbol{D} \subseteq \partial\boldsymbol{O}$ is a closed and Ahlfors–David regular portion of its boundary, and that $\boldsymbol{O}$ is locally uniform near $\partial\boldsymbol{O} \setminus \boldsymbol{D}$. The proof heavily relies on [39], which can essentially be used as a black box, but nonetheless the reader is advised to keep a copy of that paper handy.

5.4.1. The idea of Axelsson–Keith–McIntosh

On the Hilbert space $H \coloneqq \mathrm{L}^2(\boldsymbol{O})^m \times \mathrm{L}^2(\boldsymbol{O})^{dm} \times \mathrm{L}^2(\boldsymbol{O})^m$ introduce the closed operators with maximal domain

$$\text{(5.14)} \qquad \boldsymbol{\Gamma} \coloneqq \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \nabla_{\boldsymbol{D}} & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 \coloneqq \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_2 \coloneqq \begin{bmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{d} & \boldsymbol{c} \\ 0 & \boldsymbol{b} & \boldsymbol{A} \end{bmatrix},$$

where $\nabla_{\boldsymbol{D}}$ was defined in (5.10) and $-\operatorname{div}_{\boldsymbol{D}}$ is its adjoint. Then define the *perturbed Dirac operator* $\boldsymbol{\Pi_B} \coloneqq \boldsymbol{\Gamma} + \boldsymbol{B}_1\boldsymbol{\Gamma}^*\boldsymbol{B}_2$ on $\mathrm{D}(\boldsymbol{\Pi_B}) \coloneqq \mathrm{D}(\boldsymbol{\Gamma}) \cap \mathrm{D}(\boldsymbol{B}_1\boldsymbol{\Gamma}^*\boldsymbol{B}_2)$.

It follows that

$$\boldsymbol{\Pi_B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \nabla_{\boldsymbol{D}} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -\operatorname{div}_{\boldsymbol{D}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{d} & \boldsymbol{c} \\ 0 & \boldsymbol{b} & \boldsymbol{A} \end{bmatrix}, \quad \boldsymbol{\Pi_B^2} = \begin{bmatrix} \boldsymbol{L} & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Here, coefficients are identified with the corresponding multiplication operators and, owing to (5.11), the second order operator $\boldsymbol{L}$ with correct domain appears. The precise structure of the asterisked entries is not needed. The operators in (5.14) have the following properties.

(H1) $\boldsymbol{\Gamma}$ is *nilpotent*, that is, closed, densely defined and satisfies $\mathrm{R}(\boldsymbol{\Gamma}) \subseteq \mathrm{N}(\boldsymbol{\Gamma})$.

(H2) $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are defined on H. There exist $\kappa_i, K_i \in (0,\infty)$, $i = 1, 2$, such that

$$\begin{aligned} \operatorname{Re}(\boldsymbol{B}_1 U \mid U)_2 &\geq \kappa_1 \|U\|_2^2 && (U \in \mathrm{R}(\boldsymbol{\Gamma}^*)), \\ \operatorname{Re}(\boldsymbol{B}_2 U \mid U)_2 &\geq \kappa_2 \|U\|_2^2 && (U \in \mathrm{R}(\boldsymbol{\Gamma})), \\ \|\boldsymbol{B}_i U\|_2 &\leq K_i \|U\|_2 && (U \in H). \end{aligned}$$

(H3) $\boldsymbol{B}_2\boldsymbol{B}_1$ maps $\mathrm{R}(\boldsymbol{\Gamma}^*)$ into $\mathrm{N}(\boldsymbol{\Gamma}^*)$ and $\boldsymbol{B}_1\boldsymbol{B}_2$ maps $\mathrm{R}(\boldsymbol{\Gamma})$ into $\mathrm{N}(\boldsymbol{\Gamma})$.

Indeed, in (H2) we can take $\kappa_1 = 1$ and $\kappa_2 = \lambda$, see (5.9) and also (5.3). Abstract Hilbert space theory therefore yields that $\boldsymbol{\Pi_B}$ is bisectorial [16, Prop. 2.5] and that the *unperturbed Dirac operator* $\boldsymbol{\Pi} := \boldsymbol{\Gamma} + \boldsymbol{\Gamma}^*$ is self-adjoint [16, Cor. 4.3].

Suppose that $\boldsymbol{\Pi_B}$ even has a bounded H^∞-calculus on $\overline{\mathrm{R}(\boldsymbol{\Pi_B})} \subseteq H$. Then $\mathrm{D}(\sqrt{\boldsymbol{\Pi_B^2}}) = \mathrm{D}(\boldsymbol{\Pi_B})$ follows with equivalent homogeneous graph norms, see Example 1.4.12. In both operators the first component acts independently of the others and is defined on $\mathrm{D}(\sqrt{\boldsymbol{L}})$ and $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$, respectively. In the light of Proposition 1.4.8 (i), this gives $\mathrm{D}(\sqrt{\boldsymbol{L}}) = \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$ and the Kato estimate

$$\|u\|_2 + \|\nabla u\|_2 \approx \left\| \boldsymbol{\Pi_B} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \right\|_2 \approx \left\| \sqrt{\boldsymbol{\Pi_B^2}} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \right\|_2 = \|\sqrt{\boldsymbol{L}} u\|_2 \quad (u \in \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m).$$

Implicit constants depend on $\boldsymbol{L}$ only through the bound for the H^∞-calculus for $\boldsymbol{\Pi_B}$. In order to prove Theorem 5.0.1 under the additional interior thickness assumption we have to argue that $\boldsymbol{\Pi_B}$ indeed has a bounded H^∞-calculus with a bound that depends on $\boldsymbol{L}$ only through its coefficient bounds, or what is equivalent thereto by McIntosh's Theorem 1.4.10, that $\boldsymbol{\Pi_B}$ satisfies the quadratic estimate

$$\int_0^\infty \|t\boldsymbol{\Pi_B}(1+t^2\boldsymbol{\Pi_B^2})^{-1}U\|_2^2 \,\frac{\mathrm{d}t}{t} \approx \|U\|_2^2 \qquad (U \in \overline{\mathrm{R}(\boldsymbol{\Pi_B})}) \tag{5.15}$$

with the same dependency of the implicit constants.

5.4.2. Quadratic estimates for Dirac operators

There are general frameworks of perturbed Dirac operators [15, 16, 39], each of which starts from a triple of operators $(\boldsymbol{\Gamma}, \boldsymbol{B}_1, \boldsymbol{B}_2)$ on H that verifies the assumptions (H1) - (H3). Additional hypotheses (H4) - (H7) on the operators and certain geometric assumptions are required in order to obtain (5.15).

We will soon see that the operators in (5.14) verify (H4) - (H7) from [39]. For the time being, we take that for granted and discuss the geometric assumptions. There are four of them [39, Ass. 2.1]:

($\boldsymbol{O}$) Comparability $|B \cap \boldsymbol{O}| \approx |B|$ holds uniformly for all balls B of radius $\mathrm{r}(B) \leq 1$ centered in $\boldsymbol{O}$.

($\partial\boldsymbol{O}$) Comparability $\mathcal{H}^{d-1}(B \cap \partial\boldsymbol{O}) \approx r^{d-1}$ holds uniformly for all balls B of radius $\mathrm{r}(B) \leq 1$ centered in $\partial\boldsymbol{O}$.

(V) Multiplication by $\mathrm{C}_0^\infty(\mathbb{R}^d)$-functions maps $V = \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$ (the form domain) into itself and there exists a bounded extension operator $E : V \to \mathrm{W}^{1,2}(\mathbb{R}^d)^m$.

(α) For some $\alpha \in (0,1)$ the complex interpolation space $[\mathrm{L}^2(\boldsymbol{O})^m, V]_\alpha$ coincides with $\mathrm{W}^{\alpha,2}(\boldsymbol{O})^m$ up to equivalent norms.

In [39] the terminology *domain* was used for *non-empty proper open subset* and $\boldsymbol{O}$ was named Ω. Their central result [39, Thm. 3.2] is as follows.

Theorem 5.4.1. *Under the structural assumptions (H1) - (H7) and the geometric assumptions* ($\boldsymbol{O}$), ($\partial\boldsymbol{O}$), (V), (α) *the operator* $\boldsymbol{\Pi_B}$ *is bisectorial and satisfies* (5.15). *Implicit constants depend on* $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ *only through the parameters quantified in (H2).*

Let us see how this relates to our geometric assumptions. Clearly ($\boldsymbol{O}$) is just the same as (5.4). Theorem 2.1.7 yields (α) for every $\alpha \in (0, \frac{1}{2})$.

(Complex interpolation of (at most) countable products of spaces works componentwise [93, Sec. 1.18.1]). Componentwise application of Corollary 5.1.12 furnishes the extension operator in (V) and the stability property follows since multiplication by $\mathrm{C}_0^\infty(\mathbb{R}^d)$-functions is bounded on $\mathrm{W}^{1,2}(\boldsymbol{O})^m$ and maps the dense subset $\mathrm{C}_{\boldsymbol{D}}^\infty(\boldsymbol{O})^m \subseteq V$ into itself.

Our major point here is that $(\partial\boldsymbol{O})$ can be replaced by the significantly weaker assumption that $\partial\boldsymbol{O}$ is porous. By Corollary 5.1.10 our boundary $\partial\boldsymbol{O}$ has the latter property.

Fortunately, the reader does not have to go through all of [39] in order to see why this relaxation of geometric assumptions works. Indeed, as is clearly stated in that paper before Lemma 7.6, $(\partial\boldsymbol{O})$ is used only once, namely to ensure validity of the following lemma (with a constant $\hat{\eta} > 0$ that happens to be 1 under $(\partial\boldsymbol{O})$). Compare also with their Corollary 7.8.

Lemma 5.4.2. *If $\boldsymbol{O} \subseteq \mathbb{R}^d$ is open and satisfies $(\partial\boldsymbol{O})$, then for each $r_0, t_0 > 0$ there exists $C > 0$ and $\hat{\eta} > 0$ such that*

$$\left|\left\{x \in \boldsymbol{O} : |x - x_0| < r,\ \mathrm{d}(x, \mathbb{R}^d \setminus \boldsymbol{O}) \leq tr\right\}\right| \leq C t^{\hat{\eta}} r^d \tag{5.16}$$

for all $x_0 \in \overline{\boldsymbol{O}}$, $r \in (0, r_0]$ and $t \in (0, t_0]$.

Hence, our only task in relaxing $(\partial\boldsymbol{O})$ is to reprove that lemma under the mere assumption that $\partial\boldsymbol{O}$ is porous, which we will do now.

Proof of Lemma 5.4.2 assuming only that $\partial\boldsymbol{O}$ is porous. Let E be the set in (5.16). For $t \geq 1$ the trivial bound $|E| \leq |B(x_0, r)| \lesssim r^d$ is enough. Thus, we can assume $t < 1$.

To each $x \in E$ there corresponds some $x_\partial \in \partial\boldsymbol{O}$ with $|x - x_\partial| \leq tr$. Since different $x \in E$ are at distance less than $2r$ from each other, we can pick a ball B of radius $4r$ centered in $\partial\boldsymbol{O}$ that contains all x_∂. Temporarily assume $4r \leq 1$. Then we can use Lemma A.1.8 and obtain $C \geq 1$ and $0 < s < d$ such that $B \cap \partial\boldsymbol{O}$ can be covered by at most $C(4/t)^s$ balls B_i of radius tr centered in $\partial\boldsymbol{O}$. Hence, each $x \in E$ is contained in one of the balls $2B_i$ and we conclude

$$|E| \lesssim (2tr)^d \#_i \leq C 2^d 4^s t^{d-s} r^d.$$

In the case $4r > 1$ we have the same type of covering property for $B \cap \partial\boldsymbol{O}$: Indeed, first we use the Vitali lemma to cover $B \cap \partial\boldsymbol{O}$ by $10^d(4r)^d \leq 10^d(4r_0)^d$ balls of radius 1 centered in $\partial\boldsymbol{O}$ and then we use Lemma A.1.8 with balls of radius $\frac{t}{4} \leq tr$. This affects the value of C but we can still take $\hat{\eta} := d - s$. $\square$

The upshot is that Theorem 5.4.1 yields the quadratic estimates (5.15) for $\mathbf{\Pi}_{\boldsymbol{B}}$ and hence the proof is complete once we have verified (H4) - (H7).

5.4.3. The additional Dirac operator hypotheses

Here are the additional hypotheses of [39, Sect. 5] that the operators in (5.14) have to verify. It is convenient to set $n := m(d+2)$ for the number of components of a function in $H = \mathrm{L}^2(\boldsymbol{O})^m \times \mathrm{L}^2(\boldsymbol{O})^{dm} \times \mathrm{L}^2(\boldsymbol{O})^m$.

(H4) $\boldsymbol{B}_1, \boldsymbol{B}_2$ are multiplication operators with functions in $\mathrm{L}^\infty(\boldsymbol{O}; \mathcal{L}(\mathbb{C}^n))$.

(H5) For every $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ the associated multiplication operator M_φ maps $\mathrm{D}(\mathbf{\Gamma})$ into itself. The commutator $\mathbf{\Gamma} M_\varphi - M_\varphi \mathbf{\Gamma}$ with domain $\mathrm{D}(\mathbf{\Gamma})$ acts via multiplication by some $c_\varphi \in \mathrm{L}^\infty(\boldsymbol{O}; \mathcal{L}(\mathbb{C}^n))$ and its components satisfy $|c_\varphi^{i,j}(x)| \lesssim |\nabla \varphi(x)|$ for an implicit constant that does not depend on φ.

(H6) For every open ball B centered in $\boldsymbol{O}$, and for all $U \in \mathrm{D}(\mathbf{\Gamma})$, $V \in \mathrm{D}(\mathbf{\Gamma}^*)$ both with compact support in $B \cap \boldsymbol{O}$ it follows that

$$\left| \int_{\boldsymbol{O}} \mathbf{\Gamma} U \, \mathrm{d}x \right| \lesssim |B|^{\frac{1}{2}} \|U\|_2,$$
$$\left| \int_{\boldsymbol{O}} \mathbf{\Gamma}^* V \, \mathrm{d}x \right| \lesssim |B|^{\frac{1}{2}} \|V\|_2.$$

(H7) There exist $\beta, \gamma \in (0,1]$ such that the fractional powers of $\mathbf{\Pi} = \mathbf{\Gamma} + \mathbf{\Gamma}^*$ satisfy

$$\|U\|_{[H, \mathrm{W}_{\boldsymbol{D}}^{1,2}(\boldsymbol{O})^n]_\beta} \lesssim \|(\mathbf{\Pi}^2)^{\beta/2} U\|_2 \qquad (U \in \mathrm{R}(\mathbf{\Gamma}^*) \cap \mathrm{D}(\mathbf{\Pi}^2)),$$
$$\|V\|_{[H, \mathrm{W}_{\boldsymbol{D}}^{1,2}(\boldsymbol{O})^n]_\gamma} \lesssim \|(\mathbf{\Pi}^2)^{\gamma/2} V\|_2 \qquad (V \in \mathrm{R}(\mathbf{\Gamma}) \cap \mathrm{D}(\mathbf{\Pi}^2)),$$

where $[\cdot\,,\cdot]$ denotes again the complex interpolation bracket.

In (H5) we have $\mathrm{D}(\mathbf{\Gamma}) = \mathrm{W}_{\boldsymbol{D}}^{1,2}(\boldsymbol{O})^m \times \mathrm{L}^2(\boldsymbol{O})^{dm} \times \mathrm{L}^2(\boldsymbol{O})^m$. The mapping property follows from (V) and the commutator assertion from the product rule. By duality, (H5) also holds for $\mathbf{\Gamma}^*$ with commutators $\mathbf{\Gamma}^* M_\varphi - M_\varphi \mathbf{\Gamma}^* = -c_{\overline{\varphi}}^*$.

For (H6), take a unit vector $e \in \mathbb{C}^n$ and let $\varphi \in \mathrm{C}_0^\infty(\boldsymbol{O})$ be valued in $[0,1]$ with $\varphi = 1$ on $\operatorname{supp}(U)$. We have $|U \cdot \mathbf{\Gamma}^*(\varphi e)| \leq |U|$ since $\operatorname{div}_{\boldsymbol{D}}(\varphi e) =$

$\operatorname{div}(\varphi e) = 0$ on $\operatorname{supp}(U)$, see also Section 5.2. Moreover, $\varphi \boldsymbol{\Gamma} U = \boldsymbol{\Gamma} U$ follows from (H5) using $\nabla\varphi = 0$ on $\operatorname{supp}(U)$. So, we obtain

$$\left| \int_{\boldsymbol{O}} \boldsymbol{\Gamma} U \cdot e \, \mathrm{d}x \right| = \left| \int_{\boldsymbol{O}} U \cdot \boldsymbol{\Gamma}^*(\varphi e) \, \mathrm{d}x \right| \leq \int_{\boldsymbol{O}} |U| \, \mathrm{d}x \leq |B|^{1/2} \|U\|_2,$$

which suffices since e was arbitrary. For V we simply switch the roles of $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$.

In (H7) we take $\beta = 1$. By choice of U we conclude $U \in \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \times \{0\} \times \{0\}$, so we obtain from the bounded H^∞-calculus for the self-adjoint operator $\boldsymbol{\Pi}$ and Example 1.4.12 that

$$\|U\|_{\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^n} = \|\boldsymbol{\Pi} U\|_2 \approx \|(\boldsymbol{\Pi}^2)^{1/2} U\|_2. \tag{5.17}$$

We take $\gamma \in (0, \varepsilon)$ with ε as in Theorem 5.0.2. As $\boldsymbol{\Pi}^2$ corresponds to $\boldsymbol{\Pi}^2_{\boldsymbol{B}}$ with $\boldsymbol{A} = \mathbf{1}$, $\boldsymbol{b} = \boldsymbol{c}^t = \mathbf{0}$, and $\boldsymbol{d} = 1$, we discover $-\Delta_{\boldsymbol{D}} + 1$ in the upper left corner and obtain with

$$W := \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \quad \text{that} \quad V = \begin{bmatrix} 0 \\ v \\ \nabla_{\boldsymbol{D}} v \end{bmatrix} = \boldsymbol{\Pi}\, W,$$

where $v \in \mathrm{D}(-\Delta_{\boldsymbol{D}} + 1)$. We conclude

$$\begin{aligned} \|V\|_{[H, \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^n]_\gamma} \approx \|V\|_{\mathrm{W}^{\gamma,2}(\boldsymbol{O})^n} \lesssim \|v\|_{\mathrm{W}^{1+\gamma,2}_{\boldsymbol{D}}(\boldsymbol{O})} \\ \approx \|(-\Delta_{\boldsymbol{D}} + 1)^{1/2+\gamma/2} v\|_2 = \|(\boldsymbol{\Pi}^2)^{1/2+\gamma/2} W\|_2, \end{aligned}$$

where the first step is due to Theorem 2.1.7, the second step uses Lemma 5.3.6 and the third one follows from Theorem 5.0.2. Using (5.17) and the version of Example 1.4.4 for bisectorial operators, the last term compares to $\|\boldsymbol{\Pi}(\boldsymbol{\Pi}^2)^{\gamma/2} W\|_2 = \|(\boldsymbol{\Pi}^2)^{\gamma/2} V\|_2$.

The proof of Theorem 5.0.1 is now complete under the additional assumption that the underlying open set satisfies the interior thickness condition (5.4). Note that here $\boldsymbol{O}$ takes the role of O in Theorem 5.0.1. At this point in the proof, our result already fully covers all earlier results from the literature.

5.5. Elimination of the interior thickness condition

In this section we complete the proof of Theorem 5.0.1 with the strategy sketched in Step 3 from the introduction to this chapter. In the whole section we work with two triples of domain, Dirichlet part and elliptic system: (O, D, L) will be satisfying the assumptions from Theorem 5.0.1 and for $(\boldsymbol{O}, \boldsymbol{D}, \boldsymbol{L})$ we start with $\boldsymbol{O} \subseteq \mathbb{R}^d$ open and $\boldsymbol{D} \subseteq \partial\boldsymbol{O}$ closed, but further properties including interior thickness of $\boldsymbol{O}$ will be added in the course of the proof.

5.5.1. Localization of the functional calculus to invariant open subsets

Definition 5.5.1. A *good projection* is an orthogonal projection $\mathcal{Q}$ on $\mathrm{L}^2(\boldsymbol{O})$ that commutes with bounded multiplication operators and with $\nabla_{\boldsymbol{D}}$ on $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$. In this case the inclusion map from the image $\mathcal{Q}\mathrm{L}^2(\boldsymbol{O})$ of $\mathcal{Q}$ back into $\mathrm{L}^2(\boldsymbol{O})$ is denoted by $\mathcal{Q}^*$.

Note that the inclusion map from Definition 5.5.1 coincides with the adjoint of $\mathcal{Q} : \mathrm{L}^2(\boldsymbol{O}) \to \mathcal{Q}\mathrm{L}^2(\boldsymbol{O})$, which justifies the usage of the symbol $\mathcal{Q}^*$. We let good projections act componentwise on $\mathrm{L}^2(\boldsymbol{O})^m$. The concrete example the reader should have in mind is that $\mathcal{Q}$ is the multiplication by the characteristic function of the union of connected components of $\boldsymbol{O}$ if the latter fulfills the geometric requirements from Theorem 5.0.1. We shall come back to that. In fact, we will only work with two good projections later on, but we believe that the more general localization procedure that we are going to construct in this section could prove useful elsewhere.

Let $(\mathcal{Q}_i)_{i \in I}$, $I \subseteq \mathbb{N}$, be a family of pairwise orthogonal good projections which decomposes $\mathrm{L}^2(\boldsymbol{O})^m$ in the sense that

$$\mathrm{L}^2(\boldsymbol{O})^m \cong \bigotimes_i \mathcal{Q}_i \mathrm{L}^2(\boldsymbol{O})^m \quad \text{via the isomorphism} \quad S : U \mapsto (\mathcal{Q}_i U)_i.$$

The ℓ^2-tensor notation is explained in the list of notations. We remind the reader that in such a context we also use ℓ^2-tensors of operators who act componentwise on their natural domain. Since a good projection commutes with $\nabla_{\boldsymbol{D}}$ it is also bounded on $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$. Consequently, we have

$$\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \cong \bigotimes_i \mathcal{Q}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \tag{5.18}$$

via the same isomorphism S as before.

Lemma 5.5.2. *Let $\mathcal{Q}$ be a good projection. Then $\mathcal{Q}$ commutes with $\boldsymbol{L}$ in the sense that $\mathcal{Q}\boldsymbol{L} \subseteq \boldsymbol{L}\mathcal{Q}$.*

Proof. Recall the block decomposition (5.11) of $\boldsymbol{L}$. By definition, $\mathcal{Q}$ commutes with the second and third block of that decomposition. By duality and self-adjointness of the projection, $\mathcal{Q}$ also commutes with the first block and therefore with $\boldsymbol{L}$. □

The lemma shows that $\boldsymbol{L}\mathcal{Q}_i^*$ is an operator in $\mathcal{Q}_i \mathrm{L}^2(\boldsymbol{O})^m$. More precisely, it is the part of $\boldsymbol{L}$ in $\mathcal{Q}_i \mathrm{L}^2(\boldsymbol{O})^m$ with maximal domain $\mathcal{Q}_i \, \mathrm{D}(\boldsymbol{L})$. An abstract property of functional calculi stated in Proposition 1.4.8 allows us to pull the projections in and out of the functional calculus, that is

$$\mathcal{Q}_i f(\boldsymbol{L}) \subseteq f(\boldsymbol{L})\mathcal{Q}_i \qquad \text{and} \qquad f(\boldsymbol{L}\mathcal{Q}_i^*) = f(\boldsymbol{L})\mathcal{Q}_i^*. \tag{5.19}$$

As above we get in particular $\mathrm{D}(f(\boldsymbol{L}\mathcal{Q}_i^*)) = \mathcal{Q}_i \, \mathrm{D}(f(\boldsymbol{L}))$. We use these calculation rules freely in order to give the following decomposition of the functional calculus for $\boldsymbol{L}$ via good projections.

Proposition 5.5.3. *One has*

$$f(\boldsymbol{L}) = S^{-1}\Big[\bigotimes_i f(\boldsymbol{L}\mathcal{Q}_i^*)\Big] S \quad \textit{with} \quad \mathrm{D}(f(\boldsymbol{L})) = S^{-1}\Big(\bigotimes_i \mathrm{D}(f(\boldsymbol{L}\mathcal{Q}_i^*))\Big), \tag{5.20}$$

where the equality of spaces is with equivalent norms.

Proof. If $u \in \mathrm{D}(f(\boldsymbol{L}))$, then $\mathcal{Q}_i u \in \mathrm{D}(f(\boldsymbol{L}\mathcal{Q}_i^*))$ and $\mathcal{Q}_i f(\boldsymbol{L})u = f(\boldsymbol{L}\mathcal{Q}_i^*)Q_i u$ hold for all i according to (5.19). We conclude

$$Sf(\boldsymbol{L})u = (\mathcal{Q}_i f(\boldsymbol{L})u)_i = (f(\boldsymbol{L}\mathcal{Q}_i^*)Q_i u)_i = \Big[\bigotimes_i f(\boldsymbol{L}\mathcal{Q}_i^*)\Big] Su$$

and the inclusion "$\subseteq$" of operators in (5.20) follows as S is an isomorphism. Conversely, let $(u_i)_i \in \mathrm{D}(\bigotimes_i f(\boldsymbol{L}\mathcal{Q}_i^*))$. Then $u \coloneqq \sum_i u_i$ converges in the ℓ^2 sense and we have $Su = (u_i)_i$. It remains to prove $u \in \mathrm{D}(f(\boldsymbol{L}))$. From the second identity in (5.19) we get $u_i \in \mathrm{D}(f(\boldsymbol{L}))$ for every i as well as

$$f(\boldsymbol{L})\Big[\sum_{i \in I \cap \{0,\dots,n\}} u_i\Big] = \sum_{i \in I \cap \{0,\dots,n\}} f(\boldsymbol{L}\mathcal{Q}_i^*)u_i \qquad (n \in \mathbb{N}).$$

In the limit as $n \to \infty$ the term on the right-hand side converges by definition of the domain of the tensorized operator and $\sum_{i \in I \cap \{0,\dots,n\}} u_i$ tends to u. Since $f(\boldsymbol{L})$ is a closed operator, we conclude $u \in \mathrm{D}(f(\boldsymbol{L}))$. □

5.5.2. Projections coming from indicator functions

From now on we assume that $\boldsymbol{O}$ is locally uniform near $\boldsymbol{N} := \partial\boldsymbol{O} \setminus \boldsymbol{D}$ and that there is a decomposition $\boldsymbol{O} = \bigcup_i O_i$, where the O_i are pairwise disjoint open sets. Since O_i is open and closed in $\boldsymbol{O}$, it follows $\partial O_i \subseteq \partial\boldsymbol{O}$. We put $D_i := \boldsymbol{D} \cap \partial O_i$. We write $\mathcal{P}_i$ for the orthogonal projection on $\mathrm{L}^2(\boldsymbol{O})$ induced by multiplication with $\mathbf{1}_{O_i}$. We also use the zero-extension operators

$$\mathcal{E}_i : \mathrm{L}^2(O_i) \to \mathcal{P}_i\mathrm{L}^2(\boldsymbol{O}).$$

These are unitary with $\mathcal{E}_i^*$ the pointwise restriction of functions to O_i. This allows us to identify $\mathrm{L}^2(O_i)^m$ with $\mathcal{P}_i\mathrm{L}^2(\boldsymbol{O})^m$.

We start by investigating how this identification extends to $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$.

Lemma 5.5.4. *Let $\varphi \in \mathrm{C}^\infty_{D_i}(O_i)^m$. If $E\varphi \in \mathrm{C}^\infty_{D_i}(\mathbb{R}^d)^m$ is any extension, then*

$$\mathrm{d}(\operatorname{supp}(E\varphi) \cap O_i, \boldsymbol{O} \setminus O_i) > 0,$$

and $\mathcal{E}_i\varphi \in \mathrm{C}^\infty_{\boldsymbol{D}}(\boldsymbol{O})^m$ with $\nabla\mathcal{E}_i\varphi = \mathcal{E}_i\nabla\varphi$.

Proof. For the first claim let $x \in \operatorname{supp}(E\varphi) \cap O_i$ and let $z' \in \overline{\boldsymbol{O} \setminus O_i} = \overline{\boldsymbol{O}} \setminus O_i$ realize the distance of x to $\boldsymbol{O} \setminus O_i$. Hence, we can pick some $z \in \partial O_i$ on the line segment connecting x and z'. First, consider the case that $z \in D_i$. Then

$$\mathrm{d}(x, \boldsymbol{O} \setminus O_i) = |x - z'| \geq |x - z| \geq \mathrm{d}(\operatorname{supp} E\varphi, D_i) > 0.$$

Otherwise, we are in the case $z \in \boldsymbol{N}$. If $\mathrm{d}(x, \boldsymbol{O} \setminus O_i) \geq \frac{\delta}{2}$, then we are done, so let us assume $\mathrm{d}(x, \boldsymbol{O} \setminus O_i) < \frac{\delta}{2}$, so that in particular $|x - z| < \frac{\delta}{2}$. Then there is $y \in \boldsymbol{O} \setminus O_i$ such that $|x - y| < \frac{\delta}{2}$. This gives $x, y \in \boldsymbol{N}_\delta$ and by Definition 5.1.1 we can join x and y by a continuous path in $\boldsymbol{O}$. But as $x \in O_i$ and $y \in \boldsymbol{O} \setminus O_i$, this path has to cross $\partial O_i \subseteq \partial\boldsymbol{O}$, which leads to a contradiction. Consequently, with $\rho := \min(\mathrm{d}(\operatorname{supp} E\varphi, D_i), \frac{\delta}{2})$ we get

$$\mathrm{d}(\operatorname{supp}(E\varphi) \cap O_i, \boldsymbol{O} \setminus O_i) \geq \rho.$$

For the second claim we fix a smooth function χ equal to 1 outside the ρ-neighborhood of $\boldsymbol{O} \setminus O_i$ and equal to 0 on the respective $\frac{\varrho}{2}$-neighborhood. Then $\chi E\varphi \in \mathrm{C}^\infty_0(\mathbb{R}^d)^m$ vanishes on $\boldsymbol{O} \setminus O_i$, whereas on O_i we have $\chi E\varphi = E\varphi = \varphi$ by the choice of ρ. We conclude $\mathcal{E}_i\varphi = (\chi E\varphi)|_{\boldsymbol{O}}$ and $\nabla\mathcal{E}_i\varphi = \mathcal{E}_i\nabla\varphi$ on $\boldsymbol{O}$. Finally, let $x \in \boldsymbol{D}$. Then either there is $j \neq i$ with $\mathrm{d}(x, O_j) < \frac{\varrho}{2}$, in which case we have $\mathrm{d}(x, \partial\boldsymbol{O} \setminus O_i) < \frac{\varrho}{2}$ and hence $\chi = 0$ in a neighborhood of x. Else, we have $\mathrm{d}(x, O_j) \geq \frac{\varrho}{2}$ for all $j \neq i$, so that $x \in \boldsymbol{D} \subseteq \partial\boldsymbol{O}$ implies $x \in \boldsymbol{D} \cap \partial O_i = D_i$ and hence $E\varphi = 0$ holds near x. This proves $\chi E\varphi \in \mathrm{C}^\infty_{\boldsymbol{D}}(\mathbb{R}^d)^m$. Now, $\mathcal{E}_i\varphi \in \mathrm{C}^\infty_{\boldsymbol{D}}(\boldsymbol{O})^m$ follows by restriction to $\boldsymbol{O}$. □

Proposition 5.5.5. *The $\mathcal{P}_i$ are good projections that satisfy $\mathcal{P}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O}) = \mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i)$.*

Proof. By Lemma 5.5.4 we know that $\mathcal{E}_i$ is an isometry from the dense subset $\mathrm{C}^\infty_{D_i}(O_i)$ of $\mathrm{W}^{1,2}_{D_i}(O_i)$ into $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$. Therefore, $\mathcal{E}_i$ is $\mathrm{W}^{1,2}_{D_i}(O_i) \to \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ bounded. Since $\mathcal{E}_i$ maps into the range of $\mathcal{P}_i$, we arrive at $\mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i) \subseteq \mathcal{P}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$.

On the other hand, take $\varphi \in \mathrm{C}^\infty_{\boldsymbol{D}}(\boldsymbol{O})$. Then we have $\varphi|_{O_i} \in \mathrm{C}^\infty_{D_i}(O_i)$ and

$$\mathcal{P}_i \varphi = \mathcal{E}_i(\varphi|_{O_i}) \tag{5.21}$$

is in $\mathrm{C}^\infty_{\boldsymbol{D}}(\boldsymbol{O})$ due to Lemma 5.5.4 with gradient

$$\nabla \mathcal{P}_i \varphi = \mathcal{E}_i \nabla(\varphi|_{O_i}) = \mathcal{E}_i(\nabla \varphi)|_{O_i} = \mathcal{P}_i \nabla \varphi.$$

As a consequence, we get that $\mathcal{P}_i$ is bounded on $\mathrm{C}^\infty_{\boldsymbol{D}}(\boldsymbol{O})$ for the $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$-norm. By density, $\mathcal{P}_i$ is bounded on $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ and the identity above extends to the same space, thereby showing that $\mathcal{P}_i$ is a good projection. We use this to have a second look on identity (5.21). The left-hand side is bounded on $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ by the foregoing argument and the right-hand side maps into $\mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i)$ by the very first step of this proof. We conclude $\mathcal{P}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O}) \subseteq \mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i)$ by continuity. □

From the preceding proposition and (5.18) we get the decomposition

$$\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m \cong \bigotimes_i \mathcal{P}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m = \bigotimes_i \mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i)^m \tag{5.22}$$

via the usual isomorphism S. Analogously to Section 5.2, we introduce in $\mathrm{L}^2(O_i)^m$ the divergence form operator L_i with coefficients $\boldsymbol{A}|_{O_i}$, $\boldsymbol{b}|_{O_i}$, $\boldsymbol{c}|_{O_i}$, $\boldsymbol{d}|_{O_i}$ corresponding to the sesquilinear form

$$a_i : \mathrm{W}^{1,2}_{D_i}(O_i)^m \times \mathrm{W}^{1,2}_{D_i}(O_i)^m \to \mathbb{C}, \quad a_i(u,v) = \int_{O_i} \begin{bmatrix} \boldsymbol{d} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} \, \mathrm{d}x.$$

We will see momentarily that this operator is unitarily equivalent to $\boldsymbol{L}\mathcal{P}_i^*$. As in Section 5.3.3, the key lies in showing unitary equivalence for the gradients.

Lemma 5.5.6. *The operators ∇_{D_i} and $\nabla_{\boldsymbol{D}} \mathcal{P}_i^*$ are unitarily equivalent via $\mathcal{E}_i \nabla_{D_i} = \nabla_{\boldsymbol{D}} \mathcal{P}_i^* \mathcal{E}_i$.*

Proof. The two operators have the same domain $\mathrm{W}^{1,2}_{D_i}(O_i)$ by definition and Proposition 5.5.5. Moreover, they have the same action as both appearing gradients are restrictions of the respective distributional gradient, which commutes with the zero extension operator. $\square$

Proposition 5.5.7. *The operators L_i and $\boldsymbol{L}\mathcal{P}_i^*$ are unitarily equivalent via $\mathcal{E}_i L_i = \boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i$.*

Proof. First of all, we note that $\mathrm{D}(L_i)$ and $\mathrm{D}(\boldsymbol{L}\mathcal{P}_i^*\mathcal{E}_i)$ are both subsets of $\mathrm{W}^{1,2}_{D_i}(O_i)^m$. For the first operator this holds by definition, whereas for the second one it follows from $\mathrm{D}(\boldsymbol{L}) \subseteq \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$ and Proposition 5.5.5. Next, let $u, v \in \mathrm{W}^{1,2}_{D_i}(O_i)^m$ and let $Ev \in \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$ be any extension of v. Lemma 5.5.6 yields

$$
\begin{aligned}
a_i(u,v) &= \int_{\boldsymbol{O}} \begin{bmatrix} \boldsymbol{d} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \mathcal{E}_i u \\ \mathcal{E}_i \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} Ev \\ \nabla Ev \end{bmatrix}} \, \mathrm{d}x \\
&= \int_{\boldsymbol{O}} \begin{bmatrix} \boldsymbol{d} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \mathcal{P}_i^* \mathcal{E}_i u \\ \nabla(\mathcal{P}_i^* \mathcal{E}_i u) \end{bmatrix} \cdot \overline{\begin{bmatrix} Ev \\ \nabla Ev \end{bmatrix}} \, \mathrm{d}x = \boldsymbol{a}(\mathcal{P}_i^* \mathcal{E}_i u, Ev).
\end{aligned}
$$

If $u \in \mathrm{D}(L_i)$, then the left-hand side becomes $(L_i u \,|\, v)_{\mathrm{L}^2(O_i)^m} = (\mathcal{E}_i L_i u \,|\, Ev)_{\mathrm{L}^2(\boldsymbol{O})^m}$. Since Ev can be any function in $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$, we conclude that $\mathcal{P}_i^* \mathcal{E}_i u \in \mathrm{D}(\boldsymbol{L})$ with $\boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i u = \mathcal{E}_i L_i u$. Conversely, let $u \in \mathrm{D}(\boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i)$. We take $Ev = \mathcal{E}_i v$, which is admissible by Proposition 5.5.5, and the right-hand side becomes $(\boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i u \,|\, \mathcal{E}_i v)_{\mathrm{L}^2(\boldsymbol{O})^m} = (\mathcal{E}_i^* \boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i u \,|\, v)_{\mathrm{L}^2(O_i)^m}$. This proves $u \in \mathrm{D}(L_i)$ with $L_i u = \mathcal{E}_i^* \boldsymbol{L}\mathcal{P}_i^* \mathcal{E}_i u$. $\square$

Let us summarize the situation. The set $\boldsymbol{O}$ is locally uniform near $\partial\boldsymbol{O} \setminus \boldsymbol{D}$ and can be decomposed into pairwise disjoint open sets O_i. The divergence form operator $\boldsymbol{L}$ is given by coefficients $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ on $\boldsymbol{O}$ and L_i is the divergence form operator on O_i whose coefficients are obtained by restricting the coefficients of $\boldsymbol{L}$ to O_i, and which is subject to a vanishing trace condition on $D_i = \boldsymbol{D} \cap \partial O_i$. Combining all intermediate steps, we derive the following correspondence.

Proposition 5.5.8. *The following are equivalent:*

(i) $\mathrm{D}(\sqrt{\boldsymbol{L}}) = \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m$ *with* $\|\sqrt{\boldsymbol{L}}u\|_2 \approx \|u\|_{\mathrm{W}^{1,2}(\boldsymbol{O})^m}$,

(ii) $\mathrm{D}(\sqrt{L_i}) = \mathrm{W}^{1,2}_{D_i}(O_i)^m$ *with* $\|\sqrt{L_i}u\|_2 \approx \|u\|_{\mathrm{W}^{1,2}(O_i)^m}$ *for all i, where the implicit constants are independent of i.*

Proof. Proposition 5.5.7 gives $\boldsymbol{L}\mathcal{P}_i^* = \mathcal{E}_i L_i \mathcal{E}_i^*$. Then we use Proposition 5.5.3 for $f(z) \coloneqq \sqrt{z}$ and the compatibility of the functional calculus with unitary equivalences in Proposition 1.4.8.(ii) to figure out that

$$\mathrm{D}(\sqrt{\boldsymbol{L}}) = S^{-1}\Big(\bigotimes_i \mathrm{D}(\sqrt{\boldsymbol{L}\mathcal{P}_i^*})\Big) = S^{-1}\Big(\bigotimes_i \mathcal{E}_i \, \mathrm{D}(\sqrt{L_i})\Big).$$

On the other hand, we conclude from (5.22) that

$$\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m = S^{-1}\Big(\bigotimes_i \mathcal{P}_i \mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})^m\Big) = S^{-1}\Big(\bigotimes_i \mathcal{E}_i \mathrm{W}^{1,2}_{D_i}(O_i)^m\Big).$$

Both chains of equalities are topological. Now, (ii) is the same as saying that the tensor spaces on the right-hand sides coincide up to equivalent norms. Since S is an isomorphism, this is equivalent to (i). □

5.5.3. Embedding of $\boldsymbol{O}$ into an interior thick set

Now, we reverse the order of reasoning by embedding the geometric configuration (O, D) in a "fattened version" $(\boldsymbol{O}, \boldsymbol{D})$ with the same geometric quality but additionally satisfying the interior thickness condition. This is the content of the following proposition, the proof of which will occupy the rest of the section.

Proposition 5.5.9. *Let O and D be as in Theorem 5.0.1 and put $N \coloneqq \partial O \setminus D$. Then there exists an open interior thick set $\boldsymbol{O} \supseteq O$ such that $\boldsymbol{O} \setminus O$ is open and $\partial O \subseteq \partial \boldsymbol{O}$. With $\boldsymbol{D} \coloneqq \partial \boldsymbol{O} \setminus N$ one has that $\boldsymbol{D}$ is closed and Ahlfors–David regular, $D \subseteq \boldsymbol{D}$ and $\boldsymbol{O}$ is locally uniform near $\boldsymbol{N} \coloneqq N$. In particular, $\boldsymbol{D} \cap \partial O = D$.*

By assumption O is locally an (ε, δ)-domain near N. Let Σ denote a grid of open axis-parallel cubes of diameter $\frac{\delta}{8}$ in $\mathbb{R}^d$. Let Σ' contain those cubes Q from Σ for which $\overline{Q}$ intersects D but which stay away from N, say $Q \cap N_{\delta/4} = \emptyset$ for good measure. Now, put

$$\boldsymbol{O} \coloneqq O \cup \bigcup_{Q \in \Sigma'} (Q \setminus \partial O) \quad \text{and} \quad \boldsymbol{D} \coloneqq \partial \boldsymbol{O} \setminus N.$$

Clearly $\boldsymbol{O}$ is an open superset of O. Moreover, $\boldsymbol{D}$ is a closed superset of D because $\boldsymbol{O} \setminus O$ stays away from N and hence the relative openness of N in ∂O is inherited to $\partial \boldsymbol{O}$. We have to verify the following conditions:

(a) $\boldsymbol{O} \setminus O$ is open and $\partial O \subseteq \partial \boldsymbol{O}$,

(b) $\boldsymbol{O}$ is locally uniform near $\boldsymbol{N}$,

(c) $\boldsymbol{O}$ is interior thick, that is, it satisfies (5.4), and

(d) $\boldsymbol{D}$ satisfies the Ahlfors–David condition.

Then $D \subseteq \boldsymbol{D} \subseteq {}^c N$ implies $\boldsymbol{D} \cap \partial O = D$.

Proof of (a). We have $\boldsymbol{O} \setminus O = \bigcup_{Q \in \Sigma'} (Q \setminus \overline{O})$, which is an open set. Since O is open as well, $\partial O \subseteq \partial \boldsymbol{O}$ follows. □

Proof of (b). From the construction of Σ' we get $\boldsymbol{O} \cap N_{\delta/4} = O \cap N_{\delta/4}$. This already gives the ε-cigar condition with δ replaced by $\delta/4$. Since O is open and closed in $\boldsymbol{O}$, connected components of O are also connected components of $\boldsymbol{O}$ and hence satisfy the positive radius condition by assumption. All remaining connected components keep distance to N and are therefore not considered in the positive radius condition. Consequently, $\boldsymbol{O}$ is an $(\varepsilon, \frac{\delta}{4})$-domain near $\boldsymbol{N} = N$. □

The boundary ∂O is porous by Corollary 5.1.10. This implies $|\partial O| = 0$, see Lemma A.1.1. Since the cubes in Σ are all of the same size, we can also record the following observation. We will freely use these facts from now on.

Lemma 5.5.10. *Each cube $Q \in \Sigma$ is interior thick and has Ahlfors–David regular boundary, where implicit constants depend only on δ and d.*

Proof of (c). Let B be a ball of radius $r = \mathrm{r}(B) \leq 1$ with center $x \in \boldsymbol{O}$. If $x \in \overline{Q}$ for some $Q \in \Sigma'$, then $|B \cap \boldsymbol{O}| \geq |B \cap Q| \gtrsim r^d$ with implicit constant depending on δ and d. Otherwise, we must have $x \in O$. If additionally $x \in N_{\delta/2}$, then Proposition 5.1.7 yields the desired lower bound $|B \cap \boldsymbol{O}| \gtrsim r^d$.

It remains to treat the case $x \in O \setminus N_{\delta/2}$. Let Q' be a cube in the grid Σ whose closure contains x. Again, if $Q' \subseteq O$, then $|B \cap \boldsymbol{O}| \geq |B \cap Q'|$ and we are done. If not, then Q' intersects ${}^c O$ and from $x \in O$ we conclude that Q' contains some $z \in \partial O$. By the size of Q' we infer $Q' \cap N_{\delta/4} = \emptyset$. Therefore, we have $z \in D$, which implies $Q' \in \Sigma'$ and we are back in the very first case. □

We continue with the following

Lemma 5.5.11. *One has $\boldsymbol{D} = D \cup \bigcup_{Q \in \Sigma'} (\partial Q \setminus O)$.*

Proof. We show that

$$\partial \boldsymbol{O} = \partial O \cup \bigcup_{Q \in \Sigma'} (\partial Q \setminus O)$$

since then the lemma follows by intersection with ${}^cN$, taking into account $\mathrm{d}(Q, N) \geq \frac{\delta}{4}$ for $Q \in \Sigma'$.

Let $x \in \partial \boldsymbol{O}$. Then $x \notin O$ since $\boldsymbol{O}$ is open and contains O. If every neighborhood of x intersects O, then $x \in \partial O$.

Otherwise, there is some open ball B with center x that is disjoint to O. However, as every neighborhood of x intersects $\boldsymbol{O}$ by assumption, there must be some $Q' \in \Sigma'$ such that $B \cap Q' \neq \emptyset$. Since Σ is a grid, the ball B only intersects finitely many cubes in Σ'. We conclude that the sequence of balls $(\frac{1}{n}B)_{n\in\mathbb{N}}$ hits some cube $Q \in \Sigma'$ infinitely often. Thus, x is in the closure of Q. But $x \notin \boldsymbol{O}$ and $x \notin \overline{O}$ imply $x \notin Q$, hence we must have $x \in \partial Q \setminus O$. This completes the proof of the inclusion "$\subseteq$".

Conversely, let $x \in \partial O \cup \bigcup_{Q\in\Sigma'}(\partial Q \setminus O)$. If $x \in \partial O$, then $x \in \partial \boldsymbol{O}$ follows from (a). Otherwise, there is some $Q' \in \Sigma'$ such that $x \in \partial Q' \setminus O$. Since Σ is a grid of open cubes, this implies $x \notin Q$ for every $Q \in \Sigma$. Hence, we have $x \notin \boldsymbol{O}$. But each neighborhood of x intersects Q' and since the boundary of an open set has no interior points, it also intersects $Q' \setminus \partial O \subseteq \boldsymbol{O}$. Hence, we have $x \in \partial \boldsymbol{O}$. □

Proof of (d). For the rest of the section let B be some ball with center x in $\boldsymbol{D}$ and radius $r = \mathrm{r}(B) \in (0, \operatorname{diam}(\boldsymbol{D}))$. Our task is to show comparability

$$\mathcal{H}^{d-1}(\boldsymbol{D} \cap B) \approx r^{d-1} \tag{5.23}$$

and we organize the argument in the cases coming from Lemma 5.5.11. By the same lemma we have $\operatorname{diam}(\boldsymbol{D}) = \infty$ if and only if we have $\operatorname{diam}(D) = \infty$. Hence, we can use the Ahlfors—David condition for D with balls up to radius say $2\operatorname{diam}(\boldsymbol{D})$ owing to Lemma A.1.4.

Case 1: $x \in D$. The lower bound in (5.23) follows directly from the Ahlfors–David condition for D:

$$\mathcal{H}^{d-1}(\boldsymbol{D} \cap B) \geq \mathcal{H}^{d-1}(D \cap B) \gtrsim r^{d-1}.$$

For the upper bound we need to make sure that B does not intersect too many cubes in Σ'. Consider the subcollection

$$\Sigma'_B \coloneqq \{Q \in \Sigma' : \overline{Q} \cap B \neq \emptyset\}. \tag{5.24}$$

If $r \leq \delta$, then $\#\Sigma'_B \lesssim 1$ by the grid size of Σ and we obtain

$$\mathcal{H}^{d-1}\Bigg(\bigcup_{Q\in\Sigma'} \partial Q \cap B\Bigg) \leq \sum_{Q\in\Sigma'_B} \mathcal{H}^{d-1}(\partial Q \cap B) \lesssim (\#\Sigma'_B) r^{d-1} \lesssim r^{d-1}, \tag{5.25}$$

where the third step uses the upper bound in the Ahlfors–David condition for the boundaries of the cubes $Q \in \Sigma$ as in Lemma 5.5.10, even though B is not necessarily centered in ∂Q. This is not an issue because if $\partial Q \cap B$ is not empty, then B is contained in a ball with doubled radius centered in ∂Q.

If $r > \delta$, then we need the following lemma to bound the size of Σ'_B. Its proof is similar to that of Lemma 5.4.2.

Lemma 5.5.12. *Let $x \in D$, $r \in (0, 2\operatorname{diam}(\boldsymbol{D}))$, $h \in (0, r)$ and consider the set*

$$E_{r,h} \coloneqq \{y \in \mathrm{B}(x,r) : \mathrm{d}(y, D) \leq h\}.$$

Then $|E_{r,h}| \lesssim hr^{d-1}$, where the implicit constant only depends on D and d.

Proof. For convenience, put $E \coloneqq E_{r,h}$ and fix $y' \in E$. Associate to each $y \in E$ some $y_D \in D$ with $|y - y_D| \leq h$. We claim that $B \coloneqq \mathrm{B}(y'_D, 4r) \supseteq \{y_D : y \in E\}$. Indeed,

$$|y_D - y'_D| \leq |y_D - y| + |y - y'| + |y' - y'_D| < h + 2r + h < 4r.$$

Moreover, there is some $C > 0$ such that $B \cap D$ can be covered by $C(r/h)^{d-1}$ many balls B_i of radius h centered in D, see Lemma A.1.5. Next, pick $y \in E$. Then $y_D \in B \cap D$ and therefore $y_D \in B_i$ for some $i \in I$. Thus, $y \in 2B_i$ and consequently $E \subseteq \bigcup_i 2B_i$. Finally,

$$|E| \leq \sum_i |2B_i| \lesssim \#_i h^d \lesssim (r/h)^{d-1} h^d = hr^{d-1}. \qquad \square$$

Coming back to finding a substitute for (5.25) in the case $r > \delta$, we claim that $\bigcup \Sigma'_B \subseteq E_{2r,\delta/8}$. Indeed, let $Q \in \Sigma'_B$ and $y \in Q$. Since $\overline{Q}$ intersects D, we have $\mathrm{d}(y, D) \leq \operatorname{diam}(Q) = \frac{\delta}{8}$, and by definition of Σ'_B in (5.24) there is some $z \in \overline{Q} \cap B$ so that

$$|x - y| \leq |x - z| + |z - y| \leq r + \operatorname{diam}(Q) \leq r + \frac{\delta}{8} < 2r.$$

Owing to Lemma 5.5.12, we can now do the following counting argument:

$$(\#\Sigma'_B)\delta^d \approx \Big|\bigcup \Sigma'_B\Big| \leq |E_{2r,\delta/8}| \lesssim \delta r^{d-1}.$$

It follows that $\#\Sigma'_B \lesssim (r/\delta)^{d-1}$, which in turn gives

$$\mathcal{H}^{d-1}\Bigg(\bigcup_{Q \in \Sigma'} \partial Q \cap B\Bigg) \leq \sum_{Q \in \Sigma'_B} \mathcal{H}^{d-1}(\partial Q) \approx (\#\Sigma'_B)\delta^{d-1} \lesssim r^{d-1}.$$

This is the same upper bound as in the case $r < \delta$, see (5.25). Hence, in both cases the Ahlfors–David condition for D allows us to estimate

$$\mathcal{H}^{d-1}(\boldsymbol{D} \cap B) \leq \mathcal{H}^{d-1}(D \cap B) + \mathcal{H}^{d-1}\Bigg(\bigcup_{Q \in \Sigma'} \partial Q \cap B \Bigg) \lesssim r^{d-1},$$

which gives the required upper bound in (5.23).

Case 2: $x \in \partial Q \setminus O$ for some $Q \in \Sigma'$. We distinguish whether $\frac{1}{2}B$ is disjoint to D or not. So, let us first suppose that $\frac{1}{2}B \cap D \neq \emptyset$ and let $z \in \frac{1}{2}B \cap D$. Then $\mathrm{B}(z, \frac{r}{2})$ is centered in D and contained in B, so that we obtain the lower bound from the Ahlfors–David condition for D applied to $\mathrm{B}(z, \frac{r}{2})$. For the upper bound, we use Case 1 applied to $\mathrm{B}(z, 2r) \supseteq B$.

Now, suppose that $\frac{1}{2}B \cap D = \emptyset$. By construction of Σ' we have $\mathrm{d}(x, D) \leq \frac{\delta}{8}$ and $\mathrm{d}(x, N) \geq \frac{\delta}{4}$. Hence $\frac{r}{2} \leq \frac{\delta}{8}$ and $\frac{1}{2}B$ does not intersect $\partial O = D \cup N$. Since this ball is centered outside of O and does not intersect ∂O, we must have $\frac{1}{2}B \subseteq {}^c O$. This shows $\partial Q \cap \frac{1}{2}B = (\partial Q \setminus O) \cap \frac{1}{2}B$ and the lower bound in (5.23) follows from the Ahlfors–David condition for ∂Q. For the upper bound we argue as in Case 1 with radii $r \leq \delta$.

This concludes the proof of (d) and hence the proof of Proposition 5.5.9. $\square$

5.5.4. Proof of Theorem 5.0.1

We combine Theorem 5.5.8 with Proposition 5.5.9.

Proof of Theorem 5.0.1. Given O and D, construct sets $\boldsymbol{O}$ and $\boldsymbol{D}$ according to Proposition 5.5.9. To ease the connection with Section 5.5.2 we put $O_0 := O$ and $O_1 := \boldsymbol{O} \setminus O$. Then $D_0 := \boldsymbol{D} \cap \partial O = D$ and $D_1 := \boldsymbol{D} \cap \partial O_1 = \partial O_1$. We extend the coefficients A, b, c, d to coefficients $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ on $\boldsymbol{O}$. For $\boldsymbol{A}$ and $\boldsymbol{d}$ we put the corresponding identity matrix on $\boldsymbol{O} \setminus O$ to ensure ellipticity and $\boldsymbol{b}$, $\boldsymbol{c}$ are simply extended by zero. With those extended coefficients we define the operator $\boldsymbol{L}$ with form domain $\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O})$ as in (5.11). Since by (5.22) the form domain for $\boldsymbol{L}$ splits as

$$\mathrm{W}^{1,2}_{\boldsymbol{D}}(\boldsymbol{O}) \cong \mathrm{W}^{1,2}_{D_0}(O_0) \otimes \mathrm{W}^{1,2}_{D_1}(O_1),$$

we get that the coefficients for $\boldsymbol{L}$ are again elliptic in the sense of (5.3) with lower bound $\min(\lambda, 1)$ and upper bound $\max(\Lambda, 1)$. Here, Λ is an upper bound for the coefficients of L. Proposition 5.5.7 reveals that L is unitarily equivalent to $\boldsymbol{L}\mathcal{P}_0^*$. It was shown in Section 5.4 that Theorem 5.0.1 is valid on interior thick sets. Consequently, we can apply that theorem to the operator $\boldsymbol{L}$ on $\boldsymbol{O}$. But this brings us into the business of Proposition 5.5.8 and we can conclude the square root property for L. $\square$

CHAPTER 6

Kato's square root property: L^p–Theory

As in Chapter 5, we consider a second order elliptic $(m \times m)$-system L in divergence form on an open and possibly unbounded set $O \subseteq \mathbb{R}^d$, $d \geq 2$, with bounded measurable complex coefficients, formally given by

$$Lu = -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) - \sum_{i=1}^{d} \partial_i(b_i u) + \sum_{j=1}^{d} c_j \partial_j u + du.$$

Here, $(m \times m)$-system means that the coefficients are valued in $\mathcal{L}(\mathbb{C}^m)$ and that u takes its values in $\mathbb{C}^m$. Let D be a closed, possibly empty, subset of the boundary ∂O. We complement L with Dirichlet boundary conditions on D and Neumann boundary conditions on $N \coloneqq \partial O \setminus D$. The pair (O, D) is always assumed to be regular in the sense of Chapter 5, which is to say that D is Ahlfors–David regular and O is locally uniform near $\partial O \setminus D$.

To be more precise, let $V \coloneqq \mathrm{W}^{1,2}_D(O)^m$ be the $\mathrm{W}^{1,2}(O)^m$-closure of smooth functions that vanish in a neighborhood of D, then we interpret L as the maximal accretive operator in $\mathrm{L}^2(O)^m$ associated with the sesquilinear form $a : V \times V \to \mathbb{C}$ defined by

$$a(u,v) = \int_O \sum_{i,j=1}^{d} a_{ij}\partial_j u \cdot \overline{\partial_i v} + \sum_{i=1}^{d} b_i u \cdot \overline{\partial_i v} + \sum_{j=1}^{d} c_j \partial_j u \cdot \overline{v} + du \cdot \overline{v} \,\mathrm{d}x, \tag{6.1}$$

which we assume to satisfy for some $\lambda > 0$ the (inhomogeneous) *Gårding* *inequality*

$$\operatorname{Re} a(u,u) \geq \lambda(\|u\|_2^2 + \|\nabla u\|_2^2) \qquad (u \in V). \tag{6.2}$$

The operator L possesses a square root $L^{\frac{1}{2}}$ in virtue of its sectorial functional calculus and it essentially follows from Theorem 5.0.1 that

$$L^{\frac{1}{2}} : \mathrm{W}_D^{1,2}(O)^m \to \mathrm{L}^2(O)^m \quad \text{is a topological isomorphism.}$$

In this chapter we cope with the question when $L^{\frac{1}{2}}$ extends from $\mathrm{W}_D^{1,2}(O)^m \cap \mathrm{W}_D^{1,p}(O)^m$ to a $\mathrm{W}_D^{1,p}(O)^m \to \mathrm{L}^p(O)^m$ isomorphism. The rough nature of geometry and coefficients leads to the inconvenience that such an extrapolation is not possible for all $p \in (1,\infty)$ as would be the case in classical Calderón-Zygmund theory. A review on relevant counterexamples can be found in [36, Introduction].

Let us come to positive results. In the whole space situation, a pretty clear picture was given by Auscher in [5]. Up to the endpoints, the range in which $L^{\frac{1}{2}}$ extends to an isomorphism is determined by boundedness properties of the semigroup generated by $-L$ and its gradient family. Put

$$\mathcal{J} \coloneqq \{p \in [1,\infty] \colon \{\mathrm{e}^{-tL}\}_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\},$$
$$\mathcal{I} \coloneqq \{p \in [1,\infty] \colon \{\sqrt{t}\nabla \mathrm{e}^{-tL}\}_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\}.$$

The sets $\mathcal{J}$ and $\mathcal{I}$ are intervals in virtue of interpolation. For brevity, write $\mathcal{S} \coloneqq \mathcal{S}(L) \coloneqq \{\mathrm{e}^{-tL}\}_{t>0}$ and $\mathcal{N} \coloneqq \mathcal{N}(L) \coloneqq \{\sqrt{t}\nabla \mathrm{e}^{-tL}\}_{t>0}$ for the involved operator families in the sequel. The precise meaning of boundedness of these families will be clarified in Definition 6.1.1. With these families define the *critical numbers*

$$p_-(L) \coloneqq \inf \mathcal{J}, \qquad p_+(L) \coloneqq \sup \mathcal{J},$$
$$q_-(L) \coloneqq \inf \mathcal{I}, \qquad q_+(L) \coloneqq \sup \mathcal{I}.$$

For upper bounds for $p_-(L)$ and $q_-(L)$, lower bounds for $p_+(L)$ and $q_+(L)$, and relations between these numbers the reader is advised to consult [5]. Eventually, it can be shown that $L^{\frac{1}{2}}$ extends to a $\mathrm{W}^{1,p} \to \mathrm{L}^p$ isomorphism when $p_-(L) < p < q_+(L)$.

A first major step towards a satisfactory L^p-theory for the square root property in the case of mixed boundary conditions was accomplished by Auscher, Badr, Haller-Dintelmann, and Rehberg in [8]. They worked in bounded and d-regular domains with a Lipschitz boundary around the Neumann boundary part and a $(d-1)$-regular Dirichlet part. The most severe constraint in their setting is that they only allow real, scalar coefficients, in which case $p_-(L)$ is automatically 1. They showed the square root property for the interval $(1, 2+\varepsilon)$ for some $\varepsilon > 0$.

This result was refined by Egert in [36]. In there, complex systems were permitted and the isomorphism range is given by $(p_-(L), 2+\varepsilon)$.

These two results have in common that the geometry is dictated by the L^2-result from [38]. Due to our improvement on the L^2-theory in Chapter 5, an improvement in the L^p-theory is natural. In the range $p \in (p_-(L), 2)$, we obtain an L^p result only imposing the geometric constraints needed in the L^2-theory. For $p > 2$ we seek an "optimal" result in the spirit of the whole space result in [5], that is to say, the upper bound shall be given in terms of $q_+(L)$ and not merely some ε which cannot be quantified in a useful manner. In this situation we have to impose stronger requirements, namely pure Neumann boundary conditions, that O is an (ε, ∞)-domain near the full boundary ∂O, and that some of the lower-order coefficients vanish. We emphasize that our notion of (ε, ∞)-domain near ∂O forces O to be unbounded, and that this unbounded nature plays a crucial role in the proof. Our main result of this chapter then reads as follows.

Theorem 6.0.1. *Let $O \subseteq \mathbb{R}^d$ be an open set and $D \subseteq \partial O$ a closed subset of the boundary. Suppose that D is Ahlfors–David regular and that O is locally uniform near $N = \partial O \setminus D$ (Definition 5.1.1). Then the following holds:*

(i) *If $p_-(L) < p < 2$, then $L^{\frac{1}{2}}$ extends to an isomorphism $\mathrm{W}^{1,p}_D(O)^m \to \mathrm{L}^p(O)^m$.*

(ii) *If $2 < q < q_+(L)$, O is an (ε, ∞)-domain near ∂O, $D = \emptyset$, and the coefficients b and d vanish, then $L^{\frac{1}{2}}$ extends to an isomorphism $\mathrm{W}^{1,q}(O)^m \to \mathrm{L}^q(O)^m$.*

Upper and lower bounds depend on L only via its coefficient bounds and implied constants from L^p-bounds for $\mathcal{S}$ and L^q-bounds for $\mathcal{N}$, respectively.

Here, *coefficient bounds* refers to the lower bound in the ellipticity condition and the pointwise upper bound for the coefficients, see also Chapter 5.

Remark 6.0.2. In this thesis, we *only* allow systems in which each component is subject to Dirichlet boundary conditions on the same portion $D \subseteq \partial O$ of the boundary. In contrast, Egert's work [36] allows $\mathbb{W}^{1,2}_{\mathbb{D}}(O)$ (Definition 6.6.1) as the form domain, which roughly means that the kth component is subject to a vanishing trace condition on the set $D_k \subseteq \partial O$. As is explained in [35, Sec. 5.6], Steps 1 and 2 in Chapter 5 are perfectly compatible with such systems. However, the construction of $\boldsymbol{O}$ in Step 3 depends on D, hence it is not possible to perform Step 3 with varying Dirichlet parts. Therefore, the L^2-theory forces us to work in this restricted class of systems. However, the Calderón-Zygmund decomposition in Section 6.6 is not touched upon this, which is why we will perform it in full generality.

Outline of this chapter

We give a brief overview of this chapter. First, we deal with concepts of $\mathrm{L}^p \to \mathrm{L}^q$ boundedness and $\mathrm{L}^p \to \mathrm{L}^q$ off-diagonal estimates of families of operators on L^2 in Section 6.1. The highlight of this section is Proposition 6.1.5, in which L^2 off-diagonal estimates for L are established. Afterwards, we present "L^p-extrapolation" results in Section 6.2. These are divided into the cases $p < 2$, which is due to Blunck and Kunstmann, and $p > 2$, which is a very mature version of a good-λ argument. Section 6.3 presents the M$^{\mathrm{c}}$Intosh approximation, which is used to derive nice representation formulæ for $L^{\frac{1}{2}}$ and related operators. In Section 6.4, one of the L^p-extrapolation results comes into action and is used to show L^p-boundedness of the H^∞-calculus of L. These bounds are used later-on in Section 6.7.

To extend the square root isomorphism on L^2 to L^p, we essentially need lower and upper L^p-bounds. The key difficulty in the lower bounds are Riesz-transform bounds. These are dealt with in Section 6.5. For the case $p > 2$ we will need a conservation property and local Poincaré inequalities, which are also established in that section. The conservation property will explain why we need pure Neumann boundary conditions and is the reason why we have to require that certain coefficients vanish. The local Poincaré inequalities use the homogeneous extension theory from the end of Chapter 3. To get theses inequalities in a scale-invariant way, the unboundedness of O enters the scene.

The upper bounds for the square root are shown in Section 6.7. The case $p > 2$ follows easily by duality from the Riesz-transform bounds. For the case $p < 2$ we have to work harder (the lower bound $p_-(L) < 2$ is not enough for a duality argument). The argument relies on a Sobolev Calderón-Zygmund decomposition, which is performed in Section 6.6.

Finally, everything is assembled in Section 6.8.

Notation

If $B \subseteq \mathbb{R}^d$ is a ball, put $C_1(B) \coloneqq 4B$ and, if $j \geq 2$, put $C_j(B) \coloneqq 2^{j+1}B \setminus 2^j B$. Note that for $j \geq 2$ we have by the reverse triangle inequality $\mathrm{d}(B, C_j(B)) \geq 2^{j-1}\,\mathrm{r}(B)$. We employ the same convention with cubes instead of balls.

For $p, q \in (1, \infty)$, write

$$\gamma_{pq} \coloneqq d\left|\frac{1}{p} - \frac{1}{q}\right|.$$

If $q = 2$, write γ_p instead of γ_{p2}. The same goes for the case $p = 2$, in which we simply write γ_q.

The *upper Sobolev exponent* p^* of $p \in [1, \infty]$ is given as follows. If $p < d$, then $1/p^* \coloneqq 1/p - 1/d$. Otherwise, $p^* \coloneqq \infty$. The *lower Sobolev exponent* p_* is determined by $1/p_* \coloneqq 1/p + 1/d$.

In the course of this chapter, we will occasionally work with the semigroup generated by $-L^*$. The operator L^* is associated with the adjoint form

$$a^*(u, v) \coloneqq \overline{a(v, u)} \qquad (u, v \in \mathrm{W}^{1,2}_D(O)^m), \tag{6.3}$$

see [82, Prop. 1.24]. Clearly, the coefficient bounds of a and a^* coincide. The functional calculi of L and L^* are linked by Proposition 1.4.6.

6.1. Boundedness properties of operator families

In this section we investigate boundedness properties of several families of operators associated with the elliptic operator L. These families will occur in integral kernels later-on. In classical harmonic analysis, such kernels are usually assumed to satisfy at least weak size and regularity properties, see [48, Sec. 5.3.2]. Boundedness of these families replaces the size condition, whereas off-diagonal estimates permit us to do similar arguments as with kernel regularity.

Definition 6.1.1. Let $U \subseteq \mathbb{C} \setminus \{0\}$ and $\mathcal{T} \coloneqq \{T(z)\}_{z \in U}$ be a family of bounded operators $\mathrm{L}^2(\Xi)^{m_1} \to \mathrm{L}^2(\Xi)^{m_2}$, where m_1, m_2 are natural numbers and $\Xi \subseteq \mathbb{R}^d$ is a measurable set. Given $1 \leq p \leq q \leq \infty$, say that $\mathcal{T}$ is $\mathrm{L}^p \to \mathrm{L}^q$ *bounded* if there is a constant C such that for all $u \in \mathrm{L}^2(\Xi)^{m_1} \cap \mathrm{L}^p(\Xi)^{m_1}$ and $z \in U$ one has

$$\|T(z)u\|_{\mathrm{L}^q(\Xi)^{m_2}} \leq C|z|^{-\frac{\gamma_{pq}}{2}}\|u\|_{\mathrm{L}^p(\Xi)^{m_1}}. \tag{6.4}$$

Assume that in addition there is $c \in (0, \infty)$ such that whenever $E, F \subseteq \Xi$ and $\operatorname{supp}(u) \subseteq E$ the stronger estimate

$$\|T(z)u\|_{\mathrm{L}^q(F)^{m_2}} \leq C|z|^{-\frac{\gamma_{pq}}{2}} \mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{|z|}} \|u\|_{\mathrm{L}^p(E)^{m_1}} \tag{6.5}$$

holds. In this case, say that $\mathcal{T}$ satisfies $\mathrm{L}^p \to \mathrm{L}^q$ *off-diagonal estimates*. Finally, if $p = q$ in the above situation, we simply talk about L^p-*boundedness* and L^p *off-diagonal estimates*.

Remark 6.1.2. If $\mathcal{T}$ is a family that satisfies $\mathrm{L}^p \to \mathrm{L}^q$ off-diagonal estimates, we say that $c, C \in (0, \infty)$ are implied constants if the family $\mathcal{T}$ satisfies (6.5) with this choice of constants.

We start out with L^2 off-diagonal estimates of Gaffney type. The proof is an adaptation of that given in [5, Prop. 2.1]. The argument is based on a perturbation and the Crouzeix–Delyon Theorem, which we will recall momentarily.

Definition 6.1.3. Let T be an operator on a Hilbert space H and $\omega \in [0, \pi/2]$. Say that T is *m-ω-accretive* if its numerical range

$$\mathrm{W}(T) \coloneqq \Big\{ (Tu \,|\, u)_H \colon u \in \mathrm{D}(T) \Big\}$$

is contained in $\overline{\mathrm{S}_\omega^+}$ and $T + \varepsilon$ is invertible for some $\varepsilon > 0$.

A proof for the following result can be found in [53, Cor. 7.1.17].

Proposition 6.1.4 (Crouzeix–Delyon Theorem)**.** *Let T be an injective and m-ω-accretive operator on a Hilbert space H for some $\omega \in [0, \pi/2]$. If $f \in \mathrm{H}^\infty(\mathrm{S}_\psi^+)$ for some $\psi \in (\omega, \pi)$, then $f(T)$ is a bounded operator on H and one has the estimate*

$$\|f(T)\|_{H \to H} \leq \Big(2 + \frac{2}{\sqrt{3}}\Big) \|f\|_{\mathrm{H}^\infty(\mathrm{S}_\psi^+)}.$$

Let us come back to the Gaffney type estimates. For this, recall the meaning of the families $\mathcal{S}$ and $\mathcal{N}$ from the introduction to this chapter.

Proposition 6.1.5 (Gaffney estimates)**.** *Given $\varphi \in [0, {}^{\pi}\!/_{2} - \omega)$, the families $\{\mathrm{e}^{-zL}\}_{z \in \mathrm{S}_\varphi}$, $\{z\nabla \mathrm{e}^{-z^2 L}\}_{z \in \mathrm{S}_\varphi}$ and $\{zL\mathrm{e}^{-zL}\}_{z \in \mathrm{S}_\varphi}$ satisfy L^2 off-diagonal estimates, and the implied constants do only depend on L via its coefficient bounds.*

Proof. In the whole proof we abbreviate $V := \mathrm{W}^{1,2}_D(O)^m$. The symbol "$\lesssim$" will only be used for bounds independent of the function φ and the parameter ρ, which we introduce in a moment. The argument divides into three steps.
Step 1: Reduction to real times.

Let $z \in \mathrm{S}^+_\psi$ and write $z = |z|\mathrm{e}^{i\theta}$. Recall the coefficients A, b, c, and d appearing in the form a from the definition of L. We multiply these coefficients by $\mathrm{e}^{i\theta}$ to define a new form a_θ. Its coefficients then have the same upper bound as before and the lower bound (5.3) holds with λ replaced by $c_\psi \lambda$, where c_ψ is some constant depending on ψ. This latter fact follows by elementary trigonometric considerations. The L^2 realization of a_θ is $\mathrm{e}^{i\theta} L$ and we conclude that this operator is again sectorial, but of a smaller angle. We emphasize that the coefficient bounds of $\mathrm{e}^{i\theta} L$ do only depend on the coefficient bounds of L and ψ, but not on θ. Now, since the sectorial functional calculus is compatible with scaling, we get $zL = |z|(\mathrm{e}^{i\theta} L)$ and $\mathrm{e}^{-zL} = \mathrm{e}^{-|z|(\mathrm{e}^{i\theta} L)}$. Therefore, the estimate for, say, $zL\mathrm{e}^{-zL}$ follows from the estimate for the family $\{t(\mathrm{e}^{i\theta} L)\mathrm{e}^{-t(\mathrm{e}^{i\theta} L)}\}_{t>0}$.
Step 2: Perturbation of the sesquilinear form. Let $t > 0$, φ a bounded Lipschitz function with Lipschitz constant at most 1 and let $\rho > 0$. Define the form $b_\rho(u, v) := a(\mathrm{e}^{-\rho\varphi} u, \mathrm{e}^{\rho\varphi} v)$ for $u, v \in V$. Observe that the right-hand side is well-defined since the component-wise multiplication with $\mathrm{e}^{\pm\rho\varphi}$ preserves V. We have omitted the dependence on φ to ease notation and since all estimates will be independent of φ. Using the product and chain rule leads to

$$
\begin{aligned}
b_\rho(u, v) &= \int_O A\nabla u \cdot \overline{\nabla v} - \rho u A\nabla\varphi \cdot \overline{\nabla v} + \rho \overline{v} A\nabla u \cdot \overline{\nabla\varphi} + \rho^2 u\overline{v} A\nabla\varphi \cdot \overline{\nabla\varphi} \\
&\qquad + b\nabla u \overline{v} - \rho u b\nabla\varphi\overline{v} + uc\overline{\nabla v} + \rho u\overline{v} c\nabla\varphi + du\overline{v}\,\mathrm{d}x \\
&= a(u, v) + \int_O -\rho u A\nabla\varphi \cdot \overline{\nabla v} + \rho\overline{v} A\nabla u \cdot \overline{\nabla\varphi} + \rho^2 u\overline{v} A\nabla\varphi \cdot \overline{\nabla\varphi} \\
&\qquad - \rho u b\nabla\varphi\overline{v} + \rho u\overline{v} c\nabla\varphi\,\mathrm{d}x \\
&=: a(u, v) + c_\rho(u, v).
\end{aligned}
$$

Observe that b_ρ is a perturbation of a. Write Λ and λ for the upper and lower bounds of a. The upper bound can be chosen in such a way that is also dominates the coefficient functions. We claim that there is a constant $c \in (0, \infty)$ depending only on d, Λ and λ such that

$$
\text{(i)}\ \operatorname{Re} b_\rho(u, u) \geq \frac{\lambda}{2}\|u\|_V^2 - c\rho^2\|u\|_2^2, \qquad \text{(ii)}\ |b_\rho(u, u)| \leq 2\Lambda\|u\|_V^2 + c\rho^2\|u\|_2^2.
$$

In the course if this, we will furthermore see that $b_\rho(\cdot, \cdot)$ is a bounded form, but we only care for the precise constants when both arguments coincide.

It suffices to show that there exists such a constant c for which $|c_\rho(u,u)| \leq \min(\lambda/2, \Lambda)\|u\|_V^2 + c\rho^2\|u\|_2^2$. Then (ii) follows from the definition of Λ and the triangle inequality, whereas (i) is a consequence of

$$\operatorname{Re} b_\rho(u,u) = \operatorname{Re} a(u,u) + \operatorname{Re} c_\rho(u,u) \geq \operatorname{Re} a(u,u) - |c(u,u)|$$

and the definition of a lower bound as in (6.2).

We estimate the terms of c_ρ separately. The first-order terms are estimated in the same way, so we will only demonstrate the calculation on one of them. This being said, use the triangle inequality, the bound on the Lipschitz constant of φ, the Cauchy-Schwarz inequality and comparability of the 2-norm with the 1-norm to get

$$\Big|\int_O \rho u A\nabla\varphi \cdot \overline{\nabla v}\Big| \leq \rho \sum_j \int_O |u| \Big(\sum_i |a_{ij}\partial_i \varphi|\Big) |\partial_j v| \leq \sqrt{d}\rho d \Lambda \|u\|_2 \|v\|_V.$$

Now, specialize to $u = v$ and apply Young's inequality with ε to the right-hand side to deduce

$$\leq d\sqrt{d}\Lambda \frac{1}{2}\Big(\rho^2\varepsilon^{-2}\|u\|_2^2 + \varepsilon^2\|u\|_V^2\Big).$$

Choosing ε small enough so that $d\sqrt{d}\Lambda\frac{1}{2}\varepsilon^2 \leq \frac{1}{5}\min(\lambda/2, \Lambda)$ concludes this term. Note that the factor $1/5$ is because c_ρ consists of 5 terms.

We continue with the zeroth-order terms. For the term with factor ρ^2, the Cauchy-Schwarz inequality and the almost everywhere bound on $|\nabla\varphi|$ readily reveal

$$\Big|\int_O \rho^2 u\overline{v} A\nabla\varphi \cdot \overline{\nabla\varphi}\Big| \leq \rho^2 d^2 \Lambda \|u\|_2 \|v\|_2,$$

which is already fine. The zeroth-order terms with a factor ρ are estimated similarly, followed by the same argument using Young's inequality with ε as above.

Step 3: Operators associated with the perturbed form. It follows from (i) that $\operatorname{Re} b_\rho(u,u) + 2c\rho^2\|u\|_2^2$ is strictly positive, hence we can calculate with (i) and (ii) that

$$|\tan(b_\rho(u,u) + 2c\rho^2(u\,|\,u)_2)| \leq \frac{|\operatorname{Im} b_\rho(u,u) + 2c\rho^2\|u\|_2^2|}{\operatorname{Re} b_\rho(u,u) + 2c\rho^2\|u\|_2^2} \leq \frac{4\Lambda}{\lambda} + 3.$$

This means that the numerical range of the sesquilinear form $b_\rho(u,u) + 2c\rho^2(u\,|\,u)_2$ is contained in the sector $\overline{\mathrm{S}_\theta^+}$, where $\theta \coloneqq \arctan(4\Lambda/\lambda + 3)$. Also,

(i) and the Lax-Milgram Lemma reveal that $u \mapsto b_\rho(u, \cdot\,) + 2c\rho^2(u \,|\, \cdot\,)_2$ is an isomorphism $V \to V^*$.

Let L_ρ denote the L^2-realization of $b_\rho(\cdot,\cdot)$. As for b_ρ, we omit again the dependence on φ for the same reason. Then the numerical range of $L_\rho + 2c\rho^2$ is also contained in $\overline{\mathrm{S}_\theta^+}$ and, since $L_\rho + 2c\rho^2$ is the part of an isomorphism, it is itself invertible. But this means nothing else than that $L_\rho + 2c\rho^2$ is m-θ-accretive. Therefore, by Crouzeix–Delyon's theorem (Proposition 6.1.4), its $\mathrm{H}^\infty(\mathrm{S}_\psi^+)$-calculus is bounded with a universal constant for every $\psi \in (\theta, \pi)$.

Let $\psi \coloneqq \pi/4 + \theta/2 \in (\theta, \pi/2)$ and consider the functions $\mathrm{e}^{-t\mathbf{z}}$ and $t\mathbf{z}\mathrm{e}^{-t\mathbf{z}}$. First, their $\mathrm{H}^\infty(S_\psi^+)$-norm is independent of t because scaling is a bijection on S_ψ^+. Second, the norm is finite because $\psi < \pi/2$ and only depends on the coefficient bounds and dimension for they fully determine ψ. Therefore, the bounded H^∞-calculus of $L_\rho + 2c\rho^2$ yields

$$\|\mathrm{e}^{-tL_\rho}\|_{\mathrm{L}^2\to\mathrm{L}^2} + \|t(L_\rho + 2c\rho^2)\mathrm{e}^{-tL_\rho}\|_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim \mathrm{e}^{2c\rho^2 t}. \tag{6.6}$$

To deduce a similar estimate for the gradient of the semigroup, use the ellipticity property (i) and the fact that the semigroup maps into the domain of its generator to get

$$\begin{aligned}\|\nabla \mathrm{e}^{-tL_\rho}u\|_2^2 &\leq \frac{2}{\lambda}\operatorname{Re} b_\rho(\mathrm{e}^{-tL_\rho}u, \mathrm{e}^{-tL_\rho}u) + 2c\rho^2\|\mathrm{e}^{-tL_\rho}u\|_2^2 \\ &= \frac{2}{\lambda}\operatorname{Re}((L_\rho + 2c\rho^2)\mathrm{e}^{-tL_\rho}u \,|\, \mathrm{e}^{-tL_\rho}u).\end{aligned}$$

Multiply this identity by t, take absolute values and use the Cauchy-Schwarz inequality together with the bounds from (6.6) to conclude

$$\|\sqrt{t}\nabla \mathrm{e}^{-tL_\rho}u\|_2^2 \leq \frac{2}{\lambda}|(t(L_\rho + 2c\rho^2)\mathrm{e}^{-tL_\rho}u \,|\, \mathrm{e}^{-tL_\rho}u)| \leq \frac{2}{\lambda}\mathrm{e}^{4c\rho^2 t}\|u\|_2^2.$$

Next, we apply the square root to this inequality and take the supremum over $\|u\|_2 = 1$ to deduce the operator norm bound

$$\|\sqrt{t}\nabla \mathrm{e}^{-tL_\rho}\|_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim \mathrm{e}^{2c\rho^2 t},$$

that is, the operator family $\{\sqrt{t}\nabla \mathrm{e}^{-tL_\rho}\}_{t>0}$ satisfies the same bounds as the families in (6.6).

Step 4: Conclude using similarity between L and L_ρ. Fix $E, F \subseteq O$ and $u \in \mathrm{L}^2(O)$ with $\operatorname{supp}(u) \subseteq E$. We specialize φ and ρ from above. Define the bounded Lipschitz functions $\varphi_n(x) \coloneqq \min(\mathrm{d}_E(x), n)$ for $n \geq 1$. Their Lipschitz constants are at most 1, which follows from $[\mathrm{d}_E]_{\mathrm{Lip}} \leq 1$ and the fact that Lipschitz constants add up when taking the minimum of Lipschitz functions on $\mathbb{R}^d$. Also, put $\rho \coloneqq \frac{d}{4ct}$, where $d \coloneqq \mathrm{d}(E, F)$. Let $L_{\rho,n}$ be the operator from Step 2 with these choices of φ_n and ρ.

As already noted in Step 1, multiplication with $\mathrm{e}^{\pm\rho\varphi_n}$ leaves V invariant. By duality we get extensions to V^*. Note that $\mathrm{e}^{-\rho\varphi_n}$ is the inverse of $\mathrm{e}^{\rho\varphi_n}$. Reinterpreting $b_{\rho,n}(u,v) = a(\mathrm{e}^{-\rho\varphi_n}u, \mathrm{e}^{\rho\varphi_n}v)$ in this light, we see that the form operators $V \to V^*$ induced by these forms are similar. Since $\mathrm{e}^{\pm\rho\varphi_n}$ is also an isomorphism pair on L^2, this similarity inherits to L and $L_{\rho,n}$ for they are parts of the form operators.

Using the aforementioned similarity, it follows from Proposition 1.4.8 (ii) and $\mathrm{e}^{-\rho\varphi_n} = 1$ on E that

$$\mathrm{e}^{-tL}u = \mathrm{e}^{-\rho\varphi_n}\mathrm{e}^{\rho\varphi_n}\mathrm{e}^{-tL}\mathrm{e}^{-\rho\varphi_n}u = \mathrm{e}^{-\rho\varphi_n}\mathrm{e}^{-tL_{\rho,n}}u.$$

On F we have the crude estimate $|\mathrm{e}^{-\rho\varphi}| \leq \mathrm{e}^{-\rho\min(\mathrm{d}(E,F),n)}$. Hence, applying the $\mathrm{L}^2(F)$-norm to this identity gives

$$\|\mathrm{e}^{-tL}u\|_{\mathrm{L}^2(F)} \leq \mathrm{e}^{-\rho\min(\mathrm{d}(E,F),n)}\|\mathrm{e}^{-tL_{\rho,n}}u\|_2.$$

Using (6.6) and the support property of u gives

$$\lesssim \mathrm{e}^{-\rho(\min(d,n)-2c\rho t)}\|u\|_{\mathrm{L}^2(E)} \to \mathrm{e}^{-\frac{d^2}{8ct}}\|u\|_{\mathrm{L}^2(E)} \quad \text{as } n \to \infty.$$

Similarly, we derive the identity

$$\begin{aligned} tL\mathrm{e}^{-tL}u &= \mathrm{e}^{-\rho\varphi_n}tL_{\rho,n}\mathrm{e}^{-tL_{\rho,n}}u \\ &= \mathrm{e}^{-\rho\varphi_n}t(L_{\rho,n} + 2c\rho^2)\mathrm{e}^{-tL_{\rho,n}}u - \mathrm{e}^{-\rho\varphi_n}t2c\rho^2\mathrm{e}^{-tL_{\rho,n}}u. \end{aligned}$$

Taking the $\mathrm{L}^2(F)$-norm and estimating as above yields

$$\|tL\mathrm{e}^{-tL}u\|_{\mathrm{L}^2(F)} \lesssim \mathrm{e}^{-\frac{d^2}{8ct}}\Big(1 + \frac{d^2}{8ct}\Big)\|u\|_{\mathrm{L}^2(E)}.$$

Boundedness of the function $x \mapsto (1+x)\mathrm{e}^{-\frac{x}{2}}$ for positive x then leads to

$$\|tL\mathrm{e}^{-tL}u\|_{\mathrm{L}^2(F)} \lesssim \mathrm{e}^{-\frac{d^2}{16ct}}\|u\|_{\mathrm{L}^2(E)}.$$

Finally, rewrite $\sqrt{t}\nabla\mathrm{e}^{-tL}$ using similarity to $L_{\rho,n}$, apply the product rule, and estimate (taking $|\nabla\varphi| \leq 1$ into account) the appearing semigroup and gradient semigroup terms as before. Then boundedness of the auxiliary function $x \mapsto (1 + \frac{1}{4c}x)\mathrm{e}^{-\frac{x^2}{2}}$ lets us conclude as before

$$\|\sqrt{t}\nabla\mathrm{e}^{-tL}u\|_{\mathrm{L}^2(F)} \lesssim \mathrm{e}^{-\frac{d^2}{16ct}}\|u\|_{\mathrm{L}^2(E)}. \qquad \square$$

In Proposition 6.1.12 we will establish and relate various L^p and $\mathrm{L}^p \to \mathrm{L}^2$ (respectively $\mathrm{L}^2 \to \mathrm{L}^p$) bounds and off-diagonal estimates. Our last result is the L^2 anchor for this, and the next two lemmas give, on the other hand, $\mathrm{L}^p \to \mathrm{L}^2$ respectively $\mathrm{L}^2 \to \mathrm{L}^p$ bounds for $\mathcal{S}$ or $\mathcal{N}$. Then Proposition 6.1.12 will be a straightforward consequence of composition (see Lemma 6.1.9 below) and interpolation.

Lemma 6.1.6. *Let $p \in [1, 2)$. If $\mathcal{S}$ is L^p bounded, then $\mathcal{S}$ is $\mathrm{L}^p \to \mathrm{L}^2$ bounded.*

Proof. Let $u \in \mathrm{L}^2(O)^m \cap \mathrm{L}^p(O)^m$ with normalization $\|u\|_p = 1$ for convenience. Also, let $\theta \in (0, 1)$ be such that

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{2^*}, \tag{6.7}$$

which is possible due to $p < 2$. Using the classical Gagliardo-Nirenberg inequality on $\mathbb{R}^d$ and the extension operator from Theorem 3.0.2 we get the following inhomogeneous estimate on O:

$$\|v\|_{\mathrm{L}^2(O)} \lesssim \|v\|_{\mathrm{W}^{1,2}(O)}^{\theta} \|v\|_{\mathrm{L}^p(O)}^{1-\theta} \qquad (v \in \mathrm{W}^{1,2}_D(O)).$$

The estimate remains true on $\mathrm{W}^{1,2}_D(O)^m$ by component-wise application. Use this version with $v = \mathrm{e}^{-tL}u$. Squaring both sides of the inequality, using normalization of $\|u\|_p$ combined with L^p boundedness of the semigroup, applying the ellipticity assumption (6.2) and using that the semigroup maps into the domain of its generator gives

$$\begin{aligned} \|\mathrm{e}^{-tL}u\|_2^2 &\lesssim \|\mathrm{e}^{-tL}u\|_{\mathrm{W}^{1,2}(O)^m}^{2\theta} \leq \lambda^{-\theta} \operatorname{Re} a(\mathrm{e}^{-tL}u, \mathrm{e}^{-tL}u)^{\theta} \\ &\lesssim |(L\mathrm{e}^{-tL}u \,|\, \mathrm{e}^{-tL}u)_2|^{\theta}. \end{aligned}$$

The analyticity of the semigroup gives in particular that $t \mapsto \mathrm{e}^{-tL}u$ is differentiable with derivative $-L\mathrm{e}^{-tL}u$, see [53, Prop. 3.4.1 b)]. Hence, if we put $f(t) \coloneqq \|\mathrm{e}^{-tL}u\|_2^2$, then the above estimate translates into the differential inequality $f(t) \leq C(-f'(t))^{\theta}$, where $t > 0$ and $C \in (0, \infty)$ is the implied constant from above.

We may assume $f(s) \neq 0$ on $(t/2, t)$, since otherwise $f(t) = 0$ by the semigroup property and the $\mathrm{L}^p \to \mathrm{L}^2$ estimate is then trivially satisfied. Then we can take the differential inequality to the power of $1/\theta$ and divide by $f(t)$ to get $-Cf'(t)/f(t)^{1/\theta} \geq 1$. Integrating this expression from $t/2$ to t gives

$$\frac{t}{2} \leq -C \int_{t/2}^{t} \frac{f'(s)}{f(s)^{1/\theta}} \,\mathrm{d}s \leq \frac{C\theta}{1-\theta} f(t)^{1-1/\theta}.$$

Rearranging this inequality and plugging in the definition of f gives

$$\|\mathrm{e}^{-tL}u\|_2 = f(t)^{1/2} \lesssim t^{-\frac{1}{2}\frac{\theta}{1-\theta}}. \tag{6.8}$$

Write out the definition of 2^* in (6.7) and rearrange terms to see

$$\frac{\theta}{d} = (1-\theta)\Big(\frac{1}{p} - \frac{1}{2}\Big).$$

Plug this back into (6.8), then the normalization of $\|u\|_p$ yields that this is the desired $\mathrm{L}^p \to \mathrm{L}^2$ bound. □

We continue with the case $p > 2$. Here, we need one of the geometrical extra assumptions in Theorem 6.0.1 (ii), which permits the usage of global homogeneous estimates on O.

Lemma 6.1.7. *Assume that O is an (ε, ∞) domain near ∂O. Let $p \in (2,\infty)$ and assume that $\mathcal{N}$ is L^p bounded. Then $\mathcal{S}$ is $\mathrm{L}^2 \to \mathrm{L}^p$ bounded.*

Proof. Step 1: Boundedness of $\mathcal{S}$ up to 2^*. Let $t > 0$, $p \in (2, 2^*)$ and $u \in \mathrm{L}^2(O)^m$ with normalization $\|u\|_2 = 1$. The ellipticity condition (6.2), $\mathrm{e}^{-tL}u \in \mathrm{D}(L)$ and the Cauchy-Schwarz inequality yield

$$\lambda\|\mathrm{e}^{-tL}u\|^2_{\mathrm{W}^{1,2}(O)^m} \leq \operatorname{Re} a(\mathrm{e}^{-tL}u, \mathrm{e}^{-tL}u) \leq \|L\mathrm{e}^{-tL}u\|_2\|\mathrm{e}^{-tL}u\|_2. \tag{6.9}$$

By choice of p there is $\theta \in (0,1)$ such that $1/p = (1-\theta)/2 + \theta/2^*$. Using the classical Gagliardo-Nirenberg inequality and the extension operator for O (compare for the proof of Lemma 6.1.6) we get the following inhomogeneous interpolation inequality:

$$\|v\|_p \lesssim \|v\|_2^{1-\theta}\|v\|^\theta_{\mathrm{W}^{1,2}(O)^m} \qquad (v \in \mathrm{W}^{1,2}_D(O)^m).$$

Divide (6.9) by λ and combine this with the interpolation inequality applied to $v = \mathrm{e}^{-tL}u \in \mathrm{D}(L) \subseteq \mathrm{W}^{1,2}_D(O)^m$ to deduce

$$\|\mathrm{e}^{-tL}u\|_p \lesssim t^{-\theta/2}\|\mathrm{e}^{-tL}u\|_2^{1-\theta/2}\|tL\mathrm{e}^{-tL}u\|_2^{\theta/2}.$$

Now, using contractivity of the semigroup and L^2-boundedness of the family $\{tL\mathrm{e}^{-tL}\}_{t>0}$, which is a consequence of Proposition 6.1.5 (or can be deduced from the Crouzeix-Delyon theorem), and the normalization of $\|u\|_2$, this reduces to $\|\mathrm{e}^{-tL}u\|_p \lesssim t^{-\theta/2}$. If we write out the definition of 2^* in the defining equality for θ and solve the expression for θ, this reveals $\theta = \gamma_p$, which completes the proof of $\mathrm{L}^2 \to \mathrm{L}^p$ boundedness.

Step 2: Going farther using Sobolev embeddings. If $d = 2$ then $\mathrm{L}^2 \to \mathrm{L}^p$ boundedness follows directly from Step 1 since then $2^* = \infty$. Hence, we may assume $d \geq 3$. Also, we assume $p \geq 2^*$ since otherwise the claim is again trivial by Step 1. Then there is a finite sequence $q_0, \ldots, q_k$ such that $q_k^* = p$, $q_{j+1} = q_j^*$ for all j and $q_0 \in [2, 2^*)$. By Step 1, the semigroup is $\mathrm{L}^2 \to \mathrm{L}^{q_0}$ bounded. In the sequel, we show that $\{\mathrm{e}^{-tL}\}_{t>0}$ is $\mathrm{L}^{q_j} \to \mathrm{L}^{q_j^*}$ bounded for all j. Then the claim follows using composition, see Lemma 6.1.9 below.

Fix $t > 0$ and some index j. For brevity, write $q \coloneqq q_j$. By construction, $q \in [2, p] \cap [2, d]$. The latter is a consequence of $q^* \leq q_k^* = p < \infty$. Hence, $\{\sqrt{t}\nabla \mathrm{e}^{-tL}\}_{t>0}$ is L^q-bounded by interpolation of the assumption with the L^2-bound, which holds always (see Step 1). Write $C \in (0, \infty)$ for the implied constant, C depends on L via the respective L^p-bound. Since O is an (ε, ∞) domain near ∂O, the Sobolev inequality on $\mathbb{R}^d$ together with the homogeneous estimates for the extension operator in Theorem 3.9.2 yield a homogeneous Sobolev embedding on O. Then we get for $u \in \mathrm{L}^q(O)^m$ the bound

$$\|\mathrm{e}^{-tL} u\|_{\mathrm{L}^{q^*}(O)^m} \lesssim \|\nabla \mathrm{e}^{-tL} u\|_{\mathrm{L}^q(O)^m} \leq C t^{-\frac{1}{2}} \|u\|_{\mathrm{L}^q(O)^m}.$$

Since $\gamma_{qq^*} = 1$, this completes the claim. □

Corollary 6.1.8. *Assume that O is an (ε, ∞) domain near ∂O. Let $p \in (2, \infty)$ and assume that $\mathcal{N}$ is L^p bounded. Then $\mathcal{N}$ is also $\mathrm{L}^2 \to \mathrm{L}^p$ bounded.*

Proof. Write $\sqrt{2t}\nabla \mathrm{e}^{-2tL} = \sqrt{2}\sqrt{t}\nabla \mathrm{e}^{-tL}\mathrm{e}^{-tL}$. Then the claim follows from Lemma 6.1.7 and the assumption on using composition, see Lemma 6.1.9 below. □

Lemma 6.1.9 (Composition of off-diagonal estimates)**.** *Let $U \subseteq \mathbb{C}$ and let $\mathcal{S} \coloneqq \{S(z)\}_{z \in U}$, $\mathcal{T} \coloneqq \{T(z)\}_{z \in U}$ be operator families such that $\mathcal{T}$ is $\mathrm{L}^p \to \mathrm{L}^q$ bounded and $\mathcal{S}$ is $\mathrm{L}^q \to \mathrm{L}^r$ bounded for $1 \leq p \leq q \leq r \leq \infty$. Then the family $\{S(z)T(z)\}_{z \in U}$ is $\mathrm{L}^p \to \mathrm{L}^r$ bounded. If both families even satisfy the respective off-diagonal estimates, then the same is true for the product family.*

Proof. We start with the case of mere boundedness, which is fairly easy. Let $z \in U$ and C, C' the implied constants of $\mathrm{L}^q \to \mathrm{L}^r$ and $\mathrm{L}^p \to \mathrm{L}^q$ boundedness. Then

$$\|S(z)T(z)\|_{\mathrm{L}^p \to \mathrm{L}^r} \leq C|z|^{-\gamma_{qr}/2} \|T(z)\|_{\mathrm{L}^p \to \mathrm{L}^q} \leq CC'|z|^{-\gamma_{qr}/2}|z|^{-\gamma_{pq}/2}.$$

When summing up γ_{pq} and γ_{qr}, the terms $\pm 1/q$ cancel out to give γ_{pr}, which completes this part.

For the off-diagonal estimates, let $E, F \subseteq O$ be measurable and $u \in \mathrm{L}^2(O)^m$ supported in E. We assume $\mathrm{d}(E,F) > 0$ since otherwise (6.4) already implies (6.5) by the support property of u. Therefore, Lemma 6.1.10 below is applicable with ambient set O in place of Ξ. Let G denote the auxiliary set from that lemma. Moreover, write $C, c \in (0,\infty)$ and $C', c' \in (0,\infty)$ for the implied constants from the $\mathrm{L}^q \to \mathrm{L}^r$ and $\mathrm{L}^p \to \mathrm{L}^q$ off-diagonal estimates. Then, using the decomposition $\mathbf{1}_O = \mathbf{1}_G + \mathbf{1}_{^cG}$ and the properties of the set G we obtain

$$\begin{aligned}
&\|S(z)T(z)u\|_{\mathrm{L}^r(F)^m} \\
&\leq \|S(z)\mathbf{1}_G T(z)u\|_{\mathrm{L}^r(F)^m} + \|S(z)\mathbf{1}_{^cG} T(z)u\|_{\mathrm{L}^r(F)^m} \\
&\leq C|z|^{-\gamma_{qr}/2}\Big(\mathrm{e}^{-c\frac{\mathrm{d}(F,G)^2}{|z|}}\|T(z)u\|_{\mathrm{L}^q(G)} + \mathrm{e}^{-c\frac{\mathrm{d}(F,{^cG})^2}{|z|}}\|T(z)u\|_{\mathrm{L}^q(^cG)}\Big) \\
&\leq CC'|z|^{-\gamma_{qr}/2}|z|^{-\gamma_{pq}/2}\Big(\mathrm{e}^{-c\frac{\mathrm{d}(F,G)^2}{|z|}}\mathrm{e}^{-c'\frac{\mathrm{d}(E,G)^2}{|z|}} + \mathrm{e}^{-c\frac{\mathrm{d}(F,{^cG})^2}{|z|}}\mathrm{e}^{-c'\frac{\mathrm{d}(E,{^cG})^2}{|z|}}\Big)\|u\|_{\mathrm{L}^p(E)} \\
&\leq CC'|z|^{-\gamma_{pr}/2}\Big(\mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{2|z|}} + \mathrm{e}^{-c'\frac{\mathrm{d}(E,F)^2}{2|z|}}\Big)\|u\|_{\mathrm{L}^p(E)}. \qquad \square
\end{aligned}$$

Lemma 6.1.10. *Let $E, F \subseteq \Xi \subseteq \mathbb{R}^d$ be measurable sets with $\mathrm{d}(E,F) \neq 0$. Then there is an open set $G \subseteq \Xi$ such that $E \subseteq G$, $F \subseteq {^cG}$ and $\mathrm{d}(E, {^cG}), \mathrm{d}(F,G) \geq \mathrm{d}(E,F)/2$.*

Proof. Define $G \coloneqq \{x \in \Xi \colon \mathrm{d}(x,F) > \mathrm{d}(E,F)/2\}$. The inequality $\mathrm{d}(F,G) \geq \mathrm{d}(E,F)/2$ follows directly from the definition of G.

To see the analogous bound for $\mathrm{d}(E, {^cG})$, argue by contradiction and assume there were some $z \in {^cG}$ and $x \in E$ with $|x-z| < \mathrm{d}(E,F)/2 - \varepsilon$ for some $\varepsilon > 0$. By definition of G, there would exist $y \in F$ satisfying $|y-z| \leq \mathrm{d}(E,F)/2 + \varepsilon$. Then the triangle inequality yields the contradiction

$$\mathrm{d}(E,F) \leq |x-y| \leq |x-z| + |y-z| < \mathrm{d}(E,F).$$

Consequently, $\mathrm{d}(E, {^cG}) \geq \mathrm{d}(E,F)/2$.

The inclusion $E \subseteq G$ is a consequence of $\mathrm{d}(x,F) \geq \mathrm{d}(E,F) > \mathrm{d}(E,F)/2$ for $x \in E$. Note that we use here that $\mathrm{d}(E,F) \neq 0$. Finally, $x \in F$ implies $\mathrm{d}(x,F) = 0 \leq \mathrm{d}(E,F)/2$, so $F \subseteq {^cG}$. $\square$

For later use, we also treat duality of off-diagonal estimates.

Lemma 6.1.11 (Duality)**.** *Let $U \subseteq \mathbb{C}$ and let $\mathcal{T} \coloneqq \{T_z\}_{z\in U}$ be an operator family on L^2 which is $\mathrm{L}^p \to \mathrm{L}^q$ bounded for $1 \leq p \leq q \leq \infty$. Then $\mathcal{T}^* \coloneqq \{T_z^*\}_{z \in U}$ is $\mathrm{L}^{q'} \to \mathrm{L}^{p'}$ bounded. Moreover, if $\mathcal{T}$ satisfies $\mathrm{L}^p \to \mathrm{L}^q$ off-diagonal estimates, then $\mathcal{T}^*$ satisfies $\mathrm{L}^{q'} \to \mathrm{L}^{p'}$ off-diagonal estimates. In both cases, the implied constants for $\mathcal{T}$ and $\mathcal{T}^*$ coincide.*

Proof. We focus on the off-diagonal estimates since all steps needed in the verification of boundedness are included in the calculation for off-diagonal estimates.

So, let $E, F \subseteq \Xi$ be measurable and let $u \in L^{q'} \cap \mathrm{L}^2$ with support in E and write $C, c \in (0, \infty)$ for the implied constants of $\mathrm{L}^p \to \mathrm{L}^q$ boundedness of $\mathcal{T}$. Then for $z \in U$ we get using duality and the identity $\gamma_{pq} = \gamma_{q'p'}$ the estimate

$$\begin{aligned} &\|T_z^* u\|_{\mathrm{L}^{p'}(F)} = \sup_{\|v\|_p = 1} |(\mathbf{1}_F T_z^* u \,|\, v)| = \sup_{\|v\|_p=1} |(u \,|\, \mathbf{1}_E T_z(\mathbf{1}_F v))| \\ &\leq \sup_{\|v\|_p=1} \|u\|_{q'} \|T_z(\mathbf{1}_F v)\|_{\mathrm{L}^q(E)} \leq \sup_{\|v\|_p=1} C\|u\|_{q'} |z|^{-\gamma_{pq}/2} \mathrm{e}^{-c\,\mathrm{d}(E,F)^2/|z|} \|v\|_{\mathrm{L}^p(F)} \\ &\leq C|z|^{-\gamma_{q'p'}/2} \mathrm{e}^{-c\,\mathrm{d}(E,F)^2/|z|} \|u\|_{\mathrm{L}^{q'}(E)}. \end{aligned}$$ □

It (more or less) just remains to patch all the preparatory work together.

Proposition 6.1.12 ($\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates)**.** *Let $p_0 \in (1, 2)$ and $q_0 \in (2, \infty)$. Then the following holds.*

(i) *If $\mathcal{S}$ is L^{p_0} bounded, then $\mathcal{S}$, $\mathcal{N}$ and $\{tL\mathrm{e}^{-tL}\}_{t>0}$ satisfy $\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates for all $p \in (p_0, 2)$.*

(ii) *If O is an (ε, ∞) domain near ∂O and $\mathcal{N}$ is L^{q_0} bounded, then $\mathcal{S}$ and $\mathcal{N}$ satisfy $\mathrm{L}^2 \to \mathrm{L}^q$ off-diagonal estimates for all $q \in (2, q_0)$.*

Proof. We begin with the proof of (i). We first claim that these families are $\mathrm{L}^{p_0} \to \mathrm{L}^2$ bounded. For $\mathcal{S}$ this is just Lemma 6.1.6. In the other cases, we re-use the trick from Corollary 6.1.8. Write $\sqrt{2t}\nabla \mathrm{e}^{-2tL} = \sqrt{2}(\sqrt{t}\nabla \mathrm{e}^{-tL})\mathrm{e}^{-tL}$ and $2tL\mathrm{e}^{-2tL} = 2(tL\mathrm{e}^{-tL})\mathrm{e}^{-tL}$. Then the claim follows by composition of the $\mathrm{L}^p \to \mathrm{L}^2$ boundedness for $\mathcal{S}$ with the L^2-boundedness of $\mathcal{N}$ and $\{tL\mathrm{e}^{-tL}\}_{t>0}$.

In the end, we use interpolation to get off-diagonal estimates. We present this technique with all details once and use it freely afterwards. Let $p \in (p_0, 2)$ and write

$$1/p = (1-\theta)/p_0 + \theta/2 \tag{6.10}$$

for some $\theta \in (0, 1)$. Moreover, let $t > 0$ and $T(t)$ be some operator to the index t in one of the three families. Also, fix $E, F \subseteq O$ measurable. Then we get with $u \in \mathrm{L}^p(O)^m \cap \mathrm{L}^2(O)^m$ supported in E that

$$\begin{aligned} \|T(t)u\|_{\mathrm{L}^2(F)} &\leq \|T(t)\|^{1-\theta}_{\mathrm{L}^{p_0}(E) \to \mathrm{L}^2(F)} \|T(t)\|^{\theta}_{\mathrm{L}^2(E) \to \mathrm{L}^2(F)} \|u\|_{\mathrm{L}^p(E)} \\ &\leq (C')^{1-\theta} t^{-(1-\theta)\gamma_{p_0}/2} C^\theta \mathrm{e}^{-\theta c \frac{\mathrm{d}(E,F)^2}{t}} \|u\|_{\mathrm{L}^p(E)}, \end{aligned} \tag{6.11}$$

where we have used the $\mathrm{L}^{p_0} \to \mathrm{L}^2$ boundedness from above with implied constant $C' \in (0,\infty)$ and L^2 off-diagonal estimates with implied constants $C, c \in (0,\infty)$. Note that we have considered $T(t)$ as an operator from $\mathrm{L}^2(E) \to \mathrm{L}^2(F)$ to "encode" the off-diagonal bounds in the operator norm, which makes it accessible for interpolation.

Let's have a closer look at the exponent of t. Expand γ_{p_0} and use (6.10) to get

$$(1-\theta)\gamma_{p_0} = d\Big(\frac{1-\theta}{p_0} - \frac{1-\theta}{2}\Big) = d\Big(\frac{1}{p} - \frac{\theta}{2} - \frac{1}{2} + \frac{\theta}{2}\Big) = \gamma_p.$$

Hence, the scaling in (6.11) is correct, which completes the proof of (i).

The proof of (ii) is similar, but relies on Lemma 6.1.7. Indeed, the interpolation argument to obtain off-diagonal estimates works the same provided we have established $\mathrm{L}^2 \to \mathrm{L}^p$ boundedness for $\mathcal{S}$ and $\mathcal{N}$ beforehand. For $\mathcal{S}$, this is directly Lemma 6.1.7 and for $\mathcal{N}$ we argue by the same decomposition as in (i). Note that the decomposition argument now uses not only the $\mathrm{L}^2 \to \mathrm{L}^q$ boundedness of the semigroup as a non-trivial ingredient, but also the L^q boundedness of $\mathcal{N}$. Therefore, boundedness of $\{tL\mathrm{e}^{-tL}\}_{t>0}$ does not come for free as was the case in (i). □

Remark 6.1.13. Given $\varphi \in [0, \pi/2 - \omega)$, we can replace real times $t > 0$ in the conclusions of Proposition 6.1.12 by $z \in \mathrm{S}_\varphi$. This means, for instance, that in (i) we get $\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates for $\{\mathrm{e}^{-zL}\}_{z\in \mathrm{S}_\varphi}$. To see this, take $\theta \in (\varphi, \pi/2 - \omega)$ and let $z \in \mathrm{S}_\varphi$. Let w denote the intersection point between $\partial \mathrm{S}_\theta$ and the axis passing through z that is parallel to the real axis. Put $t \coloneqq z - w$. By construction, $t > 0$, $\arg(z) < \varphi$, and $|w| \le |z|$. Consequently, $t \geq |z|(\cos(\varphi) - \cos(\theta))$. Hence we can write $z = t + w$, and conclude by composition with the complex L^2 off-diagonal estimates from Proposition 6.1.5.

Lemma 6.1.14 (Exponential decay)**.** *Assume $\mathcal{S}$ is L^q-bounded for some $q \in [1,\infty]$ and $p \in (q,2)$ or $p \in (2,q)$, respectively. Then*

$$\|\mathrm{e}^{-tL}u\|_p \lesssim \mathrm{e}^{-ct}\|u\|_p \qquad (t > 0, u \in \mathrm{L}^2 \cap \mathrm{L}^p),$$

where the implicit constant depends on L via its coefficient bounds and the implied constant from L^q-boundedness of $\mathcal{S}$.

Proof. It follows from (6.2) that $L - \lambda/2$ is still m-θ-accretive for some angle $\theta \in [0, \pi/2)$ only depending on the coefficient bounds of L. Hence, the

Crozeix-Delyon theorem yields $\mathrm{e}^{t\lambda/2}\|\mathrm{e}^{-tL}\|_{\mathrm{L}^2\to\mathrm{L}^2} \leq 4$. Complex interpolation of this bound with the L^q-boundedness of $\mathcal{S}$ from the assumption gives the claim. $\square$

6.2. L^p-estimates for operators on L^2

In the following, we will present two results to establish L^p-bounds for an operator on L^2. The first one goes back to Blunck and Kunstmann [24] and deals with the case $p < 2$. The second one captures $p > 2$ and is a version of a good-λ argument established originally by Auscher, Coulhon, Duong, and Hofmann in [10], but this argument has deep roots in harmonic analysis, see the historical remarks in [5, p.2 footnote 7].

Some motivation

We believe this is a good spot to explain what we mean by "L^p-bounds for an L^2-operator", although we have already implicitly used this concept in the section right before. Consider an operator T on L^2 and the dense subspace $\mathrm{L}^2 \cap \mathrm{L}^p$ of L^p. Let $u \in \mathrm{L}^2 \cap \mathrm{L}^p$, then T is well-defined on u and Tu is in particular a measurable function. Then the question is if there is a constant $C > 0$ such that $\|Tu\|_p \leq C\|u\|_p$. If this is the case, T can be continuously extended to an extension on L^p, again with operator norm bounded by C.

Often, the operator T is given as the strong limit of bounded operators T_n on L^2 which are more regular. For example, they might be given by a nice representation formula, which already allows to show L^p-boundedness for all p. A reader who is familiar with the classical theory of singular integral operators might think of T_n as being given by a truncated kernel. In this situation, though T_n is already bounded on L^p, the constant in the L^p-bound a priori degenerates when $n \to \infty$.

In such a situation, instead of establishing L^p-bounds for the operator T directly, it suffices to show that there is a constant $C > 0$ such that $\|T_n u\|_p \leq C\|u\|_p$ for all $u \in \mathrm{L}^2 \cap \mathrm{L}^p$ *uniformly* in n. Indeed, then the bounds for T follow from Fatou's lemma, since the L^2-convergence also yields convergence of $|T_n u|^p$ to $|Tu|^p$ almost everywhere along a subsequence.

The result in Section 6.2.1 could be applied directly to T, whereas the result in Section 6.2.2 already requires that the operator is bounded on L^p. Of course, the L^p-boundedness is only used in a *qualitative* and not a *quantitative* way. Hence, this second method is definitely tailored to the situation with an approximating family $\{T_n\}_n$. However, even in the first situation it is often better to work with an approximation of good operators since the a priori L^p-boundedness makes it easier to perform certain calculations which are useful when checking the assumptions in the extrapolation theorems.

6.2.1. Beyond Calderón-Zygmund theory

The following proposition is due to Blunck and Kunstmann [24], a simplified version on $\mathbb{R}^d$ was shown in [5] and extended to arbitrary measurable subsets using an extension–restriction argument in [36, Prop. 5.2].

Proposition 6.2.1 (Blunck & Kunstmann extrapolation)**.** *Let $q \in [1,2)$, $p \in (q,2)$, $\Xi \subseteq \mathbb{R}^d$ be measurable and $T : \mathrm{L}^2(\Xi)^{m_1} \to \mathrm{L}^2(\Xi)^{m_2}$ be bounded, where $m_1, m_2 \in \mathbb{N}$. Assume there is a family $(A_r)_{r>0}$ of bounded operators on $\mathrm{L}^2(\Xi)^{m_1}$ such that for any open ball $B \subseteq \mathbb{R}^d$ with radius r and $u \in \mathrm{L}^2(\Xi)^{m_1}$ with $\operatorname{supp}(u) \subseteq B$ it holds*

$$\text{(BK1)} \qquad \left(\fint_{C_j(B)\cap\Xi} |T(1-A_r)u|^2\right)^{\frac{1}{2}} \leq g(j) r^{\frac{d}{2}-\frac{d}{q}} \left(\int_{B\cap\Xi} |u|^q\right)^{\frac{1}{q}} \quad (j \geq 2),$$

$$\text{(BK2)} \qquad \left(\int_{C_j(B)\cap\Xi} |A_r u|^2\right)^{\frac{1}{2}} \leq g(j) r^{\frac{d}{2}-\frac{d}{q}} \left(\int_{B\cap\Xi} |u|^q\right)^{\frac{1}{q}} \quad (j \geq 1).$$

If $\Sigma := \sum_j g(j) 2^{\frac{dj}{2}}$ is finite, then T admits L^p-bounds and the implied constant depends on T only via the L^2-bound for T and Σ.

Remark 6.2.2. In the situation of Proposition 6.2.1, T is actually of weak-type (q,q), which then yields the strong L^p-bounds. Since we won't need the weak-type estimate for T, we have decided to go with the simpler formulation.

Example 6.2.3. Assume $-L$ is the generator of a semigroup e^{-tL} on $\mathrm{L}^2(\Xi)^{m_1}$ that satisfies $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates. Consider the family $A_r := 1-(1-\mathrm{e}^{-r^2L})^n$ for some $n \in \mathbb{N}$ and all $r > 0$, and let B and u as in the Proposition. Clearly, A_r is bounded on $\mathrm{L}^2(\Xi)^{m_1}$. We claim that $\{A_r\}_{r>0}$ satisfies (BK2). To see this, expand

$$A_r = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \mathrm{e}^{-kr^2L}.$$

With $C, c \in (0, \infty)$ the implied constants from the $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates we readily derive

$$\Big(\int_{C_j(B)\cap\Xi} |A_r u|^2\Big)^{\frac{1}{2}} \leq C \sum_{k=1}^{n} \binom{n}{k} (kr^2)^{-\gamma_q/2} \mathrm{e}^{-c4^{j-1}/k} \Big(\int_{B\cap\Xi} |u|^q\Big)^{\frac{1}{q}}$$
$$\lesssim 2^n r^{-\gamma_q} \mathrm{e}^{-\frac{c}{n}4^{j-1}} \Big(\int_{B\cap\Xi} |u|^q\Big)^{\frac{1}{q}}.$$

So, with $g(j) \coloneqq 2^n \mathrm{e}^{-\frac{c}{n}4^{j-1}}$ the summability condition in Proposition 6.2.1 is clearly satisfied due to the double exponential decay in j.

6.2.2. Good-λ estimates

Definition 6.2.4. Let $\Xi \subseteq \mathbb{R}^d$ be measurable and f a locally integrable function on Ξ. Define the *maximal operator on* Ξ by

$$\mathcal{M}^{\Xi} f(x) = \sup_{B \ni x} \mathrm{r}(B)^{-d} \int_{B\cap\Xi} |f(y)| \, \mathrm{d}y \qquad (x \in \mathbb{R}^d).$$

Remark 6.2.5. Note that $\mathcal{M}^{\Xi} f$ coincides with $\mathcal{M}(\mathcal{E}_0 f)$ up to a dimensional constant, where $\mathcal{E}_0 f$ is the zero extension of f to $\mathbb{R}^d$.

The following proposition is a consequence of the whole space version found in [5, Thm. 1.2]. The trick is borrowed from [36] and was already mentioned above Proposition 6.2.1. One has to apply the whole space result to the operators $T' \coloneqq \mathcal{E}_0 T \mathcal{R}_\Xi$ and $A'_r \coloneqq \mathcal{E}_0 A_r \mathcal{R}_\Xi$. Thereby, it is helpful to keep Remark 6.2.5 in mind.

Proposition 6.2.6 (Good-λ extrapolation)**.** *Let $q \in (2, \infty]$, $\Xi \subseteq \mathbb{R}^d$ be measurable and $T : \mathrm{L}^2(\Xi)^{m_1} \to \mathrm{L}^2(\Xi)^{m_2}$ be bounded, where $m_1, m_2 \in \mathbb{N}$. Assume there is a family $(A_r)_{r>0}$ of bounded operators on $\mathrm{L}^2(\Xi)^{m_1}$ and a constant $C > 0$ such that for any open ball B with radius $r \coloneqq \mathrm{r}(B)$ and all $u \in \mathrm{L}^2(\Xi)^{m_1}$ it holds*

$$\text{(GL1)} \qquad \Big(\int_{B\cap\Xi} |T(1 - A_r)u|^2\Big)^{\frac{1}{2}} \leq C r^{\frac{d}{2}} \mathcal{M}^{\Xi}(|u|^2)^{1/2}(y) \qquad (y \in B),$$

$$\text{(GL2)} \qquad \Big(\int_{B\cap\Xi} |TA_r u|^q\Big)^{\frac{1}{q}} \leq C r^{\frac{d}{q}} \mathcal{M}^{\Xi}(|Tu|^2)^{1/2}(y) \qquad (y \in B).$$

Then, if $p \in (2, q)$ and T maps $\mathrm{L}^2(\Xi)^{m_1} \cap \mathrm{L}^p(\Xi)^{m_1}$ into $\mathrm{L}^p(\Xi)^{m_2}$, there is a constant $c > 0$ that depends only on dimension, q, p and C such that

$$\|Tu\|_p \leq c\|u\|_p \qquad (u \in \mathrm{L}^2(\Xi)^{m_1} \cap \mathrm{L}^p(\Xi)^{m_1}).$$

Remark 6.2.7. As already mentioned in the motivation, note that Proposition 6.2.6 already assumes that the operator is bounded on L^p. However, boundedness is only used qualitatively, and it is the aim of this result to give good quantitative bounds which allow for a limiting argument.

6.3. M^c^Intosh approximation & functional calculus

This section is dedicated to an approximation technique due to M^c^Intosh. First, we show operator-adapted approximations to the identity. For a rough operator this leads to approximations by operators with nice representation formulæ which have the a priori qualitative L^p-boundedness and are suitable to bring off-diagonal bounds into action.

Let us first introduce some terminology and relate it to the algebra of functions $\mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ introduced in Section 1.4. If f is a holomorphic function on S_φ^+, say that f is *regularly decaying in* 0 if there are $C_0, s_0, r_0 > 0$ such that $|f(z)| \leq C_0 |z|^{s_0}$ for all $z \in \mathrm{S}_\varphi^+$ with $|z| < r_0$. Similarly, say that f is *regularly decaying in* ∞ if there are $C_\infty, s_\infty, r_\infty > 0$ such that $|f(z)| \leq C_\infty |z|^{-s_\infty}$ for all $z \in \mathrm{S}_\varphi^+$ with $|z| > r_\infty$. If f is both regularly decaying in 0 and ∞, then we say that f is *regularly decaying*. Observe that f is regularly decaying if and only if $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$.

Proposition 6.3.1 (M^c^Intosh approximation). *Let T be a sectorial operator of angle $\omega \in [0, \pi)$, $\varphi \in (\omega, \pi)$ and $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. Put $c := \int_0^\infty f(t) \frac{\mathrm{d}t}{t}$ and let $(a_n)_n$, $(b_n)_n$ be sequences with $a_n \to 0$ and $b_n \to \infty$. Then*

$$\int_{a_n}^{b_n} f(tT)x \frac{\mathrm{d}t}{t} \longrightarrow cx \qquad (x \in \overline{\mathrm{D}(T) \cap \mathrm{R}(T)}\,).$$

Proof. For convenience, put $f_t := f(t\mathbf{z})$ and $F_{a,b} := \int_a^b f(t\mathbf{z}) \frac{\mathrm{d}t}{t}$ for $0 < a \leq b < \infty$ on S_φ^+. Write $C, s \in (0, \infty)$ for the implied constants coming from $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. The proof divides into several steps.

Step 1: Properties of $F_{a,b}$.

Claim 1: $F_{a,b}$ is holomorphic. Fix $0 < a \leq b < \infty$ and let $(z_n)_n$ be a sequence in S_φ^+ that converges to $z \in \mathrm{S}_\varphi^+$. One has $f_t(z_n) \to f_t(z)$ by continuity of f and $f_t(w)$ is bounded for $(t, w) \in [a, b] \times (\{z_n : n \in \mathbb{N}\} \cup \{z\})$ by continuity of f and compactness. Hence, Lebesgue's theorem implies $F_{a,b}(z_n) \to F_{a,b}(z)$, which means that $F_{a,b}$ is continuous in z.

Next, let $\Delta \subseteq \mathrm{S}_\varphi^+$ be the boundary of a compactly contained triangle. Then the Fubini-Tonelli theorem together with Cauchy's integral theorem applied to the holomorphic function f_t yield

$$\int_\Delta \int_a^b f(tz) \frac{\mathrm{d}t}{t} \, \mathrm{d}z = \int_a^b \int_\Delta f_t(z) \, \mathrm{d}z \, \frac{\mathrm{d}t}{t} = 0.$$

Altogether, Morera's theorem yields that $F_{a,b}$ is holomorphic on S_φ^+.

Claim 2: $F_{a,b}$ is regularly decaying in 0 and ∞. To see that $F_{a,b}$ decays in ∞, estimate

$$|F_{a,b}(z)| \le C \int_a^\infty t^{-s} |z|^{-s} \frac{\mathrm{d}t}{t} = C \frac{a^{-s}}{s} |z|^{-s}.$$

Similarly, $|F_{a,b}(z)| \le C \frac{b^s}{s} |z|^s$. So altogether, $F_{a,b} \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ with

$$|F_{a,b}(z)| \le \frac{C}{s} \max(a^{-s}, b^s) \min(|z|^{-s}, |z|^s). \tag{6.12}$$

Claim 3: $F_{a,b}$ is in $\mathrm{H}^\infty(\mathrm{S}_\varphi^+)$ uniformly in a and b. To see this, use a scaling by $|z|^{-1}$ to derive

$$|F_{a,b}(z)| \le \int_0^\infty |f_t(z)| \frac{\mathrm{d}t}{t} = \int_0^\infty |f(tz|z|^{-1})| \frac{\mathrm{d}t}{t} \le C \int_0^\infty \min(t^s, t^{-s}) \frac{\mathrm{d}t}{t}.$$

The integral on the right-hand side is clearly finite and independent of a, b and z. For later use, we summarize this as

$$|F_{a,b}(z)| \lesssim 1 \qquad (0 < a \le b < \infty, z \in \mathrm{S}_\varphi^+). \tag{6.13}$$

Step 2: Calculation of $F_{a,b}(T)$.

We use the Fubini-Tonelli theorem to calculate $F_{a,b}(T)$. Its application is readily justified using (6.12). So

$$\begin{aligned} 2\pi\mathrm{i} \int_a^b f(tT) \frac{\mathrm{d}t}{t} &= \int_a^b \int_< f(tz)(z-T)^{-1} \, \mathrm{d}z \, \frac{\mathrm{d}t}{t} \\ &= \int_< \int_a^b f(tz) \frac{\mathrm{d}t}{t} (z-T)^{-1} \, \mathrm{d}z = 2\pi\mathrm{i} F_{a,b}(T). \end{aligned} \tag{6.14}$$

Observe that the integral on the left-hand side with a, b replaced by a_n, b_n are precisely the integrals for which we want to show strong convergence to c times the identity on $\overline{\mathrm{D}(T) \cap \mathrm{R}(T)}$.

Step 3: Convergence of $F_{a_n,b_n}(T)x$ for $x \in \mathrm{D}(T) \cap \mathrm{R}(T)$.

For convenience, put $e := \frac{\mathbf{z}}{(1+\mathbf{z})^2}$ on S_φ^+. Evidently, $e \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$.

Claim 1: F_{a_n,b_n} converges locally uniformly to c for $n \to \infty$. We want to appeal to Vitali's theorem from complex analysis, see [53, Prop. 5.1.1].

First, we have to check that F_{a_n,b_n} is a locally bounded sequence. But we have already seen in (6.13) that F_{a_n,b_n} is actually uniformly bounded for the $\mathrm{H}^\infty(\mathrm{S}_\varphi^+)$-norm, so this can be checked. Second, F_{a_n,b_n} has to converge pointwise on a set with a limit point in S_φ^+. To see this, let $r \in (0,\infty)$, then

$$F_{a_n,b_n}(r) = \int_{a_n}^{b_n} f(tr)\,\frac{\mathrm{d}t}{t} = \int_{ra_n}^{rb_n} f(t)\,\frac{\mathrm{d}t}{t} \longrightarrow c.$$

Hence, by Vitali's theorem, there is a holomorphic function g on S_φ^+ to which F_{a_n,b_n} converges locally uniformly when $n \to \infty$. But g coincides with c on $(0,\infty)$, so the identity theorem reveals that F_{a_n,b_n} converges locally uniformly to the constant value c.

Claim 2: $\mathrm{R}(e(T)) = \mathrm{D}(T) \cap \mathrm{R}(T)$. By the basic properties of the functional calculus we can write

$$e(T) = T(1+T)^{-2} = (1+T)^{-1}T(1+T)^{-1}.$$

This shows $\mathrm{R}(e(T)) \subseteq \mathrm{D}(T) \cap \mathrm{R}(T)$. Conversely, let $x \in \mathrm{D}(T) \cap \mathrm{R}(T)$. By the latter, there is $y \in \mathrm{D}(T)$ with $x = Ty$. But since $y \in \mathrm{D}(T)$, we can also write $y = (1+T)^{-1}z$ for some z. Apply $1+T$ to deduce $z = (1+T)y = y+x \in \mathrm{D}(T)$. Consequently, there is some w such that $z = (1+T)^{-1}w$. In total,

$$x = Ty = T(1+T)^{-1}z = T(1+T)^{-2}w \in \mathrm{R}(e(T)).$$

Claim 3: $F_{a_n,b_n}(T)x$ converges to cx for $x \in \mathrm{D}(T) \cap \mathrm{R}(T)$. Put $f_n := F_{a_n,b_n}e$. Since F_{a_n,b_n} is uniformly bounded, $f_n \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ for uniform implied constants $C', s' \in (0,\infty)$. So $|f_n(z) - ce(z)|$ is uniformly integrable over the contour of a sector. Also, $f_n \to ce$ pointwise by Claim 1, so Lebesgue's theorem yields

$$\|f_n(T) - ce(T)\|_{X \to X} \leq (2\pi)^{-1} \int_{<} |f_n(z) - ce(z)|\,\frac{|\mathrm{d}z|}{|z|} \to 0.$$

Since $e(T)$ is bounded, it follows $f_n(T) = F_{a_n,b_n}(T)e(T)$. In particular, $F_{a_n,b_n}(T)$ converges to c strongly on $\mathrm{R}(e(T))$, which coincides with $\mathrm{D}(T) \cap \mathrm{R}(T)$ by Claim 2.

Step 4: Upgrade to $x \in \overline{\mathrm{D}(T) \cap \mathrm{R}(T)}$.

This follows from Step 3 with the usual $\varepsilon/3$-argument provided we have shown that $\{F_{a,b}(T)\}_{0<a\leq b<\infty}$ is uniformly bounded. To this end, rewrite

$$F_{a,b}(z) = \int_0^b f_t(z)\,\frac{\mathrm{d}t}{t} - \int_0^a f_t(z)\,\frac{\mathrm{d}t}{t} = h(bz) - h(az), \tag{6.15}$$

where $h(z) := \int_0^1 f(tz)\,\frac{\mathrm{d}t}{t}$.

We claim that $h \in \mathcal{E}(\mathrm{S}_\varphi^+)$. Some arguments are similar to arguments that we have used before, so we keep them brief. First, h decays of order $|z|^s$ in zero, where s is the order of decay of f, see Step 1. Second, $g := h - c$ is a holomorphic function that decays of order $|z|^{-s}$ in infinity. Indeed, derive the identity $g(z) = \int_1^\infty f(tz)\,\frac{\mathrm{d}t}{t}$ from the identity theorem by using the substitution $t' := \frac{t}{s}$ for any real s in the defining integral for c. Then the estimate can again be obtained by mimicking the calculation from Step 1. Proceed with the decomposition

$$h = \Big(c(1+\mathbf{z})^{-1} + g\Big) - c(1+\mathbf{z})^{-1} + c. \tag{6.16}$$

The function $c(1+\mathbf{z})^{-1} + g$ is holomorphic and decays regularly at infinity, because this is the case for each addend. But we can also write

$$c(1+\mathbf{z})^{-1} + g = (c+g) - c\mathbf{z}(1+\mathbf{z})^{-1} = h - c\mathbf{z}(1+\mathbf{z})^{-1},$$

so this function is also regularly decaying in zero. Altogether, we find that $c(1+\mathbf{z})^{-1}+g \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$, which yields $h \in \mathcal{E}(\mathrm{S}_\varphi^+)$ by the decomposition (6.16).

Using linearity of the functional calculus and scaling (see Proposition 1.4.7), this allows us to turn (6.15) into

$$F_{a,b}(T) = h_b(T) - h_a(T) = h(bT) - h(aT).$$

Moreover, (1.6) in the very same proposition on scaling shows that the operators on the right-hand side have operator norms independent of a and b, which concludes this proof. □

Remark 6.3.2. For a general function in H_0^∞ we have to work with $0 < a < b < \infty$ in Step 1 to ensure $F_{a,b} \in \mathrm{H}_0^\infty$. However, with better decay, we can do more. Consider $f := \mathbf{z}^{\frac{1}{2}}\mathrm{e}^{-\mathbf{z}}$ on S_φ^+ for some $\varphi \in (0,\pi/2)$ and define $F_a := \int_a^\infty f(t\mathbf{z})\,\frac{\mathrm{d}t}{t}$ for $a > 0$. Then, using $\Gamma(1/2) = \sqrt{\pi}$, we get for $z \in \mathrm{S}_\varphi^+$ and with a suitable c depending on φ that

$$|F_a(z)| \le \int_a^\infty (t|z|)^{\frac{1}{2}}\mathrm{e}^{-2ct|z|}\,\frac{\mathrm{d}t}{t} \le \mathrm{e}^{-ca|z|}\int_0^\infty (t|z|)^{\frac{1}{2}}\mathrm{e}^{-ct|z|}\,\frac{\mathrm{d}t}{t} = \mathrm{e}^{-ca|z|}\left(\frac{\pi}{c}\right)^{\frac{1}{2}}.$$

So indeed, $F_a \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ and Proposition 6.3.1 holds with $b_n := \infty$ for all n.

Example 6.3.3. Let T be an injective sectorial operator of angle $\omega < \pi/2$ on a reflexive Banach space X and let $\varphi \in (\omega,\pi/2)$. Then T is densely defined with dense range, see [53, Prop. 2.1.1 h)]. Hence, also $\mathrm{D}(T) \cap \mathrm{R}(T)$ is dense in X, see [53, Prop. 2.1.1 c)]. Recall $\int_0^\infty t^{\frac{1}{2}} \mathrm{e}^{-t} \frac{\mathrm{d}t}{t} = \Gamma(1/2) = \sqrt{\pi}$ and consider $f = \sqrt{\mathbf{z}}\mathrm{e}^{-\mathbf{z}} \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. Then $\int_0^\infty f(t) \frac{\mathrm{d}t}{t} = \sqrt{\pi}$ and the M$^{\mathrm{c}}$Intosh approximation (Proposition 6.3.1) in combination with Remark 6.3.2 yield

$$\frac{1}{\sqrt{\pi}} \int_{a_n}^{\infty} t^{\frac{1}{2}} T^{\frac{1}{2}} \mathrm{e}^{-tT} x \, \frac{\mathrm{d}t}{t} \to x \quad \text{when} \quad n \to \infty \tag{6.17}$$

for all $x \in X$ and all positive null sequences $(a_n)_n$.

Example 6.3.4. Put $T \coloneqq L$ and $X \coloneqq \mathrm{L}^2(O)^m$. Then Example 6.3.3 is applicable. By (6.14) we rewrite the left-hand side in (6.17) as $F_{a_n}(T)$. The substitution $t = s^2$ gives

$$F_{a_n^2}(z) = \int_{a_n^2}^\infty f_t(z) \, \frac{\mathrm{d}t}{t} = 2 \int_{a_n}^\infty f_{s^2}(z) \, \frac{\mathrm{d}s}{s} =: G_{a_n}(z).$$

Of course, $G_{a_n} \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ since this is true for $F_{a_n^2}$. Hence, plugging L into G_{a_n} and redoing the calculation from (6.14) gives

$$\frac{2}{\sqrt{\pi}} \int_{a_n}^{\infty} L^{\frac{1}{2}} \mathrm{e}^{-s^2 L} u \, \mathrm{d}s \to u \qquad \text{as} \quad n \to \infty$$

for all $u \in \mathrm{L}^2(O)^m$ and all positive null sequences $(a_n)_n$. We can always replace u by $L^{-\frac{1}{2}} u$ to derive the representation formula

$$L^{-\frac{1}{2}} u = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s^2 L} u \, \mathrm{d}s \qquad (u \in \mathrm{L}^2(O)^m)$$

in the sense of an improper Riemann integral in 0. The identity $L^{\frac{1}{2}} \mathrm{e}^{-s^2 L} L^{-\frac{1}{2}} = \mathrm{e}^{-s^2 L}$ is clear by the basic properties of the sectorial functional calculus. Similarly, we obtain with u replaced by $L^{\frac{1}{2}} u$ for $u \in \mathrm{D}(L^{\frac{1}{2}})$ that

$$L^{\frac{1}{2}} u = \frac{2}{\sqrt{\pi}} \int_0^\infty L \mathrm{e}^{-t^2 L} u \, \mathrm{d}t \qquad (u \in \mathrm{D}(L^{\frac{1}{2}})) \tag{6.18}$$

in the sense of an improper Riemann integral. Note that we have used $L\mathrm{e}^{-t^2 L} = [\mathbf{z}^{\frac{1}{2}} \mathrm{e}^{-t^2 \mathbf{z}} \mathbf{z}^{\frac{1}{2}}](L) \supseteq L^{\frac{1}{2}} \mathrm{e}^{-s^2 L} L^{\frac{1}{2}}$ and the domain of the operator on the right-hand side coincides with $\mathrm{D}(L^{\frac{1}{2}})$.

Apply Lemma 6.1.14 to the representation of $L^{-\frac{1}{2}}$ to arrive at the following

Corollary 6.3.5. *Assume* $\mathcal{S}$ *is* L^q*-bounded for some* $q \in (1,\infty)$ *and* $p \in (q,2)$ *or* $p \in (2,q)$*, respectively. Then*

$$\|L^{-\frac{1}{2}}u\|_p \lesssim \|u\|_p \qquad (u \in \mathrm{L}^2(O)^m \cap \mathrm{L}^p(O)^m),$$

where the implicit constant depends on L *via its coefficient bounds and the implied constant from* L^q*-boundedness of* $\mathcal{S}$*.*

Proof. Use the representation for $L^{-\frac{1}{2}}$ from the example above and the exponential decay from Lemma 6.1.14 with implied constants $C, c \in (0,\infty)$ to derive

$$\|L^{-\frac{1}{2}}u\|_p \leq \frac{2C}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-cs^2}\, \mathrm{d}s \|u\|_p.$$

The integral in s is finite and depends on L only via ellipticity and the implied constants from L^q-boundedness of $\mathcal{S}$. □

6.4. Bounded H^∞-calculus

The Crouzeix–Delyon Theorem yields that the H^∞-calculus of L on L^2 is bounded. As a first application of the extrapolation theorem of Blunck and Kunstmann, we moreover derive L^p bounds for the H^∞-calculus on $\mathrm{L}^p \cap \mathrm{L}^2$ in the case $p < 2$ and assuming suitable off-diagonal decay. These bounds are extended to $p > 2$ using duality. The result of this section is interesting on its own, but will also be needed in the proof of Proposition 6.7.1.

Proposition 6.4.1. *Assume that* $\mathcal{S}$ *satisfies* $\mathrm{L}^p \to \mathrm{L}^2$ *off-diagonal estimates for some* $p \in (1,2)$ *or* $\mathrm{L}^2 \to \mathrm{L}^p$ *off-diagonal estimates if* $p \in (2,\infty)$*. Then there exists a constant* $C > 0$ *such that one has for all* $\varphi \in (\omega,\pi)$ *the estimate*

$$\|f(L)\|_p \leq C\|f\|_\infty \|u\|_p \qquad (f \in \mathrm{H}^\infty(\mathrm{S}_\varphi^+), u \in \mathrm{L}^2 \cap \mathrm{L}^p). \tag{6.19}$$

The constant C *depends on* L *only via its coefficient bounds and the implied constants from the off-diagonal estimates.*

Proof. **Step 1**: Reduction to $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ with $\|f\|_\infty = 1$. Let $f \in \mathrm{H}^\infty(\mathrm{S}_\varphi^+)$ and define $e \coloneqq \mathbf{z}(1+\mathbf{z})^{-2}$. The normalization of the norm simply follows by scaling. For convenience, put $c \coloneqq \|e\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)}$.

Now consider the sequence $f_n := fe_n$, where $e_n := e^{\frac{1}{n}}$. Since e_n has decay of order $\frac{1}{n}$ in 0 and ∞, $f_n \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. Moreover, the sequence of functions $\mathbf{z}^{\frac{1}{n}}$ converges pointwise to 1 as $n \to \infty$, hence f_n converges pointwise to f, and

$$\|f_n\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)} \leq \|f\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)} c^{\frac{1}{n}} \lesssim 1. \tag{6.20}$$

Since the $\mathrm{H}^\infty(\mathrm{S}_\varphi^+)$-calculus of L is bounded on $\mathrm{L}^2(O)^m$, we derive from (6.20) that the family $\{f_n(L)\}_n$ of operators on L^2 is uniformly bounded. Hence, the Convergence Lemma (see [53, Prop. 5.1.4 b)], dense domain and range are implied by the properties of L) shows that $f_n(L)$ converges strongly to $f(L)$ on $\mathrm{L}^2(O)^m$. So, if (6.19) holds for all $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$, we get for $u \in \mathrm{L}^2(O)^m \cap \mathrm{L}^p(O)^m$ from Fatou's lemma and with C_φ the implied constant from (6.20) that

$$\begin{aligned}\|f(L)u\|_p &\leq \liminf_k \|f_{n_k}(L)u\|_p \leq \liminf_k C\|f_{n_k}\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)}\|u\|_p \\ &\leq CC_\varphi\|u\|_p.\end{aligned}$$

Density of $\mathrm{L}^2(O)^m \cap \mathrm{L}^p(O)^m$ in $\mathrm{L}^p(O)^m$ concludes this step.
Step 2: Reduction to $p \in (1,2)$, where $\mathcal{S}$ satisfies $\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates. Assume that $\mathcal{S}$ satisfies $\mathrm{L}^2 \to \mathrm{L}^p$ off-diagonal estimates for some $p \in (2,\infty)$. Duality yields that the dual family $\mathcal{S}^* = \mathcal{S}(L^*)$ satisfies $\mathrm{L}^2 \to \mathrm{L}^{p'}$ off-diagonal estimates, see Lemma 6.1.11. Put $g := f^*$, then duality, Proposition 1.4.6 and the assumption of this step yield

$$\begin{aligned}\|f(L)\|_{\mathrm{L}^p\to\mathrm{L}^p} &= \|f(L)^*\|_{\mathrm{L}^{p'}\to\mathrm{L}^{p'}} = \|g(L^*)\|_{\mathrm{L}^{p'}\to\mathrm{L}^{p'}} \leq C\|g\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)} \\ &= C\|f\|_{\mathrm{H}^\infty(\mathrm{S}_\varphi^+)}.\end{aligned}$$

Step 3: The case $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$, $\|f\|_\infty = 1$ and $p \in (1,2)$ where $\mathcal{S}$ satisfies $\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates. We appeal to Proposition 6.2.1 with $T := f(L)$ and $A_r := 1-(1-\mathrm{e}^{-r^2L})^n$, where n is such that $\gamma_p + 2n > d/2$. The smoothing operator A_r was discussed in Example 6.2.3 and $f(L)$ is a bounded operator on $\mathrm{L}^2(O)^m$ by definition of the functional calculus of L.

It remains to verify (BK1). For this, let B be a ball of radius r and $u \in \mathrm{L}^2(O)^m$ with $\operatorname{supp} u \subseteq B$. Since $1 - A_r = [(1-\mathrm{e}^{-r^2\mathbf{z}})^n](L)$ is bounded, $T(1-A_r)$ can be rewritten as $[f(1-\mathrm{e}^{-r^2\mathbf{z}})^n](L)$. Using Lemma 6.4.2 below, write

$$T(1-A_r)u = \int_{\Gamma_+} \eta_+(z)\mathrm{e}^{-zL}u\,\mathrm{d}z - \int_{\Gamma_-} \eta_-(z)\mathrm{e}^{-zL}u\,\mathrm{d}z. \tag{6.21}$$

For $j \geq 2$, take the $\mathrm{L}^2(C_j(B)\cap O)$-norm in (6.21) and use the $\mathrm{L}^p \to \mathrm{L}^2$ off-diagonal estimates (note that these are valid on $\Gamma_\pm$ taking Remark 6.1.13 into account) with implied constants $C, c \in (0,\infty)$ to obtain

$$\Big(\int_{C_j(B)\cap O} |T(1-A_r)u|^2\Big)^{\frac{1}{2}} \leq C(I_{j,+} + I_{j,-})\|u\|_{\mathrm{L}^p(B\cap O)},$$

where

$$I_{j,\pm} \coloneqq \int_{\Gamma_\pm} |\eta_\pm(z)||z|^{-\gamma_p/2}\mathrm{e}^{-c4^{j-1}r^2/|z|}\,\mathrm{d}|z|. \tag{6.22}$$

To complete the proof, we have to show that $I_{j,\pm} \leq g(j)r^{-\gamma_p}$ with $g(j)$ summable against $2^{dj/2}$. We proceed by establishing good estimates for $\eta_\pm$. For this, we will use the estimate

$$|1-\mathrm{e}^{-r^2\xi}|^n \lesssim \min(1, r^{2n}|\xi|^n) \qquad (\xi \in \gamma_\pm), \tag{6.23}$$

which is a consequence of the complex path integral and that the exponential function is its own complex primitive.

Recall $\eta_\pm(z) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pm} \mathrm{e}^{z\xi}g(\xi)\,\mathrm{d}\xi$ with $g(\xi) \coloneqq f(\xi)(1-\mathrm{e}^{-r^2\xi})^n$. The normalization of f and (6.23) give the pointwise bound $|g(\xi)| \leq \mathrm{e}^n\min(1, r^{2n}|\xi|^n)$ for $\xi \in \Gamma_\pm$. Moreover, for $z \in \gamma_\pm$ and $\xi \in \Gamma_\pm$ one has

$$|\arg(z\xi)| = |\pm\pi/2 \mp \theta \pm \eta| = \pi/2 + (\nu - \theta) > \pi/2.$$

Consequently, using the substitution $u = |z|s$, one has

$$\begin{aligned}
|\eta_\pm(z)| &\lesssim \int_0^\infty \mathrm{e}^{-c|z|s}g(s\mathrm{e}^{\pm\nu})\,\mathrm{d}s \lesssim \int_0^\infty \mathrm{e}^{-c|z|s}\min(1, \ell^{2n}s^n)\,\mathrm{d}s\\
&= \int_0^{r^{-2}} \mathrm{e}^{-c|z|s}r^{2n}s^n\,\mathrm{d}s + \int_{r^{-2}}^\infty \mathrm{e}^{-c|z|s}\,\mathrm{d}s \qquad (6.24)\\
&= |z|^{-1}\Big(\int_0^{\frac{|z|}{r^2}} \mathrm{e}^{-cu}r^{2n}|z|^{-n}u^n\,\mathrm{d}u + \int_{\frac{|z|}{r^2}}^\infty \mathrm{e}^{-cu}\,\mathrm{d}u\Big).
\end{aligned}$$

In the case $r^{2n}|z|^{-n} \leq 1$ we continue with

$$\begin{aligned}
&= |z|^{-1}\Big(\int_0^{\frac{|z|}{r^2}} \mathrm{e}^{-cu}r^{2n}|z|^{-n}u^n\,\mathrm{d}u + \int_{\frac{|z|}{r^2}}^\infty \mathrm{e}^{-cu}\left(\frac{|z|}{r^2}\right)^n r^{2n}|z|^{-n}\,\mathrm{d}u\Big)\\
&\leq |z|^{-1}r^{2n}|z|^{-n}\int_0^\infty \mathrm{e}^{-cu}u^n\,\mathrm{d}u.
\end{aligned}$$

Otherwise, in the case $r^{2n}|z|^{-n} \geq 1$, proceed in (6.24) with

$$\begin{aligned}
&\leq |z|^{-1}\Big(r^{2n}|z|^{-n}\int_0^{\frac{|z|}{r^2}} u^n\,\mathrm{d}u + \int_{\frac{|z|}{r^2}}^\infty \mathrm{e}^{-cu}\,\mathrm{d}u\Big)\\
&\lesssim |z|^{-1}\Big(\frac{|z|}{r^2} + 1\Big) \leq 2|z|^{-1}.
\end{aligned}$$

In summary, this shows $|\eta_\pm(z)| \lesssim |z|^{-1}\min(1, r^{2n}|z|^{-n})$. Plug this back into (6.22), split the integral according to the cases from the minimum, and use the substitution $s = t4^{-(j-1)}r^{-2}$ to derive

$$
\begin{aligned}
I_{j,\pm} &\lesssim \int_0^\infty \min(1, r^{2n}t^{-n}) t^{-\gamma_p/2} \mathrm{e}^{-c4^{j-1}r^2/t} \frac{\mathrm{d}t}{t} \\
&= \int_0^{r^2} t^{-\gamma_p/2} \mathrm{e}^{-c4^{j-1}r^2/t} \frac{\mathrm{d}t}{t} + \int_{r^2}^\infty r^{2n} t^{-n-\gamma_p/2} \mathrm{e}^{-c4^{j-1}r^2/t} \frac{\mathrm{d}t}{t} \\
&\leq \mathrm{e}^{-c4^{j-1}} \int_0^{r^2} t^{-\gamma_p/2} \mathrm{e}^{-c4^{j-1}r^2/t} \frac{\mathrm{d}t}{t} + \int_{r^2}^\infty r^{2n} t^{-n-\gamma_p/2} \mathrm{e}^{-c4^{j-1}r^2/t} \frac{\mathrm{d}t}{t} \\
&\leq r^{-\gamma_p} \Big(\mathrm{e}^{-c4^{j-1}} \int_0^\infty s^{-\gamma_p/2} \mathrm{e}^{-c/s} \frac{\mathrm{d}s}{s} + 4^{-(j-1)(n+\gamma_p/2)} \int_0^\infty s^{-n-\gamma_p/2} \mathrm{e}^{-c/s} \frac{\mathrm{d}s}{s} \Big).
\end{aligned}
$$

The integrals in both terms are finite and depend only on γ_p and implicit constants from the off-diagonal estimates. Also, the factor $r^{-\gamma_p}$ appears in both terms as desired. Finally, summability against $2^{dj/2}$ is clear for the first term due to the double exponential decay in j, whereas the second term decays sufficiently fast by the constraint on n introduced right at the beginning of this proof. □

The following lemma was used in the proposition above. A proof can be found in [36, Lemma 5.1]. The assertion can be seen as a special case of compatibility between the sectorial calculus of the generator of an analytic semigroup and its Phillips calculus, see [53, Sec. 3.3] for more information.

Lemma 6.4.2. *Let $\omega < \theta < \nu < \varphi < \pi/2$ and $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$. Let $\Gamma_\pm$ parameterize the rays from 0 to ∞ of angle $\pm\pi/2 - \theta$ and $\gamma_\pm$ the corresponding rays of angle $\pm\nu$. Then one has the identity*

$$
f(L) = \int_{\Gamma_+} \mathrm{e}^{-zL} \eta_+(z) \,\mathrm{d}z - \int_{\Gamma_-} \mathrm{e}^{-zL} \eta_-(z) \,\mathrm{d}z,
$$

where

$$
\eta_\pm(z) \coloneqq \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm} \mathrm{e}^{z\xi} f(\xi) \,\mathrm{d}\xi.
$$

6.5. Riesz transform

We consider the operator $\nabla L^{-\frac{1}{2}}$ called the *Riesz transform*. In the case of classical Fourier analysis, the Fourier multiplier with symbol $-\mathrm{i}\frac{\xi_j}{|\xi|}$ is called the

jth Riesz transform. Recall that the Fourier symbols of ∂_j and Δ are given by $\mathrm{i}\xi$ and $-|\xi|^2$. Then it immediately becomes apparent that the jth Riesz transform is just the jth component of the operator $\nabla L^{-\frac{1}{2}}$ with $L = -\Delta$, which motivates the nomenclature above.

In Theorem 5.0.1 we have seen that $\mathrm{D}(L^{\frac{1}{2}}) = \mathrm{W}^{1,2}_D(O)^m$ and that $L^{-\frac{1}{2}}$ is a topological isomorphism from $\mathrm{L}^2(O)^m$ to $\mathrm{W}^{1,2}_D(O)^m$. This shows in particular that the Riesz transform on $\mathrm{L}^2(O)$ is well-defined and bounded.

The aim of this section is to establish L^p-bounds for the Riesz transform. Owing to Proposition 6.2.1, we derive these bounds in the case $p < 2$ in Section 6.5.1. The arguments for this case are in a straightforward accordance with the material in [36, Sec. 6]. The situation for $p > 2$ is more difficult. In the pure Neumann case $D = \emptyset$ we modify the arguments from [5, Sec. 4.1.2] to get a result in our geometric situation. This is carried out in Section 6.5.4. As a preparation, we will establish local Poincaré inequalities in Section 6.5.2, which heavily use the properties of the extension operator from Theorem 3.9.2. Also, we will need a conservation property. It will be shown in Section 6.5.3 and is the reason why we have to restrict ourself to the pure Neumann case along with further restrictions already mentioned in the main result of this chapter.

6.5.1. The case $p < 2$

Proposition 6.5.1. *Suppose that $\mathcal{N}$ satisfies $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates for some $q \in (1, 2)$ and let $p \in (q, 2)$. Then*

$$\|\nabla L^{-\frac{1}{2}} u\|_p \lesssim \|u\|_p \qquad (u \in \mathrm{L}^p(O)^m \cap \mathrm{L}^2(O)^m),$$

and the bound depends on L only via its coefficient bounds and the implied constants in the off-diagonal estimates.

Proof. To derive L^p-boundedness of the Riesz transform, we employ Proposition 6.2.1 with $T = \nabla L^{-\frac{1}{2}}$. For brevity, write $\gamma \coloneqq \gamma_q$. Define the operator family $\{A_r\}_{r>0}$ as in Example 6.2.3 by $A_r \coloneqq 1 - (1 - \mathrm{e}^{-r^2 L})^n$ but with the additional constraint $\gamma + 2n > d/2$ on n, compare with Step 3 in the proof of Proposition 6.4.1. Condition (BK2) was already checked in that example.

To establish (BK1), let $B \subseteq \mathbb{R}^d$ be some open ball of radius $r > 0$ and $u \in \mathrm{L}^2(O)^m$ with support in B. We derive a useful representation formula for T first. Note that the Bochner integrals $\frac{2}{\sqrt{\pi}} \int_\varepsilon^\infty L^{\frac{1}{2}} \mathrm{e}^{-t^2 L} v \, \mathrm{d}t$ exist in $\mathrm{L}^2(O)^m$ and converge to v as $\varepsilon \to 0$ for all $v \in \mathrm{L}^2(O)^m$ according to Example 6.3.4.

By L^2-boundedness of the Riesz transform,

$$\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \nabla \mathrm{e}^{-t^2L} v \,\mathrm{d}t = \nabla L^{-\frac{1}{2}} \frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty L^{\frac{1}{2}} \mathrm{e}^{-t^2L} v \,\mathrm{d}t \to \nabla L^{-\frac{1}{2}} v \quad \text{as } \varepsilon \to 0.$$

Specify $v = (1-A_r)u$ to derive the identity

$$\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \nabla \mathrm{e}^{-t^2L}(1-A_r)u \,\mathrm{d}t \to T(1-A_r)u \quad \text{as } \varepsilon \to 0. \tag{6.25}$$

We want to rewrite the integrand in the left-hand side of (6.25). To this end, using the functional calculus of L and the binomial theorem, we get

$$\begin{aligned}
\mathrm{e}^{-t^2L}(1-\mathrm{e}^{-r^2L})^n &= [\mathrm{e}^{-t^2\mathbf{z}}(1-\mathrm{e}^{-r^2\mathbf{z}})^n](L) \\
&= \Big[\sum_{k=0}^n \binom{n}{k}(-1)^k \mathrm{e}^{-(t^2+kr^2)\mathbf{z}}\Big](L) \\
&= \sum_{k=0}^n \binom{n}{k}(-1)^k \mathrm{e}^{-(t^2+kr^2)L}.
\end{aligned}$$

Keeping the definition of A_r in mind, plug this into the left-hand side of (6.25) to arrive at

$$\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \sum_{k=0}^n \binom{n}{k}(-1)^k \nabla \mathrm{e}^{-(t^2+kr^2)L} u \,\mathrm{d}t. \tag{6.26}$$

We claim that the limit $\varepsilon \to 0$ of (6.26) exists in $\mathrm{L}^2(C_j(B)\cap O)$. Indeed, this follows from Lebesgue's theorem if the integral $\int_0^\infty \|\nabla \mathrm{e}^{-(t^2+kr^2)L}u\|_{\mathrm{L}^2(C_j(B)\cap O)}\,\mathrm{d}t$ is finite for $k = 1,\ldots,n$. We use the L^2 off-diagonal estimates for $\mathcal{N}$ from Proposition 6.1.5 and the transformation $s = (t^2+kr^2)^{-1}$ to calculate

$$\begin{aligned}
\int_0^\infty \|\nabla \mathrm{e}^{-(t^2+kr^2)L}u\|_{\mathrm{L}^2(C_j(B)\cap O)}\,\mathrm{d}t &\lesssim \int_0^\infty (t^2+kr^2)^{-1}\mathrm{e}^{-\frac{c4^{j-1}r^2}{t^2+kr^2}}\|u\|_2\,\mathrm{d}t \\
&\leq \frac{1}{2}\int_0^\infty s^{-\frac{1}{2}}\mathrm{e}^{-cr^2s}\,\mathrm{d}s\;\|u\|_2,
\end{aligned}$$

which is indeed finite. Consequently, keeping (6.25) in mind, we derive in $\mathrm{L}^2(C_j(B)\cap O)$ the identity

$$T(1-A_r)u = \int_0^\infty \sum_{k=0}^n \binom{n}{k}(-1)^k \nabla \mathrm{e}^{-(t^2+kr^2)L}u\,\mathrm{d}t.$$

Now, apply the substitution $s = t^2/r^2 + k$ to transform the right-hand side into

$$\int_0^\infty \sum_{k=0}^n \binom{n}{k}(-1)^k \mathbf{1}_{(0,\infty)}(r^2(s-k))\nabla \mathrm{e}^{-r^2sL}u\frac{r}{2\sqrt{s-k}}\,\mathrm{d}s.$$

Using the auxiliary function

$$h(s) := \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \mathbf{1}_{(0,\infty)}(r^2(s-k))\frac{1}{\sqrt{s-k}},$$

this can be rewritten as

$$T(1-A_r)u = \int_0^\infty h(s) r\nabla \mathrm{e}^{-r^2 sL} u \,\mathrm{d}s.$$

For $j \geq 2$, we apply the $\mathrm{L}^2(C_j(B)\cap O)$-norm to our representation of $T(1-A_r)$ above and apply $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates for $\mathcal{N}$ with implied constants $C, c \in (0,\infty)$ to derive

$$\Bigl(\int_{C_j(B)\cap O} |T(1-A_r)u|^2\Bigr)^{\frac{1}{2}} \leq Cr^{-\gamma}\int_0^\infty |h(s)| s^{-\gamma/2-1/2}\mathrm{e}^{-c4^{j-1}/s}\,\mathrm{d}s \|u\|_{\mathrm{L}^q(B\cap O)}.$$

We split the integral on the right-hand side into a local integral and a global integral. To be precise, write

$$\int_0^\infty |h(s)| s^{-\gamma/2-1/2}\mathrm{e}^{-c4^{j-1}/s}\,\mathrm{d}s = \int_0^{4n} \dots \,\mathrm{d}s + \int_{4n}^\infty \dots \,\mathrm{d}s =: I_0 + I_\infty.$$

We begin with an estimate for I_0. We want to extract exponential decay (for this, we need that s is small) by splitting the exponential term in the integral, and remain with a finite term. Observe that $s^{-\gamma/2-1/2}\mathrm{e}^{-\frac{c}{2s}}$ remains bounded for $s \to 0$ by l'Hôpital's rule, and the bound on $(0,4n)$ depends only on γ and n. Thus, we get

$$\begin{aligned} I_0 &\leq \mathrm{e}^{-\frac{c4^{j-1}}{8n}}\int_0^{4n} |h(s)| s^{-\gamma/2-1/2}\mathrm{e}^{-\frac{c}{2s}}\,\mathrm{d}s \lesssim \mathrm{e}^{-\frac{c4^{j-1}}{8n}}\sum_{k=0}^{n}\binom{n}{k}\int_0^{4n}\frac{\mathbf{1}_{(0,\infty)}(s-k)}{\sqrt{s-k}}\,\mathrm{d}s \\ &\leq \mathrm{e}^{-\frac{c4^{j-1}}{8n}}\sum_{k=0}^{n}\binom{n}{k}\int_0^{4n} s^{-\frac{1}{2}}\,\mathrm{d}s \leq \mathrm{e}^{-\frac{c4^{j-1}}{8n}} 2^n \int_0^{4n} s^{-\frac{1}{2}}\,\mathrm{d}s. \end{aligned}$$

The integral on the right-hand side is finite, so we have bounded I_0 by a term with double exponential decay in j.

To estimate I_∞, we derive a better representation formula for h using the residue theorem. As a primer, calculate for fixed $0 \leq k \leq n$ that

$$\begin{aligned}\binom{n}{k}(-1)^k &= n! \frac{1}{(-1)\cdots(-k)} \frac{1}{1\cdots(n-k)} \\ &= n! \prod_{j=0}^{k-1} \frac{1}{j-k} \prod_{j=k+1}^{n} \frac{1}{j-k} \\ &= n!(-1)^n \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{k-j}.\end{aligned}$$

Plug this back into the definition of h and note that for $s > 4n$ the indicator functions in the definition of h all evaluate to 1, to get

$$h(s) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (s-k)^{-\frac{1}{2}} = \frac{1}{2} n! (-1)^n \sum_{k=0}^{n} \prod_{j\neq k} \frac{1}{k-j} \frac{1}{\sqrt{s-k}}.$$

Now, note that the function

$$\frac{1}{\mathbf{z}(\mathbf{z}-1)\dots(\mathbf{z}-n)} \frac{1}{\sqrt{s-\mathbf{z}}}$$

has poles of order 1 in $k = 0, \dots, n$ with corresponding residues $\prod_{j\neq k} \frac{1}{k-j} \frac{1}{\sqrt{s-k}}$. Therefore, the residue theorem reveals

$$\frac{(-1)^n n!}{2\pi\mathrm{i}} \int_{|z|=s/2} \frac{1}{z(z-1)\dots(z-n)} \frac{1}{\sqrt{s-z}} \, \mathrm{d}z = 2h(s).$$

Due to $s > 4n$ and $|z| = s/2$ on the contour of integration, we have by the reverse triangle inequality $|z-k| \geq s/2 - n \geq s/4$ for $k = 0, \dots, n$. Also, $|s-z| \geq s - s/2 = s/2$, so

$$|h(s)| \lesssim \int_{|z|=s/2} s^{-n-3/2} \, |\mathrm{d}z| \lesssim s^{-n-1/2}.$$

Use this and the substitution $s = 4^{j-1}t$ to estimate I_∞ according to

$$I_\infty \lesssim \int_{4n}^{\infty} s^{-n-\gamma/2} \mathrm{e}^{-c4^{j-1}/s} \frac{\mathrm{d}s}{s} \leq 4^{-(j-1)(n+\gamma/2)} \int_0^\infty t^{-n-\gamma/2} \mathrm{e}^{-c/t} \frac{\mathrm{d}t}{t}.$$

The integral on the right-hand side is finite, so I_∞ decays like $4^{-(j-1)(n+\gamma/2)}$. By our constraint on n, this shows sumability of $I_1 + I_\infty$ against $2^{dj/2}$. This establishes (BK1). $\square$

6.5.2. Local Poincaré inequalities

We show local Poincaré inequalities for $\mathrm{W}^{1,p}_D(O)$, which roughly means that we estimate the L^p-norm of a function on a ball centered in ∂O against its gradient norm on a slightly enlarged ball. If the ball is not centered in the Dirichlet part D, we have to subtract the mean value to arrive at such an estimate. The crucial point for a *local* inequality is that the implied constant in this bound scales like the radius of the ball. In this thesis we will only use the case $D = \emptyset$, but it felt natural to include the general case for the sake of improving the general theory of mixed boundary conditions.

The starting point for these estimates is Theorem 3.9.2. However, we will need properties beyond those implied by Assumption 3.1.1, like the corkscrew condition near N coming from Proposition 5.1.7 or Remark 5.1.8, and Ahlfors–David regularity of D. Note that if $\delta = \infty$ in 5.1.1, then the same is true for Assumption 3.1.1, see Remark 5.1.6.

Proposition 6.5.2 (Local Poincaré inequality, Neumann case)**.** *There are constants $R \in (0, \infty]$ and $c \in [1, \infty)$ such that for any ball B centered in N with radius at most R it holds*

$$\|f - (f)_{B\cap O}\|_{\mathrm{L}^p(B\cap O)} \lesssim \mathrm{r}(B)\|\nabla f\|_{\mathrm{L}^p(cB\cap O)} \qquad (f \in \mathrm{W}^{1,p}_D(O)).$$

Here, $(f)_{B\cap O}$ is the mean value of f over $B \cap O$. If O is locally an (ε, ∞)-domain near N, then $R = \infty$.

Proof. For brevity, put $r := \mathrm{r}(B)$. We use the extension operator $\mathcal{E}$ from Theorem 3.9.2. The constant R is given by the radius bound in that result and c is the enlargement factor in there. The structure of the radius bounds gives $R = \infty$ in the situation when O is locally an (ε, ∞)-domain near N.

Then, split

$$\begin{aligned}\|f - (f)_{B\cap O}\|_{\mathrm{L}^p(B\cap O)} &\leq \|\mathcal{E}f - (\mathcal{E}f)_B\|_{\mathrm{L}^p(B)} + \|(\mathcal{E}f)_B - (f)_{B\cap O}\|_{\mathrm{L}^p(B)} \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Let $S \subseteq B$ with $|S| > 0$. Since B is convex, we can use the classical Poincaré inequality

$$\|g - (g)_S\|_{\mathrm{L}^p(B)} \lesssim r\frac{|B|}{|S|}\|\nabla g\|_{\mathrm{L}^p(B)} \qquad (g \in \mathrm{W}^{1,p}(B)), \tag{6.27}$$

see [47, Lem. 7.12 & Lem. 7.16]. Combined with Theorem 3.9.2, estimate

$$\mathrm{I} \lesssim r\|\nabla\mathcal{E}f\|_{\mathrm{L}^p(B)} \lesssim r\|\nabla f\|_{\mathrm{L}^p(cB\cap O)}.$$

For the other term, use Jensen's inequality, (6.27) with $S := B \cap O$, the corkscrew condition (either from Proposition 5.1.7 if δ is finite or Remark 5.1.8 otherwise), and Theorem 3.9.2 as above, to give

$$\begin{aligned}\mathrm{II} &= \Big\| \fint_B [\mathcal{E} f - (f)_{B\cap O}] \Big\|_{\mathrm{L}^p(B)} \leq \|\mathcal{E} f - (f)_{B\cap O}\|_{\mathrm{L}^p(B)} \\ &\lesssim r \frac{|B|}{|B \cap O|} \|\nabla \mathcal{E} f\|_{\mathrm{L}^p(B)} \approx r \|\nabla \mathcal{E} f\|_{\mathrm{L}^p(B)} \lesssim r \|\nabla f\|_{\mathrm{L}^p(cB\cap O)}. \qquad \square\end{aligned}$$

The following result is the analogue for balls centered in the Dirichlet part.

Proposition 6.5.3 (Local Poincaré inequality, Dirichlet case)**.** *There are constants $R \in (0, \infty]$ and $c \in [1, \infty)$ such that for any ball B centered in D with radius at most R it holds*

$$\|f\|_{\mathrm{L}^p(B\cap O)} \lesssim \mathrm{r}(B) \|\nabla f\|_{\mathrm{L}^p(cB\cap O)} \qquad (f \in \mathrm{W}^{1,p}_D(O)). \tag{6.28}$$

If O is locally an (ε, ∞)-domain near N, then $R = \infty$.

Before we turn to the proof of the local Poincaré inequality on balls centered in D, we need a comparison result for capacity and Hausdorff content.

The case $p \leq d$ in the following proposition can be obtained from [2, Thm. 5.1.13]; A simplified version can also be found in [35, Thm. 1.2.32]. The case $p > d$ is treated in [35, Lem. 1.2.10 & Rem. 1.2.8], though the uniformity of the constant is only implicit in the proof by translation in the $\mathrm{L}^{p'}$-norm for the convolution $G_\alpha * \delta_x$.

Proposition 6.5.4 (Lower bound for $\mathrm{C}_{1,p}$)**.** *Let $1 < p < \infty$. Then there is a constant $A > 0$ such that for all $E \subseteq \mathbb{R}^d$ compact and non-empty one has*

$$\mathcal{H}^{d-1}_\infty(E) \leq A\mathrm{C}_{1,p}(E) \quad (p \leq d) \qquad \textit{and} \quad A^{-1} \leq \mathrm{C}_{1,p}(E) \qquad (p > d).$$

Proof of Theorem 6.5.3. As in the proof of Proposition 6.5.2, the parameters R and c come from Theorem 3.9.2. Also, write again $\mathcal{E}$ for that extension operator. We follow the usual strategy to transform to a reference geometry. Our preparatory work allows us to control the "perturbation" in the Dirichlet part.

Let $B = \mathrm{B}(x, r)$ be a ball with $x \in D$ and $0 < r \leq R$. By continuity it suffices to show (6.28) for $f \in \mathrm{C}^\infty_D(O)$. Put $g(y) := \mathcal{E} f(x + ry)$ for $y \in \mathrm{B}(0, 1)$. By consistency of the extension operator and Sobolev embeddings, g has a continuous representative which vanishes everywhere on $K := r^{-1}((D \cap \overline{B}) - x)$.

We claim that $\mathrm{C}_{1,p}(K) \gtrsim 1$. Indeed, if $p > d$ then this follows directly from Proposition 6.5.4. Otherwise, note first that $\mathcal{H}^{d-1}_\infty$ and $\mathcal{H}^{d-1}$ are comparable under Ahlfors–David regularity (the calculation is similar to that in Lemma A.2.4). Then, we use that proposition in conjunction with the dilation property of the Hausdorff measure [97, Thm. 28.1], translation invariance and Ahlfors–David regularity to give

$$\mathrm{C}_{1,p}(K) \gtrsim \mathcal{H}^{d-1}_\infty(K) \approx \mathcal{H}^{d-1}(K) = r^{1-d}\mathcal{H}^{d-1}(\overline{B} \cap D) \approx 1.$$

Now, we can show (6.28). Using the extension operator and the transformation rule, we deduce

$$\|f\|_{\mathrm{L}^p(B\cap O)} \leq \|\mathcal{E}f\|_{\mathrm{L}^p(B)} = r^{d/p}\|g\|_{\mathrm{L}^p(\mathrm{B}(0,1))}.$$

Applying Lemma 6.5.5 below with $\Xi \coloneqq \mathrm{B}(0,1)$ and using $\mathrm{C}_{1,p}(K) \gtrsim 1$ leads to

$$\lesssim r^{d/p}\mathrm{C}_{1,p}(K)^{-1/p}\|\nabla g\|_{\mathrm{L}^p(\mathrm{B}(0,1))} \lesssim r^{d/p}\|\nabla g\|_{\mathrm{L}^p(\mathrm{B}(0,1))}.$$

Next, apply the chain rule to the definition of g and transform back to B to deduce

$$= r^{d/p+1}\|[\nabla\mathcal{E}f](ry + x)\|_{\mathrm{L}^p(\mathrm{B}(0,1))} = r\|\nabla\mathcal{E}f\|_{\mathrm{L}^p(B)}.$$

Finally, Theorem 3.9.2 lets us conclude

$$\lesssim r\|\nabla f\|_{\mathrm{L}^p(cB\cap O)}. \qquad \square$$

Lemma 6.5.5 (Simplified version of [2, Cor. 8.2.2])**.** *Let $\Xi \subseteq \mathbb{R}^d$ be open, bounded and convex. Then there is a constant $A > 0$ such that if $K \subseteq \Xi$ is a compact subset with $\mathrm{C}_{1,p}(K) > 0$ and $f \in \mathrm{W}^{1,p}(\Xi)$ has a continuous representative that vanishes identically on K, then*

$$\|f\|_{\mathrm{L}^p(\Xi)} \leq A\mathrm{C}_{1,p}(K)^{-1/p}\|\nabla f\|_{\mathrm{L}^p(\Xi)}.$$

6.5.3. Conservation property

Loosely speaking, the conservation property states “$\mathrm{e}^{-tL}\mathbf{1} = \mathbf{1}$”. Since the semigroup maps into $\mathrm{W}^{1,2}_D(O)^m$, this identity forces us to work with pure Neumann boundary conditions. On unbounded domains, such an identity cannot hold true in the L^2-sense because the constant 1 is not even in that space. But we can nevertheless show the following adjoint version.

Proposition 6.5.6 (Conservation property)**.** *Suppose $D = \emptyset$, that the coefficients b and d of L vanish, and let $u \in \mathrm{L}^2(O)^m$ be compactly supported. Then $\mathrm{e}^{-tL^*}u \in \mathrm{L}^1(O)^m$ and $\int_O u = \int_O \mathrm{e}^{-tL^*}u$ holds for all $t > 0$.*

Proof. Say u is supported in the ball B with radius $r = \mathrm{r}(B)$. Let ψ be a smooth, compactly supported cutoff function for B and put $\psi_n(x) := \psi(x/2^n)$ for $x \in \mathbb{R}^d$. Also, keep in mind that the form domain of a and a^* reduces to $\mathrm{W}^{1,2}(O)^m$ by $D = \emptyset$. The proof divides into several steps.

Step 1: $\mathrm{e}^{-tL^*}u \in \mathrm{L}^1$. This is a straightforward consequence of L^2 off-diagonal estimates for $\mathcal{S}(L^*)$ and Hölder's inequality. Write $C, c \in (0,\infty)$ for the implied constants and estimate

$$\begin{aligned}\int_O |\mathrm{e}^{-tL^*}u| &= \int_{4B\cap O} |\mathrm{e}^{-tL^*}u| + \sum_{j\geq 2}\int_{C_j(B)\cap O} |\mathrm{e}^{-tL^*}u| \\ &\lesssim Cr^{d/2}\Big(1 + \sum_{j\geq 2} 2^{jd/2}\mathrm{e}^{-c4^{j-1}r^2/t}\Big)\|u\|_2.\end{aligned}$$

The series in j is convergent, which completes this step.

Step 2: $\mathrm{e}^{-tL^*}u \to u$ in L^1 as $t \to 0$. As above, we employ a dyadic splitting of the area of integration and use Hölder's inequality to find

$$\int_O |\mathrm{e}^{-tL^*}u - u| \lesssim r^{d/2}\|\mathrm{e}^{-tL^*}u - u\|_2 + \sum_{j\geq 2}\int_{C_j(B)\cap O} |\mathrm{e}^{-tL^*}u - u|.$$

The first term is fine by the strong continuity of the semigroup on L^2. For the other term, keep in mind that $u = 0$ on $C_j(B)$ for $j \geq 2$, and assume $t \leq 1$. Then, off-diagonal estimates allow us to derive

$$\begin{aligned}&\int_{C_j(B)\cap O} |\mathrm{e}^{-tL^*}u - u| = \int_{C_j(B)\cap O} |\mathrm{e}^{-tL^*}u| \\ &\leq Cr^{d/2}2^{jd/2}\mathrm{e}^{-c4^{j-1}r^2/t}\|u\|_2 \leq Cr^{d/2}2^{jd/2}\mathrm{e}^{-c4^{j-1}r^2/2}\mathrm{e}^{-c2r^2/t}\|u\|_2.\end{aligned}$$

Then the sum over j is convergent, so we conclude

$$\sum_{j\geq 2}\int_{C_j(B)\cap O} |\mathrm{e}^{-tL^*}u - u| \lesssim r^{d/2}\mathrm{e}^{-c2r^2/t}\|u\|_2,$$

which vanishes for $t \to 0$.

Step 3: $t \mapsto a^*(\mathrm{e}^{-tL^*}u, \psi_n)$ is a uniformly bounded sequence in $\mathrm{L}^\infty(0,\infty)$ which goes pointwise to 0 as $n \to \infty$. Recall the definition of a^* in (6.3). We have $\psi_n \in \mathrm{W}^{1,2}(O)^m$ by construction. Here, we need the pure Neumann boundary conditions to be able to plug ψ_n into a^*. Moreover, $\mathrm{e}^{-tL^*}u \in$

$\mathrm{D}(L^*) \subseteq \mathrm{W}^{1,2}(O)^m$. Hence, since $b = d = 0$, we get from the definition (6.1) of a the identity

$$
\begin{aligned}
a^*(\mathrm{e}^{-tL^*}u, \psi_n) &= \overline{a(\psi_n, \mathrm{e}^{-tL^*}u)} \\
&= \int_O \sum_{i,j=1}^d \partial_i \mathrm{e}^{-tL^*}u \cdot \overline{a_{ij}\partial_j \psi_n} + \sum_{j=1}^d \mathrm{e}^{-tL^*}u \cdot \overline{c_j \partial_j \psi_n}.
\end{aligned} \tag{6.29}
$$

Due to $\psi_n = 1$ on $2^n B$, the gradient terms of ψ_n are supported outside of $2^n B$. This is the reason why we had to restrict ourself to the case where b and d vanish. Next, we take absolute values in (6.29) and use L^2 off-diagonal estimates for $\mathcal{N}$ and $\mathcal{S}$ with implied constants $C, c \in (0, \infty)$ to deduce

$$|a^*(\mathrm{e}^{-tL^*}u, \psi_n)| \lesssim C(t^{-1/2} + 1)\mathrm{e}^{-c4^{n-1}/t^2} \|u\|_2.$$

For fixed t, this goes to zero when $n \to \infty$, and $t^{-1/2}\mathrm{e}^{-c4^{n-1}/t^2} \leq t^{-1/2}\mathrm{e}^{-c/t^2}$ yields a uniform bound.

Step 4: $\int_0 \mathrm{e}^{tL^*}u$ is constant in t. Let $0 < s < t < \infty$. Since $\mathrm{e}^{-tL^*}u, \mathrm{e}^{-sL^*}u \in \mathrm{L}^1$ by Step 1 and ψ_n is uniformly bounded and goes pointwise to 1, Lebesgue's theorem yields

$$\int_O \mathrm{e}^{-tL^*}u - \int_O \mathrm{e}^{-sL^*}u = \lim_{n\to\infty} \Big(\int_O \mathrm{e}^{-tL^*}u\psi_n - \int_O \mathrm{e}^{-sL^*}u\psi_n \Big).$$

By analyticity of the semigroup, $\partial_\tau \mathrm{e}^{-\tau L^*} = -L^*\mathrm{e}^{-\tau L^*}$ holds. This derivative interchanges with the bounded functional $v \mapsto \int_O v\psi_n$ on $\mathrm{L}^2(O)$, hence

$$= \lim_{n\to\infty} \int_s^t \partial_\tau \int_O \mathrm{e}^{-\tau L^*}u\psi_n \,\mathrm{d}\tau = \lim_{n\to\infty} - \int_s^t a^*(\mathrm{e}^{-\tau L^*}u, \psi_n)\,\mathrm{d}\tau.$$

The limit is zero due to Lebesgue's theorem and Step 3.

Step 5: $\int_O \mathrm{e}^{-tL^*}u = \int_O u$ for all $t > 0$. Apply the limit $s \to 0$ to the identity from Step 4 and use the strong continuity from Step 2 to deduce

$$\int_O \mathrm{e}^{-tL^*}u = \lim_{s\to 0} \int_O \mathrm{e}^{-sL^*}u = \int_O u. \qquad \square$$

6.5.4. The case $p > 2$

Proposition 6.5.7. *Assume $D = \emptyset$, that the coefficients b and d of L vanish, and suppose that $\mathcal{N}$ satisfies $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates for some $q \in (2, \infty)$ and let $p \in (2, q)$. Then $\nabla L^{-\frac{1}{2}}$ is bounded on $\mathrm{L}^p(O)^m$, and the bound depends on L only via its coefficient bounds and the implied constants in the off-diagonal estimates.*

Proof. Note first that it suffices to show uniform L^q-bounds for the truncated operators $T_\varepsilon \coloneqq \int_\varepsilon^\infty t^{\frac{1}{2}} \nabla \mathrm{e}^{-tL} u \frac{\mathrm{d}t}{t}$ for $\varepsilon > 0$, which is a consequence of the strong convergence of $\frac{1}{\sqrt{\pi}} T_\varepsilon \to \nabla L^{-\frac{1}{2}}$ discussed in Example 6.3.3 and the discussion in Section 6.2. To show L^q-bounds independent of ε, we appeal to the extrapolation result from Proposition 6.2.6. As before, define the operator family $\{A_r\}_{r>0}$ by $A_r \coloneqq 1 - (1 - \mathrm{e}^{-r^2 L})^n$, where $n > d/4$.

Step 1: Verify assumption (GL1). Since the L^2-norm is taken over B, we use the decomposition $u = \sum_{j \geq 1} \mathbf{1}_{C_j(B)} u$ to bring off-diagonal decay into business. For brevity, let us put $u_j \coloneqq \mathbf{1}_{C_j(B)} u$. Note that the sum converges in L^2 by Lebesgue's theorem. Hence, by L^2-continuity of T_ε with constant independent of ε (which is a consequence of L^2-boundedness of the Riesz transform and uniform bounds for the McIntosh approximation, see the final step of its proof), and the triangle inequality we have

$$\Big(\int_{B \cap O} |T_\varepsilon (1 - A_r) u|^2 \Big)^{\frac{1}{2}} \leq \sum_{j \geq 1} \| T_\varepsilon (1 - A_r) u_j \|_{\mathrm{L}^2(B \cap O)}. \tag{6.30}$$

It suffices to show the bounds

$$\| T_\varepsilon (1 - A_r) u_j \|_{\mathrm{L}^2(B \cap O)} \leq r^{d/2} g(j) \Big(\fint_{2^{j+1} B \cap O} |u|^2 \Big)^{\frac{1}{2}} \tag{6.31}$$

for $j \geq 1$, where $g(j)$ is summable over $j \geq 1$. Indeed, the mean value integrals can be bounded by $\mathcal{M}^O(|u|^2)^{1/2}(y)$ for $y \in B$ and the sum over j in (6.30) then reduces to an absolute constant.

For $j = 1$, L^2-boundedness of $\{T_\varepsilon(1 - A_r)\}_{\varepsilon, r > 0}$ immediately gives

$$\| T_\varepsilon (1 - A_r) u_1 \|_{\mathrm{L}^2(B \cap O)} \lesssim \| u \|_{\mathrm{L}^2(C_1(B) \cap O)} \lesssim r^{d/2} \Big(\fint_{4B \cap O} |u|^2 \Big)^{\frac{1}{2}}.$$

Now let $j \geq 2$. Recall the identity

$$T_\varepsilon (1 - A_r) = \int_\varepsilon^\infty t^{\frac{1}{2}} \nabla \mathrm{e}^{-tL} (1 - A_r) \, \frac{\mathrm{d}t}{t} = \int_\varepsilon^\infty t^{\frac{1}{2}} \nabla f(L) \, \frac{\mathrm{d}t}{t}, \tag{6.32}$$

where we use the auxiliary function $f(\mathbf{z}) \coloneqq \mathrm{e}^{-t\mathbf{z}} (1 - \mathrm{e}^{-r^2 \mathbf{z}})^n$. Let $\omega < \theta < \nu < \varphi < \pi/2$, then clearly $f \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi)$, so that, owing to Lemma 6.4.2, we can write

$$f(L) = \int_{\Gamma_+} \mathrm{e}^{-zL} \eta_+(z) \, \mathrm{d}z - \int_{\Gamma_-} \mathrm{e}^{-zL} \eta_-(z) \, \mathrm{d}z,$$

where

$$\eta_\pm(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm} \mathrm{e}^{z\xi} f(\xi) \,\mathrm{d}\xi \qquad (z \in \Gamma_\pm),$$

and $\Gamma_\pm$, and $\gamma_\pm$ are defined as in Lemma 6.4.2. We want to derive pointwise estimates for $|\eta_\pm|$ as in the proof of Proposition 6.4.1. To this end, we seek good pointwise bounds the integrand $\mathrm{e}^{z\xi} f(\xi) = \mathrm{e}^{\xi(z-t)}(1-\mathrm{e}^{-r^2\xi})^n$ in the definition of $\eta_\pm$. For its second factor, $(1-\mathrm{e}^{-r^2\xi})^n$, we can rely on (6.23), so we only need to control $\mathrm{e}^{\xi(z-t)}$. From $|\arg(z-t)| \geq |\arg(z)|$ one gets

$$|\arg(\xi(z-t))| \geq |\arg(\xi) + \arg(z-t)| \geq |\arg(\xi) + \arg(z)| = {}^{\pi}/_{2} + (\nu - \theta).$$

As $\nu - \theta > 0$, it follows $|\mathrm{e}^{\xi(z-t)}| \leq \mathrm{e}^{-c|\xi||z-t|}$ with $c > 0$ depending on $\nu - \theta$. Finally, since $t > 0$ and $-z \in \overline{\mathrm{S}}_{\nu+\pi/2}$, the reverse triangle inequality on sectors lets us conclude $|\mathrm{e}^{\xi(z-t)}| \leq \mathrm{e}^{-c|\xi|(|z|+|t|)}$ for some different $c > 0$ as above. Now we can essentially repeat the calculation in the proof of Proposition 6.4.1 to derive the bound

$$|\eta_\pm(z)| \lesssim (|z|+t)^{-1} \min(1, r^{2n}(|z|+t)^{-n}) \qquad (z \in \Gamma_\pm),$$

the only difference is that we use the transformation $u = s(|z|+t)$ instead of $u = |z|s$ after splitting the integrals.

Let us come back to (6.31) with $j \geq 2$. We use identity (6.32), insert the definition of $f(L)$, and commute ∇ with the inner integral, to give

$$\|T_\varepsilon(1-A_r)u_j\|_{\mathrm{L}^2(B\cap O)} \leq 2 \int_\varepsilon^\infty \int_{\Gamma_\pm} t^{\frac{1}{2}} |z|^{-\frac{1}{2}} |\eta_\pm(z)| \|z^{\frac{1}{2}} \nabla \mathrm{e}^{-zL} u_j\|_{\mathrm{L}^2(B\cap O)} \,\mathrm{d}|z| \,\frac{\mathrm{d}t}{t}.$$

Using the bound on $|\eta_\pm|$, and L^2 off-diagonal estimates, we control the last quantity by

$$\int_\varepsilon^\infty \int_{\Gamma_\pm} t^{\frac{1}{2}} |z|^{-\frac{1}{2}} \mathrm{e}^{-c\frac{4^j r^2}{|z|}} (|z|+t)^{-1} r^{2n} (|z|+t)^{-n} \,\mathrm{d}|z| \,\frac{\mathrm{d}t}{t} \|u_j\|_{\mathrm{L}^2(O)}. \tag{6.33}$$

Next, we bound the inner integral. To ease notation, write $\beta \coloneqq \frac{4^j r^2}{t}$, and $|z| = s$. Split the integral into $s \leq t$ and $s \geq t$. In the first case, we use the trivial bounds $(s+t)^{-1} \leq s^{-1}$, and $(s+t)^{-n} \leq t^{-n}$, to give

$$\int_0^t \mathrm{e}^{-c\frac{4^j r^2}{s}} t^{\frac{1}{2}} s^{-\frac{1}{2}} (s+t)^{-1} r^{2n} (s+t)^{-n} \,\mathrm{d}s \leq \beta^n 4^{-jn} \int_0^t \mathrm{e}^{-c\frac{4^j r^2}{s}} t^{\frac{1}{2}} s^{-\frac{1}{2}} \,\frac{\mathrm{d}s}{s}.$$

Using the transformation $u = \frac{4^j r^2}{s}$, and recalling the definition of β, continue with

$$\beta^n 4^{-jn} \int_0^t \mathrm{e}^{-c\frac{4^j r^2}{s}} t^{\frac{1}{2}} s^{-\frac{1}{2}} \,\frac{\mathrm{d}s}{s} \le \beta^n 4^{-jn} \beta^{-\frac{1}{2}} \int_\beta^\infty \mathrm{e}^{-cu} u^{\frac{1}{2}} \,\frac{\mathrm{d}s}{s} \tag{6.34}$$

To estimate the integral in u, we can either neglect the factor e^{-cu} to get an estimate against $\beta^{\frac{1}{2}}$, or we split $c = c_1 + c_2$ with $c_1, c_2 > 0$ to give

$$\int_\beta^\infty \mathrm{e}^{-cu} u^{-\frac{1}{2}} \,\frac{\mathrm{d}u}{u} \le \mathrm{e}^{-c_1\beta} \int_\beta^\infty \mathrm{e}^{-c_2 u} u^{-\frac{1}{2}} \,\frac{\mathrm{d}u}{u} \lesssim \mathrm{e}^{-c_1\beta}.$$

In summary, as a consequence of $\beta^n \mathrm{e}^{-c_1\beta} \lesssim 1$ for $\beta \ge 1$, we can bound the term on the right-hand side of (6.34) by

$$\beta^n 4^{-jn} \min(\mathrm{e}^{-c_1\beta}, \beta^{\frac{1}{2}}) \lesssim 4^{-jn} \min(\beta^{-\frac{1}{2}}, \beta^n).$$

In the second case, when $s \ge t$, use the inequalities $(s+t)^{-1} \le s^{-1}$, and $(s+t)^{-n} \le s^{-n}$, to give

$$\int_t^\infty t^{\frac{1}{2}} s^{-\frac{1}{2}} \mathrm{e}^{-c\frac{4^j r^2}{s}} (s+t)^{-1} r^{2n} (s+t)^{-n} \,\mathrm{d}s \le \int_t^\infty \mathrm{e}^{-c\frac{4^j r^2}{s}} t^{\frac{1}{2}} s^{-\frac{1}{2}} \left(\frac{r^2}{s}\right)^n \frac{\mathrm{d}s}{s}.$$

We use again the transformation $u = \frac{4^j r^2}{s}$ to derive the identity

$$\begin{aligned} \int_t^\infty \mathrm{e}^{-c\frac{4^j r^2}{s}} t^{\frac{1}{2}} s^{-\frac{1}{2}} \left(\frac{r^2}{s}\right)^n \frac{\mathrm{d}s}{s} &= \int_0^\beta \mathrm{e}^{-cu} t^{\frac{1}{2}} (4^j r^2)^{-\frac{1}{2}} u^{\frac{1}{2}+n} 4^{-jn} \,\frac{\mathrm{d}u}{u} \\ &= \beta^{-\frac{1}{2}} 4^{-jn} \int_0^\beta \mathrm{e}^{-cu} u^{\frac{1}{2}+n} \,\frac{\mathrm{d}u}{u}. \end{aligned}$$

The integral in u can be controlled by $\min(\beta^{\frac{1}{2}+n}, 1)$, either by neglecting e^{-cu}, or by using integrability of $\mathrm{e}^{-cu} u^{n-\frac{1}{2}}$ over $(0, \infty)$, so that we get in summary

$$\int_t^\infty t^{\frac{1}{2}} s^{-\frac{1}{2}} \mathrm{e}^{-c\frac{4^j r^2}{s}} (s+t)^{-1} r^{2n} (s+t)^{-n} \,\mathrm{d}s \lesssim 4^{-jn} \min(\beta^n, \beta^{-\frac{1}{2}}).$$

Going back to (6.33), this yields the bound

$$\|T_\varepsilon(1 - A_r) u_j\|_{\mathrm{L}^2(B\cap O)} \lesssim 4^{-jn} (2^j r)^{-\gamma} \int_\varepsilon^\infty \min(\beta^{\frac{\gamma}{2}+n}, \beta^{-\frac{1}{2}}) \,\frac{\mathrm{d}t}{t} \|u_j\|_{\mathrm{L}^2(O)}.$$

The integral in t is bounded by a universal constant, which can be seen by splitting the integral at height $4^j r^2$. Also

$$\|u_j\|_{\mathrm{L}^2(O)} \lesssim 2^{\frac{jd}{2}} r^{\frac{d}{2}} \left(⨍_{2^{j+1}B\cap O} |u|^2 \right)^{\frac{1}{2}},$$

so that in the end we get (6.31) with $g(j) \coloneqq 2^{j(d/2-2n)}$, which is indeed summable in j by the constraint on n. This concludes the proof of (GL1).

Step 2: Verify assumption (GL2). We continue with (GL2). Here, we need the conservation property and local Poincaré inequalites, so this is the part of the proof that brings the extra restrictions on coefficients and boundary conditions into play. For the rest of the proof, conservation property will always refer to Proposition 6.5.6 and local Poincaré inequality to the Neumann version Proposition 6.5.2. Note that the local Poincaré inequality works for balls B with $\frac{1}{2}B \cap \overline{O} \neq \emptyset$. To see this, distinguish whether the ball is properly contained in O or not. If this is the case, use the convex version stated in (6.27). Otherwise, use an auxiliary boundary ball centered in an intersection point of ∂O with B of radius $2\,\mathrm{r}(B)$. To introduce the mean value on this auxiliary ball, an estimate similar to that of term II in the proof of the local Poincaré inequality can be used (since $\frac{1}{2}B \cap \overline{O} \neq \emptyset$, the corkscrew condition can be applied to a ball centered in ∂O of radius $\mathrm{r}(B)/2$).

To ease notation, we will denote integrals over $B \cap O$ only by $\int_B$ and assume tacitly that the integrand is extended outside O by zero.

Reductions. Let us start with three reductions. For this, take an arbitrary ball B in $\mathbb{R}^d$ of radius $r = \mathrm{r}(B)$ and let $u \in \mathrm{L}^2(O)^m$. First, we may assume that $\frac{1}{2}B$ hits $\overline{O}$. Indeed, if B does not hit $\overline{O}$, (GL2) is void, and otherwise we can replace B by $2B$ in (GL2). Second, it suffices to show the claim with A_r replaced by e^{-kr^2L} for all $k = 1, \dots, n$ in virtue of the expansion of A_r. Third, with $v \coloneqq \int_\varepsilon^\infty \mathrm{e}^{-t^2L} u \,\mathrm{d}t$ it suffices to show

$$\|\nabla \mathrm{e}^{-kr^2L} v\|_{\mathrm{L}^q(B)} \leq c r^{d/2}\, \mathcal{M}^O(|\nabla v|^2)^{1/2}(y) \tag{6.35}$$

for all $y \in B$. Indeed, since the defining integral for v converges absolutely in $\mathrm{W}^{1,2}(O)^m$ by exponential decay and gradient estimates of the semigroup, pull $\nabla \mathrm{e}^{-kr^2L}$ into the integral and commute the semigroup terms to conclude on the one hand that $\nabla \mathrm{e}^{-kr^2L} v = T_\varepsilon \mathrm{e}^{-kr^2L} u$. On the other hand, $\nabla v = T_\varepsilon u$, which completes this reduction step.

To estimate the left-hand side of (6.35) we test with $g \in \mathrm{C}_0^\infty(B \cap O)$. Calculate first with the conservation property that

$$0 = \int_B \nabla g = \int_{\mathbb{R}^d} \mathrm{e}^{-kr^2L^*} \nabla g,$$

where the semigroup is supposed to act componentwise. The integrand is integrable by Step 1 in the proof of the conservation property, so Lebesgue's theorem yields

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \int_{C_j(B)} \mathrm{e}^{-kr^2 L^*} \nabla g.$$

The same stays true if we multiply the right-hand side by any constant. Hence

$$\begin{aligned}
\int_B \nabla \mathrm{e}^{-kr^2 L} v g &= \lim_{N\to\infty} -\sum_{j=1}^{N} \int_{C_j(B)} v \mathrm{e}^{-kr^2 L^*} \operatorname{div} g \\
&= \lim_{N\to\infty} -\sum_{j=1}^{N} \int_{\mathbb{R}^d} (\mathbf{1}_{C_j(B)}[v-(v)_{4B}]) \mathrm{e}^{-kr^2 L^*} \operatorname{div} g \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \int_B \nabla \mathrm{e}^{-kr^2 L}[\mathbf{1}_{C_j(B)}(v-(v)_{4B})] g.
\end{aligned}$$

In accordance with our convention for integrals we write $(v)_{4B}$ for the mean value of the zero extension $\mathcal{E}_0 v$ of v over $4B$. Finally, use this identity to get by duality

$$\begin{aligned}
\|\nabla \mathrm{e}^{-kr^2 L} v\|_{\mathrm{L}^q(B)} &= \sup_{\substack{g\in \mathrm{C}_0^\infty(B) \\ \|g\|_{q'}=1}} \Big| \lim_{N\to\infty} \sum_{j=1}^{N} \int_B \mathrm{e}^{-kr^2 L} \nabla (\mathbf{1}_{C_j(B)}[v-(v)_{4B}]) g \Big| \\
&\leq \liminf_{N\to\infty} \sum_{j=1}^{N} \Big\| \nabla \mathrm{e}^{-kr^2 L}(\mathbf{1}_{C_j(B)}[v-(v)_{4B}]) \Big\|_{\mathrm{L}^q(B)}.
\end{aligned} \tag{6.36}$$

We derive suitable bounds for the terms of the sum on the right-hand side of (6.36) and start with the case $j=1$. Using that $\mathcal{N}$ is $\mathrm{L}^2 \to \mathrm{L}^q$ bounded and the local Poincaré inequality, we readily get

$$\begin{aligned}
\|\nabla \mathrm{e}^{-kr^2 L}(\mathbf{1}_{4B}[v-(v)_{4B}])\|_{\mathrm{L}^q(B)} &\lesssim r^{-1}(r^2)^{-\gamma_q/2} \|v-(v)_{4B}\|_{\mathrm{L}^2(4B\cap O)} \\
&\lesssim r^{-\gamma_q} \|\nabla v\|_{\mathrm{L}^2(c4B\cap O)} \lesssim r^{d/q} \mathcal{M}^O(|\nabla v|^2)^{1/2}(y)
\end{aligned}$$

for all $y \in B$. We continue with $j \geq 2$. We seek the bound

$$\Big\| \nabla \mathrm{e}^{-kr^2 L}(\mathbf{1}_{C_j(B)}[v-(v)_{4B}]) \Big\|_{\mathrm{L}^q(B)} \lesssim g(j) r^{d/q} \mathcal{M}^O(|\nabla v|^2)^{1/2}(y) \tag{6.37}$$

for summable $g(j)$ and all $y \in B$. Using a telescoping sum we get the decomposition

$$v-(v)_{4B} = v-(v)_{2^{j+1}B} + \sum_{\ell=2}^{j} (v)_{2^\ell B} - (v)_{2^{\ell+1}B} =: \mathrm{I} + \mathrm{II}.$$

We start with term I. Using the $\mathrm{L}^2 \to \mathrm{L}^q$ off-diagonal decay of $\mathcal{N}$ with implied constants $C, c \in (0,\infty)$, calculate

$$\|\nabla \mathrm{e}^{-kr^2 L}(\mathbf{1}_{C_j(B)}[v - (v)_{2^{j+1}B}])\|_{\mathrm{L}^q(B)} \lesssim r^{-1} r^{-\gamma_q} \mathrm{e}^{-c4^{j-1}} \|v - (v)_{2^{j+1}B}\|_{\mathrm{L}^2(C_j(B))}.$$

Now estimate the $\mathrm{L}^2(C_j(B))$-norm by the $\mathrm{L}^2(2^{j+1}B)$-norm and apply the local Poincaré inequality to obtain

$$\lesssim r^{-\gamma_q} \mathrm{e}^{-c4^{j-1}} \|\nabla v\|_{\mathrm{L}^2(c2^{j+1}B)} \lesssim r^{d/q} \mathrm{e}^{-c4^{j-1}} 2^{(j+1)d/2} \,\mathcal{M}^O(|\nabla v|^2)^{1/2}(y)$$

for $y \in B$. We continue with the estimate corresponding to II. As a primer, calculate using Jensen's inequality and the local Poincaré inequality

$$\begin{aligned} |(v)_{2^\ell B} - (v)_{2^{\ell+1}B}|^2 &= \Big| \fint_{2^\ell B} v - (v)_{2^{\ell+1}B} \Big|^2 \leq \fint_{2^\ell B} |v - (v)_{2^{\ell+1}B}|^2 \\ &\lesssim \fint_{2^{\ell+1}B} |v - (v)_{2^{\ell+1}B}|^2 \lesssim 4^\ell r^2 2^{-\ell d} r^{-d} \|\nabla v\|^2_{\mathrm{L}^2(c2^{\ell+1}B)}. \end{aligned}$$

Take the square root and an estimate against the maximal operator to obtain

$$|(v)_{2^\ell B} - (v)_{2^{\ell+1}B}| \lesssim 2^\ell r \,\mathcal{M}^O(|\nabla v|^2)^{1/2}(y) \tag{6.38}$$

for all $y \in B$. Let us come back to the actual estimate. Using off-diagonal bounds and Hölder's inequality we find

$$\begin{aligned} &\sum_{\ell=1}^{j} \|\nabla \mathrm{e}^{-kr^2 L}(\mathbf{1}_{C_j(B)}[(v)_{2^\ell B} - (v)_{2^{\ell+1}B}])\|_{\mathrm{L}^q(B)} \\ &\lesssim \sum_{\ell=1}^{j} r^{-1} r^{-\gamma_q} \mathrm{e}^{-c4^{j-1}} |C_j(B)|^{1/2} |(v)_{2^\ell B} - (v)_{2^{\ell+1}B}|. \end{aligned}$$

Plugging in (6.38) leads to

$$\begin{aligned} &\lesssim \sum_{\ell=1}^{j} 2^\ell r^{-\gamma_q} \mathrm{e}^{-c4^{j-1}} 2^{jd/2} r^{d/2} \,\mathcal{M}^O(|\nabla v|^2)^{1/2}(y) \\ &\lesssim r^{d/q} \mathrm{e}^{-c4^{j-1}} 2^{j+1} 2^{jd/2} \,\mathcal{M}^O(|\nabla v|^2)^{1/2}(y) \end{aligned}$$

for all $y \in B$. In total, we have obtained (6.37) with $g(j) = \mathrm{e}^{-c4^{j-1}} 2^{(j+1)d}$. Since $g(j)$ is summable, we obtain from (6.36) the bound

$$\|\nabla \mathrm{e}^{-kr^2 L} v\|_{\mathrm{L}^q(B)} \lesssim r^{d/q} \,\mathcal{M}^O(|\nabla v|^2)^{1/2}(y)$$

for $y \in B$, which completes the proof of (GL2). □

6.6. A Calderón-Zygmund decomposition for Sobolev functions

Calderón-Zygmund decompositions are a classical tool in harmonic analysis. The idea is to decompose a function on all scales into good and bad functions, where good means small in a certain sense and the bad functions are at least localized in a good manner. The classical Calderón-Zygmund decomposition happens at the level of Lebesgue spaces. Here, we show a decomposition in $\mathbb{W}^{1,p}_{\mathbb{D}}$-spaces. A decomposition in homogeneous Sobolev spaces on the whole space was performed in [5, Lem. 4.12]. This construction was modified for mixed boundary conditions on domains in [8, Sec. 7], but only in the case $m = 1$. A refinement for $\mathbb{C}^m$-valued functions was performed in [36]. In this section, we first show a version for spaces $\mathbb{W}^{1,p}_{\mathbb{D}}(\mathbb{R}^d)$ which only uses $(d-1)$-regularity of D_k for each k. A version on domains then follows on *any* open set for which an extension operator is at hand.

Definition 6.6.1. Let $p \in [1, \infty]$, $m \in \mathbb{N}$, $\Xi \subseteq \mathbb{R}^d$ open and $E_k \subseteq \overline{\Xi}$ for $k = 1, \ldots, m$. With the array $\mathbb{E} \coloneqq (E_k)_{k=1}^m$ define the space

$$\mathbb{W}^{1,p}_{\mathbb{E}}(\Xi) \coloneqq \bigotimes_{k=1}^{m} \mathrm{W}^{1,p}_{E_k}(\Xi),$$

equipped with the subspace topology inherited from $\mathrm{W}^{1,p}(\Xi)^m$. Moreover, introduce the abbreviation $\| \cdot \|_{\mathbb{W}^{1,p}(\Xi)}$ for the norm on $\mathbb{W}^{1,p}_{\mathbb{E}}(\Xi)$.

Lemma 6.6.2 (Sobolev Calderón-Zygmund – whole space)**.** *Let $D_k \subseteq \mathbb{R}^d$ be closed and $(d-1)$-regular for $k = 1, \ldots, m$, and let $1 < p < \infty$. For every $u \in \mathbb{W}^{1,p}_{\mathbb{D}}(\mathbb{R}^d)$ and every $\alpha > 0$ there exist an (at most) countable index set J, a family of cubes $(Q_j)_{j \in J}$ and functions $g, b_j : \mathbb{R}^d \to \mathbb{C}^m$ for $j \in J$ such that the following holds.*

(i) *$u = g + \sum_j b_j$ holds pointwise almost everywhere,*

(ii) *the family $(Q_j)_j$ is locally finite, and every $x \in \mathbb{R}^d$ is contained in at most 12^d cubes,*

(iii) $\sum_j |Q_j| \lesssim \frac{1}{\alpha^p} \|u\|^p_{\mathbb{W}^{1,p}(\mathbb{R}^d)}$,

(iv) *$g \in \mathbb{W}^{1,\infty}_{\mathbb{D}}(\mathbb{R}^d)$ with $\|g\|_{\mathbb{W}^{1,\infty}(\mathbb{R}^d)} \lesssim \alpha$,*

(v) *$b_j \in \mathbb{W}^{1,p}_{\mathbb{D}}(\mathbb{R}^d)$ with $\|b_j\|_{\mathbb{W}^{1,p}(\mathbb{R}^d)} \lesssim \alpha |Q_j|^{\frac{1}{p}}$ for every j,*

(vi) $\|g\|_{\mathbb{W}^{1,p}(\mathbb{R}^d)} + \|\sum_{j\in J'} b_j\|_{\mathbb{W}^{1,p}(\mathbb{R}^d)} \lesssim \|u\|_{\mathbb{W}^{1,p}(\mathbb{R}^d)}$ *for all* $J' \subseteq J$,

(vii) b_j *is compactly supported in* Q_j *for every* j,

(viii) *if* $1 < q < \infty$, $u \in \mathbb{W}^{1,q}_{\mathbb{D}}(\mathbb{R}^d)$ *and* $J' \subseteq J$, *then* $\sum_{j\in J'} b_j$ *converges unconditionally in* $\mathbb{W}^{1,q}_{\mathbb{D}}(\mathbb{R}^d)$.

Before coming to the proof, let us shortly state the situation on O as a corollary.

Corollary 6.6.3 (Sobolev Calderón-Zygmund – open set). *Let* $O \subseteq \mathbb{R}^d$ *be open,* $D_k \subseteq \partial O$ *be closed and* $(d-1)$*-regular for* $k = 1, \dots, m$ *such that* O *is locally a uniform domain near* $\partial O \setminus D_k$ *for all* k*, and let* $1 < p < \infty$. *Then for every* $u \in \mathbb{W}^{1,p}_{\mathbb{D}}(O)$ *and every* $\alpha > 0$ *there exist an (at most) countable index set* J, *a family of cubes* $(Q_j)_{j\in J}$ *and functions* $g, b_j \colon O \to \mathbb{C}^m$ *for* $j \in J$ *such that the following holds.*

(i) $u = g + \sum_j b_j$ *holds pointwise almost everywhere,*

(ii) *the family* $(Q_j)_{j\in J}$ *is locally finite, and every* $x \in O$ *is contained in at most* 12^d *cubes,*

(iii) $\sum_{j\in J} |Q_j| \lesssim \frac{1}{\alpha^p}\|u\|^p_{\mathbb{W}^{1,p}(O)}$,

(iv) $g \in \mathrm{Lip}_{\mathbb{D}}(O)$ *with* $\|g\|_{\mathrm{Lip}(O)} \lesssim \alpha$,

(v) $b_j \in \mathbb{W}^{1,p}_{\mathbb{D}}(O)$ *with* $\|b_j\|_{\mathbb{W}^{1,p}(O)} \lesssim \alpha|Q_j|^{\frac{1}{p}}$ *for every* $j \in J$,

(vi) *if* $p < d$, *then* $b_j \in \mathrm{L}^q(O)$ *for* $q \in [p, p^*]$ *with* $\|b_j\|_q \lesssim \alpha|Q_j|^{1/p+(1-\theta)/d}$, *where* $\theta \in [0,1]$ *is such that* $1/q = (1-\theta)/p + \theta/p^*$,

(vii) $\|g\|_{\mathbb{W}^{1,p}(O)} + \|\sum_{j\in J'} b_j\|_{\mathbb{W}^{1,p}(O)} \lesssim \|u\|_{\mathbb{W}^{1,p}(O)}$ *for all* $J' \subseteq J$,

(viii) b_j *is supported in* $Q_j \cap O$ *for every* j,

(ix) *if* $1 < q < \infty$, $u \in \mathbb{W}^{1,q}_{\mathbb{D}}(O)$ *and* $J' \subseteq J$, *then* $\sum_{j\in J'} b_j$ *converges unconditionally in* $\mathbb{W}^{1,q}_{\mathbb{D}}(O)$.

Proof. Using the extension operators for $\mathrm{W}^{1,p}_{D_k}(O)$ componentwise on $\mathbb{W}^{1,p}_{\mathbb{D}}(O)$ we get an extension $U \coloneqq \mathcal{E}u \in \mathbb{W}^{1,p}_{\mathbb{D}}(\mathbb{R}^d)$ of u. We apply the whole-space Calderón-Zygmund decomposition (Proposition 6.6.2) to U to obtain an index set J, cubes $(Q_j)_{j\in J}$ and functions $G, B_j : \mathbb{R}^d \to \mathbb{C}^m$ satisfying the properties in that proposition. Define $g = G|_O$ and $b_j = B_j|_O$.

Most properties listed in this corollary are clear by definition, we only comment on the exceptions. For (iii) and (vii), use boundedness of the extension operator. In (iv), note that $\mathrm{W}^{1,\infty}_D(\mathbb{R}^d) = \mathrm{Lip}_D(\mathbb{R}^d) \subseteq \mathrm{Lip}_D(O)$ by Rademacher's theorem.

Finally, to see (vi) we interpolate two bounds. First, we derive with the Poincaré inequality on cubes (take the compact support assumption (vii) into account) followed by (v) the estimate

$$\|B_j\|_{\mathrm{L}^p(\mathbb{R}^d)^m} \lesssim |Q_j|^{1/d} \|B_j\|_{\mathbb{W}^{1,p}(\mathbb{R}^d)} \lesssim \alpha |Q_j|^{1/p+1/d}.$$

Second, the Sobolev inequality in combination with (v) gives the bound $\|b_j\|_{p^*} \lesssim \alpha |Q_j|^{1/p}$. Therefore, if $1/q = (1-\theta)/p + \theta/p^* = 1/p - \theta/d$, then $\|b_j\|_q \lesssim \alpha |Q_j|^{1/p+(1-\theta)/d}$ by the interpolation inequality. $\square$

Now we return to the proof of the $\mathbb{R}^d$ version.

Proof of Proposition 6.6.2. For brevity we omit $\mathbb{R}^d$ in the notation for function spaces. If $1 \leq k \leq m$ then we write $u^{(k)}$ for the kth component of u. Implicit constants expressed by the symbol "$\lesssim$" are always meant to be independent of the scale α. The proof divides into 6 steps.

Step 1: Adapted maximal function and Whitney decomposition. To start with, we consider the set

$$U := \Big\{ x \in \mathbb{R}^d \colon \mathcal{M}\Big(|u|^p + |\nabla u|^p + \sum_{k=1}^m \Big| \frac{u^{(k)}}{\mathrm{d}_{D_k}} \Big|^p \Big)(x) > \alpha^p \Big\}.$$

Since the maximal function is upper semi-continuous, U is open. If U is empty, we put $g = u$, so that we don't need bad functions at all. Then (i) is fulfilled by construction and all other assertions except (iv) are void. To show (iv), we use the definition of U and the fact that a function is dominated by its maximal function almost everywhere (which is an easy consequence of Lebesgue's differentiation theorem) to conclude for almost every $x \in \mathbb{R}^d$ that

$$|g(x)|^p + |\nabla g(x)|^p \leq \mathcal{M}\Big(|u|^p + |\nabla u|^p + \sum_{k=1}^m \Big| \frac{u^{(k)}}{\mathrm{d}_{D_k}} \Big|^p \Big)(x) \leq \alpha^p.$$

This already yields $g \in \mathbb{W}^{1,\infty}$ with the desired norm estimate. Moreover, since u has vanishing trace on D by assumption, $g \in \mathbb{W}^{1,\infty}_{\mathbb{D}}$ follows by consistency of the trace operator.

Otherwise, consider the closed set $F := \mathbb{R}^d \setminus U$. We claim that F is a proper subset of the Euclidean space, since then we can decompose U using a

Whitney decomposition. For the construction of the Whitney decomposition and further properties, the reader can consult [22, Lem. 5.1]. Indeed, it follows from the weak-type estimate for the maximal operator and Hardy's inequality (Proposition 4.3.1) that

$$|U| \lesssim \frac{1}{\alpha^p}\Big(\|u\|_{\mathbb{W}^{1,p}}^p + \sum_{k=1}^m \int_{\mathbb{R}^d} \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^p\Big) \lesssim \frac{1}{\alpha^p}\|u\|_{\mathbb{W}^{1,p}}^p < \infty. \tag{6.39}$$

Starting from a Whitney decomposition of U and enlarging all cubes by the factor $\frac{9}{8}$, we arrive at a family of cubes $(Q_j)_j$ with the properties

(a) $Q_j \subseteq U$ for every j,

(b) $(Q_j)_j$ is locally finite,

(c) $8\sqrt{d}Q_j \cap F \neq \emptyset$,

(d) $\sum_j \mathbf{1}_{Q_j} \leq 12^d$,

(e) $Q_j \cap Q_k \neq \emptyset$ implies $\dfrac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q_k)} \leq 4$.

Indeed, (a) follows immediately from the properties of Whitney cubes. For (b), fix $x \in U$ and employ a counting argument on $\mathrm{B}(x, \mathrm{d}(x, F)/2)$. For (c), calculate using the properties of Whitney cubes that a non-enlarged cube has to be scaled by the factor $9\sqrt{d}$ to hit F. Then the claim for Q_j follows by definition. Property (d) can be looked up in [22, Lem. 5.2 (c)]. Finally, for (e), check that the enlarged cubes intersect if and only if the original cubes intersect. Then, the claim follows from [22, Lem. 5.2 (a)].

Properties (b) and (d) yield (ii). For brevity, put $d_j \coloneqq \operatorname{diam}(Q_j)$. Moreover, using that we have enlarged the Whitney cubes above, we can construct a partition of unity $(\varphi_j)_j$ on U such that

$$\operatorname{supp}(\varphi_j) \subseteq Q_j \quad \text{and} \quad \|\varphi_j\|_\infty + d_j\|\nabla\varphi_j\|_\infty \lesssim 1. \tag{6.40}$$

We conclude this step with the proof of (iii), which follows readily from (a), (d) and (6.39) with the calculation

$$\sum_j |Q_j| \leq \int_U \sum_j \mathbf{1}_{Q_j} \leq 12^d|U| \lesssim \frac{1}{\alpha^p}\|u\|_{\mathbb{W}^{1,p}}^p.$$

Step 2: Estimates for u on cubes. In this short but crucial step we derive estimates for u using the maximal function, which will turn useful in the estimates for good and bad functions later on. For convenience, put $Q_j^* \coloneqq 8\sqrt{d}Q_j$.

Fix j and pick $z \in Q_j^* \cap F$, which is possible owing to (c). Then it follows from the definition of F that

$$\begin{aligned}
\int_{Q_j} |u|^p + |\nabla u|^p + \sum_{k=1}^m \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^p &\lesssim \frac{|Q_j|}{|Q_j^*|}\int_{Q_j^*} |u|^p + |\nabla u|^p + \sum_{k=1}^m \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^p \\
&\leq |Q_j|\,\mathcal{M}\Big(|u|^p + |\nabla u|^p + \sum_{k=1}^m \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^p\Big)(z) \qquad (6.41)\\
&\leq \alpha^p |Q_j|.
\end{aligned}$$

Step 3: Definition of good and bad functions. Fix j and $1 \leq k \leq m$. Say that Q_j is k-*usual* if $\mathrm{d}(Q_j, D_k) \geq d_j$ and that Q_j is k-*special* if $\mathrm{d}(Q_j, D_k) < d_j$. The nomenclature is motivated as follows. In the k-usual situation we can define and estimate the bad functions using a Poincaré argument and, hence, don't rely on the usage of boundary conditions. In the k-special case we will benefit from the Hardy term in the maximal function. Since different components of u are subject to different Dirichlet conditions, this also explains the coupling between j and k in the classification of the cubes.

That being said, define the bad function b_j on Q_j componentwise via

$$b_j^{(k)} \coloneqq \varphi_j(u^{(k)} - u_{Q_j}^{(k)}) \quad \text{if } Q_j \text{ is } k\text{-usual}, \qquad b_j^{(k)} \coloneqq \varphi_j u^{(k)} \quad \text{otherwise}.$$

Here, $u_{Q_j}^{(k)}$ denotes the mean value of $u^{(k)}$ over Q_j. Put $g \coloneqq u - \sum_j b_j$, then the validity of (i) is by definition. Note that there is no issue of convergence according to (d). Furthermore, property (vii) holds by construction.

Step 4: Taking care of the bad functions. Fix some j and $1 \leq k \leq m$. First, we consider the case that Q_j is k-usual. Using the product rule and (6.40), start with

$$\begin{aligned}
\|b_j^{(k)}\|_{\mathrm{W}^{1,p}}^p &= \int_{\mathbb{R}^d} |\varphi_j(u^{(k)} - u_{Q_j}^{(k)})|^p + |\nabla\varphi_j(u^{(k)} - u_{Q_j}^{(k)})|^p + |\varphi_j \nabla u^{(k)}|^p \\
&\lesssim \int_{Q_j} |u^{(k)}|^p + |u_{Q_j}^{(k)}|^p + \frac{1}{d_j^p}|u^{(k)} - u_{Q_j}^{(k)}|^p + |\nabla u^{(k)}|^p.
\end{aligned}$$

Using Jensen's inequality for the second term and Poincare's inequality for the third term, we obtain

$$\lesssim \int_{Q_j} |u^{(k)}|^p + |\nabla u^{(k)}|^p.$$

This completes this estimate owing to (6.41) from Step 2.

To the contrary, assume that Q_j is k-special. Note that if $y \in Q_j$, then $\mathrm{d}_{D_k}(y) \leq \mathrm{d}(Q_j, D_k) + d_j$. So, by definition of k-special cubes, it follows $2d_j > \mathrm{d}_{D_k}(y)$ for all $y \in Q_j$, which allows us to estimate

$$\begin{aligned}
\|b_j^{(k)}\|_{\mathrm{W}^{1,p}}^p &= \int_{\mathbb{R}^d} |\varphi_j u^{(k)}|^p + |\nabla \varphi u^{(k)}|^p + |\varphi \nabla u^{(k)}|^p \\
&\lesssim \int_{Q_j} |u^{(k)}|^p + \frac{1}{d_j^p} |u^{(k)}|^p + |\nabla u^{(k)}|^p \\
&\lesssim \int_{Q_j} |u^{(k)}|^p + \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^p + |\nabla u^{(k)}|^p.
\end{aligned}$$

Then we conclude using Step 2 as before.

To complete the proof of (v), we have to ensure the vanishing trace condition on $\mathbb{D}$. In any case, $\varphi_j u^{(k)} \in \mathrm{W}^{1,p}_{D_k}$ by assumption on $u^{(k)}$. In the k-usual case, note that $\varphi_j u^{(k)}_{Q_j}$ is a compactly supported Lipschitz function which vanishes on D_k by the support property of φ_j and the fact that Q_j is a k-usual cube. Hence, this term also lies in the correct space.

Step 5: Convergence of the series of bad functions. Let $1 < q < \infty$ be such that $u \in \mathbb{W}^{1,q}_{\mathbb{D}}$ and let $J' \subseteq J$ be an arbitrary subcollection. Observe that $\nabla \sum_{j \in J'} b_j = \sum_{j \in J'} \nabla b_j$ due to (b). Hence, using (d) in combination with the equivalence of ℓ^p-norms on finite sets and the calculations from Step 4, we get

$$\begin{aligned}
\|\sum_{j \in J'} b_j\|_{\mathbb{W}^{1,q}}^q &= \int_{\mathbb{R}^d} |\sum_{j \in J'} b_j|^q + |\sum_{j \in J'} \nabla b_j|^q \\
&\lesssim \sum_{j \in J'} \int_{Q_j} |b_j|^q + |\nabla b_j|^q \\
&\lesssim \int_{\mathbb{R}^d} \sum_{j \in J'} \mathbf{1}_{Q_j} \Big(|u^{(k)}|^q + \Big|\frac{u^{(k)}}{\mathrm{d}_{D_k}}\Big|^q + |\nabla u^{(k)}|^q \Big).
\end{aligned}$$

The function $\sum_{j \in J'} \mathbf{1}_{Q_j}$ is globally bounded by (d) and the Hardy term is under control owing to Hardy's inequality from Theorem 4.0.3. As a by-product, with $J' = J$ this gives the estimate for $\|b\|_{\mathbb{W}^{1,p}}$ in (vi). Of course, the estimate for $\|g\|_{\mathbb{W}^{1,p}}$ is then a trivial consequence of $g = u - b$, which completes (iv). Next, assume that J inherits its ordering from the natural numbers. Then, if we replace J' by $J'_m \coloneqq J' \cap \{m, m+1, \dots\}$, the sequence of functions $\sum_{j \in J'_m} \mathbf{1}_{Q_j}$ converges pointwise to zero, so that we can appeal to Lebesgue's theorem to conclude that the partial sums of $\sum_{j \in J'} b_j$ form a Cauchy sequence. Hence, the series converges in $\mathbb{W}^{1,q}_{\mathbb{D}}$. That the convergence is unconditional follows by a similar argument (take two rearrangements and subtract their partial sums from each other).

Step 6: Controlling the good function. Fix again some $1 \leq k \leq m$. On F we have that g and u coincide by construction, so that the full Sobolev estimate on F follows the same lines as in Step 1 for the case $U = \emptyset$. So from now on, fix $x \in U$. First, we show $|g^{(k)}(x)|^p \lesssim \alpha^p$, which completes the non-gradient estimates in (iv). Write $J_{u,x}$ and $J_{s,x}$ for the index sets of k-usual and k-special cubes which contain x. Both sets contain at most 12^d elements according to (d). Using that $(\varphi_j)_j$ is a partition of unity on U, we calculate

$$
\begin{aligned}
g^{(k)}(x) &= u^{(k)}(x) - \sum_j b_j^{(k)}(x) \\
&= u^{(k)}(x) - \sum_{j \in J_{u,x}} \varphi_j(x)(u^{(k)}(x) - u_{Q_j}^{(k)}) - \sum_{j \in J_{s,x}} \varphi_j(x) u^{(k)}(x) \\
&= \sum_{j \in J_{u,x}} \varphi_j(x) u_{Q_j}^{(k)}.
\end{aligned}
$$

Using the comparability of ℓ^p norms on finite sets, Jensen's inequality, Step 2 and the bound on $\# J_{u,x}$ we derive

$$
\begin{aligned}
|g^{(k)}(x)|^p &\lesssim \sum_{j \in J_{u,x}} |u_{Q_j}^{(k)}|^p \leq \sum_{j \in J_{u,x}} \Big(\fint_{Q_j} |u^{(k)}| \Big)^p \lesssim \sum_{j \in J_{u,x}} \fint_{Q_j} |u^{(k)}|^p \\
&\lesssim \alpha^p.
\end{aligned} \tag{6.42}
$$

Note that $\sum_j \nabla \varphi_j(x) = 0$ holds since $(\varphi_j)_j$ is a partition of unity and the sum is locally finite. Then a similar representation for the gradient follows using the product rule and reads

$$
\nabla g^{(k)}(x) = \sum_{j \in J_{u,x}} u_{Q_j}^{(k)} \nabla \varphi_j(x).
$$

Instead of estimating this term directly, put

$$
h(x) \coloneqq \sum_{j \in J_x} u_{Q_j}^{(k)} \nabla \varphi_j(x) \quad \text{and} \quad h_s(x) \coloneqq \sum_{j \in J_{s,x}} u_{Q_j}^{(k)} \nabla \varphi_j(x).
$$

Here, J_x is the collection of cubes that contain x and coincides with the union of $J_{u,x}$ and $J_{s,x}$. Consequently, $\nabla g^{(k)}(x) = h(x) - h_s(x)$, and the bound for g will follow from $|h(x)|^p + |h_s(x)|^p \lesssim \alpha^p$. We show this latter claim in the following.

With the same arguments as for (6.42), but taking the observation $d_j \gtrsim \mathrm{d}_{D_k}$ on k-special cubes from Step 4 into account, estimate

$$
|h_s(x)|^p \lesssim \sum_{j \in J_{s,x}} \Big(\frac{1}{d_j} \fint_{Q_j} |u^{(k)}| \Big)^p \lesssim \sum_{j \in J_{s,x}} \fint_{Q_j} \Big| \frac{u^{(k)}}{\mathrm{d}_{D_k}} \Big|^p \lesssim \alpha^p.
$$

To estimate $|h(x)|^p$, fix some j_0 such that $x \in Q_{j_0}$, which exists by construction of $(Q_j)_j$. If Q_j is any cube that contains x, then the sizes of Q_j and Q_{j_0} are comparable by (e). In particular, there is a factor $c > 0$ that does not depend on j such that $Q_j \subseteq cQ_{j_0}$. Furthermore, assume that $c \geq 8\sqrt{d}$, so that $Q^*_{j_0} \coloneqq cQ_{j_0}$ intersects F according to (c). Now, extend the defining sum of $h(x)$ using the fact that $\sum_j \nabla\varphi(x) = 0$ to obtain

$$h(x) = \sum_{j \in J_x} u^{(k)}_{Q_j} \nabla\varphi_j(x) = \sum_{j \in J_x} \Big(u^{(k)}_{Q_j} - u^{(k)}_{Q^*_{j_0}}\Big)\nabla\varphi_j(x).$$

As before, estimate

$$|h(x)|^p \lesssim \sum_{j \in J_x} \frac{1}{d_j^p} |u^{(k)}_{Q_j} - u^{(k)}_{Q^*_{j_0}}|^p. \tag{6.43}$$

We proceed by estimating a fixed term in that sum, so pick some $j \in J_x$. Using that $u^{(k)}_{Q^*_{j_0}}$ is just a constant, $|Q^*_{j_0}| \approx |Q_j|$, $Q_j \subseteq Q^*_{j_0}$ and Jensen's inequality, calculate

$$|u^{(k)}_{Q_j} - u^{(k)}_{Q^*_{j_0}}|^p = \left| \fint_{Q_j} u^k(y) - u^{(k)}_{Q^*_{j_0}} \,\mathrm{d}y \right|^p \lesssim \fint_{Q^*_{j_0}} |u^k(y) - u^{(k)}_{Q^*_{j_0}}|^p \,\mathrm{d}y.$$

Owing to the classical Poincaré inequality on $Q^*_{j_0}$ and the comparability $\operatorname{diam}(Q^*_{j_0}) \approx d_j$ we further estimate

$$\lesssim \operatorname{diam}(Q^*_{j_0})^p \fint_{Q^*_{j_0}} |\nabla u^{(k)}|^p \approx d_j^p \fint_{Q^*_{j_0}} |\nabla u^{(k)}|^p.$$

Plugging this back into (6.43), the factors d_j^p cancel out. The mean value integral can be estimated against α^p using the maximal operator trick from Step 2. Finally, the bound on $\#J_x$ lets us conclude the estimates for (iv). □

6.7. Boundedness of the square root

We are going to show (upper) L^p-bounds for the square root. These complement the lower bounds coming from the Riesz transform already shown. The case $p > 2$ is fairly easy by a duality argument with the Riesz transform and is presented in Section 6.7.2. The argument in the case $p < 2$ is much more difficult. Of course we could also argue by duality in this case, but we would have to invest L^p-boundedness for $\mathcal{N}^*$, which is a stronger requirement than merely L^p-bounds for $\mathcal{S}$. The proof is again closely oriented at [36, Prop. 8.1]. Besides the necessary changes to treat unbounded sets, we have also corrected a minor flaw that appears in the case $d = 2$ in that paper.

6.7.1. The case $p < 2$: weak-type argument

The following proposition provides L^p-bounds for $L^{\frac{1}{2}}$ using a weak-type estimate. The Calderón-Zygmund decomposition from Section 6.6 is an essential tool in the proof.

Proposition 6.7.1. *Let $q \in (1, 2]$ be such that there exists $s \in [q, q^*)$ such that $\mathcal{S}$ is L^s-bounded, and let $p \in (q, 2)$. Then one has the estimate*

$$\|L^{\frac{1}{2}} u\|_p \lesssim \|u\|_{\mathrm{W}^{1,p}(O)^m} \qquad (u \in \mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,p}_D(O)^m),$$

where the implicit constant depends on L only via its coefficient bounds and implied constants coming from the L^s-bounds for $\mathcal{S}$.

Remark 6.7.2. There are two remarks in order:

(i) On $O = \mathbb{R}^d$ one can permit $q = 1$. In our case, this is not possible because the Sobolev Calderón-Zygmund decomposition is only available in the reflexive range, which is an artifact of Hardy's inequality.

(ii) Note that the boundedness assumption for $\mathcal{S}$ is strictly weaker than that for the Riesz transform in Proposition 6.5.1.

Proof. Throughout, we will often omit the underlying set or number of components in the notation of function spaces if that makes the notation easier, for example we will write $\mathrm{W}^{1,2}$ instead of $\mathrm{W}^{1,2}(O)^m$.

The representation formula (6.18) allows us to write $L^{\frac{1}{2}}$ as an improper Riemann integral. To be more precise, let $u \in \mathrm{W}^{1,2}_D(O)^m = \mathrm{D}(L^{\frac{1}{2}})$, then

$$L^{\frac{1}{2}} u = \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \int_{2^{-n}}^{\infty} L\mathrm{e}^{-t^2 L} u \,\mathrm{d}t, \tag{6.44}$$

where the limit is taken in $\mathrm{L}^2(O)^m$. With this, we can divide the proof into three steps. First, we show a weak-type estimate for the truncated operators $T_n := \int_{2^{-n}}^{\infty} L\mathrm{e}^{-t^2 L} \,\mathrm{d}t$ that is uniform in n. This step is subdivided into several intermediate claims. Second, we conclude uniform L^p-bounds for T_n and $p \in (q, 2)$ using complex interpolation. Finally, we conclude the L^p-bound for $L^{\frac{1}{2}}$.

Step 1: Weak-type estimate for T_n. Let $u \in \mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,p}_D(O)^m$ and $\alpha > 0$. We want to show the weak-type estimate

$$\Big|\Big\{x \in O \colon |T_n u(x)| > \alpha\Big\}\Big| \lesssim \alpha^{-q} \|u\|^q_{\mathrm{W}^{1,q}}, \tag{6.45}$$

where the implicit constant is independent of n. Using Corollary 6.6.3, perform a Calderón-Zygmund decomposition of u, that is to say, decompose $u = g + \sum_j b_j$. Put $b \coloneqq \sum_j b_j$, then $b \in \mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,q}_D(O)^m$ by property (ix) in the corollary. For brevity, we will from now on only write (i)-(ix) for the respective properties of the Calderón-Zygmund decomposition without any further specification.

Split estimate into good and bad part. As usual in Calderón-Zygmund theory, split the left-hand side of (6.45) into

$$\Big|\Big\{x\colon |T_n u(x)| > \alpha\Big\}\Big| \leq \Big|\Big\{x\colon |T_n g(x)| > \frac{\alpha}{2}\Big\}\Big| + \Big|\Big\{x\colon |T_n b(x)| > \frac{\alpha}{2}\Big\}\Big|. \tag{6.46}$$

Note that $x \in O$, but we omit this quantification to save space.

Handling the good part. The first term on the right-hand side is for free from the L^2-theory. Indeed, Tchebychev's inequality and the L^2-estimate for the square root (which also yields uniform $\mathrm{W}^{1,2}_D(O)^m \to \mathrm{L}^2(O)^m$ bounds for $\{T_n\}_n$, compare with the beginning of the proof of Proposition 6.5.1) yield

$$\Big|\Big\{x\colon |T_n g(x)| > \frac{\alpha}{2}\Big\}\Big| \lesssim \alpha^{-2}\|T_n g\|_2^2 \lesssim \alpha^{-2}\|g\|^2_{\mathrm{W}^{1,2}}.$$

To proceed, use the interpolation inequality for Sobolev spaces (which follows from the interpolation inequality for Lebesgue spaces applied to function and gradient separately) with $\theta = \frac{2-q}{2}$, (iv) and (vii) to conclude

$$\|g\|^2_{\mathrm{W}^{1,2}} \lesssim \alpha^{2-q}\|g\|^q_{\mathrm{W}^{1,q}} \lesssim \alpha^{2-q}\|u\|^q_{\mathrm{W}^{1,q}}.$$

Combining the last two estimates completes the estimate for the good function term on the right-hand side of (6.46).

Reduction to finitely many bad functions. Let $J_0 \subseteq J$ be finite. Split $b = (b - \sum_{j\in J_0} b_j) + \sum_{j \in J_0} b_j$ and use Tchebychev's inequality and the L^2-boundedness of $\{T_n\}_n$ to deduce

$$\Big|\Big\{x\colon |T_n b(x)| > \frac{\alpha}{2}\Big\}\Big| \lesssim \Big|\Big\{x\colon |T_n\Big(\sum_{j\in J_0} b_j\Big)(x)| > \frac{\alpha}{4}\Big\}\Big| + \frac{16}{\alpha^2}\Big\|b - \sum_{j\in J_0} b_j\Big\|_2^2.$$

Thus, an estimate of the first term against $\alpha^{-q}\|u\|^q_{W^{1,q}}$ with bound independent of J_0 yields the claim because the second term tends to zero by (ix) when making J_0 sufficiently large. As a slight abuse of notation, we still write b for the sum $\sum_{j\in J_0} b_j$. Moreover, in the further course of this proof, the index j will only range over the finite index set J_0 and not anymore over J.

Decomposition of the bad part. Denote the sidelength of Q_j by ℓ_j and write r_j for the largest dyadic number less than ℓ_j. The integrand in the definition of T_n is integrable in the $\mathrm{L}^2 \to \mathrm{L}^2$ operator norm away from 0, which is a consequence of the boundedness of the family $\{tL\mathrm{e}^{-tL}\}_{t>0}$ on L^2. Hence, we can split the defining integral of T_n into a local integral from 2^{-n} to $r_j \vee 2^{-n}$ and a global integral from $r_j \vee 2^{-n}$ to ∞, and both integrals again yield bounded operators on L^2. Furthermore, since $\sum_j b_j$ is a finite sum by the previous reduction, we can pull the summation out of the local and global integrals. Altogether, this gives

$$
\begin{aligned}
&\Big|\Big\{x \in O \colon \Big|\Big[\int_{2^{-n}}^{\infty} L\mathrm{e}^{-t^2L} b \,\mathrm{d}t\Big](x)\Big| > \frac{\alpha}{2}\Big\}\Big| \\
&\leq \Big|\Big\{\Big|\Big[\int_{2^{-n}}^{r_j \vee 2^{-n}} L\mathrm{e}^{-t^2L} b \,\mathrm{d}t\Big](x)\Big| > \frac{\alpha}{4}\Big\}\Big| + \Big|\Big\{\Big|\Big[\int_{r_j \vee 2^{-n}}^{\infty} L\mathrm{e}^{-t^2L} b \,\mathrm{d}t\Big](x)\Big| > \frac{\alpha}{4}\Big\}\Big| \\
&= \Big|\Big\{\Big|\Big[\sum_j \int_{2^{-n}}^{r_j \vee 2^{-n}} L\mathrm{e}^{-t^2L} b_j \,\mathrm{d}t\Big](x)\Big| > \frac{\alpha}{4}\Big\}\Big| + \Big|\Big\{\Big|\Big[\sum_j \int_{r_j \vee 2^{-n}}^{\infty} L\mathrm{e}^{-t^2L} b_j \,\mathrm{d}t\Big](x)\Big| > \frac{\alpha}{4}\Big\}\Big| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

Let us mention that making the truncation at a dyadic scaling and not at height ℓ_j directly will only become handy when estimating the global integral later on.

Handling the local integral. We continue with an estimate for the local integral I. Assume $r_j > 2^{-n}$ since otherwise the term is trivial. The next step is again classical in Calderón-Zygmund theory. We split $O = \bigcup_j (4Q_j \cap O) \cup (O \setminus \bigcup_j 4Q_j)$, and use additivity of the measure and Tchebychev's inequality to get

$$
\mathrm{I} \lesssim \Big|\bigcup_j 4Q_j\Big| + \alpha^{-2}\Big\|\mathbf{1}_{O\setminus \bigcup_j 4Q_j} \sum_j \int_{2^{-n}}^{r_j \vee 2^{-n}} L\mathrm{e}^{-t^2L} b_j \,\mathrm{d}t\Big\|_2^2. \tag{6.47}
$$

Using subadditivity of the measure and (iii), conclude

$$
\Big|\bigcup_j 4Q_j\Big| \lesssim \sum_j |Q_j| \lesssim \alpha^{-q} \|u\|_{\mathrm{W}^{1,q}}^q. \tag{6.48}
$$

Thus, the first term on the right-hand side of (6.47) is fine. To estimate the L^2-norm, we test with $v \in \mathrm{L}^2(O)^m$ satisfying $\|v\|_2 = 1$ and use the triangle inequality to deduce

$$
\Big|\int_O \mathbf{1}_{O\setminus\cup_j 4Q_j} \sum_j \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2L} b_j \,\mathrm{d}t \, \overline{v} \,\mathrm{d}x\Big| \leq \sum_j \int_{O\setminus 4Q_j} \Big|\int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2L} b_j \,\mathrm{d}t\Big| |v| \,\mathrm{d}x. \tag{6.49}
$$

Note that we only ignore the enlarged cube in which b_j is supported. The reason is that we have introduced this restriction of integration area to bring off-diagonal estimates into business. Hence, it is no surprise that we further decompose $O\backslash 4Q_j$ into the annuli $C_k(Q_j) \cap O$, where j is fixed for the moment. With the Cauchy-Schwarz inequality, this gives

$$\begin{aligned} &\int_{O\backslash 4Q_j} \Big| \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \, \mathrm{d}t \Big| |v| \, \mathrm{d}x \\ &\leq \sum_{k \geq 2} \| \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \, \mathrm{d}t \|_{\mathrm{L}^2(C_k(Q_j)\cap O)} \|\mathcal{E}_0 v\|_{\mathrm{L}^2(C_k(Q_j))}. \end{aligned} \tag{6.50}$$

Here, $\mathcal{E}_0 v$ is the zero extension of v. For the latter factor, continue with

$$\|\mathcal{E}_0 v\|^2_{\mathrm{L}^2(C_k(Q_j))} \lesssim 2^{kd} \ell_j^d \fint_{C_k(Q_j)} |\mathcal{E}_0 v|^2 \, \mathrm{d}x \leq 2^{kd} \ell_j^d \mathcal{M}(|\mathcal{E}_0 v|^2)(y) \qquad (y \in Q_j).$$

For the other factor, we claim that we can chose $q \leq r \leq \min(2, q^*)$ for which the family $\{tL\mathrm{e}^{-tL}\}_{t>0}$ satisfies $\mathrm{L}^r \to \mathrm{L}^2$ off-diagonal estimates. Indeed, if q^* is larger than 2, we can simply put $r = 2$, which is admissible by Proposition 6.1.5. Otherwise, owing to the assumption on q, fix $s \in [q, q^*)$ for which $\mathcal{S}$ is L^s-bounded. Then, for any $r \in (s, q^*]$, we can rely on $\mathrm{L}^r \to \mathrm{L}^2$ off-diagonal estimates for the family $\{tL\mathrm{e}^{-tL}\}_{t>0}$ thanks to Proposition 6.1.12 (i). (Note for later use that this choice of r also yields $\mathrm{L}^r \to \mathrm{L}^2$ off-diagonal estimates for $\mathcal{S}$.) In either case, we assume that the off-diagonal estimates are with respect to the distance $\mathrm{d}_\infty(\cdot,\cdot)$ instead of $\mathrm{d}(\cdot,\cdot)$ and we write $C, c \in (0,\infty)$ for the implied constants. For the integrand of the local integral, this together with the support property of b_j and the definition of r_j results in the bound

$$\|L\mathrm{e}^{-t^2 L} b_j\|_{\mathrm{L}^2(C_k(Q_j)\cap O)} \leq C t^{-\gamma_r - 2} \mathrm{e}^{-c 4^{k-1} r_j^2 / t^2} \|b_j\|_r.$$

Together with the bound (vi) for $\|b_j\|_r$ this gives

$$\begin{aligned} \Big\| \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \, \mathrm{d}t \Big\|_{\mathrm{L}^2(C_k(Q_j)\cap O)} &\leq \int_{2^{-n}}^{r_j} \|L\mathrm{e}^{-t^2 L} b_j\|_{\mathrm{L}^2(C_k(Q_j)\cap O)} \, \mathrm{d}t \\ &\lesssim \alpha \ell_j^{d/q+1-\theta} \int_{2^{-n}}^{r_j} t^{-\gamma_r - 2} \mathrm{e}^{-c4^{k-1} r_j^2/t^2} \, \mathrm{d}t, \end{aligned}$$

where $\theta = \gamma_{qr}$. Using the substitution $s = 4^k r_j^2 / t^2$ and $r_j \approx \ell_j$, this results using the definition of θ in

$$\begin{aligned} &\leq \frac{1}{2} \alpha \ell_j^{d/q+1-\theta} (2^k r_j)^{-\gamma_r - 1} \int_{4^k}^{\infty} s^{(\gamma_r - 1)/2} \mathrm{e}^{-cs/4} \, \mathrm{d}s \\ &\lesssim \alpha \ell_j^{d/2} 2^{-k(\gamma_r+1)} \int_{4^k}^{\infty} s^{(\gamma_r-1)/2} \mathrm{e}^{-cs/4} \, \mathrm{d}s. \end{aligned}$$

Next, we use the lower bound on s to extract exponential decay by splitting the exponential term. We have seen this trick several times in the course of this chapter. This yields

$$\lesssim \alpha \ell_j^{d/2} 2^{-k(\gamma_r+1)} \mathrm{e}^{-c4^k/8} \int_0^\infty s^{(\gamma_r-1)/2} \mathrm{e}^{-cs/8} \,\mathrm{d}s.$$

The integral in s is finite and does not depend on j or k. We can plug this estimate back into (6.50) to arrive at

$$\int\limits_{O \setminus 4Q_j} \Big| \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t \Big| |v| \,\mathrm{d}x \lesssim \alpha \ell_j^d \sum_{k \geq 2} 2^{k(d/2-\gamma_r-1)} \mathrm{e}^{-c4^k/8} \mathcal{M}(|\mathcal{E}_0 v|^2)(y)^{\frac{1}{2}},$$

where $y \in Q_j$. The sum in k is convergent due to the double exponential decay in k. Next, we average this over Q_j (which does not affect the left-hand side) to get

$$\begin{aligned} \Big\| \int_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t \Big\|_{\mathrm{L}^2(C_k(Q_j) \cap O)} &\lesssim \alpha \ell_j^d \fint_{Q_j} \mathcal{M}(|\mathcal{E}_0 v|^2)(y)^{\frac{1}{2}} \,\mathrm{d}y \\ &= \alpha \int_{Q_j} \mathcal{M}(|\mathcal{E}_0 v|^2)(y)^{\frac{1}{2}} \,\mathrm{d}y. \end{aligned}$$

Summing this bound over j yields a bound for (6.49), namely

$$\begin{aligned} \Big| \int_O \mathbf{1}_{O \setminus \cup_j 4Q_j} \sum_j \int\limits_{2^{-n}}^{r_j} L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t \, \overline{v} \,\mathrm{d}x \Big| &\lesssim \alpha \sum_j \int_{Q_j} \mathcal{M}(|\mathcal{E}_0 v|^2)^{\frac{1}{2}} \,\mathrm{d}y \\ &\lesssim \alpha \int_{\cup_j Q_j} \mathcal{M}(|\mathcal{E}_0 v|^2)^{\frac{1}{2}} \,\mathrm{d}y, \end{aligned}$$

where we have used the finite overlap property (ii) of the Q_j in the final step. We apply Kolmogorov's inequality with $q = 1/2$ (see [48, Ex. 2.1.5]), followed by (6.48) and the normalization of $\|v\|_2$ to get

$$\lesssim \alpha \Big| \bigcup_j Q_j \Big|^{\frac{1}{2}} \Big\| |\mathcal{E}_0 v|^2 \Big\|_1^{\frac{1}{2}} \lesssim \alpha^{1-q/2} \|u\|_{\mathrm{W}^{1,q}}^{q/2} \|v\|_2 = \alpha^{1-q/2} \|u\|_{\mathrm{W}^{1,q}}^{q/2}.$$

Square this and multiply by α^{-2} as in (6.47) to get a bound against $\alpha^{-q} \|u\|_{\mathrm{W}^{1,q}}^q$ for the whole local integral.

Controlling the global integral. As a preparation, consider the function $F = \int_1^\infty \mathbf{z}\mathrm{e}^{-t^2 \mathbf{z}} \,\mathrm{d}t$. Recall the notation F_t for $t > 0$. For $r > 0$ the substitution $t = sr$ yields

$$F_{r^2} = \int_1^\infty r^2 \mathbf{z} \mathrm{e}^{-s^2 r^2 \mathbf{z}} \,\mathrm{d}s = r \int_1^\infty \mathbf{z}(sr) \mathrm{e}^{-(sr)^2 \mathbf{z}} \frac{\mathrm{d}s}{s} = r \int_r^\infty \mathbf{z} t \mathrm{e}^{-t^2 \mathbf{z}} \frac{\mathrm{d}t}{t}.$$

Consequently, scaling and Fubini's theorem (justified by $F \in \mathrm{H}_0^\infty(\mathrm{S}_\varphi^+)$ for all $\varphi \in (0, \pi/2)$, which can be seen by a calculation similar to that in Remark 6.3.2) reveal

$$F(r^2 L) = F_{r^2}(L) = r \int_r^\infty L\mathrm{e}^{-t^2 L} \,\mathrm{d}t.$$

We will use this representation to decompose the global integral into a square function. Also, introduce for each integer k the set $J_k = \{j \in J_0 \colon r_j \vee 2^{-n} = 2^k\}$, which groups bad functions supported on the same dyadic scale, so to say. Recall that J_0 and therefore all J_k are finite, and only finitely many J_k are non-empty. Hence, we can write

$$\sum_{j \in J_0} \int\limits_{r_j \vee 2^{-n}}^{\infty} L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t = \sum_k \sum_{j \in J_k} \int\limits_{r_j \vee 2^{-n}}^{\infty} L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t = \sum_k F(4^k L)\Big(\sum_{j \in J_k} 2^{-k} b_j\Big)$$

without worrying about convergence, rearrangement or whatsoever. Finally, as for the local integral, chose $q \leq r \leq \min(2, q^*)$ such that $\mathcal{S}$ satisfies $\mathrm{L}^r \to \mathrm{L}^2$ off-diagonal estimates.

With these preparations at hand, we come back to term II for the global integral and estimate it first using Tchebychev's inequality to get

$$\text{(6.51)} \quad \mathrm{II} \lesssim \frac{1}{\alpha^r} \Big\| \sum_{j \in J^0} \int_{r_j \vee 2^{-n}}^\infty L\mathrm{e}^{-t^2 L} b_j \,\mathrm{d}t \Big\|_r^r = \frac{1}{\alpha^r} \Big\| \sum_k F(4^k L)\Big(\sum_{j \in J_k} 2^{-k} b_j\Big) \Big\|_r^r.$$

Since $\mathcal{S}$ satisfies $\mathrm{L}^r \to \mathrm{L}^2$ off-diagonal estimates, the H^∞-calculus of L on L^2 admits L^r-bounds by Proposition 6.4.1, so that Lemma 6.7.3 below is applicable, and turns (6.51) into the square function

$$\lesssim \frac{1}{\alpha^r} \Big\| \Big(\sum_k \Big| \sum_{j \in J_k} 2^{-k} b_j \Big|^2 \Big)^{\frac{1}{2}} \Big\|_r^r = \frac{1}{\alpha^r} \int_O \Big(\sum_k \Big| \sum_{j \in J_k} 2^{-k} b_j \Big|^2 \Big)^{\frac{r}{2}}.$$

Use $r/2 \leq 1$, and the finite overlap (ii) of the Q_j together with the support condition (viii) for the bad functions to get

$$\leq \frac{1}{\alpha^r} \int_O \sum_k \Big| \sum_{j \in J_k} 2^{-k} b_j \Big|^r \lesssim \frac{1}{\alpha^r} \sum_k \sum_{j \in J_k} 2^{-kr} \int_O |b_j|^r.$$

Using $2^k \geq r_j \gtrsim \ell_j$ for $j \in J_k$ and the L^r-estimate for bad functions (vi) further gives

$$\lesssim \frac{1}{\alpha^r} \sum_{j \in J^0} \ell_j^{-r} \alpha^r |Q_j|^{r/q - \theta r/d} \ell_j^r,$$

where $\theta \in [0,1]$ is appropriately chosen. Canceling out some factors and using $r/q - \theta r/d = 1$, which is easily seen by a rearrangement of the definition of θ, this becomes with (iii)

$$= \sum_{j \in J^0} |Q_j| \lesssim \alpha^{-q} \|u\|_{\mathrm{W}^{1,q}}^q.$$

Bringing everything together, this completes the proof of the weak-type estimate (6.45).

Step 2: L^p-bounds for T_n by interpolation. For $p \in (1,\infty)$, introduce the *Marcinkiewicz space*

$$\mathrm{L}^{p,\infty}(O) \coloneqq \Big\{ f \in \mathrm{L}^1(O) + \mathrm{L}^\infty(O) \colon \|f\|_{\mathrm{L}^{p,\infty}(O)} < \infty \Big\},$$

where

$$\|f\|_{\mathrm{L}^{p,\infty}(O)} \coloneqq \sup_{\alpha > 0} \alpha \Big| \Big\{ x \in O \colon |f(x)| > \alpha \Big\} \Big|^{\frac{1}{p}},$$

see [93, Sec. 1.18.6, Lemma]. Note that $\| \cdot \|_{\mathrm{L}^{p,\infty}}$ is only a quasi-norm, which means that the triangle inequality only holds up to a constant. If $1 < q < 2$, $\theta \in (0,1)$ and $1/p \coloneqq (1-\theta)/q + \theta/2$, then we get the interpolation identity

$$(\mathrm{L}^{q,\infty}(O), \mathrm{L}^2(O))_{\theta,p} = \mathrm{L}^p(O), \tag{6.52}$$

see [93, Sec. 1.18.6, Thm. 2] and take [93, Sec. 1.18.6, Lemma] into account. Here, we use that the real interpolation method also works with quasi-Banach spaces, that is to say, complete quasi-normed spaces. So, identity (6.52) has to be understood as an equality of sets with an equivalence between the quasi-norm on the left-hand side and the norm on the right-hand side.

With this nomenclature, the weak-type estimate (6.45) can be rephrased to

$$\|T_n\|_{\mathrm{L}^{q,\infty}(O)^m} \lesssim \|u\|_{\mathrm{W}^{1,q}(O)^m} \qquad (u \in \mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,q}_D(O)^m),$$

with an implicit constant independent of n. Using Lemma 2.2.8, $\mathrm{W}^{1,2}_D(O) \cap \mathrm{W}^{1,q}_D(O)$ is dense in $\mathrm{W}^{1,q}_D(O)$. Therefore, since $\mathrm{L}^{q,\infty}(O)$ is complete, T_n extends to a bounded operator $\mathrm{W}^{1,q}_D(O)^m \to \mathrm{L}^{q,\infty}(O)^m$ with bound independent of n. Also, $T_n : \mathrm{W}^{1,2}_D(O)^m \to \mathrm{L}^2(O)^m$. This is because T_n decomposes into the square root and a uniformly bounded approximation of the identity on L^2, and for the former we have shown the respective bound in Theorem 5.0.1.

Hence, we get from Proposition 1.1.1 (applied with $\mathcal{R} = |_O$ and $\mathcal{E}$ from Theorem 3.0.2), real interpolation, and (6.52) that T_n is a map

$$\mathrm{W}^{1,p}_D(O)^m = (\mathrm{W}^{1,q}_D(O)^m, \mathrm{W}^{1,2}_D(O)^m)_{\theta,p} \to (\mathrm{L}^{q,\infty}(O)^m, \mathrm{L}^2(O)^m)_{\theta,p} = \mathrm{L}^p(O)^m,$$

where we have used the compatibility of (real) interpolation with finite products, and equality is up to equivalent (quasi-)norms. Consequently,

$$\|T_n u\|_{\mathrm{L}^p(O)^m} \lesssim \|u\|_{\mathrm{W}^{1,p}(O)^m} \qquad (u \in \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m), \tag{6.53}$$

where the implicit constants do not depend on n.

Step 3: Conclusion of the proof. Let $p \in (q, 2)$, then estimate (6.53) and identity (6.44) along with Fatou's lemma readily reveal

$$\|L^{\frac{1}{2}} u\|_q \leq \liminf_n \frac{2}{\pi} \|T_n u\|_q \lesssim \|u\|_{\mathrm{W}^{1,q}(O)^m} \quad (u \in \mathrm{W}^{1,q}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m).$$

The correct dependence of the implicit constant on L was tracked throughout the proof. □

Lemma 6.7.3 ([36, Lem. 8.2]). *Let $p \in (1, \infty)$, $\Xi \subseteq \mathbb{R}^d$ be measurable and $m \in \mathbb{N}$. Also, let T be an injective sectorial operator on $\mathrm{L}^2(\Xi)^m$ such that for some $\psi \in (0, \pi)$ it holds*

$$\|f(T)u\|_p \leq C_\psi \|f\|_\infty \|u\|_p \qquad (f \in \mathrm{H}^\infty(\mathrm{S}^+_\psi), u \in \mathrm{L}^2(\Xi)^m \cap \mathrm{L}^p(\Xi)^m).$$

If $f \in \mathrm{H}^\infty_0(\mathrm{S}^+_\psi)$, then there is a constant $C \in (0, \infty)$ that depends on f and ψ such that

$$\Big\| \sum_{k \in \mathbb{Z}} f(4^k T) u_k \Big\|_p \leq C C_\psi \Big\| \Big(\sum_{k \in \mathbb{Z}} |u_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

for every sequence $(u_k)_{k \in \mathbb{Z}} \subseteq \mathrm{L}^2(\Xi)^m \cap \mathrm{L}^p(\Xi)^m$ for which the right-hand side is finite.

6.7.2. The case $p > 2$: duality with the Riesz transform

Proposition 6.7.4. *Assume that O is an (ε, ∞)-domain near ∂O and let $q \in (2, \infty)$ be such that $\mathcal{N}$ is L^q-bounded and let $p \in (2, q)$. Then one has the estimate*

$$\|L^{\frac{1}{2}} u\|_p \lesssim \|u\|_{\mathrm{W}^{1,p}(O)} \qquad (u \in \mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,p}_D(O)^m),$$

where the implicit constant depends on L only via its coefficient bounds and implied constants from L^q-bounds.

Proof. Since $\mathcal{N}$ is L^q bounded, it follows from Proposition 6.1.12 (ii) that $\mathcal{S}$ satisfies $\mathrm{L}^2 \to \mathrm{L}^p$ off-diagonal estimates. Using duality (see Lemma 6.1.11), $\mathcal{S}^* = \mathcal{S}(L^*)$ satisfies $\mathrm{L}^{p'} \to \mathrm{L}^2$ off-diagonal estimates. The implied constants are the same as the implied constants coming from the $\mathrm{L}^2 \to \mathrm{L}^p$ off-diagonal estimates for $\mathcal{S}$. As usual, decompose operators in $\mathcal{N}^* = \mathcal{N}(L^*)$ as $\sqrt{2}(\sqrt{t}\nabla \mathrm{e}^{-tL^*})\mathrm{e}^{-tL^*}$. The L^2 off-diagonal estimates for $\mathcal{N}(L^*)$ provided by Proposition 6.1.5 have good implied constants since L and L^* have the same coefficient bounds. Then, use the composition technique from Lemma 6.1.9 to conclude $\mathrm{L}^{p'} \to \mathrm{L}^2$ off-diagonal estimates for $\mathcal{N}^*$. Consequently, the Riesz transform $\nabla(L^*)^{-\frac{1}{2}}$ admits $\mathrm{L}^{p'}$-bounds by Proposition 6.5.1 applied with p' instead of p.

To bring this into action, we claim the following two facts for $u \in \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m$ and $v \in \mathrm{W}^{1,p'}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m$:

$$(L^{\frac{1}{2}}u \,|\, (L^*)^{\frac{1}{2}}v)_2 = a(u,v) \quad \text{and} \quad |a(u,v)| \lesssim \|u\|_{\mathrm{W}^{1,p}(O)^m}\|v\|_{\mathrm{W}^{1,p'}(O)^m},$$

where the implicit constant depends on dimension and the upper coefficient bound for L. The latter claim follows from Hölder's inequality and boundedness of the coefficients. For the former, start with $u \in \mathrm{D}(L)$ and $v \in \mathrm{W}^{1,2}_D(O)^m$. Using the Kato result on L^2 for L^* we get $v \in \mathrm{D}((L^*)^{\frac{1}{2}})$. Hence, since L is obtained from a by the form method, this implies

$$(L^{\frac{1}{2}}u \,|\, (L^*)^{\frac{1}{2}}v)_2 = (Lu \,|\, v)_2 = a(u,v). \tag{6.54}$$

The right-hand side is a bounded sesquilinear form on $\mathrm{W}^{1,2}_D(O)^m$ and the same is true for the left-hand side since $L^{\frac{1}{2}}$ and $(L^*)^{\frac{1}{2}}$ are topological isomorphisms from $\mathrm{W}^{1,2}_D(O)^m \to \mathrm{L}^2(O)^m$. To conclude, notice that $\mathrm{D}(L)$ is dense in $\mathrm{D}(L^{\frac{1}{2}}) = \mathrm{W}^{1,2}_D(O)^m$, so identity (6.54) extends to $u \in \mathrm{W}^{1,2}_D(O)^m$ by continuity.

With these preparations, the proof is almost done. Let $u \in \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m$ and $v \in \mathrm{L}^{p'}(O) \cap \mathrm{L}^2(O)$. Decompose $v = (L^*)^{\frac{1}{2}}(L^*)^{-\frac{1}{2}}v$. Then,

$$|(L^{\frac{1}{2}}u \,|\, v)| = |a(u,(L^*)^{-\frac{1}{2}}v)| \lesssim \|u\|_{\mathrm{W}^{1,p}}\|(L^*)^{-\frac{1}{2}}v\|_{\mathrm{W}^{1,p'}} \lesssim \|u\|_{\mathrm{W}^{1,p}}\|v\|_{p'},$$

where we have upgraded the boundedness of the Riesz transform in $\mathrm{L}^{p'}$ to a full $\mathrm{W}^{1,p}$-bound for $(L^*)^{-\frac{1}{2}}$ using Corollary 6.3.5. Thus, testing yields the claim. □

6.8. Proof of Theorem 6.0.1

Now the proof of the main theorem of this chapter is essentially a combination of the results from Sections 6.5 and 6.7, mixed with some standard arguments.

Proof of Theorem 6.0.1. We start with the proof of (i). Chose $p_-(L) < r < q < p$. On the one hand, since $\mathcal{S}$ is L^q-bounded, Proposition 6.7.1 is applicable (with $s = q$) and yields the L^p-bound

$$\|L^{\frac{1}{2}} u\|_p \lesssim \|u\|_{\mathrm{W}^{1,p}(O)} \qquad (u \in \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m). \tag{6.55}$$

Since $\mathrm{W}^{1,p}_D(O)^m$ and $\mathrm{L}^p(O)^m$ are Banach spaces and $\mathrm{W}^{1,2}_D(O)^m \cap \mathrm{W}^{1,p}_D(O)^m$ is dense in $\mathrm{W}^{1,p}_D(O)^m$, the bound (6.55) reveals that $L^{\frac{1}{2}}$ has a continuous extension to $\mathrm{W}^{1,p}_D(O)^m$, again denoted by $L^{\frac{1}{2}}$. The upper bound for $L^{\frac{1}{2}}$ has the correct dependence on L according to Proposition 6.7.1.

On the other hand, since $\mathcal{S}$ is L^r-bounded by choice of r, it follows from Proposition 6.1.12 that $\mathcal{N}$ satisfies $\mathrm{L}^q \to \mathrm{L}^2$ off-diagonal estimates. Hence, Proposition 6.5.1 yields

$$\|\nabla L^{-\frac{1}{2}} u\|_p \lesssim \|u\|_p \qquad (u \in \mathrm{L}^p(O)^m \cap \mathrm{L}^2(O)^m).$$

Combine this bound with the L^p-bound from Corollary 6.3.5 to deduce the inhomogeneous Sobolev bound

$$\|L^{-\frac{1}{2}} u\|_{\mathrm{W}^{1,p}(O)^m} \lesssim \|u\|_p \qquad (u \in \mathrm{L}^p(O)^m \cap \mathrm{L}^2(O)^m). \tag{6.56}$$

To show that $L^{\frac{1}{2}}$ is surjective, let $u \in \mathrm{L}^p(O)^m$ and let $(u_n)_n$ be a sequence in $\mathrm{L}^p(O)^m \cap \mathrm{L}^2(O)^m$ that converges to u with respect to the L^p norm. The Cauchy property of $(u_n)_n$ inherits to the sequence $(L^{-\frac{1}{2}} u_n)_n$ in $\mathrm{W}^{1,p}_D(O)^m$ in virtue of the estimate (6.56). Consequently, there is $v \in \mathrm{W}^{1,p}_D(O)^m$ to which $L^{-\frac{1}{2}} u_n$ converges. Use the L^2 identity $u_n = L^{\frac{1}{2}} L^{-\frac{1}{2}} u_n$, the continuity of $L^{\frac{1}{2}}$ on $\mathrm{W}^{1,p}_D(O)^m$ and the aforementioned convergence of $L^{-\frac{1}{2}} u_n$ to deduce $u_n \to L^{\frac{1}{2}} v$ in L^p. But u_n converges also to u by construction, so $u = L^{\frac{1}{2}} v$, which shows ontoness.

To see that $L^{\frac{1}{2}}$ is injective on $\mathrm{W}^{1,p}_D(O)^m$, let $u \in \mathrm{W}^{1,p}_D(O)^m$ with $L^{\frac{1}{2}} u = 0$ and let $(u_n)_n \subseteq \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m$ be an approximating sequence for u with respect to the $\mathrm{W}^{1,p}(O)^m$-topology. Use (6.56) with $L^{\frac{1}{2}} u_n$ and continuity to get

$$\|u_n\|_{\mathrm{W}^{1,p}(O)^m} \lesssim \|L^{\frac{1}{2}} u_n\|_p \to \|L^{\frac{1}{2}} u\|_p = 0.$$

Thus, $(u_n)_n$ is a null sequence and $u = 0$ as desired.

Finally, an application of (6.56) with $L^{\frac{1}{2}} u$ for $u \in \mathrm{W}^{1,p}_D(O)^m \cap \mathrm{W}^{1,2}_D(O)^m$ shows that the lower bound depends as expected on L according to Proposition 6.5.1 and Corollary 6.3.5.

Assertion (ii) follows by the very same argument but with Proposition 6.7.4 instead of Proposition 6.7.1 and Proposition 6.5.7 instead of Proposition 6.5.1. □

APPENDIX A

Appendix

A.1. Porous sets

We provide a streamlined approach to the geometry of porous sets. All this is known to the experts but some results require going through existing literature in a rather opaque way. The reader may look up relevant definitions in Section 1.3.

Lemma A.1.1. *Every porous set $E \subseteq \mathbb{R}^d$ is a Lebesgue null set.*

Proof. By Remark 1.3.24, each ball B centered in E contains a ball of comparable radius that does not intersect E. Hence, there is $\delta \in (0,1)$ depending only on E such that

$$\frac{|B \cap E|}{|B|} \leq 1 - \delta.$$

By Lebesgue's differentiation theorem this implies $\mathbf{1}_E = 0$ almost everywhere. $\square$

We recall the Vitali covering lemma that will be used frequently in the following, see [58, Thm. 1.2].

Lemma A.1.2. *Let $\{B_i\}_{i \in I}$ be a family of open balls with uniformly bounded radii. Then there exists a subfamily $\{B_j\}_{j \in J}$ of disjoint balls such that*

$$\bigcup_i B_i \subseteq \bigcup_j 5B_j.$$

Corollary A.1.3. *Let $E \subseteq \mathbb{R}^d$ and $0 < r \leq R < \infty$. For any ball B of radius R the set $E \cap B$ can be covered by $10^d(R/r)^d$ ball of radius r centered in $E \cap B$.*

Proof. Consider the covering $\{\mathrm{B}(x, r/5)\}_{x \in B \cap E}$ of $B \cap E$. We find a disjoint subfamily $\{B_i\}_{i \in I}$ such that $B \cap E \subseteq \cup_{i \in I} 5B_i$. We denote by $\#_i$ the cardinality of I and calculate

$$\#_i c_d (r/5)^d = |\bigcup_{i \in I} B_i| \leq |2B| = c_d 2^d R^d,$$

where c_d is the measure of the unit ball. This shows $\#_i \leq 10^d (R/r)^d$. □

We continue with the simple observation that the radius bound by 1 in the definition of ℓ-regularity is arbitrary.

Lemma A.1.4. *Let $E \subseteq \mathbb{R}^d$ and $0 < \ell \leq d$. If for some $M \in (0, \infty)$ there is comparability $\mathcal{H}^\ell(B \cap E) \approx \mathrm{r}(B)^\ell$ uniformly for all open balls B of radius $\mathrm{r}(B) \leq M$ centered in E, then the same is true for any $M \in (0, \infty)$.*

Proof. Suppose we have uniform comparability for balls up to radius $\mathrm{r}(B) \leq m$. Given $M > m$, we need to extend it to balls B centered in E of radius $\mathrm{r}(B) \leq M$. Let $c \coloneqq m/M$. The calculation

$$\frac{m^\ell \mathrm{r}(B)^\ell}{M^\ell} \lesssim \mathcal{H}^l(cB \cap E) \leq \mathcal{H}^\ell(B \cap E)$$

gives the lower estimate. For the upper one, we cover $B \cap E$ by $10^d/c^d$ balls of radius $c\mathrm{r}(B)$ centered in $B \cap E$ according to Corollary A.1.3 and conclude $\mathcal{H}^\ell(B \cap E) \lesssim \mathrm{r}(B)^\ell$. □

We come to computing the Assouad dimensions of Ahlfors-regular sets.

Lemma A.1.5. *Let $E \subseteq \mathbb{R}^d$ be ℓ-regular for some $0 < \ell \leq d$ and let $M < \infty$. There exist constants $c, C > 0$ such that, if $x \in E$ and $0 < r \leq R < M$, then in order to cover $E \cap \mathrm{B}(x, R)$ by balls of radius r centered in E, at least $c(R/r)^\ell$ and at most $C(R/r)^\ell$ balls are needed. If E is unbounded and uniformly ℓ-regular, then this also holds for $M = \infty$.*

Proof. Let $\{B_i\}_{i \in I}$ be *some* cover of $E \cap \mathrm{B}(x, R)$ by balls of radius r. We use Lemma A.1.4 to calculate

$$R^\ell \lesssim \mathcal{H}^\ell(B(x, R) \cap E) \leq \mathcal{H}^\ell(\cup_{i \in I} B_i \cap E) \leq \sum_{i \in I} \mathcal{H}^\ell(B_i \cap E) \lesssim \#_i r^\ell,$$

which shows $\#_i \gtrsim (R/r)^\ell$ and gives the constant c. As for C, we select a subfamily of disjoint balls B_j from the covering $\{\mathrm{B}(x, r/5)\}_{x \in B \cap E}$ of $B \cap E$. Then we estimate, using Lemma A.1.4,

$$\#_j (r/5)^\ell \lesssim \sum_{j \in J} \mathcal{H}^\ell(B_j \cap E) \leq \mathcal{H}^\ell(2B \cap E) \lesssim (2R)^\ell$$

and conclude $\#_j \lesssim (R/r)^\ell$. $\square$

Proposition A.1.6. *Let $E \subseteq \mathbb{R}^d$ be uniformly ℓ-regular. It follows that $\underline{\dim}_{\mathcal{AS}}(E) = \overline{\dim}_{\mathcal{AS}}(E) = \ell$.*

Proof. We can rephrase Lemma A.1.5 in the language of Definition 1.3.13. It precisely asserts that $\ell \in \underline{\mathcal{AS}}(E) \cap \overline{\mathcal{AS}}(E)$. Hence, we get $\underline{\dim}_{\mathcal{AS}}(E) \geq \ell$ and $\overline{\dim}_{\mathcal{AS}}(E) \leq \ell$. The claim follows since $\underline{\dim}_{\mathcal{AS}}(E) \leq \overline{\dim}_{\mathcal{AS}}(E)$ holds for any set E. Indeed, given $\lambda \in \underline{\mathcal{AS}}(E)$ and $\mu \in \overline{\mathcal{AS}}(E)$ we have $(R/r)^\lambda \lesssim (R/r)^\mu$ for all $0 < r < R < \operatorname{diam}(E)$ and hence $\lambda \leq \mu$. $\square$

We turn to porosity. The following result was already mentioned in the introduction to Chapter 2.

Lemma A.1.7. *Let $E \subseteq F \subseteq \mathbb{R}^d$. If F is ℓ-regular and E is m-regular with $0 < m < \ell \leq d$, then E is porous in F. Likewise, if $\overline{\dim}_{\mathcal{AS}}(E) < \underline{\dim}_{\mathcal{AS}}(F)$, then E is uniformly porous in F.*

Proof. We begin with the first claim. Lemma A.1.5 yields some $C \geq 1$ such that, if $x \in E$ and $0 < r \leq R \leq 1$, then at most $C(2R/r)^m$ balls of radius r centered in E are needed to cover $E \cap \mathrm{B}(x, 2R)$. It also yields some $c > 0$ such that at least $c(R/(2r))^\ell$ balls of radius $2r$ centered in F are needed to cover $F \cap \mathrm{B}(x, R)$. We use this observation with $r = \kappa R$, where $\kappa \in (0, 1)$ satisfies $c/(2\kappa)^\ell > C(2/\kappa)^m$. This is possible due to $m < \ell$.

Let $\{B_i\}_{i \in I}$ be a family of $\#_i \leq C(2/\kappa)^m$ balls of radius r centered in E that cover $E \cap \mathrm{B}(x, 2R)$. By choice of κ the balls $\{2B_i\}_{i \in I}$ cannot cover $F \cap \mathrm{B}(x, R)$. Pick $y \in F \cap \mathrm{B}(x, R)$ that is not contained in any of the $2B_i$. By construction we have $\mathrm{B}(y, r) \subseteq \mathbb{R}^d \setminus \cup_{i \in I} B_i$ but due to $r < R$ we also have $\mathrm{B}(y, r) \subseteq \mathrm{B}(x, 2R)$ and hence $E \cap \mathrm{B}(y, r) \subseteq \cup_i B_i$. Thus, we must have $E \cap \mathrm{B}(y, r) = \emptyset$ and conclude that E is porous in F.

The proof of the second claim is identical, but we do not assume $R \leq 1$ and have the covering properties for some $m \in \overline{\mathcal{AS}}(E)$ and $\ell \in \underline{\mathcal{AS}}(F)$ with $m < \ell$ by assumption. $\square$

Lemma A.1.8. *If $E \subseteq \mathbb{R}^d$ is porous, then there exist $C \geq 1$ and $0 < s < d$ such that, given $x \in E$ and $0 < r < R \leq 1$, there is a covering of $E \cap B(x, R)$ by $C(R/r)^s$ balls of radius r centered in E. Moreover, if E is uniformly porous, then $\overline{\dim}_{\mathcal{AS}}(E) < d$.*

Proof. We only show the porous case since the uniform case again just follows by dropping all restrictions on the radii. In the following all cubes are closed and axis-aligned. We can equivalently replace balls by cubes and radii by side lengths in the definition of porosity and Assouad dimension. Likewise, it suffices to establish the claim of the lemma with cubes.

In view of Remark 1.3.24 we can fix $n \in \mathbb{N}$ such that for every cube $Q \subseteq \mathbb{R}^d$ there is a cube $Q' \subseteq Q \setminus E$ of sidelength $\ell(Q') = \ell(Q)/n$. We fix a cube Q centered in E of side length $R \leq 1$. Let $0 < r \leq R$ and fix $k \in \mathbb{N}$ such that $R/(2n)^{k+1} \leq r < R/(2n)^k$. We claim that we can cover Q by $((2n)^d - 1)^{k+1}$ closed cubes of side length $R/(2n)^{k+1}$. Put $s := \log((2n)^d - 1)/\log(2n) < d$. Then

$$((2n)^d - 1)^{k+1} = (2n)^s (2n)^{ks} < (2n)^s (R/r)^s$$

shows the assertion.

For the claim we start with $k = 1$. There is a cube $Q' \subseteq Q \setminus E$ of side length R/n. Then there is a cube Q'' in the grid of $(2n)^d$ cubes with sidelength $R/(2n)$ covering Q that is contained in Q'. This means that we only need $(2n)^d - 1$ cubes of side length $R/(2n)$ to cover E. We conclude by applying this argument inductively on each cube of the previous covering. $\square$

Combining the uniform cases of the two preceding lemmas lets us re-obtain a result of Luukkainen [73, Thm 5.2]. Note that $\underline{\dim}_{\mathcal{AS}}(\mathbb{R}^d) = d$ due to Proposition A.1.6.

Proposition A.1.9. *A set $E \subseteq \mathbb{R}^d$ is uniformly porous if and only if its upper Assouad dimension is strictly less than d.*

We can use the non-uniform cases to show that some open sets are of class $\mathcal{D}^t$. The argument is a slight adaption of [71, Thm. 4.2].

Proposition A.1.10. *Let $O \subseteq \mathbb{R}^d$ be open. If ∂O is porous, then $O \in \mathcal{D}^t$ for some $t \in (0, 1)$. If ∂O is ℓ-regular for some $0 < \ell < d$, then $O \in \mathcal{D}^t$ for all $t \in (0, \max\{1, d - \ell\})$.*

Proof. If ∂O is porous, then we pick $C \geq 1$ and $0 < s < d$ according to Lemma A.1.8 such that for each $j \geq 0$ and for any ball B with radius $r \leq 1$ centered in ∂O we can cover $B \cap \partial O$ by at most $C2^{js}$ balls of radius $r2^{-j}$. If ∂O is ℓ-regular, then Lemma A.1.5 guarantees that we can take $s = \ell$. In any case, fix $\max(s, d-1) < u < d$. Put $E_j \coloneqq \{x \in B : \mathrm{d}(x, \partial O) \leq r2^{-j}\}$ and $A_j \coloneqq E_j \setminus E_{j+1}$. By construction, the covering property for $B \cap \partial O$ implies that we can cover E_j by at most $C2^{js}$ balls of radius $r2^{-(j-1)}$. The d-regularity of the Lebesgue measure then implies

$$|A_j| \leq |E_j| \lesssim 2^{js} r^d 2^{d-jd}. \tag{A.1}$$

We use that $\{A_j\}_{j \geq 0}$ is a disjoint cover of $B \setminus \partial O$, comparability $\mathrm{d}(x, \partial O) \approx r2^{-j}$ on A_j, estimate (A.1), and $s < u$ to calculate

$$\begin{aligned} \int_{B \setminus \partial O} \mathrm{d}(y, \partial E)^{u-d} \, \mathrm{d}y \leq \sum_j \int_{A_j} \mathrm{d}(y, \partial O)^{u-d} \, \mathrm{d}y &\lesssim \sum_j |A_j| 2^{dj-uj} r^{u-d} \\ &\lesssim \sum_j r^u 2^{j(s-u)} \lesssim r^u. \end{aligned}$$

Setting $t \coloneqq d - u \in (0,1)$, we write this in the form

$$\sup_{x \in \partial O} \sup_{0 < r \leq 1} r^{t-d} \int_{\mathrm{B}(x,r) \setminus \partial O} \mathrm{d}(y, \partial O)^{-t} < \infty,$$

which just means that $O \in \mathcal{D}^t$. In the case of ℓ-regular boundary, every $u \in (\max\{\ell, d-1\}, 1)$ and thus every $t \in (0, \max\{1, d-\ell\})$ was admissible in the proof. $\square$

A.2. Background on Hardy's inequality

In the sequel we provide the essential notions and calculations to derive Proposition 4.3.2 from the material in [70]. The Hausdorff measure, Hausdorff content and related notions were presented in Section 1.3.

Definition A.2.1. Let $E \subseteq \mathbb{R}^d$ be closed. Call E *uniformly p-fat* if there exists a constant $b > 0$ such that

$$\mathrm{cap}_p(E \cap \overline{\mathrm{B}}(x,r), \mathrm{B}(x,2r)) \geq br^{d-p}.$$

Here, $\mathrm{cap}_p(\cdot,\cdot)$ is the relative p-capacity. A definition is provided in [70, Sec. 2].

Definition A.2.2. Let $O \subseteq \mathbb{R}^d$ be open. Then O satisfies the *inner boundary density condition* with exponent λ if there exists a constant $c > 0$ such that for all $x \in O$ holds

$$\mathcal{H}^\lambda_\infty(\mathrm{B}(x, 2\,\mathrm{d}_{\partial O}(x)) \cap O) \geq c\,\mathrm{d}_{\partial O}(x)^\lambda.$$

It is shown in [70, Thm. 1] that ${}^c O$ is uniformly p-fat if O satisfies the inner boundary density condition with exponent $\lambda > d - p$. The result there is formulated for domains but an inspection of the proof shows that connectedness is superfluous. Moreover, the constant for the uniform p-fatness condition depends on O only via λ and c.

Now Hardy's inequality follows from the uniformly p-fat complement condition and the constant depends on O only via b, or by the previous consideration, via λ and c. This follows since the constants were explicitly traced in [54] and this observation was also confirmed by the author of [70].

To relate the inner boundary density condition with Ahlfors–David regularity, we introduce the notion of ℓ-thickness.

Definition A.2.3. Let $E \subseteq \mathbb{R}^d$ be a Borel set and $0 < \ell \leq d$. Call E *ℓ-thick* if

$$\forall x \in E, 0 < r < \operatorname{diam}(E)\colon \quad \mathcal{H}^\ell_\infty(\mathrm{B}(x, r) \cap E) \gtrsim r^\ell. \tag{A.2}$$

Lemma A.2.4. *Let $E \subseteq \mathbb{R}^d$ be Ahlfors–David regular. Then E is $(d-1)$-thick and the implicit constant in* (A.2) *depends only on the implicit constants in the Ahlfors–David regularity condition.*

Proof. Let $F \subseteq E$ be a Borel set and let $x_j \in F$, $0 < r_j \leq \operatorname{diam}(E)$ such that $\bigcup_j \mathrm{B}(x_j, r_j)$ covers F. Then

$$\sum_j r_j^{d-1} \gtrsim \sum_j \mathcal{H}^{d-1}(\mathrm{B}(x_j, r_j) \cap E) \geq \mathcal{H}^{d-1}(F \cap E) = \mathcal{H}^{d-1}(F).$$

Taking the infimum over all such coverings yields $\mathcal{H}^{d-1}_\infty(F) \gtrsim \mathcal{H}^{d-1}(F)$. Now if we take F to be $\mathrm{B}(x, r) \cap E$ the claim follows. □

Lemma A.2.5. *Let $O \subseteq \mathbb{R}^d$ be open. If ∂O is λ-thick and either O is bounded or ∂O is unbounded, then E satisfies the inner boundary density condition with exponent λ. Again, the constant c in the inner boundary density condition depends only on the implicit constant in the definition of λ-thickness.*

Proof. Let $x \in O$ and $y \in \mathrm{B}(x, 2\,\mathrm{d}_{\partial O}(x))$, then

$$\mathcal{H}^{\lambda}_{\infty}(\mathrm{B}(x, 2\,\mathrm{d}_{\partial O}(x)) \cap \partial O) \geq \mathcal{H}^{\lambda}_{\infty}(\mathrm{B}(y, \mathrm{d}_{\partial O}(x)) \cap \partial O).$$

If O is bounded, then so is the function $\mathrm{d}_{\partial O}$ on O and we can apply the λ-thickness condition to get the desired lower bound. If ∂O is unbounded, then all radii are permitted in the λ-thickness condition and we are also done. □

Remark A.2.6. If O is unbounded and ∂O is bounded, we can obtain Hardy's inequality by considering the auxiliary set $\mathrm{B}(x, 2\,\mathrm{diam}(\partial O))$ for some $x \in \partial O$. The key steps for this argument are contained in the proof of Theorem 4.0.3.

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List of notations

General

$\lvert \cdot \rvert$	Euclidean norm (on finite-dimensional spaces) or Lebesgue measure ....
$\#(A)$	size of the (countable) set A ....
$a \vee b, a \wedge b$	Maximum and minimum of the numbers a and b ....
$\operatorname{supp}(\cdot)$	Support of a function or distribution ....
$\lesssim, \gtrsim, \approx$	Comparable from above, comparable from below, or equivalent ....
$\cdot$	anonymous argument of a function ....
$\operatorname{len}(\gamma)$	length of the (rectifiable) path γ ....
$\lvert\alpha\rvert$	$\alpha_1 + \cdots + \alpha_d$, length of the multiindex α ....
$(f)_A$, $\fint_A f$	mean value of f over A, where $\lvert A\rvert > 0$ ....
$\Gamma(z)$	$\int_0^\infty t^{\mathbf{z}-1} \mathrm{e}^{-t} \,\mathrm{d}t$, the Gamma function ....
c_d	measure of the unit ball in dimension d ....

Calculus

∇	gradient operator ....
∂_j	jth partial derivative ....

∂^α	shorthand for $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex
∇^ℓ	$(\partial^\alpha)_{\lvert\alpha\rvert=\ell}$, vector of derivatives of order ℓ
Δ	Laplacian

Exponents

p'	Hölder-conjugate exponent of p
p^*	upper Sobolev exponent defined via $1/p^* = 1/p - 1/d$ for $p < d$, otherwise ∞
p_*	lower Sobolev exponent defined via $1/p_* = 1/p + 1/d$
γ_{pq}	shorthand for $d\left\lvert\frac{1}{p} - \frac{1}{q}\right\rvert$
γ_p	shorthand for γ_{p2} or γ_{2p}

Sets

${}^cA$	complement of A
$\overline{A}$	closure of A
∂A	boundary of A
$\mathring{A}$	interior of A
$\mathrm{diam}(A)$	diameter of A
$\mathrm{d}(A, B)$	(semi) distance between A and B
$\mathrm{d}_\infty(A, B)$	(semi) distance between A and B with respect to $\lvert \cdot \rvert_\infty$
$\mathrm{d}(x, E), \mathrm{d}_E(x)$	shorthand for the function $x \mapsto \mathrm{d}(\{x\}, E)$
$\mathbf{1}_A$	indicator function of A
$\mathrm{B}(x, r)$	open ball centered in x with radius r
$\mathrm{r}(B)$	the radius of a ball B

Interpolation

Function spaces

$\mathcal{R}_E, \mathcal{E}_E$	Jonsson–Wallin restriction and extension operators on the $(d-1)$-regular set E	25
$\mathcal{E}_0$	zero extension operator	
$O \perp D$	Dirichlet cylinder over O	55

Geometry and dimensions

$\overline{\mathcal{AS}}(E), \underline{\mathcal{AS}}(E)$	Covering parameters for the Assouad dimension of E	9
$\overline{\dim}_{\mathcal{AS}}, \underline{\dim}_{\mathcal{AS}}$	upper and lower Assouad dimension	9
$\dim_{\mathcal{H}}$	Hausdorff dimension	10
$\operatorname{codim}_{\mathcal{H}}$	Hausdorff co-dimension	11
$\mathcal{D}^t$	Sickel class	29
k_{Ξ}	quasihyperbolic distance	67
$\gamma_{x,y}$	quasihyperbolic geodesic connecting x with y	67

$\mathbf{W}_D^{k,p}$-extension

$\mathcal{W}_i$	interior cubes	78
$\mathcal{W}_e$	exterior cubes	79
Q^*	reflected cube associated to Q	82
$\mathcal{G}, \Sigma, \Sigma'$	collections of fixed-size cubes	96
$\hat{R}, \hat{\hat{R}}$	enlarged cubes	97
$(f)_Q$	polynomial adapted to f on Q	87
$F_{j,k}, F_{P,j}$	connecting chains	83
$F(Q_j), F_P(Q)$	extended chains	88
$\mathcal{E}_A f$	zero extension of f to A, usually $A \in \mathcal{W}_i$	90
κ	shorthand for $\mathrm{d}(\operatorname{supp}(f), D)$ in smooth approximation	95

Potential theory

Operator theory

Functional calculus

Elliptic operator

Dirac operators

Harmonic analysis

Index

Curriculum Vitæ

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