# Fractional Bessel Integrals and Derivatives on Semi-axes

**E. L. Shishkina and S. M. Sitnik**

**Abstract** In this paper we study fractional powers of the Bessel differential operator. The fractional powers are defined explicitly in the integral form without use of integral transforms in its definitions. Some general properties of the fractional powers of the Bessel differential operator are proved and some are listed. Among them are different variations of definitions, relations with the Mellin and Hankel transforms, group property, evaluation of resolvent integral operator in terms of the Wright or generalized Mittag–Leffler functions. At the end, some topics are indicated for further study and possible generalizations. Also the aim of the paper is to attract attention and give references to not widely known results on fractional powers of the Bessel differential operator. This class of fractional operators is in close connection with transmutation theory and classic transmutational operators. We also study connections of Bessel fractional operators with different kinds of integral transforms.

**Keywords** Fractional Bessel operator · Hypergeometric function · Hankel transform

## 1 Introduction

We study the differential Bessel operator in the form

$$B_\nu := D^2 + \frac{\nu}{x} D, \qquad \nu \geq 0, \quad D := \frac{d}{dx}, \tag{1}$$

E. L. Shishkina (✉)
University of Information Technology and Management in Rzeszow, Rzeszow, Poland
e-mail: shishkina@amm.vsu.ru

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru



V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_28

and its fractional powers $(B_\nu)^\alpha$, $\alpha \in \mathbb{R}$. This operator has essential role in the theory of differential equations both as a radial part of the Laplace operator and also as involved in partial differential equations with Bessel operators. Such equations were called $B$-elliptic, $B$-hyperbolic and $B$-parabolic by I.A. Kipriyanov and intensively studied by his scientific school and many others researchers, now the term "Laplace–Bessel equations" is also used. For equations with Bessel operators and related topics cf. [1–3].

Of course fractional powers of the Bessel operator (1) were studied in many papers. But in the most of them fractional powers were defined implicitly as a power function multiplication under Hankel transform. This definition via integral transforms leads to many restrictions. Just imagine that for the classical Riemann–Liouville fractional integrals we have to work only with its definitions via Laplace or Mellin transforms and nothing more without explicit integral representations. If it would be true, then 99% of classical "Bible" [4] and other books on fractional calculus would be empty as they mostly use explicit integral definitions! But for fractional powers of the Bessel operator at most papers implicit definitions via Hankel transform are still used.

Of course such situation is not natural and in some papers different approaches to step closer to explicit formulas were studied. Let us mention that in [5] explicit formulas were derived as compositions of Erdélyi–Kober fractional integrals [4] on distribution spaces, in this monograph results on fractional powers of Bessel and related operators are gathered of McBride's and earlier papers. An important step was done in [6] in which explicit definitions were derived in terms of the Gauss hypergeometric functions with different applications to PDE, we also use basic formulas from [6] in this paper. The most general study was fulfilled by I. Dimovski and V. Kiryakova [7–10] for the more general class of hyper-Bessel differential operators related to the Obrechkoff integral transform. They constructed explicit integral representations of the fractional powers of these operators by using Meijer $G$-functions as kernels, and also intensively and successfully used for this the theory of transmutations. Note that in this and others fields of theoretical and applied mathematics, the methods of transmutation theory are very useful and productive and for some problems are even irreplaceable (see e.g. [11]). In [12, 13] simplified representations for fractional powers of the Bessel operator were derived with Legendre functions as kernels, and based on them general definitions were simplified and unified with standard fractional calculus notation as in [4], and also important generalized Taylor formulas were proved which mix integer powers of Bessel operators (instead of derivatives in the classical Taylor formula) with fractional power of the Bessel operator as integral remainder term, cf. also [14, 15].

This class of fractional operators is in close connection with transmutation theory and classic transmutational operators such as Sonine and Poisson ones [16, 17]. We also study connections of Bessel fractional operators with different kinds of integral transforms: Hankel, Mellin, Erd'elyi–Kober, Mejer, integral transforms with Wittaker, Wright, Mittag–Leffler and hypergeometric kernels.

# 2 Definitions

## 2.1 Special Functions and Integral Transforms

In this subsection we give definitions of some special functions. Special functions enable us to introduce integral transforms connected with these functions such that a new problem can be attacked within a known framework, usually in the context of differential equations and their generalizations.

Let start with normalized Bessel functions.

The symbol $j_\alpha$ is used for the normalized Bessel function:

$$j_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} J_\alpha(t), \qquad j_\alpha(0) = 1, \qquad j'_\alpha(0) = 0, \tag{2}$$

where $J_\alpha(t)$ is the Bessel function of the first kind of order $\alpha$ (see [18]):

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

Function $J_\alpha$ first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel (see [18]).

Using formulas 9.1.27 from [19] we obtain that the function $j_\nu(t)$ is an eigenfunction of a linear operator $B_\nu$:

$$(B_\nu)_t \, j_{\frac{\nu-1}{2}}(\tau t) = -\tau^2 j_{\frac{\nu-1}{2}}(\tau t). \tag{3}$$

We also will need some other normalized Bessel functions. Normalized Bessel functions of the second kind $y_\alpha$ is

$$y_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} Y_\alpha(t), \tag{4}$$

where $Y_\alpha$ is the Bessel functions of the second kind. Function $Y_\alpha$ for non-integer $\alpha$ is related to $J_\alpha$ by:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

In the case of integer order $n$, the function $Y_n$ is defined by taking the limit as a non-integer $\alpha$ tends to $n$,

$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x).$$

Normalized modified Bessel functions of the first and second kind $I_\alpha(x)$ and $K_\alpha(x)$ are defined by

$$i_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} I_\alpha(t), \qquad k_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} K_\alpha(t), \tag{5}$$

where modified Bessel functions of the first and second kind $I_\alpha(x)$ and $K_\alpha(x)$ are

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \qquad K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)},$$

when $\alpha$ is not an integer and when $\alpha$ is an integer, then the limit is used.

Next we consider generalized hypergeometric functions which have many particular special functions as special cases, such as elementary functions, Bessel functions, and the classical orthogonal polynomials.

A generalized hypergeometric function is defined as a power series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

The functions of the form ${}_0F_1(; a; z)$ are called confluent hypergeometric limit functions and are closely related to Bessel functions $J_\alpha$ and $I_\alpha$. The relationships are

$$J_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(; \alpha+1; -\tfrac{x^2}{4}\right),$$

$$I_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(; \alpha+1; \tfrac{x^2}{4}\right)$$

or

$${}_0F_1\left(; \alpha+1; -\tfrac{x^2}{4}\right) = j_\alpha(x), \qquad {}_0F_1\left(; \alpha+1; \tfrac{x^2}{4}\right) = i_\alpha(x).$$

Beside we need function ${}_1F_2(; a; z)$. It is known (see [20]) that for $\alpha > 0$, $\xi \geq 0$, $t > 0$

$$\int_0^t \left(t^2 - u^2\right)^{\alpha-1} u^{1-\gamma} J_\gamma(u\xi) dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1} \alpha \Gamma(\gamma+1)} {}_1F_2\left(1; \alpha+1, \gamma+1; -\frac{t^2 \xi^2}{4}\right)$$

and for $\gamma < 2, \alpha > 0, \xi \geq 0, t > 0$

$$\int_0^t \left(t^2 - u^2\right)^{\alpha-1} u^{1-\gamma} I_\gamma(u\xi)dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1}\alpha\Gamma(\gamma+1)} {}_1F_2\left(1; \alpha+1, \gamma+1; \frac{t^2\xi^2}{4}\right).$$

Next we present Whittaker functions which are appear in kernel of integral transform connected with fractional Bessel integral.

Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are special solutions of Whittaker's equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0.$$

They are modified forms of the of Kummer's confluent hypergeometric functions were introduced by Edmund Taylor Whittaker by

$$M_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} M\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z\right),$$

$$W_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z\right), \tag{6}$$

where

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a; b; z).$$

and

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a+1-b, 2-b, z).$$

are Kummer's functions.

The Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are the same as those with opposite values of $\mu$, in other words considered as a function of $\mu$ at fixed $\kappa$ and $z$ they are even functions. When $\kappa$ and $z$ are real, the functions give real values for real and imaginary values of $\mu$.

## 2.2 *Integral Transforms*

In this subsection we give definitions of integral transforms which can be used in dealing with differential equations with fractional Bessel derivatives on semi-axes.

The Mellin transform of a function $f:\mathbb{R}_+\to\mathbb{C}$ is the function $f^*$ defined by

$$f^*(s)=\mathcal{M}f(s)=\int\limits_0^\infty x^{s-1}f(x)dx,$$

where $s=\sigma+i\tau\in\mathbb{C}$, provided that the integral exists.

Following to [21] as space of originals we choose the space $P_a^b$, $-\infty<a<b<\infty$ which is the linear space of $\mathbb{R}_+\to\mathbb{C}$ functions such that $x^{s-1}f(x)\in L_1(\mathbb{R}_+)$ for every $s\in\{p\in\mathbb{C}:a\leq \operatorname{Re}p\leq b\}$.

If additionally $f^*(c+i\tau)\in L_1(\mathbb{R})$ with respect to $\tau$ then complex inversion formula holds:

$$\left\{\mathcal{M}^{-1}\varphi\right\}(x)=f(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}x^{-s}\varphi(s)\,ds.$$

For functions $f{\in}L_1^\nu(\mathbb{R}_+)$ the Hankel transform of order $\frac{\nu-1}{2}>-\frac{1}{2}$ is

$$F_\nu[f](\xi)=\widehat{f}(\xi)=\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,f(x)x^\nu\,dx.$$

Let $f{\in}L_1^\nu(\mathbb{R}_+)$ and of bounded variation in a neighborhood of a point $x$ of continuity of $f$. Then for $\nu>0$ the inversion formula

$$F_\nu^{-1}[\widehat{f}](x)=f(x)=\frac{2^{1-\nu}}{\Gamma^2\left(\frac{\nu+1}{2}\right)}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,\widehat{f}(\xi)\xi^\nu\,d\xi$$

holds.

For functions $f$ the integral transforms involving Bessel function $K_{\frac{\nu-1}{2}}$, $\nu\geq 1$ as kernel is the Meijer transform defined by

$$\mathcal{K}_\nu[f](\xi)=F(\xi)=\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,f(x)x^\nu\,dx.$$

Let $f\in L_1^{loc}(\mathbb{R}_+)$ and $f(t)=o\left(t^{\beta-\frac{\nu}{2}}\right)$ as $t\to+0$ where $\beta>\frac{\nu}{2}-2$ if $\nu>1$ and $\beta>-1$ if $\nu=1$. Furthermore let $f(t)=O(e^{at})$ as $t\to+\infty$. Then its Meijer exists a.e. for $\operatorname{Re}\xi>a$ (see [21, p. 94]).

If $0<\nu<2$ and $F(\xi)$ is analytic on the half-plane $H_a=\{p\in\mathbb{C}:\mathrm{p}\geq a$, $a\leq 0$ and $s^{\frac{\nu}{2}-1}F(\xi)\to$, $|\xi|\to+\infty$, uniformly with respect to $\arg s$ then for any number $c$, $c > a$ the inverse transform $\mathcal{K}_\nu^{-1}$ is

$$\mathcal{K}_\nu^{-1}[\widehat{f}](x) = f(x) = \frac{1}{\pi i}\int\limits_{c-i\infty}^{c+i\infty} \widehat{f}(\xi) i_{\frac{\nu-1}{2}}(x\xi)\xi^\nu d\xi.$$

Generalized Whittaker transform is

$$(W^k_{\rho,\gamma} f)(x) = \int\limits_0^\infty (xt)^k e^{\frac{x^2t^2}{2}} W_{\rho,\gamma}(x^2t^2) f(t)dt$$

with $\rho, \gamma \in \mathbb{C}$ and $k \in \mathbb{R}$, containing the Whittaker function (6) in the kernel.

## 2.3 Fractional Bessel Integrals and Derivatives on Semi-axes

In this section we give definition of the fractional Bessel integrals on semi-axes following to [5, 6, 12, 13, 22, 23].

**Definition 1** Let $f$ is integrable by $(0, \infty)$ with the weight $\rho(x)$, $\alpha > 0$. The integrals

$$(B^{-\alpha}_{\nu,0+} f)(x) = (IB^\alpha_{\gamma,0+} f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{y}{x}\right)^\nu \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) f(y)dy \tag{7}$$

and

$$(B^{-\alpha}_{\nu,-} f)(x) = (IB^\alpha_{\gamma,-} f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy \tag{8}$$

are called **left-sided fractional Bessel integral** and **right-sided fractional Bessel integral** on semi-axis $[0, \infty)$ of order $\alpha$, accordingly. In the case of the integral (7) the weight $\rho(x) = x^{4\alpha+\nu}$ and in the case of the integral (7) the weight $\rho(x) = x^{4\alpha}$.

In Definition 1 function ${}_2F_1(a, b; c; z)$ is the hypergeometric function defined for $|z| < 1$ by the power series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$$

For complex argument $z$ with $|z| \geq 1$ function ${}_2F_1(a,b;c;z)$ can be analytically continued along any path in the complex plane that avoids the branch points 1 and infinity.

Using formula 22 p. 64 from [24] of the form

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) $$

we can rewrite (7) as

$$ (B^{-\alpha}_{\nu,0+}f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_0^x \left(\frac{x^2-y^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy. \tag{9} $$

In [12, 13, 15] was shown that formulas (7) and (8) can be simplified using formula 15.4.7 p. 561 from [19]

$$ {}_2F_1(a,b;2b;z) = $$

$$ = 2^{2b-1}\Gamma\left(b+\frac{1}{2}\right) z^{\frac{1}{2}-b}(1-z)^{\frac{1}{2}\left(b-a-\frac{1}{2}\right)} P^{\frac{1}{2}-b}_{a-b-\frac{1}{2}}\left[\left(1-\frac{z}{2}\right)\frac{1}{\sqrt{1-z}}\right]. $$

So, we can write for $\alpha > 0$

$$ (B^{-\alpha}_{\nu,0+}f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_0^x (x^2-y^2)^{\alpha-\frac{1}{2}} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\nu}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy $$

and

$$ (B^{-\alpha}_{\nu,-}f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_x^\infty (y^2-x^2)^{\alpha-\frac{1}{2}} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\nu}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy $$

where $f(x) \in L_1(0,\infty)$. Here the kernels of the fractional Bessel integrals on semi-axes are expressed using two-parameter Legendre functions instead of three-parameter Gauss hypergeometric functions.

Next we give some known facts proved in [22, 23].

### 2.3.1 Basic Properties of the Fractional Bessel Integrals on Semi-axes

1. For $\nu = 0$ we have

$$(B_{0,0+}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_0^x (x-y)^{2\alpha-1} f(y)dy = (I_{0+}^{2\alpha} f)(x), \tag{10}$$

$$(B_{0,-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^\infty (y-x)^{2\alpha-1} f(y)dy = (I_{-}^{2\alpha} f)(x), \tag{11}$$

where $I_{0+}^{2\alpha}$ is the left-sided Riemann-Liouville fractional integrals (see formula 5.1 on p. 94 in [4]) and $I_{-}^{2\alpha}$ is the Liouville fractional integral (see formula 5.3 on p. 94 in [4]).

2. When $\alpha{=}1$ if $\lim\limits_{x\to+0} g(x){=}0$, $\lim\limits_{x\to+0} g'(x){=}0$ the left-sided fractional Bessel integral on semi-axis is the left inverse to the differential Bessel operator

$$(B_{\nu,a+}^{-1} B_\nu g(x))(x) = g(x)$$

and when $\alpha{=}1$ if $\lim\limits_{x\to+\infty} g(x){=}0$, $\lim\limits_{x\to+\infty} g'(x){=}0$ the right-sided fractional Bessel integral $B_{\nu,-}^{-1}$ is the left inverse to the differential Bessel operator

$$(B_{\nu,-}^{-1} B_\nu g(x))(x) = g(x).$$

3. The formula for integration by parts is valid on proper functions:

$$\int\limits_0^\infty f(x)(B_{\nu,0+}^{-\alpha} g)(x) x^\nu dx = \int\limits_0^\infty g(x)(B_{\nu,-}^{-\alpha} f)(x) x^\nu dx. \tag{12}$$

**Definition 2** Let $\alpha > 0$. The **left-sided fractional Bessel derivative** and **right-sided fractional Bessel derivative** on semi-axis $[0, \infty)$ of order $\alpha$ are defined by the next equalities, accordingly

$$(B_{\gamma,0+}^{\alpha} f)(x) = (DB_{\gamma,0+}^{\alpha} f)(x) = B_\gamma^n (IB_{\gamma,0+}^{n-\alpha} f)(x), \qquad n = [\alpha] + 1 \tag{13}$$

and

$$(B_{\gamma,-}^{\alpha} f)(x) = (DB_{\gamma,-}^{\alpha} f)(x) = B_\gamma^n (IB_{\gamma,-}^{n-\alpha} f)(x), \qquad n = [\alpha] + 1. \tag{14}$$

In [5] spaces adapted to work with operators of the form $B^{\alpha}_{\gamma,0+}$ and $B^{\alpha}_{\gamma,-}$, $\alpha\in\mathbb{R}$ were introduced:

$$F_p=\left\{\varphi\in C^{\infty}(0,\infty): x^k\frac{d^k\varphi}{dx^k}\in L^p(0,\infty)\,\text{for}\,k=0,1,2,\dots\right\},\qquad 1\leq p<\infty,$$

$$F_{\infty}=\left\{\varphi\in C^{\infty}(0,\infty): x^k\frac{d^k\varphi}{dx^k}\to 0\,\text{as}\,x\to 0+\,\text{ and as}\,x\to\infty\,\text{for}\,k=0,1,2,\dots\right\}$$

and

$$F_{p,\mu}=\left\{\varphi: x^{-\mu}\varphi(x)\in F_p\right\},\qquad 1\leq p\leq\infty,\qquad \mu\in\mathbb{C}.$$

We present here two theorems that are special cases of theorems from [5].

**Theorem 1** *Let $\alpha\in\mathbb{R}$. For all $p,\mu$ and $\nu>0$ such that $\mu\neq\frac{1}{p}-2m$, $\gamma\neq\frac{1}{p}-\mu-2m+1$, $m=1,2\dots$ the operator $B^{\alpha}_{\gamma,0+}$ is a continuous linear mapping from $F_p,\mu$ into $F_{p,\mu-2\alpha}$. If also $2\alpha\neq\mu-\frac{1}{p}+2m$ and $\gamma-2\alpha\neq\frac{1}{p}-\mu-2m+1$, $m=1,2\dots$, then $B^{\alpha}_{\gamma,0+}$ a homeomorphism from $F_p,\mu$ onto $F_{p,\mu-2\alpha}$ with inverse $B^{-\alpha}_{\gamma,0+}$.*

**Theorem 2** *Let $\alpha\in\mathbb{R}$. For all $p,\mu$ and $\gamma>0$ such that $\mu\neq\frac{1}{p}-2m+1$, $\gamma\neq\frac{1}{p}-\mu-2m$, $m=1,2\dots$ the operator $B^{\alpha}_{\gamma,-}$ is a continuous linear mapping from $F_{q,-\mu+2\alpha}$ into $F_{q,\mu}$, where $\frac{1}{q}=1-\frac{1}{p}$. If also $2\alpha\neq\mu-\frac{1}{p}+2m-1$ and $\gamma+2\alpha\neq\mu-\frac{1}{p}+2m$, $m=1,2\dots$, then $B^{\alpha}_{\gamma,-}$ a homeomorphism from $F_{q,-\mu+2\alpha}$ onto $F_{q,-\mu}$ with inverse $B^{-\alpha}_{\gamma,-}$.*

## 3 Factorisation

Following [6] and [5] we present next results.

Let $\mathrm{Re}\,(2\eta+\mu)+2>1/p$, and $\varphi\in F_{p,\mu}$. For $\mathrm{Re}\,\alpha>0$, we define $I^{\eta,\alpha}_2\varphi$ by formula

$$I^{\eta,\alpha}_2\varphi(x)=\frac{2}{\Gamma(\alpha)}\,x^{-2\eta-2\alpha}\int\limits_0^x(x^2-u^2)^{\alpha-1}u^{2\eta+1}\varphi(u)du. \tag{15}$$

Let $\mathrm{Re}\,(2\eta-\mu)>-1/p$, and $\varphi\in F_{p,\mu}$. For $\mathrm{Re}\,\alpha>0$, we define $K^{\eta,\alpha}_2\varphi$ by formula

$$K^{\eta,\alpha}_2\varphi(x)=\frac{2}{\Gamma(\alpha)}\,x^{2\eta}\int\limits_x^{\infty}(u^2-x^2)^{\alpha-1}u^{1-2(\eta+\alpha)}\varphi(u)du. \tag{16}$$

The definitions are extended to $\operatorname{Re}\alpha \leq 0$ by means of the formulas

$$I_2^{\eta,\alpha}\varphi = (\eta+\alpha+1)I_2^{\eta,\alpha+1}\varphi + \frac{1}{2}I_2^{\eta,\alpha+1}\, x\frac{d\varphi}{dx} \tag{17}$$

and

$$K_2^{\eta,\alpha}\varphi = (\eta+\alpha)K_2^{\eta,\alpha+1}\varphi - \frac{1}{2}K_2^{\eta,\alpha+1}\, x\frac{d\varphi}{dx}. \tag{18}$$

**Theorem 3** *The next factorizations of* (7) *and* (8) *are valid*

$$(B_{\nu,0+}^{-\alpha}\varphi)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{u}{x}\right)^{\nu}\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{u^2}{x^2}\right)\varphi(u)du =$$

$$= \left(\frac{x}{2}\right)^{2\alpha}\, I_2^{\frac{\nu-1}{2},\alpha} I_2^{0,\alpha}\varphi, \tag{19}$$

$$(B_{\nu,-}^{-\alpha}f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy =$$

$$= 2^{-2\alpha} K_2^{\frac{1-\nu}{2},\alpha} K_2^{0,\alpha} x^{2\alpha}\varphi \tag{20}$$

*where*

$$I_2^{0,\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\, x^{-2\alpha}\int\limits_0^x (x^2-u^2)^{\alpha-1}u\varphi(u)du,$$

$$I_2^{\frac{\nu-1}{2},\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\, x^{1-\nu-2\alpha}\int\limits_0^x (x^2-u^2)^{\alpha-1}u^{\nu}\varphi(u)du,$$

$$K_2^{0,\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\int\limits_x^\infty (u^2-x^2)^{\alpha-1}u^{1-2\alpha}\varphi(u)du.$$

$$K_2^{\frac{1-\nu}{2},\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\, x^{1-\nu}\int\limits_x^\infty (u^2-x^2)^{\alpha-1}u^{\nu-2\alpha}\varphi(u)du.$$

***Proof*** We have

$$(B_{\nu,0+}^{-\alpha}\varphi)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_0^x\left(\frac{u}{x}\right)^{\nu}\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du=$$

$$=2^{-2\alpha}x^{2\alpha}I_2^{\frac{\nu-1}{2},\alpha}I_2^{0,\alpha}\varphi=$$

$$=\frac{2^{1-2\alpha}x^{2\alpha}}{\Gamma(\alpha)}I_2^{\frac{\nu-1}{2},\alpha}y^{-2\alpha}\int\limits_0^y(y^2-u^2)^{\alpha-1}u\varphi(u)du=$$

$$=\frac{2^{2-2\alpha}x^{2\alpha}}{\Gamma^2(\alpha)}x^{-\nu+1-2\alpha}\int\limits_0^x(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy\int\limits_0^y(y^2-u^2)^{\alpha-1}u\varphi(u)du=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_0^x u\varphi(u)du\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy.$$

Let find

$$\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy=\{y^2=t\}=\frac{1}{2}\int\limits_{u^2}^{x^2}(t-u^2)^{\alpha-1}(x^2-t)^{\alpha-1}t^{\frac{\nu-1}{2}-\alpha}dt=$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\nu-1}{}_2F_1\left(\alpha+\frac{1-\nu}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right).$$

Using formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

we obtain

$${}_2F_1\left(\alpha+\frac{1-\nu}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right)={}_2F_1\left(\alpha,\alpha+\frac{1-\nu}{2};2\alpha;1-\frac{x^2}{u^2}\right)=$$

$$=\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right)=\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)$$

and

$$\int_u^x (y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy =$$

$$= \frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\nu-1}\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right) =$$

$$= \frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{\nu-1}x^{-2\alpha}\,{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right).$$

Finally

$$(B^{-\alpha}_{\nu,0+}\varphi)(x) = \frac{2^{2(1-2\alpha)}\sqrt{\pi}}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}x^{1-\nu-2\alpha}\int_0^x \left(x^2-u^2\right)^{2\alpha-1}u^{\nu}$$

$${}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du.$$

Applying the duplication formula

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) = 2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha)$$

we obtain

$$(B^{-\alpha}_{\nu,0+}\varphi)(x) = \frac{2^{1-2\alpha}}{\Gamma(2\alpha)}x^{1-\nu-2\alpha}\int_0^x \left(x^2-u^2\right)^{2\alpha-1}u^{\nu}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du =$$

$$= \frac{1}{\Gamma(2\alpha)}\int_0^x \left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1}\left(\frac{u}{x}\right)^{\nu}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du.$$

which gives (19).

Now we proof (20). We have

$$B^{-\alpha}_{\nu,-}\varphi = 2^{-2\alpha}K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi =$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)}K_2^{\frac{1-\nu}{2},\alpha}\int_y^{\infty}(u^2-y^2)^{\alpha-1}u\varphi(u)du =$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_x^\infty (y^2-x^2)^{\alpha-1}y^{\nu-2\alpha}dy\int\limits_y^\infty (u^2-y^2)^{\alpha-1}u\varphi(u)du=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_x^\infty u\varphi(u)du\int\limits_x^u (u^2-y^2)^{\alpha-1}(y^2-x^2)^{\alpha-1}y^{\nu-2\alpha}dy.$$

For inner integral we have

$$\int\limits_x^u (y^2-x^2)^{\alpha-1}(u^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy=$$

$$=\frac{1}{2}\frac{2^{1-2\alpha}\sqrt{\pi}\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left(u^2-x^2\right)^{2\alpha-1}x^{\nu-1}u^{-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)$$

and

$$B_{\nu,-}^{-\alpha}\varphi=\frac{2^{1-2\alpha}}{\Gamma^2(\alpha)}\frac{2^{1-2\alpha}\sqrt{\pi}\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}x^{1-\nu}\times$$

$$\times\int\limits_x^\infty\left(u^2-x^2\right)^{2\alpha-1}x^{\nu-1}u^{-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)u\varphi(u)du=$$

$$=\frac{2^{1-2\alpha}}{\Gamma(2\alpha)}\int\limits_x^\infty\left(u^2-x^2\right)^{2\alpha-1}u^{1-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)\varphi(u)du=$$

$$=\frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty\left(\frac{u^2-x^2}{2u}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right)\varphi(u)du.$$

Which coincides with formula (20).

The proof is complete. □

## 4 Resolvent for Fractional Powers of the Bessel Differential Operator

We consider resolvents for integral operators at standard setting, cf. [25]. For any linear operator $A$ on some Banach space $\Phi$ let us consider the equation

$$(A-\lambda I)\,g=f;\qquad \lambda\in C;\qquad f,g\in\Phi, \tag{21}$$

and its solution as resolvent operator due to the well-known formula from [25]

$$g=R_\lambda f=(A-\lambda I)^{-1}f=-(\lambda I-A)^{-1}f=-\frac{1}{\lambda}\left(I-\frac{1}{\lambda}A\right)^{-1}f$$

$$=-\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}A\right)^{k}f=-\frac{1}{\lambda}f-\frac{1}{\lambda}\left(\sum_{k=1}^{\infty}\frac{A^k}{\lambda^k}f\right). \tag{22}$$

Note that if integral representations are known for all powers $A^k$, then an integral representation for the resolvent is readily following from (21), of course if the series are convergent. In this way it is possible to get resolvent operators for the Riemann–Liouville fractional integrals, known as the Hille–Tamarkin formula [4] (in fact first proved by M.M. Dzhrbashyan in [26]), and also for the Erdélyi–Kober fractional integrals but we omit it here.

**Theorem 4** *For a resolvent operator of* $(B_{\nu,-}^{-\alpha})$ *the next formula is valid*

$$R_\lambda f=-\frac{1}{\lambda}f-\frac{1}{\lambda^2}\int\limits_x^{+\infty}f(y)\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}dy\int\limits_0^1 t^{\alpha-1}(1-t)^{\alpha-1}$$

$$\times\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\nu-1}{2}}E_{(\alpha,\alpha),(\alpha,\alpha)}\left(\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right)dt,$$

*with the Wright or generalized (multi-index) Mittag–Leffler function*

$$E_{(1/\rho_i),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+k/\rho_1)\dots\Gamma(\mu_m+k/\rho_m)}, \tag{23}$$

*cf. [10, 27–32].*

***Proof*** Let us consider

$$(B_{\nu,-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy.$$

Using the group property or index law, we have

$$(B_{\nu,-}^{-\alpha} f)^k = B_{\nu,-}^{-\alpha k} f.$$

Then from (22) we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda^k} B_{\nu,-}^{-\alpha k} f\right) = -\frac{1}{\lambda} f - \frac{1}{\lambda}\Bigg(\sum_{k=1}^{\infty} \frac{1}{\lambda^k \Gamma(2\alpha k)}$$

$$\times \int\limits_x^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} {}_2F_1\left(\alpha k+\frac{\nu-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right) f(y)dy\Bigg)$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\int\limits_x^{+\infty} f(y)dy \sum_{k=1}^{\infty}\left[\frac{1}{\lambda^k\Gamma(2\alpha k)}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1}\right.\right.$$

$$\left.\left.\times\, {}_2F_1\left(\alpha k+\frac{\nu-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right)\right]\right).$$

Using the integral representation for the hypergeometric function for $c-a-b>0$:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\int\limits_x^{+\infty} f(y)dy \int\limits_0^1 \sum_{k=1}^{\infty} \frac{1}{\lambda^k\Gamma^2(\alpha k)}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} t^{\alpha k-1}(1-t)^{\alpha k-1}$$

$$\times\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha k-\frac{\nu-1}{2}} dt$$

$$= \{k = p+1\} = -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_x^{+\infty} f(y) dy \int\limits_0^1 \sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1} \Gamma^2(\alpha(p+1))} \left( \frac{y^2 - x^2}{2y} \right)^{2\alpha(p+1)-1}$$

$$\times t^{\alpha(p+1)-1} (1-t)^{\alpha(p+1)-1} \left( 1 - \left( 1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha(p+1) - \frac{\nu-1}{2}} dt$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_x^{+\infty} f(y) \left( \frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_0^1 t^{\alpha-1} (1-t)^{\alpha-1} \left( 1 - \left( 1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}}$$

$$\times \sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1} \Gamma^2(\alpha(p+1))} \left( \frac{y^2 - x^2}{2y} \right)^{2\alpha p} t^{\alpha p} (1-t)^{\alpha p} \left( 1 - \left( 1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha p} dt$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int\limits_x^{+\infty} f(y) \left( \frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_0^1 t^{\alpha-1} (1-t)^{\alpha-1} \left( 1 - \left( 1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}}$$

$$\times \sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha + \alpha p)} \left[ \frac{1}{\lambda} \left( \frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right]^p dt. \tag{24}$$

The function in (24) is a special case of the Wright generalized hypergeometric function defined above as (23). So it follows

$$\sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha + \alpha p)} \left[ \frac{1}{\lambda} \left( \frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right]^p$$

$$= E_{(\alpha,\alpha),(\alpha,\alpha)} \left( \frac{1}{\lambda} \left( \frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right),$$

and we finally derive

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int\limits_x^{+\infty} f(y) \left( \frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_0^1 t^{\alpha-1} (1-t)^{\alpha-1}$$

$$\times \left( 1 - \left( 1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}} E_{(\alpha,\alpha),(\alpha,\alpha)} \left( \frac{1}{\lambda} \left( \frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right) dt.$$

□

# 5 Integral Transforms

Integral transform maps the original space into or onto the image space. Wherein usually difficult operations in the original space are converted in general into simple operations in the image space. For example, the Fourier transform converts a derivative of order $n$ into multiplication by the $n$ power of the variable with some constant. This is the reason that the Fourier transform is beneficial to use for solution to differential equations. Since the Hankel transform applied to a Bessel operator of order $n$ gives multiplication of a Hankel image of a function by the $2n$ power of the variable with some constant this transform is used instead of the Fourier transform when differential equation with the Bessel operator is solved. But the action of Hankel transform to the fractional Bessel derivatives of order $\alpha$ on semi-axes gives multiplication of the $2\alpha$ power of the variable by not a Hankel image of a function with some constant (see Theorem 7). In this section we collect some integral transforms which can be used to solve differential equations the fractional Bessel derivatives on semi-axes.

## 5.1 The Mellin Transform

Using the following formula 2.21.1.11 from [33, p. 265] of the form

$$\int_0^z x^{\alpha-1}(z-x)^{c-1}\,{}_2F_1\left(a,b;c;1-\frac{x}{z}\right)dx = z^{c+\alpha-1}\Gamma\left[\begin{matrix} c, & \alpha, & c-a-b+\alpha \\ c-a+\alpha, & & c-b+\alpha \end{matrix}\right], \tag{25}$$

$$z>0,\ \operatorname{Re} c>0,\ \operatorname{Re}(c-a-b+\alpha)>0,$$

we prove next theorems.

**Theorem 5** *Let $\alpha>0$. Mellin transforms of the $IB^{\alpha}_{\nu,-}$ and the $IB^{\alpha}_{\nu,0+}$ are*

$$\mathcal{M}IB^{\alpha}_{\nu,-}f(s) = \frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{matrix} \frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2} \\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2} \end{matrix}\right] f^*(2\alpha+s), \quad s>\nu-1, \quad IB^{\alpha}_{\nu,-}f\in P^b_a, \tag{26}$$

$$\mathcal{M}IB^{\alpha}_{\nu,0+}f(s) = \frac{1}{2^{2\alpha}}\Gamma\left[\begin{matrix} \frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha \\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2} \end{matrix}\right] f^*(2\alpha+s), \quad 2\alpha+s<2, \quad IB^{\alpha}_{\nu,0+}f\in P^b_a. \tag{27}$$

***Proof*** Let start from the definitions

$$((IB^{\alpha}_{\nu,-}f)(x))^*(s)=\int\limits_0^{\infty}x^{s-1}(IB^{\alpha}_{\nu,-}f)(x)dx=$$

$$=\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}x^{s-1}dx\int\limits_x^{+\infty}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy$$

$$=\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}f(y)(2y)^{1-2\alpha}dy\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx.$$

Using (25) let us find inner integral for $s>\nu-1$

$$\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx$$

$$=\frac{y^{4\alpha+s-2}}{2}\,\Gamma\left[\begin{matrix}2\alpha, & \frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & & \alpha+\frac{s}{2}\end{matrix}\right].$$

We obtain

$$((IB^{\alpha}_{\nu,-}f)(x))^*(s)=\frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{matrix}\frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2}\end{matrix}\right]\int\limits_0^{\infty}f(y)y^{2\alpha+s-1}dy=$$

$$=\frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{matrix}\frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2}\end{matrix}\right]f^*(2\alpha+s).$$

Similarly we have

$$((IB^{\alpha}_{\nu,0+}f)(x))^*(s)=\int\limits_0^{\infty}x^{s-1}(B^{-\alpha}_{\nu,0+}f)(x)dx=$$

$$=\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}x^{s-1}dx\int\limits_0^{x}\left(\frac{y}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy=$$

$$=\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}f(y)y^{\nu}dy\int\limits_y^{\infty}\left(\frac{1}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)x^{s-1}dx.$$

Let find inner integral

$$\int\limits_y^\infty \left(\frac{1}{x}\right)^\nu \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)x^{s-1}dx=$$

$$=2^{1-2\alpha}\int\limits_y^\infty \left(\frac{1}{x}\right)^{2\alpha-s+\nu} \left(x^2-y^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)dx=\left\{\frac{1}{x}=t\right\}=$$

$$=2^{1-2\alpha}\int\limits_0^{1/y} t^{\nu-2\alpha-s}\left(1-t^2y^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-t^2y^2\right)dt=\{ty=z\}=$$

$$=2^{1-2\alpha}y^{2\alpha+s-\nu-1}\int\limits_0^1 z^{\nu-2\alpha-s}\left(1-z^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-z^2\right)dz=\{z^2=s\}=$$

$$=\frac{1}{2^{2\alpha}}y^{2\alpha+s-\nu-1}\int\limits_0^1 s^{\frac{\nu-s-1}{2}-\alpha}(1-s)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-s\right)ds.$$

Using (25) we get for $2\alpha+s<2$

$$\int\limits_0^1 s^{\frac{\nu-s-1}{2}-\alpha}(1-s)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-s\right)ds$$
$$=\frac{1}{2^{2\alpha}}y^{2\alpha+s-\nu-1}\Gamma\left[\begin{array}{cc}2\alpha, & \frac{\nu-s+1}{2}-\alpha,\ 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]$$

and

$$((B^{-\alpha}_{\nu,0+}f)(x))^*(s)=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc}\frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]\int\limits_0^\infty f(y)y^{2\alpha+s-1}dy=$$

$$=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc}\frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]f^*(2\alpha+s).$$

This complete the proof. □

In order to obtain formulas for Mellin transform of fractional Bessel derivatives on semi-axes we should proof next statement.

**Lemma 1** *Let $B_\nu^n f \in P_a^b$ then for $n \in \mathbb{N}$*

$$\mathcal{M} B_\nu^n f(s) = 2^{2n}\Gamma\begin{bmatrix} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2} \end{bmatrix} f^*(s-2n). \tag{28}$$

***Proof*** Using formulas for Mellin transform from [34] we get

$$\mathcal{M} f'(s) = (1-s)\mathcal{M} f(s-1), \qquad \mathcal{M}\frac{1}{x} f(s) = \mathcal{M} f(s-1),$$

$$\mathcal{M}\frac{1}{x} f'(s) = (\mathcal{M} f'(t-1))(s) = (2-s)\mathcal{M} f(s-2),$$

$$\mathcal{M} f''(s) = (2-s)(1-s)\mathcal{M} f(s-2),$$

$$\mathcal{M} B_\nu f(s) = (2-s)(1-s) f^*(s-2)+\nu(2-s) f^*(s-2) = (2-s)(1-s+\nu) f^*(s-2).$$

So

$$\mathcal{M} B_\nu f(s) = (2-s)(1-s+\nu) f^*(s-2). \tag{29}$$

Applying the formula (29) $n$ times we obtain

$$\mathcal{M} B_\nu^n f(s) = (2-s)(4-s)\dots(2n-s)(1-s+\nu)(3-s+\nu)\dots(2n-1-s+\nu) f^*(s-2n).$$

Since

$$(2-s)(4-s)\dots(2n-s) = 2^n\left(1-\frac{s}{2}\right)\left(2-\frac{s}{2}\right)\dots\left(n-\frac{s}{2}\right) = 2^n\left(1-\frac{s}{2}\right)_n = \frac{2^n\Gamma\left(n+1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$$

and

$$(1-s+\nu)(3-s+\nu)\dots(2n-1-s+\nu)$$
$$= 2^n\left(\frac{1-s+\nu}{2}\right)\left(\frac{1-s+\nu}{2}+1\right)\dots\left(\frac{1-s+\nu}{2}+n-1\right) =$$
$$= 2^n\left(\frac{1-s+\nu}{2}\right)_n = \frac{2^n\Gamma\left(\frac{1-s+\nu}{2}+n\right)}{\Gamma\left(\frac{1-s+\nu}{2}\right)},$$

then

$$\mathcal{M}B_\nu^n f(s) = 2^{2n}\frac{\Gamma\left(n+1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\nu}{2}+n\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\nu}{2}\right)} f^*(s-2n) =$$

$$= 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right] f^*(s-2n).$$

It completes the proof □

**Theorem 6** *Let* $\alpha>0$, $n=[\alpha]+1$. *Mellin transforms of the* $DB_{\nu,-}^\alpha$ *and the* $DB_{\nu,0+}^\alpha$ *are*

$$\mathcal{M}DB_{\nu,-}^\alpha f(s) = 2^{2\alpha}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2} \\ \frac{s}{2}-\alpha-\frac{\nu-1}{2}, & \frac{s}{2}-\alpha\end{array}\right] f^*(s-2\alpha),\quad s-2n>\nu-1,\quad IB_{\nu,-}^{n-\alpha}f\in P_a^b, \tag{30}$$

$$\mathcal{M}DB_{\nu,0+}^\alpha f(s) = 2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s+1}{2}+\alpha \\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right] f^*(s-2\alpha),\ 2\alpha-2n+s<2,\ \ IB_{\nu,0+}^{n-\alpha}f\in P_a^b. \tag{31}$$

***Proof*** Applying (26) and (28) we obtain

$$((DB_{\nu,-}^\alpha f)(x))^*(s) = ((B_\nu^n(IB_{\nu,-}^{n-\alpha}f(x))^*(s) =$$

$$= 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]((IB_{\nu,-}^{n-\alpha}f(x))^*(s-2n) =$$

$$= 2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]\Gamma\left[\begin{array}{cc} \frac{s}{2}-n, & \frac{s}{2}-n-\frac{\nu-1}{2} \\ \frac{s}{2}-\alpha-\frac{\nu-1}{2}, & \frac{s}{2}-\alpha\end{array}\right] f^*(s-2\alpha). \tag{32}$$

Using the formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},\qquad z\notin\mathbb{Z}$$

in the numerator we get

$$\Gamma\left(1+n-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-n\right) = \frac{\pi}{\sin(\frac{s}{2}-n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{s}{2})\pi},$$

$$\Gamma\left(\frac{1-s+\nu}{2}+n\right)\Gamma\left(\frac{s-\nu+1}{2}-n\right) = \Gamma\left(1-\frac{1-s+\nu}{2}-n\right)\Gamma\left(\frac{1-s+\nu}{2}+n\right) =$$

$$= \frac{\pi}{\sin(\frac{1-s+\nu}{2}+n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{1-s+\nu}{2})\pi}.$$

So

$$\frac{(-1)^n\pi}{\Gamma\left(\frac{1-s+\nu}{2}\right)\sin(\frac{1-s+\nu}{2})\pi}=(-1)^n\Gamma\left(\frac{1+s-\nu}{2}\right),$$

$$\frac{(-1)^n\pi}{\Gamma\left(1-\frac{s}{2}\right)\sin(\frac{s}{2})\pi}=(-1)^n\Gamma\left(\frac{s}{2}\right).$$

Substituting the found expressions in (32) we obtain (30).

Similarly, using (27) and (28) we

$$((DB^{\alpha}_{\nu,0+}f)(x))^*(s)=((B^n_\nu(IB^{n-\alpha}_{\nu,0+}f(x))^*(s)=$$

$$=2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n\\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]((IB^{n-\alpha}_{\nu,0+}f(x))^*(s-2n)=$$

$$=2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n\\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s+1}{2}+\alpha\\ 1-\frac{s}{2}+n, & \frac{\nu-s+1}{2}+n\end{array}\right]f^*(s-2\alpha)=$$

$$=2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s}{2}+\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]f^*(s-2\alpha).$$

□

## 5.2 *The Hankel Transform*

**Theorem 7** *Let* $B^{-\alpha}_{\nu,0+}\varphi$, $B^{-\alpha}_{\nu,-}\varphi\in L^\nu_1(\mathbb{R}_+)$, *then*

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)\left[\cos(\alpha\pi)j_{\frac{\nu-1}{2}}(\xi t)-\sin(\alpha\pi)y_{\frac{\nu-1}{2}}(\xi t)\right]t^\nu dt,$$

$$4\alpha-2<\nu<4-2\alpha, \tag{33}$$

$$F_\nu[(B^{-\alpha}_{\nu,-}\varphi)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},\alpha}(t\xi)\varphi(t)t^\nu dt, \tag{34}$$

*where*

$$j^1_{\frac{\nu-1}{2},\alpha}(t\xi)=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{(t\xi)^{\frac{\nu-1}{2}}}J^1_{\frac{\nu-1}{2},\alpha}(t\xi),$$

$$J^1_{\frac{\nu-1}{2},\alpha}(t\xi)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\nu+1}{2}+\alpha+n\right)}\left(\frac{t\xi}{2}\right)^{2n+\frac{\nu-1}{2}+2\alpha}.$$

***Proof*** Using factorization formula (19) and denoting $g(x)=I_2^{0,\alpha}\varphi(x)$ we obtain

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,(B^{-\alpha}_{\nu,0+}\varphi)(x)x^\nu\,dx=$$

$$=\frac{1}{2^{2\alpha}}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\ I_2^{\frac{\nu-1}{2},\alpha}I_2^{0,\alpha}\varphi(x)x^{2\alpha+\nu}\,dx=\frac{1}{2^{2\alpha}}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\ I_2^{\frac{\nu-1}{2},\alpha}g(x)x^{2\alpha+\nu}\,dx=$$

$$=\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx\int\limits_0^x(x^2-u^2)^{\alpha-1}u^\nu g(u)du=$$

$$=\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty u^\nu g(u)du\int\limits_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx.$$

Let consider inner integral

$$\int\limits_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int\limits_u^\infty(x^2-u^2)^{\alpha-1}J_{\frac{\nu-1}{2}}(x\xi)\,x^{1-\frac{\nu-1}{2}}\,dx.$$

Using the formula 2.12.4.17 from [20] of the form

$$\int\limits_a^\infty x^{1-\rho}(x^2-a^2)^{\beta-1}J_\rho(cx)dx=2^{\beta-1}a^{\beta-\rho}c^{-\beta}\Gamma(\beta)J_{\rho-\beta}(ac),$$

$$a,c,\beta>0;\qquad(2\beta-\rho)<3/2$$

we obtain for $4\alpha - \nu < 2$

$$\int\limits_u^\infty (x^2-u^2)^{\alpha-1} J_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx = 2^{\alpha-1} u^{\alpha-\frac{\nu-1}{2}} \xi^{-\alpha} \Gamma(\alpha) J_{\frac{\nu-1}{2}-\alpha}(u\xi)$$

and

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi) = \frac{2^{\frac{\nu-1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty u^{\alpha+\frac{\nu+1}{2}} J_{\frac{\nu-1}{2}-\alpha}(u\xi) g(u) du =$$

$$= \frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du \int\limits_0^u (u^2-y^2)^{\alpha-1} y\varphi(y) dy =$$

$$= \frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty y\varphi(y) dy \int\limits_y^\infty (u^2-y^2)^{\alpha-1} u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du.$$

Let calculate inner integral using the formula 2.12.4.17 from [20] of the form

$$\int\limits_a^\infty x^{1+\rho}(x^2-a^2)^{\beta-1} J_\rho(cx) dx = 2^{\beta-1} a^{\beta+\rho} c^{-\beta} \Gamma(\beta)[\cos(\beta\pi) J_{\rho+\beta}(ac) - \sin(\beta\pi) Y_{\rho+\beta}(ac)],$$

$$a, c, \beta > 0; \qquad (2\beta+\rho) < 3/2$$

we obtain

$$\int\limits_y^\infty (u^2-y^2)^{\alpha-1} u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du = 2^{\alpha-1} y^{\frac{\nu-1}{2}} \xi^{-\alpha} \Gamma(\alpha)[\cos(\alpha\pi) J_{\frac{\nu-1}{2}}(\xi y) - \sin(\alpha\pi) Y_{\frac{\nu-1}{2}}(\xi y)]$$

for $2\alpha + \nu < 4$ and

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi) =$$

$$= \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+2\alpha}} \int\limits_0^\infty y^{\frac{\nu+1}{2}} \varphi(y)[\cos(\alpha\pi) J_{\frac{\nu-1}{2}}(\xi y) - \sin(\alpha\pi) Y_{\frac{\nu-1}{2}}(\xi y)] dy =$$

$$= \xi^{-2\alpha} \int\limits_0^\infty \varphi(t) \left[\cos(\alpha\pi) j_{\frac{\nu-1}{2}}(\xi t) - \sin(\alpha\pi) y_{\frac{\nu-1}{2}}(\xi t)\right] t^\nu dt.$$

So (33) is proved.

Now let consider (34). Let $g(x) = K_2^{0,\alpha} x^{2\alpha}\varphi(x)$. Using factorization (20) we get

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi) = 2^{-2\alpha}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\, x^\nu\, K_2^{\frac{1-\nu}{2},\alpha} K_2^{0,\alpha} x^{2\alpha}\varphi(x)dx =$$

$$= 2^{-2\alpha}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\, x^\nu\, K_2^{\frac{1-\nu}{2},\alpha} g(x)dx = \frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\, xdx \int\limits_x^\infty (u^2-x^2)^{\alpha-1}u^{\nu-2\alpha}g(u)du =$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty g(u)u^{\nu-2\alpha}du \int\limits_0^u j_{\frac{\nu-1}{2}}(x\xi)(u^2-x^2)^{\alpha-1}xdx.$$

Using the formula 2.12.4.7 from [20] of the form

$$\int\limits_0^a x^{1-\rho}(a^2-x^2)^{\beta-1}J_\rho(cx)dx = \frac{2^{1-\rho}a^{\beta-\rho}}{c^\beta\Gamma(\rho)}s_{\rho+\beta-1,\beta-\rho}(ac),$$

$$a > 0; \qquad \operatorname{Re}\beta > 0$$

we obtain for inner integral

$$\int\limits_0^u (u^2-x^2)^{\alpha-1} j_{\frac{\nu-1}{2}}(x\xi)\, x\, dx = \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int\limits_0^u (u^2-x^2)^{\alpha-1} J_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx =$$

$$= \frac{\Gamma(\alpha)}{2\Gamma(\alpha+1)}u^{2\alpha}\, {}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right).$$

So

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi) = \frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)g(u)u^\nu du =$$

$$= \frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)u^\nu K_2^{0,\alpha}u^{2\alpha}\varphi(u)du =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)} \int\limits_0^\infty {}_1F_2\left(1; \alpha+1, \frac{\nu+1}{2}; -\frac{u^2\xi^2}{4}\right) u^\nu du \int\limits_u^\infty (t^2-u^2)^{\alpha-1} t\varphi(t)dt =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)} \int\limits_0^\infty t\varphi(t)dt \int\limits_0^t (t^2-u^2)^{\alpha-1} {}_1F_2\left(1; \alpha+1, \frac{\nu+1}{2}; -\frac{u^2\xi^2}{4}\right) u^\nu du.$$

Using Wolfram Mathematica we obtain

$$\int\limits_0^t (t^2-u^2)^{\alpha-1} {}_1F_2\left(1; \alpha+1, \frac{\nu+1}{2}; -\frac{u^2\xi^2}{4}\right) u^\nu du =$$

$$= \frac{\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\alpha+\frac{\nu+1}{2}\right)} t^{2\alpha+\nu-1} {}_1F_2\left(1; \alpha+1, \alpha+\frac{\nu+1}{2}; -\frac{t^2\xi^2}{4}\right)$$

and

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi) =$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)} \int\limits_0^\infty \varphi(t)\, t^{2\alpha+\nu} {}_1F_2\left(1; \alpha+1, \alpha+\frac{\nu+1}{2}; -\frac{t^2\xi^2}{4}\right) dt.$$

Since

$${}_1F_2\left(1; \alpha+1, \alpha+\frac{\nu+1}{2}; -\frac{t^2\xi^2}{4}\right) = \Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)$$
$$\sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\alpha+\frac{\nu+1}{2}+n\right)} \left(\frac{t\xi}{2}\right)^{2n}$$

and the Wright function through which the Hankel transform of $B_{\nu,-}^{-\alpha}\varphi$ is expressed in [22] is given by

$$J^1_{\frac{\nu-1}{2},\alpha}(t\xi) = \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\nu+1}{2}+\alpha+n\right)} \left(\frac{t\xi}{2}\right)^{2n+\frac{\nu-1}{2}+2\alpha}$$

we obtain

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi)=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)\,t^{\frac{\nu+1}{2}}\,J^1_{\frac{\nu-1}{2},\alpha}(t\xi)dt=\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},\alpha}(t\xi)\varphi(t)t^\nu dt.$$

The (34) is proved.

□

Since $F_\nu[(B_\nu^n\varphi)](\xi)=(-1)^n\xi^{2n}F_\nu[\varphi](\xi)$ we obtain for $B^\alpha_{\nu,0+}\varphi$, $B^\alpha_{\nu,-}\varphi\in L_1^\nu(\mathbb{R}_+)$

$$F_\nu[(B^\alpha_{\nu,0+}\varphi)(x)](\xi)=F_\nu[(B_\nu^nB_{\nu,0+}^{-(n-\alpha)}\varphi)(x)](\xi)=(-1)^n\xi^{2n}F_\nu[B_{\nu,0+}^{-(n-\alpha)}\varphi(x)](\xi)=$$

$$=(-1)^n\xi^{2\alpha}\int\limits_0^\infty\varphi(t)\left[\cos((n-\alpha)\pi)j_{\frac{\nu-1}{2}}(\xi t)-\sin((n-\alpha)\pi)y_{\frac{\nu-1}{2}}(\xi t)\right]t^\nu dt,$$

$$n=[\alpha]+1,\qquad 4(n-\alpha)-2<\nu<4-2(n-\alpha)$$

and

$$F_\nu[(B^\alpha_{\nu,-}\varphi)(x)](\xi)=F_\nu[(B_\nu^nB_{\nu,-}^{-(n-\alpha)}\varphi)(x)](\xi)=(-1)^n\xi^{2n}F_\nu[B_{\nu,-}^{-(n-\alpha)}\varphi(x)](\xi)=$$

$$=(-1)^n\xi^{2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},n-\alpha}(t\xi)\varphi(t)t^\nu dt,\qquad n=[\alpha]+1.$$

## 5.3 The Meijer Transform

**Theorem 8** *The Meijer transforms of $B^{-\alpha}_{\nu,0+}$, $B^{-\alpha}_{\nu,-}$ for proper functions are*

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\xi^{-2\alpha}\mathcal{K}_\nu\varphi(\xi),\tag{35}$$

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,-}\varphi)(x)](\xi)=\tag{36}$$

$$=\frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}\int\limits_0^\infty\varphi(t)t^{2\alpha+\nu}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)dt-$$

$$-\frac{\pi 2^{\nu-2\alpha-2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)}\xi^{1-\nu}\int\limits_0^\infty\varphi(t)t^{2\alpha+1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right)dt.$$

***Proof*** We start with (35). Let $g(x) = I_2^{0,\alpha}\varphi(x)$. Then using the factorization (19) we obtain

$$\mathcal{K}_\nu[(B_{\nu,0+}^{-\alpha}\varphi)(x)](\xi) = \int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,(B_{\nu,0+}^{-\alpha}\varphi)(x)x^\nu\,dx =$$

$$= \frac{1}{2^{2\alpha}}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\; I_2^{\frac{\nu-1}{2},\alpha} I_2^{0,\alpha}\varphi(x)x^{2\alpha+\nu}\,dx = \frac{1}{2^{2\alpha}}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\; I_2^{\frac{\nu-1}{2},\alpha} g(x)x^{2\alpha+\nu}\,dx =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx\int\limits_0^x (x^2-u^2)^{\alpha-1}u^\nu g(u)du =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty u^\nu g(u)du\int\limits_u^\infty (x^2-u^2)^{\alpha-1}k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx.$$

Let consider the inner integral. Using the formula 2.16.3.7 from [20] of the form

$$\int\limits_a^\infty x^{1\pm\rho}(x^2-a^2)^{\beta-1}K_\rho(cx)dx = 2^{\beta-1}a^{\beta\pm\rho}c^{-\beta}\Gamma(\beta)K_{\rho\pm\beta}(ac), \qquad a, c, \beta > 0 \tag{37}$$

we get

$$\int\limits_u^\infty (x^2-u^2)^{\alpha-1}k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx = \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int\limits_u^\infty (x^2-u^2)^{\alpha-1}K_{\frac{\nu-1}{2}}(x\xi)\,x^{1-\frac{\nu-1}{2}}\,dx =$$

$$= \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\cdot 2^{\alpha-1}u^{\alpha-\frac{\nu-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\nu-1}{2}-\alpha}(u\xi)$$

and

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\frac{2^{\frac{\nu-1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty u^{\alpha+\frac{\nu+1}{2}}K_{\frac{\nu-1}{2}-\alpha}(u\xi)g(u)du=$$

$$=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du\int\limits_0^u(u^2-t^2)^{\alpha-1}t\varphi(t)dt=$$

$$=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty t\varphi(t)dt\int\limits_t^\infty(u^2-t^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du.$$

Using again (37) we can write

$$\int\limits_t^\infty(u^2-t^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du=2^{\alpha-1}t^{\frac{\nu-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\nu-1}{2}}(t\xi)$$

and

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\cdot 2^{\alpha-1}\xi^{-\alpha}\Gamma(\alpha)\int\limits_0^\infty\varphi(t)K_{\frac{\nu-1}{2}}(t\xi)t^{\frac{\nu+1}{2}}$$

$$dt=\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)k_{\frac{\nu-1}{2}}(t\xi)t^\nu dt=$$

$$=\xi^{-2\alpha}\mathcal{K}_\nu\varphi.$$

Now let prove (36). Let $g(x)=K_2^{0,\alpha}x^{2\alpha}\varphi(x)$. Then using the factorization (20) we obtain

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,-}\varphi)(x)](\xi)=\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,(B^{-\alpha}_{\nu,0-}\varphi)(x)x^\nu\,dx=$$

$$=2^{-2\alpha}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,x^\nu\,K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi(x)\,dx=$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)} \int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\, x dx \int\limits_x^\infty (u^2 - x^2)^{\alpha-1} u^{\nu-2\alpha} g(u) du =$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)} \int\limits_0^\infty u^{\nu-2\alpha} g(u) du \int\limits_0^u (u^2 - x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx.$$

Let consider the inner integral

$$\int\limits_0^u (u^2-x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx = \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}} \int\limits_0^u (u^2-x^2)^{\alpha-1} K_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx.$$

Using the formula 2.16.3.3 from [20] of the form

$$\int\limits_0^a x^{1-\rho}(a^2 - x^2)^{\beta-1} K_\rho(cx) dx = \frac{\pi 2^{\beta-2} a^{\beta-\rho}}{c^\beta \sin\rho\pi} \Gamma(\beta) I_{\beta-\rho}(ac) +$$

$$+ \frac{a^{2\beta} c^\nu}{2^{\rho+2}\beta} \Gamma(-\rho)\, {}_1F_2\left(1; \rho+1, \beta; \frac{a^2c^2}{4}\right), \qquad a, \beta > 0, \rho < 1$$

we obtain for $\nu < 3$

$$\int\limits_0^u (u^2 - x^2)^{\alpha-1} K_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx = \frac{\xi^{\frac{\nu-1}{2}}\Gamma\left(\frac{1-\nu}{2}\right)}{2^{\frac{\nu+3}{2}}\alpha} u^{2\alpha}\, {}_1F_2\left(1; \alpha+1, \frac{\nu}{2} + \frac{1}{2}; \frac{u^2\xi^2}{4}\right) -$$

$$- \frac{\pi 2^{\alpha-2}\Gamma(\alpha)}{\xi^\alpha \cos\left(\frac{\pi\nu}{2}\right)} u^{\alpha+\frac{1-\nu}{2}} I_{\alpha+\frac{1-\nu}{2}}(u\xi)$$

and

$$\int\limits_0^u (u^2 - x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx = \frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha} u^{2\alpha}\, {}_1F_2\left(1; \alpha+1, \frac{\nu}{2} + \frac{1}{2}; \frac{u^2\xi^2}{4}\right) -$$

$$- \frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}} \cos\left(\frac{\pi\nu}{2}\right)} u^{\alpha+\frac{1-\nu}{2}} I_{\alpha+\frac{1-\nu}{2}}(u\xi)$$

So

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,-}\varphi)(x)](\xi)=$$

$$=\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)g(u)du-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)g(u)du\right]=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}t\varphi(t)dt-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}t\varphi(t)dt\right]=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty t\varphi(t)dt\int\limits_0^t(t^2-u^2)^{\alpha-1}u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty t\varphi(t)dt\int\limits_0^t(t^2-u^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du\right].$$

Using Wolfram Mathematica we obtain

$$\int\limits_0^t(t^2-u^2)^{\alpha-1}u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du=\frac{\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}t^{2\alpha+\nu-1}$$

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)$$

and

$$\int\limits_0^t(t^2-u^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du=\frac{2^{\frac{\nu-3}{2}-\alpha}\Gamma(\alpha)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)}t^{2\alpha}\xi^{\alpha-\frac{\nu}{2}+\frac{1}{2}}$$

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right),\qquad 2\alpha<\nu+3.$$

Finally

$$\mathcal{K}_\nu[(B_{\nu,-}^{-\alpha}\varphi)(x)](\xi)=$$

$$=\frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}\int\limits_0^\infty \varphi(t)t^{2\alpha+\nu}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)dt-$$

$$-\frac{\pi 2^{\nu-2\alpha-2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)}\xi^{1-\nu}\int\limits_0^\infty \varphi(t)t^{2\alpha+1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right)dt.$$

□

Since $\mathcal{K}_\nu[(B_\nu^n\varphi)](\xi)=\xi^{2n}\mathcal{K}_\nu[\varphi](\xi)$ we obtain for proper functions

$$\mathcal{K}_\nu[(B_{\nu,0+}^{\alpha}\varphi)(x)](\xi)=\xi^{2\alpha}\mathcal{K}_\nu\varphi(\xi).$$

## 5.4 Generalized Whittaker Transform

**Theorem 9** *The generalized Whittaker transform of $B_{\nu,0+}^{-\alpha}$ for proper functions is*

$$\left(W_{\rho,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}}B_{\nu,0+}^{-\alpha}f\right)(x)=C(\nu,\alpha,\rho)x^{-2\alpha}\left(W_{\rho+\alpha,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}}f\right)(x),$$

*where*

$$C(\nu,\alpha,\rho)=\frac{\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}.$$

***Proof*** We have

$$\left(W_{\rho,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}}B_{\nu,0+}^{-\alpha}f\right)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_0^\infty (xt)^{\frac{\nu-1}{2}}e^{\frac{x^2t^2}{2}}W_{\rho,\frac{\nu-1}{4}}(x^2t^2)dt\times$$

$$\times\int\limits_0^t\left(\frac{y}{t}\right)^\nu\left(\frac{t^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)f(y)dy=$$

$$= \frac{x^{\frac{\nu-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^\nu dy\int\limits_y^\infty t^{\frac{\nu-1}{2}-\nu-2\alpha+1}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t^2)$$

$${}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)dt.$$

Using formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

we obtain

$${}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)=\left(\frac{y}{t}\right)^{1-\nu-2\alpha}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)$$

and

$$\left(W^{\frac{\nu-1}{2}}_{\rho,\frac{\nu-1}{4}}B^{-\alpha}_{\nu,0+}f\right)(x)=$$

$$= \frac{x^{\frac{\nu-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^{1-2\alpha}dy\int\limits_y^\infty t^{\frac{\nu-1}{2}}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t^2)$$

$${}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)dt.$$

Let consider an inner integral. We have

$$\int\limits_y^\infty t^{\frac{\nu-1}{2}}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t^2)\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)dt=\{t^2\to t,\ y^2=p\}=$$

$$=\frac{1}{2}\int\limits_p^\infty t^{\frac{\nu-1}{4}-\frac{1}{2}}e^{\frac{x^2t}{2}}(t-p)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t)\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t}{p}\right)dt.$$

Using formula 2.21.8.2 from [33] of the form

$$\int_p^\infty t^{\frac{a+b-c-1}{2}}(t-p)^{c-1}e^{\frac{\sigma t}{2}}W_{\rho,\frac{a+b-c}{2}}(\sigma t)\,{}_2F_1\left(a,b;c;1-\frac{t}{p}\right)dt=$$

$$=\frac{p^{\frac{a+b-1}{2}}}{\sigma^{\frac{c}{2}}}\frac{\Gamma(c)\,\Gamma\left(\frac{a-b-c+1}{2}-\rho\right)\Gamma\left(\frac{b-a-c+1}{2}-\rho\right)}{\Gamma\left(\frac{a+b-c+1}{2}-\rho\right)\Gamma\left(\frac{c-a-b+1}{2}-\rho\right)}e^{\frac{\sigma p}{2}}W_{\rho+\frac{c}{2},\frac{a-b}{2}}(\sigma p),$$

$$p,\,\mathrm{Re}\,c>0,\,\mathrm{Re}\,(c+2\rho)<1-|\mathrm{Re}(a-b)|;\;|\arg\sigma|<\frac{3\pi}{2}$$

we obtain

$$a=\alpha+\frac{\nu-1}{2},\;b=\alpha,\;c=2\alpha,\;\sigma=x^2,\;2\alpha+2\rho<1-\left|\frac{\nu-1}{2}\right|$$

and

$$\frac{1}{2}\int_p^\infty t^{\frac{\nu-1}{4}-\frac{1}{2}}e^{\frac{x^2t}{2}}(t-p)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t)\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t}{p}\right)dt=$$

$$=\frac{1}{2}\frac{p^{\alpha+\frac{\nu-3}{4}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}e^{\frac{x^2p}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2p)=$$

$$=\frac{1}{2}\frac{y^{2\alpha+\frac{\nu-3}{2}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)=$$

$$=A(\nu,\alpha,\rho)x^{-2\alpha}y^{2\alpha+\frac{\nu-3}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2),$$

where

$$A(\nu,\alpha,\rho)=\frac{1}{2}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}$$

Then

$$\left(W^{\frac{\nu-1}{2}}_{\rho,\frac{\nu-1}{4}} B^{-\alpha}_{\nu,0+} f\right)(x) =$$

$$= A(\nu,\alpha,\rho)\frac{x^{\frac{\nu-1}{2}-2\alpha}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^{\frac{\nu-1}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)dy =$$

$$= C(\nu,\alpha,\rho)x^{-2\alpha}\int\limits_0^\infty f(y)(xy)^{\frac{\nu-1}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)dy =$$

$$= C(\nu,\alpha,\rho)x^{-2\alpha}\left(W^{\frac{\nu-1}{2}}_{\rho+\alpha,\frac{\nu-1}{4}} f\right)(x),$$

where

$$C(\nu,\alpha,\rho) = \frac{\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}.$$

□

It is worth mentioning that the mapping property of the transmutation operators allowing one to obtain the images of the powers of the independent variable without knowledge of the transmutation operator itself [35] can be used for further study of fractional powers of Bessel operator.

## References

1. R.W. Carroll, *Singular and Degenerate Cauchy Problems* (Academic, New York, 1976)
2. I.A. Kipriyanov, *Singular Elliptic Boundary Value Problems* (Nauka, Moscow, 1997, in Russian)
3. M.I. Matiychuk, *Parabolic Singular Boundary-Value Problems* (National Academy of Sciences of Ukraine, Kiev, 1999, in Ukrainian)
4. S.G. Samko, A.A. Kilbas, O.L. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach Science Publishers, Amsterdam, 1993)
5. A.C. McBride, Fractional powers of a class of ordinary differential operators. Proc. Lond. Math. Soc. **3**(45), 519–546 (1982)
6. I.G. Sprinkhuizen-Kuyper, A fractional integral operator corresponding to negative powers of a certain second-order differential operator. J. Math. Anal. Appl. **75**, 674–702 (1979)
7. I. Dimovski, Operational calculus for a class of differential operators. C. R. Acad. Bulg. Sci. **19**(12), 1111–1114 (1966)
8. I. Dimovski, On an operational calculus for a differential operator. C. R. Acad. Bulg. Sci. **21**(6), 513–516 (1968)

9. I.H. Dimovski, V.S. Kiryakova, Transmutations, convolutions and fractional powers of Bessel-type operators via Meijer's $G$-function, in *"Complex Analysis and Applications '83" (Proceedings of the International Conference Varna 1983)*, Sofia (1985), pp. 45–66
10. V. Kiryakova, *Generalized Fractional Calculus and Applications*. Pitman Research Notes in Mathematics, vol. 301 (Longman Scientific & Technical, Harlow, Co-publ. Wiley, New York, 1994)
11. I. Dimovski, *Convolutional Calculus* (Kluwer, Dordrecht, 1990)
12. S.M. Sitnik, On explicit definitions of fractional powers of the Bessel differential operator and its applications to differential equations. Reports of the Adyghe (Circassian) International Academy of Sciences, vol. 12(2), pp. 69–75, 2010 (in Russian)
13. S.M. Sitnik, Fractional integrodifferentiations for differential Bessel operator, in *Proceedings of the International Symposium "The Equations of Mixed Type and Related Problems of the Analysis and Informatics", Nalchik* (2004, in Russian), pp. 163–167
14. S.M. Sitnik, A short survey of recent results on Buschman–Erdélyi transmutations. J. Inequal. Spec. Funct. **8**(1), 140–157 (2017). Special issue To honor Prof. Ivan Dimovski's contributions
15. S.M. Sitnik, Transmutations and applications: a survey (2010), 141 pp. arXiv: 1012.3741v1
16. V.V. Katrakhov, S.M. Sitnik, The transmutation method and boundary-value problems for singular elliptic equations. Contemp. Math. Fundam. Dir. **4**(2), 211–426 (2018, in Russian)
17. S.M. Sitnik, E.L. Shishkina, *The Transmutation Operators Method for Differential Equations with Bessel Operators* (Fizmatlit, Moscow, 2019, in Russian)
18. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922)
19. M. Abramowitz, I.A. Stegun , *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Applied Mathematics Series (Dover, New York, 1983)
20. A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol. 2, Special Functions* (Gordon & Breach Science Publishers, New York, 1990)
21. H.J.Glaeske, A.P. Prudnikov, K.A. Skornik, *Operational Calculus and Related Topics* (Chapman and Hall/CRC, New York, 2006)
22. E.L. Shishkina, S.M. Sitnik, On fractional powers of Bessel operators. J. Inequal. Spec. Funct. **8**(1), 49–67 (2017)
23. S.M. Sitnik, E.L. Shishkina, On fractional powers of the Bessel operator on semiaxis. Sib. Elektron. Mat. Izv. **15**, 1–10 (2018)
24. A. Erdelyi, Transformation of hypergeometric integrals by means of fractional integration by parts. Q. J. Math. Oxf. **10**, 176–189 (1939)
25. A.N. Kolmogorov, S.V. Fomin, *Introductory Real Analysis* (Prentice-Hall, Englewood Cliffs, 1970)
26. M.M. Dzhrbashyan, *Integral Transforms and Representations of Functions in Complex Plane* (Nauka, Moscow, 1966, in Russian)
27. V.S. Kiryakova, Multiple (multi-index) Mittag–Leffler functions and relations to generalized fractional calculus. J. Comput. Appl. Math. **118**(1–2), 241–259 (2000)
28. V.S. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. Comput. Math. Appl. **59**(3), 1128–1141 (2010)
29. V.S. Kiryakova, The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus. Comput. Math. Appl. **59**(5), 1885–1895 (2010)
30. R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag–Leffler Functions, Related Topics and Applications* (Springer, Berlin, 2014)
31. J. Paneva-Konovska, *From Bessel to Multi-index Mittag–Leffler Functions* (World Scientific, London, 2016)
32. N. Virchenko, V. Gaidei, *Classic and Generalized Multi-parameters Function* (National Academy of Sciences of Ukraine, Kiev, 2008, in Ukranian)
33. A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol. 3, More Special Functions* (Gordon & Breach Science Publishers, New York, 1990)
34. V.A. Ditkin, A.P. Prudnikov, *Integral Transforms and Operational Calculus* (Pergamon, London, 1965)
35. V.V. Kravchenko, *Pseudoanalytic Function Theory* (Birkhäuser Verlag, Basel, 2009)