On Some Generalizations of the Properties of the Multidimensional Generalized Erdélyi–Kober Operators and Their Applications

Sh. T. Karimov and S. M. Sitnik

Abstract In this paper we investigate the composition of a multidimensional generalized Erdélyi–Kober operator with differential operators of high order. In particular, with powers of the differential Bessel operator. Applications of proved properties to solving the Cauchy problem for a multidimensional polycaloric equation with a Bessel operator are shown. An explicit formula for solving the formulated problem is constructed. In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM).

Keywords Fractional integrals and derivatives · Multidimensional Erdélyi–Kober operators · Bessel differential operator · Multidimensional polycaloric equation · Cauchy problem

MSC 26A33, 35K50, 35K67

1 Introduction

Various modifications and generalizations of the classical Riemann–Liouville operators of fractional integration and differentiation are widely used in theory and applications. Such modifications include, particularly, the Erdélyi–Kober operators [1, 2]. These operators turned to be very useful in application to integral and differential equations as well as in other issues of science and technology [3, 4]. Their various modifications, generalizations, and applications can be found in works of Erdélyi [5, 6], Sneddon [7, 8], and Kyriakova [9].

Sh. T. Karimov
Fergana State University, Fergana, Uzbekistan

S. M. Sitnik (✉)
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

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One-dimensional generalized Erdélyi–Kober operator with the Bessel function in the kernel and its applications were considered by Lowndes [10, 11].

Modifications of fractional integration of Erdélyi–Kober type for two and many variables have been studied in [12–18] and others. A survey of some studies on this topic can be found in [3, 4, 9].

In [18] a multidimensional generalized Erdélyi–Kober operator was introduced in the form

$$J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{\lambda_1,\lambda_2,\dots,\lambda_n} \begin{pmatrix} \alpha_1, \ \alpha_2, \ \dots, \ \alpha_n \\ \eta_1, \ \eta_2, \ \dots, \ \eta_n \end{pmatrix} f(x)$$

$$= J_{\lambda_1}^{x_1}(\eta_1, \alpha_1) J_{\lambda_2}^{x_2}(\eta_2, \alpha_2) \cdots J_{\lambda_n}^{x_n}(\eta_n, \alpha_n) f(x)$$

$$= \left[\prod_{k=1}^{n} \frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)}\right] \int_0^{x_1}\int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2\eta_k+1} \left(x_k^2 - t_k^2\right)^{\alpha_k-1} \right.$$

$$\left. \times \ \bar{J}_{\alpha_k-1}\left(\lambda\sqrt{x_k^2 - t_k^2}\right)\right] f(t_1, \ t_2, \ \dots, \ t_n) dt_1 dt_2 \ \dots \ dt_n, \tag{1.1}$$

where $\lambda, \alpha, \eta \in R^n$, $\alpha_k > 0$, $\eta_k \geq -1/2$, $k = \overline{1, n}$; $\Gamma(\alpha)$ is the Euler gamma function; $\bar{J}_\nu(z)$ is Bessel-Clifford function expressed through the Bessel function $J_\nu(z)$, using the formula $\bar{J}_\nu(z) = \Gamma(\nu + 1)(z/2)^{-\nu} J_\nu(z)$ and $J_{\lambda_k}^{x_k}(\eta_k, \alpha_k)$ is a particular Erdélyi–Kober integral of α_k-order of kth variable

$$J_{\lambda_k}^{x_k}(\eta_k, \alpha_k) f(x) = \frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)} \int_0^{x_k} (x_k^2 - t^2)^{\alpha_k-1} \bar{J}_{\alpha_k-1}\left(\lambda_k\sqrt{x_k^2 - t^2}\right)$$

$$\times t^{2\eta_k+1} f(x_1, x_2, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n) dt.$$

In this paper we also study the basic properties of the operator (1.1) and show that the inverse operator has the form

$$J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{i\lambda} \begin{pmatrix} -\alpha \\ \eta + \alpha \end{pmatrix} f(x)$$

$$= 2^{n-|m|} \left[\prod_{k=1}^{n} \frac{x_k^{-2\eta_k}}{\Gamma(m_k - \alpha_k)} \left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{m_k}\right] \int_0^{x_1}\int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2(\eta_k+\alpha_k)+1} \right.$$

$$\left. \times \ \left(x_k^2 - t_k^2\right)^{m_k-1-\alpha_k} \bar{I}_{m_k-1-\alpha_k}\left(\lambda\sqrt{x_k^2 - t_k^2}\right)\right] f(t_1, \ t_2, \ \dots, \ t_n) dt_1 dt_2 \ \dots \ dt_n, \tag{1.2}$$

where $\alpha_k > 0$, $m_k = [\alpha_k] + 1$, $\eta_k \geq -1/2$, $k = \overline{1, n}$, $\bar{I}_\nu(z) = \Gamma(\nu + 1)(z/2)^{-\nu} I_\nu(z)$, $I_\nu(z)$ is the Bessel function of the imaginary argument. $m = (m_1, m_2, \dots, m_n)$ is a multi-index, and $|m| = m_1 + m_2 + \dots + m_n$ is its length.

Taking into account $\bar{J}_\nu(0) = 1$, in the limit for $\lambda_k \to 0$, $k = \overline{1, n}$, we obtain

$$J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{0,0,\dots,0} \begin{pmatrix} \alpha_1, \alpha_2, \dots, \alpha_n \\ \eta_1, \eta_2, \dots, \eta_n \end{pmatrix} f(x)$$

$$= \prod_{k=1}^{n} \left[\frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)} \right] \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2\eta_k+1} \left(x_k^2 - t_k^2 \right)^{\alpha_k - 1} \right] f(t) dt_1 \dots dt_n, \tag{1.3}$$

This operator is a multidimensional analog of the ordinary (not generalized) Erdélyi–Kober operator. In this case, the inverse operator (1.2) takes the following form:

$$J_0^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = 2^{n-|m|} \left[\prod_{k=1}^{n} \frac{x_k^{-2\eta_k}}{\Gamma(m_k - \alpha_k)} \left(\frac{1}{x_k} \frac{\partial}{\partial x_k} \right)^{m_k} \right]$$

$$\int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2(\eta_k+\alpha_k)+1} \left(x_k^2 - t_k^2 \right)^{m_k - 1 - \alpha_k} \right] f(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n. \tag{1.4}$$

In addition, in [18] the following theorem is proved:

Theorem 1.1 *Let* $\alpha_k > 0, \eta_k \geq -1/2$; $f(x) \in C^2(\Omega^n)$; $\lim\limits_{x_k \to 0} x_k^{2\eta_k+1} f_{x_k}(x) = 0$, $k = \overline{1, n}$. *Then the transmutation formula holds:*

$$(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2) J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} B_{\eta_k}^{x_k} f(x), \ k = \overline{1, n},$$

in particular, if $\lambda_k = 0$, $k = \overline{1, n}$, *then*

$$B_{\eta_k+\alpha_k}^{x_k} J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} B_{\eta_k}^{x_k} f(x), \ k = \overline{1, n},$$

where $\Omega^n = \prod\limits_{k=1}^{n} (0, b_k) = (0, b_1) \times (0, b_2) \times \dots \times (0, b_n)$ *be the Cartesian product,* $b_k > 0$, $k = \overline{1, n}$.

This Theorem implies

Corollary 1.1 *Suppose that the conditions of Theorem 1.1 are satisfied. Then*

$$\sum_{k=1}^{n}\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\sum_{k=1}^{n}\left[B_{\eta_k}^{x_k}\right]f(x),$$

in particular, if $\eta_k=-1/2,\ k=\overline{1,n}$, *then*

$$\sum_{k=1}^{n}\left[B_{\alpha_k-1/2}^{x_k}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta f(x),$$

where $\Delta f(x)\equiv\sum\limits_{k=1}^{n}\left[\partial^2 f(x)/\partial x_k^2\right]$ *is the multidimensional Laplace operator.*

Similarly, we can prove the validity of the following theorem.

Theorem 1.2 *Let* $\alpha_k>0,\ \eta_k\geq-1/2,\ k=\overline{1,n},\ f(x)\in C^{2n}(\Omega^n)$, $x_k^{2\eta_k+1}B_{\eta_k}^{x_k}f(x)$ *are integrable in a neighborhood of* $x_k=0$ *and* $\lim\limits_{x_k\to 0}x_k^{2\eta_k+1}f_{x_k}(x)=0,\ k=\overline{1,\ n}$. *Then*

$$\prod_{k=1}^{n}(B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2)J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\prod_{k=1}^{n}B_{\eta_k}^{x_k}f(x),$$

in particular, if $\eta_k=-1/2,\ k=\overline{1,n}$, *then*

$$\prod_{k=1}^{n}\left[B_{\alpha_k-(1/2)}^{x_k}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\frac{\partial^{2n}f(x)}{\partial x_1^2\partial x^2\ldots\partial x_n^2}.$$

The proof of Theorem 1.2 is analogous to the proof of Theorem 1.1.

In this paper, these properties are generalized for an iterated Bessel differential operator of high order. The results obtained are applied to the investigation of problems for higher-order multi-dimensional partial differential equations with singular coefficients.

In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM), cf. [19–21].

2 Generalization of the Properties of the Generalized Erdelyi–Kober Operator

Let $\left[B_{\eta_k}^{x_k}\right]^0 = E$, where E is the unit operator, $\left[B_{\eta_k}^{x_k}\right]^{m_k} = \left[B_{\eta_k}^{x_k}\right]^{m_k-1}\left[B_{\eta_k}^{x_k}\right]$ is the m_k th power of the operator $B_{\eta_k}^{x_k}$, $k = \overline{0, n}$.

Theorem 2.1 *Let* $\alpha_k > 0$, $\eta_k \geq -1/2$; $f(x) \in C^{2m_0}(\Omega^n)$; $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{p_k+1} f(x)$ *functions are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^{p_k} f(x) = 0$, $p_k = \overline{0, m_k - 1}$, $k = \overline{1, n}$. *Then*

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{m_k} J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[B_{\eta_k}^{x_k}\right]^{m_k} f(x),\ k = \overline{1, n}, \tag{2.1}$$

where $m_0 = \max\{m_1, m_2, \ldots, m_n\}$.

We note that Theorem 2.1 is also true in the case when some or all of the $\lambda_k = 0$, $k = \overline{1, n}$.

Proof Theorem 2.1 can be proved by the method of mathematical induction on m_k, $k = \overline{1, n}$. We arbitrarily fix $k \in N$, where N is the set of natural numbers. The proof of (2.1) for a fixed k and $m_k = 1$ is given in Theorem 1.1. Assume that equality (2.1) holds for $m_k = l_k$ and prove that it holds for $m_k = l_k + 1$.

From equality

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k+1} J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = \left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k} J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)$$

by the induction hypothesis, if the conditions of Theorem 2.1 are satisfied, we have

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k} J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)$$

$$= \left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right] J_\lambda \begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[B_{\eta_k}^{x_k}\right]^{l_k} f(x).$$

In the last equality, applying Theorem 1.1 to the functions $\left[B_{\eta_k}^{x_k}\right]^{l_k} f(x)$, under the conditions $\lim\limits_{x_k\to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^{l_k} f(x) = 0$, $k = \overline{1, n}$, we obtain the validity of formula (2.1). □

Corollary 2.1 *Suppose that the conditions of Theorem 2.1 are satisfied. Then*

$$\sum_{k=1}^{n}\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]^{m_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\sum_{k=1}^{n}\left[B_{\eta_k}^{x_k}\right]^{m_k} f(x),$$

in addition, if $f(x)\in C^{2|m|}(\Omega^n)$, *then*

$$\prod_{k=1}^{n}\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]^{m_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\prod_{k=1}^{n}\left[B_{\eta_k}^{x_k}\right]^{m_k} f(x). \qquad (2.2)$$

Theorem 2.2 *Let* $\alpha_k>0,\ \eta_k\geq -1/2,\quad q\in N;\quad f(x)\in C^{2q}(\Omega^n);$ *the functions* $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{l+1} f(x)$ *are integrable in a neighborhood of the origin and*

$$\lim_{x_k\to 0} x_k^{2\eta_k+1}\frac{\partial}{\partial x_k}\left[B_{\eta_k}^{x_k}\right]^{l} f(x)=0,\, l=\overline{0,q-1},\, k=\overline{1,\ n}.$$

Then

$$\left[\sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right)\right]^{q} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[\sum_{k=1}^{n}B_{\eta_k}^{x_k}\right]^{q} f(x).$$

This Theorem is proved using the polynomial formula

$$\left[\sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right)\right]^{q}=\sum_{|m|=q}\frac{q!}{m!}\prod_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right)^{m_k}$$

and with the use of equality (2.2), where $m!=m_1!m_2!\dots m_n!$

Corollary 2.2 *Suppose that the conditions of Theorem 2.1 are satisfied. Then for* $\eta_k=-1/2,\ k=\overline{1,n}$,

$$\left[\sum_{k=1}^{n}\left(B_{\alpha_k-1/2}^{x_k}+\lambda_k^2\right)\right]^{q} J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f(x)=J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta^q f(x),$$

in particular, for $\lambda_k=0$, *we have the equality*

$$\Delta_B^q J_0\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f(x)=J_0\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta^q f(x).$$

Let $L^{(y)}$ be a linear differential operator of order $l\in N$ independent of variable $x=(x_1,\ x_2,\ \dots,\ x_n)$ in the variable $y=(y_1,y_2,\ \dots,\ y_s)\in R^s$.

Theorem 2.3 *Let* $\alpha_k > 0,\ \eta_k \geq -1/2,\ k = \overline{1, n}$, $q \in N;\ f(x, y) \in C_{x,y}^{2q,lq}(\Omega^n \times \Omega^s)$, *the functions* $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{j+1} f(x, y)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k \to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^j f(x, y) = 0,\ j = \overline{0, q-1},\ k = \overline{1, n}$. *Then*

$$\left[L^{(y)} \pm \sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right)\right]^q J_\lambda^{(x)}\binom{\alpha}{\eta} f(x, y)$$

$$= J_\lambda^{(x)}\binom{\alpha}{\eta}\left[L^{(y)} \pm \sum_{k=1}^{n} B_{\eta_k}^{x_k}\right]^q f(x, y),$$

where the superscripts in the operators mean the variables by which these operators operate.

Proof Using the binomial formula, we obtain

$$\left[L^{(y)} \pm \sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right)\right]^q J_\lambda^{(x)}\binom{\alpha}{\eta} f(x, y)$$

$$= \sum_{j=0}^{q} C_q^j (\pm 1)^j \left(L^{(y)}\right)^{q-j}\left[\sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right)\right]^j J_\lambda^{(x)}\binom{\alpha}{\eta} f(x, y),$$

$C_k^j = k!/[j!(k-j)!]$ is binomial coefficients.

Next, applying Theorem 2.2, we have

$$J_\lambda^{(x)}\binom{\alpha}{\eta}\sum_{j=0}^{q} C_q^j (\pm 1)^j \left(L^{(y)}\right)^{q-j}\left[\sum_{k=1}^{n} B_{\eta_k}^{x_k}\right]^j f(x, y)$$

$$= J_\lambda^{(x)}\binom{\alpha}{\eta}\left[L^{(y)} \pm \sum_{k=1}^{n} B_{\eta_k}^{x_k}\right]^q f(x, y).$$

□

Corollary 2.3 *Suppose that the conditions of Theorem 2.3 are satisfied. If*

$$L^{(y)} = -\sum_{k=\omega+1}^{\omega+\sigma}\left(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right),$$

where $x = (x_1, x_2, \dots, x_\omega)$, $y = (x_{\omega+1}, x_{\omega+2}, \dots, x_{\omega+\sigma})$, $\omega + \sigma = n$, *then*

$$\left[\sum_{k=1}^{\omega}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right) - \sum_{k=\omega+1}^{\omega+\sigma}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right)\right]^q J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x)$$

$$= J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[\sum_{k=1}^{\omega} B^{x_k}_{\eta_k} - \sum_{k=\omega+1}^{\omega+\sigma} B^{x_k}_{\eta_k}\right]^q f(x), \quad \omega + \sigma = n.$$

Let $[D^{x_k}_{\eta_k}]^0 = E$, $D^{x_k}_{\eta_k} \equiv x_k^{-2\eta_k}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)x_k^{2\eta_k}$ and $[D^{x_k}_{\eta_k}]^{m_k} = [D^{x_k}_{\eta_k}]^{m_k-1}D^{x_k}_{\eta_k}$ the degree of an operator $D^{x_k}_{\eta_k}$ that is representable in the form $[D^{x_k}_{\eta_k}]^{m_k} = x_k^{-2\eta_k}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{m_k} x_k^{2\eta_k}$, m_k be nonnegative integers, $k = \overline{1, n}$.

Theorem 2.4 *If* $\alpha_k > 0$, $\eta_k \geq -(1/2)$, $k = \overline{1, n}$, $f(x) \in C^{m_0}(\Omega^n)$, *the functions* $x_{x_k}^{2\eta_k+1}[D^{x_k}_{\eta_k}]^{l_k+1} f(x)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to 0} x_k^{2\eta_k}[D^{x_k}_{\eta_k}]^{l_k} f(x) = 0$, $l_k = \overline{0, m_k - 1}$, $k = \overline{1, n}$, *then*

$$[D^{x_k}_{\eta_k+\alpha_k}]^{m_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}]^{m_k} f(x), \ \ k = \overline{1, n}, \tag{2.3}$$

where $m_0 = \max\{m_1, m_2, \dots, m_n\}$.

Proof This Theorem is also proved using the method of mathematical induction on m_k, $k = \overline{1, n}$. Arbitrarily fix $k \in N$. The proof of formula (2.3) for $m_k = 1$, $k = \overline{1, n}$ is given in [22, 23], according to which we have

$$[D^{x_k}_{\eta_k+\alpha_k}] J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}] f(x), \ \ k = \overline{1, n}. \tag{2.4}$$

Suppose that (2.3) holds for $m_k = l_k$ and we prove that it holds for $m_k = l_k + 1$.

$$[D^{x_k}_{\eta_k+\alpha_k}]^{l_k+1} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = [D^{x_k}_{\eta_k+\alpha_k}][D^{x_k}_{\eta_k+\alpha_k}]^{l_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x). \tag{2.5}$$

By the induction hypothesis, if the conditions of Theorem 2.4 are satisfied, we have

$$[D^{x_k}_{\eta_k+\alpha_k}]^{l_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}]^{l_k} f(x).$$

Then the equality (2.5) takes the form

$$[D^{x_k}_{\eta_k+\alpha_k}]^{l_k+1} J_\lambda \binom{\alpha}{\eta} f(x) = [D^{x_k}_{\eta_k+\alpha_k}] J_\lambda \binom{\alpha}{\eta} [D^{x_k}_{\eta_k}]^{l_k} f(x).$$

Further, applying formula (2.4) to the functions $[D^{x_k}_{\eta_k}]^{l_k} f(x)$, under the conditions $\lim\limits_{x_k\to 0} x_k^{2\eta_k} [D^{x_k}_{\eta_k}]^{l_k} f(x) = 0$, we obtain the validity of formula (2.3). □

Corollary 2.4 *Suppose that the conditions of Theorem 2.4 are satisfied, then*

$$\prod_{k=1}^{n} [D^{x_k}_{\eta_k+\alpha_k}]^{m_k} J_\lambda \binom{\alpha}{\eta} f(x) = J_\lambda \binom{\alpha}{\eta} \prod_{k=1}^{n} [D^{x_k}_{\eta_k}]^{m_k} f(x).$$

Theorem 2.5 *Let* $0 < \alpha_k < 1$, $\eta_k \geq -1/2$; $g(x) \in C^{2p}(\Omega^n)$; $\dfrac{\partial}{\partial x_k} \left[B^{x_k}_{\eta_k+\alpha_k}\right]^l g(x)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to 0} x_k^{2(\eta_k+\alpha_k)+1} \dfrac{\partial}{\partial x_k} \left[B^{x_k}_{\eta_k+\alpha_k}\right]^l$ $g(x) = 0$, $l = \overline{0, p-1}$, $p \in N$, $k = \overline{1, n}$. *Then*

$$\left[B^{x_k}_{\eta_k} - \lambda_k^2\right]^p J_\lambda^{-1} \binom{\alpha}{\eta} g(x) = J_\lambda^{-1} \binom{\alpha}{\eta} \left[B^{x_k}_{\eta_k+\alpha_k}\right]^p g(x), \; k = \overline{1, n},$$

or

$$\left[B^{x_k}_{\eta_k}\right]^p J_\lambda^{-1} \binom{\alpha}{\eta} g(x) = J_\lambda^{-1} \binom{\alpha}{\eta} \left[B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right]^p g(x), \;\; k = \overline{1, n},$$

in particular, if $\lambda_k = 0$, *then*

$$\left[B^{x_k}_{\eta_k}\right]^p J_0^{-1} \binom{\alpha}{\eta} g(x) = J_0^{-1} \binom{\alpha}{\eta} \left[B^{x_k}_{\eta_k+\alpha_k}\right]^p g(x), \;\; k = \overline{1, n}.$$

The proof of the Theorem 2.5 is analogous to the proof of Theorem 2.1.

Corollary 2.5 *Suppose that the conditions of Theorem 2.5 are satisfied, then*

$$\left[\sum_{k=1}^{n} \left(B^{x_k}_{\eta_k} - \lambda_k^2\right)\right]^p J_\lambda^{-1} \binom{\alpha}{\eta} g(x) = J_\lambda^{-1} \binom{\alpha}{\eta} \left[\sum_{k=1}^{n} B^{x_k}_{\eta_k+\alpha_k}\right]^p g(x),$$

or

$$\left[\sum_{k=1}^{n} B^{x_k}_{\eta_k}\right]^p J_\lambda^{-1} \binom{\alpha}{\eta} g(x) = J_\lambda^{-1} \binom{\alpha}{\eta} \left[\sum_{k=1}^{n} \left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right)\right]^p g(x).$$

If the conditions $\lim\limits_{x_k\to 0} x_k^{2\alpha_k} \dfrac{\partial}{\partial x_k}\left[B_{\alpha_k-(1/2)}^{x_k}\right]^l g(x) = 0$, $l = \overline{0, p-1}$, $k = \overline{1, n}$, are satisfied, then the last equality for $\lambda_k = 0$, $\eta_k = -(1/2)$, $k = \overline{0, m-1}$, implies the validity of equality

$$\Delta^p J_0^{-1}\begin{pmatrix} \alpha \\ -1/2 \end{pmatrix} g(x) = J_0^{-1}\begin{pmatrix} \alpha \\ -1/2 \end{pmatrix} \Delta_B^p g(x) \tag{2.6}$$

where $\Delta^p = \left[\sum\limits_{k=1}^{n} \dfrac{\partial^2}{\partial x_k^2}\right]^p$ is pth power of the multidimensional Laplace operator, and

$$\Delta_B^p = \left[\sum_{k=1}^{n}\left(B_{\alpha_k-(1/2)}^{x_k}\right)\right]^p = \left[\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k}{x_k}\frac{\partial}{\partial x_k}\right)\right]^p.$$

We note that the Theorems proved allow us to reduce higher-order equations with singular coefficients to polyharmonic, polycaloric, and polywave equations, and thereby to establish and investigate the correct initial and boundary value problems for such equations.

3 Applications

The results obtained are applicable to the construction of the solution of the analogue of the Cauchy problem for a multidimensional polycaloric equation with the Bessel operator.

Singular parabolic equations with Bessel operator belong to the class of equations degenerating on the boundary of the domain with respect to the space variables. These equations are often encountered in applications. Thus, in the mathematical simulation of numerous problems of heat transfer in immobile media (solids), the problems of diffusion boundary layer [24], and the problems of propagation of heat in process of injection of hot liquids in oil pools [25], we get singular parabolic equations with Bessel operator.

Degenerating equations and equations with singular coefficients form an important field of the contemporary theory of partial differential equations. Numerous works are devoted to the study of these equations. In this field, an important place is occupied by the initial and boundary-value problems for parabolic equations with Bessel operator. The theory of classical solutions to the Cauchy problem for singular parabolic equations of the second order was developed in [26–30]. The Cauchy problem for singular parabolic equations in the classes of distributions and in the classes of generalized functions of the type S' was studied in [31, 32]. However, the initial and boundary value problems for equations with Bessel operators of high orders are studied quite poorly.

In the domain $\Omega = \{(x,t) : x \in R_+^n,\ t \in R_+^1\}$, where $x = (x_1, x_2, \dots, x_n)$ be a point of the n-dimensional Euclidean space R^n, $R_+^n = \left\{x \in R^n :\ x_k > 0,\ k = \overline{1,n}\right\}$, we consider the problem of finding the solution $u(x,t)$ of the equation

$$L_\gamma^m(u) \equiv \left(\frac{\partial}{\partial t} - \Delta_B\right)^m u(x,t) = 0, \quad (x,t) \in \Omega, \tag{3.1}$$

satisfying the initial conditions

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \varphi_k(x),\ \ x \in R_+^n,\ \ k = \overline{0, m-1} \tag{3.2}$$

and homogeneous boundary conditions

$$\left.\frac{\partial^{2k+1} u}{\partial x_j^{2k+1}}\right|_{x_j=0} = 0, \quad t > 0, \quad j = \overline{1,n}, \quad k = \overline{0, m-1}, \tag{3.3}$$

where $\Delta_B = \sum\limits_{k=1}^{n} B_{\gamma_k}^{x_k}$, $B_{\gamma_k}^{x_k} = \partial^2/\partial x_k^2 + [(2\gamma_k+1)/x_k](\partial/\partial x_k)$ is the Bessel operator acting on variable x_k; $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in R^n$, $\gamma_k \in R$, $\gamma_k > -1/2$, $k = \overline{1,n}$, m is a natural number; $\varphi_k(x)$, $k = \overline{0, m-1}$ given differentiable functions.

We note that in the problems of the general theory of partial differential equations containing the Bessel operator with one or more variables, the main investigation apparatus is the corresponding integral Fourier–Bessel transform. Unlike traditional methods, here we apply the properties of the multidimensional Erdélyi–Kober operator to solve the problem.

Suppose that the solution of the Eq. (3.1) satisfying conditions (3.2) and (3.3) exists. We seek this solution in the form

$$u(x,t) = J_0^{(x)}\begin{pmatrix}\alpha \\ \eta\end{pmatrix} U(x,t), \tag{3.4}$$

where $\alpha, \eta \in R^n$, $\alpha_k = \gamma_k + (1/2) > 0$, $\eta_k = -(1/2)$, $k = \overline{1,n}$, and $U(x,t)$ is an unknown function differentiable sufficiently many times, and $J_0^{(x)}\begin{pmatrix}\alpha \\ \eta\end{pmatrix}$ is a multidimensional Erdélyi–Kober operator of fractional order (1.3) acting on a variable $x \in R^n$.

Substituting (3.4) into the boundary conditions (3.3), and then into Eq. (3.1) and the initial conditions (3.2), and using Theorem 2.3 for $L^{(t)} \equiv \partial/\partial t$, we arrive at the problem of determination of the solution $U(x,t)$ of the equation

$$\left(\frac{\partial}{\partial t}-\Delta\right)^m U(x,t)=0,\ \ (x,t)\in\Omega, \tag{3.5}$$

satisfying the initial conditions

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0}=\Phi_k(x),\ \ x\in R^n,\ \ k=\overline{0,m-1}, \tag{3.6}$$

and the homogeneous boundary conditions

$$\left.\frac{\partial^{2k+1}U}{\partial x_j^{2k+1}}\right|_{x_j=0}=0,t>0,\quad j=\overline{1,n},\quad k=\overline{0,m-1}, \tag{3.7}$$

where $\Phi_k(x)=J_0^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\varphi_k(x)$, $\eta_k=-(1/2)$, $(k=\overline{0,m-1})$, $J_0^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}$ is the inverse operator (1.4).

By using the boundary conditions (3.7), we extend the functions $\Phi_k(x)$ evenly to $x_k<0$, $(k=\overline{0,m-1})$ and denote the extended functions by $\tilde{\Phi}_k(x)$. Then in the domain $\tilde{\Omega}=\{(x,y): x\in R^n,\ \ t>0\}$ we obtain the problem of finding a solution of Eq. (3.5) satisfying the initial conditions

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0}=\tilde{\Phi}_k(x),\ \ x\in R,\ \ k=\overline{0,m-1}, \tag{3.8}$$

We introduce the notation $W_0(x,t)=U(x,t)$ and $W_k(x,t)=\left(\frac{\partial}{\partial t}-\Delta\right)^k W_0$. In this notation, the problem (3.5) and (3.8) is equivalent to the problem of determination the functions $W_k(x,t)$, $k=\overline{0,m-1}$, satisfying the system of equations

$$\begin{cases}\dfrac{\partial W_k}{\partial t}-\Delta W_k=W_{k+1},\ \ (x,t)\in\tilde{\Omega},\ \ k=\overline{0,m-2},\\ \dfrac{\partial W_{m-1}}{\partial t}-\Delta W_{m-1}=0,\ \ (x,t)\in\tilde{\Omega}\end{cases} \tag{3.9}$$

with the initial conditions

$$W_k(x,0)=F_k(x),\ \ x\in R^n,\quad k=\overline{0,m-1}, \tag{3.10}$$

where

$$F_k(x) = \sum_{j=0}^{k} (-1)^{k-j} C_k^j \Delta^{k-j} \tilde{\Phi}_j(x), \ \ k = \overline{0, m-1}. \tag{3.11}$$

For the solution of problem (3.9) and (3.10), we use the following lemma.

Lemma 3.1 *If* $g(x) \in L_1(R^n)$, *then the equality*

$$\int_0^t \frac{d\tau}{\left(2\sqrt{\pi(t-\tau)}\right)^n} \int_{R^n} \exp\left[-\frac{|x-y|^2}{4(t-\tau)}\right] \left\{ \frac{1}{\left(2\sqrt{\pi\tau}\right)^n} \int_{R^n} g(\eta) \exp\left[-\frac{(y-\eta)^2}{4\tau}\right] d\eta \right\} dy$$
$$= \frac{t}{\left(2\sqrt{\pi t}\right)^n} \int_{R^n} g(\eta) \exp\left[-\frac{|\eta - x|^2}{4t}\right] dy. \tag{3.12}$$

Proof In view of the uniform convergence of the improper integrals on the left-hand side of equality (3.12), we can change the order of integration with respect to η and y. Then we take the inner integral by the formula [33]

$$\int_{-\infty}^{+\infty} \exp\left[-p\xi^2 - q\xi\right] d\xi = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right), \quad \operatorname{Re} p > 0,$$

we obtain

$$\prod_{j=1}^{n} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x_j - y_j)^2}{4(t-\tau)} - \frac{(y_j - \eta_j)^2}{4\tau}\right] dy_j$$
$$= \left[2\frac{\sqrt{\pi}}{\sqrt{t}}\sqrt{\tau(t-\tau)}\right]^n \exp\left[-\frac{|\eta - x|^2}{4t}\right]. \tag{3.13}$$

Substituting (3.13) into the left-hand side of (3.12), after reducing such terms, we obtain the assertion of Lemma 3.1. □

We now successively solve each equation in system (3.9) starting from the last equation. By using the initial conditions (3.10) and Lemma 3.1, we determine the solution of problem (3.9) and (3.10). In view of the relation $W_0(x,t) = U(x,t)$, we obtain the solution of problem (3.5)–(3.7) in the form

$$U(x,t) = \left(2\sqrt{\pi t}\right)^{-n} \sum_{k=0}^{m-1} \frac{t^k}{k!} \int_{R^n} F_k(s) \exp\left[-\frac{|s-x|^2}{4t}\right] ds, \tag{3.14}$$

where $F_k(x)$ ($k = \overline{0, m-1}$) are known functions given by equalities (3.11).

In view of the evenness of the functions $F_k(x)$, $k = \overline{0, m-1}$, we can rewrite equality (3.14) in the form

$$U(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} U_k(x,t), \tag{3.15}$$

where

$$U_k(x,t) = \int_{R_+^n} F_k(s) G(x,t,s) ds, \tag{3.16}$$

$$G(x,s,t) = \prod_{j=1}^{n} G_0(x_j, s_j, t),$$

$$G_0(x_j, s_j, t) = \frac{1}{2\sqrt{\pi t}} \left\{ \exp\left[-\frac{(s_j - x_j)^2}{4t} \right] + \exp\left[-\frac{(s_j + x_j)^2}{4t} \right] \right\}.$$

To analyze the behavior of the functions $F_k(x)$, $k = \overline{0, m-1}$, it is necessary to perform certain transformations. To this end, we prove the following lemma:

Lemma 3.2 *Suppose that the functions* $\varphi_j(x) \in C^{2(m-j)-1}(R_+^n)$, $j = \overline{0, m-1}$, *are continuous and bounded, and that all derivatives of the functions* $\varphi_j(x)$, *up to the order* $2(m-j)-1$, $j = \overline{0, m-1}$ *inclusively, are equal to zero for* $x_k = 0$, $k = \overline{1, n}$. *Then the equalities*

$$\lim_{x_k \to 0} x_k^{2\alpha_k} \frac{\partial}{\partial x_k} \left[B_{\alpha_k - (1/2)}^{x_k} \right]^l \varphi_j(x) = 0, k = \overline{1, n}, l = \overline{0, m-1}, j = \overline{0, m-1}, \tag{3.17}$$

$$\lim_{x_k \to 0} [B_{\gamma_k}^{x_k}]^i \varphi_{j-i}(x) = 0, \;\; k = \overline{1, n}, \;\; i = \overline{0, j}, \;\; j = \overline{0, m-1}, \tag{3.18}$$

are true.

Proof By induction, we can prove the following equality:

$$\left(\frac{1}{x} \frac{d}{dx} \right)^p h(x) = \sum_{j=1}^{p} (-1)^{j+1} A_{pj} \frac{h^{(p-j+1)}(x)}{x^{p+j-1}}, \tag{3.19}$$

where A_{pj} are constants given by the recurrence relations

$$A_{(p+1)1} = A_{p1} = 1, \;\; p \ge 1, \;\; A_{(p+1)j} = (p+j-1) A_{p(j-1)} + A_{pj}, \;\; p \ge 2, \;\; j = \overline{2, p},$$

$$A_{(p+1)(p+1)} = (2p-1) A_{pp} = (2p-1)!!, \;\; p \ge 1.$$

We rewrite (3.17) in the form

$$\lim_{x_k\to 0} H(x) = \lim_{x_k\to 0} x_k^{2\alpha_k} \frac{\partial}{\partial x_k}\left[B^{x_k}_{\alpha_k-(1/2)}\right]^l \varphi_j(x)$$

$$= \lim_{x_k\to 0} x_k^{1+2\alpha_k} \sum_{q=0}^{l} C_l^q (2\alpha_k)^{l-q} \left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{l-q+1} \frac{\partial^{2q}\varphi_j(x)}{\partial x_k^{2q}}$$

Taking (3.19) into account, we have

$$\lim_{x_k\to 0} H(x)$$

$$= \sum_{q=0}^{l} C_l^q (2\alpha_k)^{l-q} \sum_{j=1}^{l-q+1} (-1)^{j+1} A_{(l-q+1)j} \lim_{x_k\to 0} \frac{\partial^{l-q-j+2}\varphi_j(x)/\partial x_k^{l-q-j+2}}{x_k^{l-q+j-1-2\alpha_k}}$$

Applying the L'Hospital rule $l-q+j$ times [34] to the last equality and taking into account the condition of the Lemma 3.2, we obtain

$$\lim_{x_k\to 0} \frac{\partial^{l-q-j+2}\varphi_j(x)/\partial x_k^{l-q-j+2}}{x_k^{l-q+j-1-2\alpha_k}} = \frac{\lim_{x_k\to 0} x_k^{1+2\alpha_k}[\partial^{2l+2q}\varphi_j(x)/\partial x_k^{2l+2q}]}{(l-q+j)!} = 0.$$

This proves of (3.17). Equality (3.18) is proved similarly. □

We now transform the functions $F_k(x)$, $k=\overline{0,m-1}$. By virtue of Lemma 3.2, the functions $\Phi_k(x)$ satisfy all conditions of Theorem 2.5. Therefore, taking into account formula (2.6), equality (3.11) for $x_k > 0$, $k=\overline{1,n}$ can be represented in the form

$$F_k(x) = J_0^{-1}\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f_k(x),\ \ k=\overline{0,m-1}, \tag{3.20}$$

where

$$f_k(x) = \sum_{j=0}^{k} (-1)^j C_k^j \Delta_B^j \varphi_{k-j}(x),\ \ k=\overline{0,m-1}. \tag{3.21}$$

Taking into account the form of the inverse operator (1.4) for $m_j=1$, $j=\overline{1,n}$, equality (3.20) can be represented as $F_k(x) = [\partial^n/(\partial x_1 \partial x_2 \ldots \partial x_n)]\bar{F}_k(x)$, where

$$\bar{F}_k(x) = \prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n}\prod_{j=1}^{n}\left[(x_j^2-s_j^2)^{-\alpha_j}s_j^{2\alpha_j}\right] f_k(s)ds_1 ds_2\ldots ds_n.$$

We note that, by Lemma 3.2, it follows from (3.21) that the functions $f_k(x)$, $k = \overline{0, m-1}$ for $x_j \geq 0$, are continuous, bounded, and $f_k(x)|_{x_j=0} = 0$, so that from the last equality we have

$$\bar{F}_k(x)\big|_{x_j=0} = 0, \quad j = \overline{1, n}, \;\; k = \overline{0, m-1}. \tag{3.22}$$

Taking (3.22) into account in (3.16), we integrate by parts. Then, substituting in this equality the value of the functions $\bar{F}_k(x)$, we obtain

$$U_k(x,t) = -\prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\int_{R_+^n} f_k(s)\prod_{j=1}^{n}\left[s_j^{2\alpha_j} G_1(x_j, s_j, t)\right] ds, \tag{3.23}$$

where

$$G_1(x_j, s_j, t) = \int_{s_j}^{+\infty} (y_j^2 - s_j^2)^{-\alpha_j} \frac{\partial}{\partial y_j} G_0(x_j, y_j, t) dy_j. \tag{3.24}$$

Let us calculate the integral (3.24). Applying the formula [33, p. 451]

$$\int_0^{+\infty} e^{-a\lambda^2}\cos(b\lambda)d\lambda = \sqrt{\frac{\pi}{4a}}\exp\left[-\frac{b^2}{4a}\right], \quad \operatorname{Re} a > 0,$$

function $G_0(x_j, y_j, t)$ can be represented in the form

$$G_0(x_j, y_j, t) = \frac{2}{\pi}\int_0^{+\infty} e^{-t\lambda^2}\cos(x_j\lambda)\cos(y_j\lambda)d\lambda.$$

We find the derivative with respect to y_j and substitute the obtained expression for the function G_{0y} in (3.24). Then we use the uniform convergence of integrals and change the order of integration. Taking the inner integral with the help of the Mehler–Sonine formula [35, p. 93], we get

$$G_1(x_j, s_j, t)$$

$$= -\frac{2^{(1/2)-\alpha_j}}{\sqrt{\pi}}\Gamma(1-\alpha_j)s_j^{(1/2)-\alpha}\int_0^{+\infty} e^{-t\lambda^2}\lambda^{\alpha_j+(1/2)} J_{\alpha_j-(1/2)}(\lambda s_j)\cos(x_j\lambda)d\lambda. \tag{3.25}$$

Now, substituting (3.23) into (3.15), and its in (3.4), after changing the order of integration, we obtain

$$u(x,t) = -\prod_{j=1}^{n}\left[\frac{2x_j^{1-2\alpha_j}}{\Gamma(\alpha_j)\Gamma(1-\alpha_j)}\right]\sum_{k=0}^{m-1}\frac{t^k}{k!}\int_{R_+^n} f_k(s)\prod_{j=1}^{n} s_j^{2\alpha_j} G_2(x_j,s_j,t)ds \tag{3.26}$$

where

$$G_2(x_j,s_j,t) = \int_0^{x_j} (x_j^2-\xi_j^2)^{\alpha_j-1}\, G_1(\xi_j,s_j,t)d\xi_j \tag{3.27}$$

We substitute the expression (3.25) for the function G_1 in (3.27) and change the order of integration. Then, using the Poisson formula [35, p. 93], we compute the inner integral. As a result, we find

$$G_2(x_j,s_j,t)$$
$$= -\frac{1}{2}\Gamma(\alpha_j)\Gamma(1-\alpha_j)\left(\frac{s_j}{x_j}\right)^{(1/2)-\alpha_j}\int_0^{\infty} e^{-t\lambda^2} J_{\alpha_j-(1/2)}(s_j\lambda)J_{\alpha_j-(1/2)}(x_j\lambda)\lambda d\lambda.$$

Further, taking into account the following formula [35, p. 60]

$$\int_0^{\infty} e^{-t\lambda^2} J_\nu(s\lambda)J_\nu(x\lambda)\lambda d\lambda = \frac{1}{2t}e^{-\frac{x^2+s^2}{4t}} I_\nu\left(\frac{xs}{2t}\right),$$

$\mathrm{Re}\,\nu > -1,\ \mathrm{Re}\,t > 0$, we have

$$G_2(x_j,s_j,t) = -\frac{1}{4t}\Gamma(\alpha_j)\Gamma(1-\alpha_j)\left(\frac{s_j}{x_j}\right)^{(1/2)-\alpha_j} e^{-\frac{x_j^2+s_j^2}{4t}} I_{\alpha_j-(1/2)}\left(\frac{x_j s_j}{2t}\right). \tag{3.28}$$

Substituting (3.28) into (3.26) and taking into account both and $\alpha_j = \gamma_j + 1/2 < 1$, $\gamma_j > -1/2$, $j = \overline{1,n}$, we find the final form of the solution of the Eq. (3.1) for $|\gamma_j| < 1/2$, $j = \overline{1,n}$, satisfying conditions (2.5) and (2.6):

$$u(x,t) = \frac{1}{(2t)^n}\prod_{j=1}^{n} x_j^{-\gamma_j}\sum_{k=0}^{m-1}\frac{t^k}{k!}\int_{R_+^n} f_k(s)G(x,s,t)ds, \tag{3.29}$$

where $f_k(x) = \sum\limits_{j=0}^{k} (-1)^j C_k^j \Delta_B^{k-j} \varphi_j(x)$,

$$G(x, s, t) = \prod_{j=1}^{n} \left\{ s_j^{\gamma_j+1} \exp\left[-\frac{x_j^2 + s_j^2}{4t} \right] I_{\gamma_j}\left(\frac{x_j s_j}{2t}\right) \right\}$$

$$= \prod_{j=1}^{n} \left[s_j^{\gamma_j+1} I_{\gamma_j}\left(\frac{x_j s_j}{2t}\right) \right] \exp\left[-\frac{|x|^2 + |s|^2}{4t} \right], \quad |x|^2 = \sum_{j=1}^{n} x_j^2. \tag{3.30}$$

A direct verification shows that the following theorem holds.

Theorem 3.1 *Let* $|\gamma_j| < 1/2$, $j = \overline{1, n}$, *and the functions* $\varphi_j(x) \in C^{2(m-j)-1}(R_+^n)$, $j = \overline{0, m-1}$ *are continuous, bounded, and all derivatives of the functions* $\varphi_j(x)$, *up to the order* $2(m-j) - 1$, $j = \overline{0, m-1}$ *inclusively, are equal to zero for* $x_k = 0$, $k = \overline{1, n}$. *Then the function* $u(x, t)$, *defined by* (3.29)*, is a classical solution of equation* $L_\gamma^m(u) = 0$, *satisfying conditions* (3.2) *and* (3.3).

Appendix: Integral Transform Composition Method (ITCM) in Transmutation Theory: How It Works

In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM).

Below we give a brief survey and outline some applications of the integral transforms composition method (ITCM) for obtaining transmutations via integral transforms. It is possible to derive wide range of transmutation operators by this method. Classical integral transforms are involved in the integral transforms composition method (ITCM) as basic blocks, among them are Fourier, sine and cosine-Fourier, Hankel, Mellin, Laplace and some generalized transforms. The ITCM and transmutations obtaining by it are applied to deriving connection formulas for solutions of singular differential equations and more simple non-singular ones. We consider well-known classes of singular differential equations with Bessel operators, such as classical and generalized Euler–Poisson–Darboux equation and the generalized radiation problem of A. Weinstein. Methods of this paper are applied to more general linear partial differential equations with Bessel operators, such as multivariate Bessel-type equations, GASPT (Generalized Axially Symmetric Potential Theory) equations of Weinstein, Bessel-type generalized wave equations with variable coefficients,ultra B-hyperbolic equations and others. So with many results and examples the main conclusion of this paper is illustrated: the integral transforms composition method (ITCM) of constructing transmutations is very important and effective tool also for obtaining connection formulas and explicit representations of solutions to a wide class of singular differential equations, including ones with Bessel operators.

What is ITCM and How It Works?

In transmutation theory explicit operators were derived based on different ideas and methods, often not connecting altogether. So there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations.

In this section we give such general method for constructing transmutation operators. We call this method *integral transform composition method* or shortly ITCM. The method is based on the representation of transmutation operators as compositions of basic integral transforms. The integral transform composition method (ITCM) gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua-Erdelyi-Lowndes, Buschman-Erdelyi, Sonin-Katrakhov and Poisson-Katrakhov ones, cf. [36–45, 63–65] as well as the classes of elliptic, hyperbolic and parabolic transmutation operators introduced by Carroll [37–39].

The formal algorithm of ITCM is the next. Let us take as input a pair of arbitrary operators A, B, and also connecting with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{A.1}$$

where t is a dual variable, g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$.

Then the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas:

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \quad P = F_A^{-1} w(t) F_B \tag{A.2}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \quad PB = AP. \tag{A.3}$$

A formal checking of (A.3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (A.2) in an explicit integral form, as well as the choice of domains of operators P and S.

Let us list the main advantages of Integral Transform Composition Method (ITCM).

- Simplicity—many classes of transmutations are obtained by explicit formulas from elementary basic blocks, which are classical integral transforms.
- ITCM gives by a unified approach all previously explicitly known classes of transmutations.
- ITCM gives by a unified approach many new classes of transmutations for different operators.
- ITCM gives a unified approach to obtain both direct and inverse transmutations in the same composition form.
- ITCM directly leads to estimates of norms of direct and inverse transmutations using known norm estimates for classical integral transforms on different functional spaces.
- ITCM directly leads to connection formulas for solutions to perturbed and unperturbed differential equations.

An obstacle for applying ITCM is the next one: we know acting of classical integral transforms usually on standard spaces like L_2, L_p, C^k, variable exponent Lebesgue spaces [46] and so on. But for application of transmutations to differential equations we usually need some more conditions hold, say at zero or at infinity. For these problems we may first construct a transmutation by ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (A.2) of course are not new for integral transforms and its applications to differential equations. But ITCM is new when applied to transmutation theory! In other fields of integral transforms and connected differential equations theory compositions (A.2) for the choice of classical Fourier transform leads to famous pseudo-differential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t) = (\pm it)^{-s}$ we obtain fractional integrals on the whole real axis, for $w(t) = |x|^{-s}$ we obtain M.Riesz potential, for $w(t) = (1 + t^2)^{-s}$ in formulas (A.2) we obtain Bessel potential and for $w(t) = (1 \pm it)^{-s}$ - modified Bessel potentials [3].

The next choice for ITCM algorithm,

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad w(t) = j_\nu(st) \tag{A.4}$$

leads to generalized translation operators of Delsart [47–49], for this case we have to choose in ITCM algorithm defined by (A.1)–(A.2) the above values (A.4) in which B_ν is the Bessel operator, H_ν is the Hankel transform, j_ν is the normalized (or "small") Bessel function. In the same manner other families of operators commuting with a given one may be obtained by ITCM for the choice $A = B$, $F_A = F_B$ with arbitrary functions $g(t), w(t)$ (generalized translation commutes with the Bessel operator). In case of the choice of differential operator A as quantum oscillator and connected integral transform F_A as fractional or quadratic Fourier transform [50] we may obtain by ITCM transmutations also for this case [43]. It is possible to apply ITCM instead of classical approaches for obtaining fractional powers of Bessel operators [43, 51–54].

Direct applications of ITCM to multidimensional differential operators are obvious, in this case t is a vector and $g(t), w(t)$ are vector functions in (A.1)–(A.2). Unfortunately for this case we know and may derive some new explicit transmutations just for simple special cases. But among them are well-known and interesting classes of potentials. In case of using ITCM by (A.1)–(A.2) with Fourier transform and $w(t)$—positive definite quadratic form we come to elliptic Riesz potentials [3, 55]; with $w(t)$—indefinite quadratic form we come to hyperbolic Riesz potentials [3, 55, 56]; with $w(x,t) = (|x|^2 - it)^{-\alpha/2}$ we come to parabolic potentials [3]. In case of using ITCM by (A.1)–(A.2) with Hankel transform and $w(t)$ - quadratic form we come to elliptic Riesz B-potentials [57, 58] or hyperbolic Riesz B-potentials [59]. For all above mentioned potentials we need to use distribution theory and consider for ITCM convolutions of distributions, for inversion of such potentials we need some cutting and approximation procedures, cf. [56, 59]. For this class of problems it is appropriate to use Schwartz or/and Lizorkin spaces for probe functions and dual spaces for distributions.

So we may conclude that the method we consider in the paper for obtaining transmutations—ITCM is effective, it is connected to many known methods and problems, it gives all known classes of explicit transmutations and works as a tool to construct new classes of transmutations. Application of ITCM needs the next three steps.

Step 1. For a given pair of operators A, B and connected generalized Fourier transforms F_A, F_B define and calculate a pair of transmutations P, S by basic formulas (A.1)–(A.2).

Step 2. Derive exact conditions and find classes of functions for which transmutations obtained by step 1 satisfy proper intertwining properties.

Step 3. Apply now correctly defined transmutations by steps 1 and 2 on proper classes of functions to deriving connection formulas for solutions of differential equations.

The next part of this article is organized as follows. First we illustrate step 1 of the above plan and apply ITCM for obtaining some new and known transmutations. For step 2 we prove a general theorem for the case of Bessel operators, it is enough to solve problems to complete strict definition of transmutations. And after that we give an example to illustrate step 3 of applying obtained by ITCM transmutations to derive formulas for solutions of a model differential equation.

Application of ITCM to Index Shift B–Hyperbolic Transmutations

In this section we apply ITCM to obtain integral representations for index shift B-hyperbolic transmutations. It corresponds to step 1 of the above plan for ITCM algorithm.

Let us look for the operator T transmuting the Bessel operator B_ν into the same operator but with another parameter B_μ. To find such a transmutation we use ITCM with Hankel transform. Applying ITCM we obtain an interesting and important family of transmutations, including index shift B-hyperbolic transmutations, "descent" operators, classical Sonine and Poisson-type transmutations, explicit integral representations for fractional powers of the Bessel operator, generalized translations of Delsart and others.

So we are looking for an operator $T_{\nu,\mu}^{(\varphi)}$ such that

$$T_{\nu,\mu}^{(\varphi)} B_\nu = B_\mu T_{\nu,\mu}^{(\varphi)} \tag{A.5}$$

in the factorized due to ITCM form

$$T_{\nu,\mu}^{(\varphi)} = H_\mu^{-1}\Big(\varphi(t) H_\nu\Big), \tag{A.6}$$

where H_ν is a Hankel transform. Assuming $\varphi(t) = Ct^\alpha$, $C \in \mathbb{R}$ does not depend on t and $T_{\nu,\mu}^{(\varphi)} = T_{\nu,\mu}^{(\alpha)}$ we can derive the following theorem.

Theorem A.1 *Let f be a proper function for which the composition (A.6) is correctly defined,*

$$\operatorname{Re}(\alpha + \mu + 1) > 0, \quad \operatorname{Re}\left(\alpha + \frac{\mu - \nu}{2}\right) < 0.$$

Then for transmutation operator $T_{\nu,\mu}^{(\alpha)}$ obtained by ITCM and such that

$$T_{\nu,\mu}^{(\alpha)} B_\nu = B_\mu T_{\nu,\mu}^{(\alpha)}$$

the next integral representation is true

$$\begin{aligned}
&\left(T_{\nu,\mu}^{(\alpha)} f\right)(x)\\
&= C\frac{2^{\alpha+3}\Gamma(\frac{\alpha+\mu+1}{2})}{\Gamma(\frac{\mu+1}{2})}\Bigg[\frac{x^{-1-\mu-\alpha}}{\Gamma(-\frac{\alpha}{2})}\\
&\quad\times\int_0^x f(y)\,{}_2F_1\Big(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\Big) y^\nu dy + \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\nu-\mu-\alpha}{2})}\\
&\quad\times\int_x^\infty f(y)\,{}_2F_1\Big(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\Big) y^{\nu-\mu-\alpha-1} dy\Bigg].
\end{aligned} \tag{A.7}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Corollary A.1 *Let $f \in L^2(0,\infty)$, $\alpha = -\mu$; $\nu = 0$. For $\mu > 0$ we obtain the operator*

$$\left(T_{0,\mu}^{(-\mu)} f\right)(x) = \frac{2\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}\Gamma(\mu/2)} x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu}{2}-1} dy, \tag{A.8}$$

such that

$$T_{0,\mu}^{(-\mu)} D^2 = B_\mu T_{0,\mu}^{(-\mu)} \tag{A.9}$$

and $T_{0,\mu}^{(-\mu)} 1 = 1$,

The operator (A.8) is the well-known Poisson operator (see [47]). We will use conventional symbol

$$\mathcal{P}_x^\mu f(x) = C(\mu) x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu}{2}-1} dy, \tag{A.10}$$

$$\mathcal{P}_x^\mu 1 = 1, \quad C(\mu) = \frac{2\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\mu}{2})}.$$

We remark that if $u = u(x,t)$, $x, t \in \mathbb{R}$, $u(x,0) = f(x)$ and $u_t(x,0) = 0$, then

$$\mathcal{P}_t^\mu u(x,t)|_{t=0} = f(x), \quad \frac{\partial}{\partial t} \mathcal{P}_t^\mu u(x,t)\Big|_{t=0} = 0. \tag{A.11}$$

Indeed, we have

$$\mathcal{P}_t^\mu u(x,t)|_{t=0} = C(\mu) t^{1-\mu} \int_0^t u(x,y)(t^2-y^2)^{\frac{\mu}{2}-1} dy\Big|_{t=0}$$
$$= C(\mu) \int_0^1 u(x,ty)|_{t=0} (1-y^2)^{\frac{\mu}{2}-1} dy = f(x)$$

and

$$\frac{\partial}{\partial t} \mathcal{P}_t^\mu u(x,t)\Big|_{t=0} = C(\mu) \int_0^1 u_t(x,ty)|_{t=0} (1-y^2)^{\frac{\mu}{2}-1} dy = 0.$$

Corollary A.2 *For $f \in L^2(0,\infty)$, $\alpha = \nu - \mu$; $-1 < \operatorname{Re}\nu < \operatorname{Re}\mu$ we obtain the first "descent" operator*

$$\left(T_{\nu,\mu}^{(\nu-\mu)} f\right)(x) = \frac{2\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu-\nu}{2})\Gamma(\frac{\nu+1}{2})} x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy. \tag{A.12}$$

such that

$$T_{\nu,\mu}^{(\nu-\mu)} B_\nu = B_\mu T_{\nu,\mu}^{(\nu-\mu)}, \quad T_{\nu,\mu}^{(\nu-\mu)} 1 = 1.$$

Corollary A.3 *Let $f \in L_{1,w}$ with the weight function $w(y) = |y|^{\operatorname{Re}\nu - \operatorname{Re}\mu}$, $\alpha = 0$, $-1 < \operatorname{Re}\mu < \operatorname{Re}\nu$. In this case we obtain the second "descent" operator:*

$$\left(T_{\nu,\mu}^{(0)} f\right)(x) = \frac{2\Gamma(\nu-\mu)}{\Gamma^2(\frac{\nu-\mu}{2})} \int_x^\infty f(y)(y^2-x^2)^{\frac{\nu-\mu}{2}-1} y\, dy. \tag{A.13}$$

In [44] the formula (A.13) was obtained as a particular case of Buschman-Erdelyi operator of the third kind but with different constant:

$$\left(T_{\nu,\mu}^{(0)} f\right)(x) = \frac{2^{1-\frac{\nu-\mu}{2}}}{\Gamma(\frac{\nu-\mu}{2})} \int_x^\infty f(y) y \left(y^2-x^2\right)^{\frac{\nu-\mu}{2}-1} dy. \tag{A.14}$$

As might be seen in the form (A.13) as well as (A.14) the operator $T_{\nu,\mu}^{(0)}$ does not depend on the values ν and μ but only on the difference between ν and μ.

Corollary A.4 *Let $f \in L^2(0,\infty)$, $\operatorname{Re}(\alpha+\nu+1) > 0$, $\operatorname{Re}\alpha < 0$. If we take $\mu = \nu$ in (A.7) we obtain the operator*

$$\left(T_{\nu,\nu}^{(\alpha)} f\right)(x) = \frac{2^{\alpha+3}\Gamma(\frac{\alpha+\nu+1}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(\frac{\nu+1}{2})} \Big[x^{-1-\nu-\alpha} \int_0^x f(y) \times {}_2F_1\Big(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\Big) y^\nu dy + \int_x^\infty f(y) {}_2F_1\Big(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{x^2}{y^2}\Big) y^{-\alpha-1} dy \Big] \tag{A.15}$$

which is an explicit integral representation of the negative fractional power α of the Bessel operator: B_ν^α.

So it is possible and easy to obtain fractional powers of the Bessel operator by ITCM. For different approaches to fractional powers of the Bessel operator and its explicit integral representations cf. [9, 43, 51–54, 60–62].

Theorem A.2 *If we apply ITCM with* $\varphi(t) = j_{\frac{\nu-1}{2}}(zt)$ *in* (A.6) *and with* $\mu = \nu$ *then the operator*

$$\begin{aligned}&\left(T^{(\varphi)}_{\nu,\nu} f\right)(x)\\ &= {}^{\nu}T^z_x f(x) = H_\nu^{-1}\big[j_{\frac{\nu-1}{2}}(zt) H_\nu[f](t)\big](x)\\ &= \frac{2^\nu \Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}(4xz)^{\nu-1}\Gamma(\frac{\nu}{2})}\int_{|x-z|}^{x+z} f(y)y[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1}dy\end{aligned} \tag{A.16}$$

coincides with the generalized translation operator (see [47–49]), for which the next properties are valid

$$^{\nu}T^z_x (B_\nu)_x = (B_\nu)_z \, {}^{\nu}T^z_x, \tag{A.17}$$

$$^{\nu}T^z_x f(x)|_{z=0} = f(x), \quad \frac{\partial}{\partial z}\, {}^{\nu}T^z_x f(x)\Big|_{z=0} = 0. \tag{A.18}$$

More frequently used representation of generalized translation operator ${}^{\nu}T^x_z$ is (see [47–49])

$$^{\nu}T^z_x f(x) = C(\nu)\int_0^\pi f(\sqrt{x^2+z^2-2xz\cos\varphi})\sin^{\nu-1}\varphi d\varphi, \tag{A.19}$$

$$C(\nu) = \left(\int_0^\pi \sin^{\nu-1}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\;\Gamma(\frac{\nu}{2})}.$$

It is easy to see that it is the same as ours.

So it is possible and easy to obtain generalized translation operators by ITCM, and its basic properties follows immediately from ITCM integral representation.

Application of Transmutations Obtained by ITCM to Integral Representations of Solutions to Hyperbolic Equations with Bessel Operators

Let us solve the problem of obtaining transmutations by ITCM (step 1) and justify integral representation and proper function classes for it (step 2). Now consider applications of these transmutations to integral representations of solutions to hyperbolic equations with Bessel operators (step 3). For simplicity we consider model equations, for them integral representations of solutions are mostly known. More complex problems need more detailed and spacious calculations. But even for these model problems considered below application of the transmutation method based on ITCM is new, it allows more unified and simplified approach to hyperbolic equations with Bessel operators of EPD/GEPD types.

Standard approach for solving differential equations is to find its general solution first, and then substitute given functions to find particular solutions. Here we will show how to obtain general solution of EPD type equation using transmutation operators.

Proposition A.3 *A general solution of the equation*

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \quad u = u(x, t; \mu) \tag{A.20}$$

for $0 < \mu < 1$ *is represented in the form*

$$u = \int_0^1 \frac{\Phi(x + t(2p - 1))}{(p(1 - p))^{1-\frac{\mu}{2}}}\, dp + t^{1-\mu} \int_0^1 \frac{\Psi(x + t(2p - 1))}{(p(1 - p))^{\mu/2}}\, dp, \tag{A.21}$$

with a pair of arbitrary functions Φ, Ψ.

Proof First, we consider the wave equation when $a = 1$,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{A.22}$$

A general solution to this equation has the form

$$F(x + t) + G(x - t), \tag{A.23}$$

where F and G are arbitrary functions. Applying operator (A.10) (obtained by ITCM) by variable t we obtain that one solution to the Eq. (A.20) is

$$u_1 = 2C(\mu) \frac{1}{t^{\mu-1}} \int_0^t [F(x + z) + G(x - z)](t^2 - z^2)^{\frac{\mu}{2}-1}\, dz.$$

Let us transform the resulting general solution as follows

$$u_1 = \frac{C(\mu)}{t^{\mu-1}} \int_{-t}^t \frac{F(x + z) + F(x - z) + G(x + z) + G(x - z)}{(t^2 - z^2)^{1-\frac{\mu}{2}}}\, dz.$$

Introducing a new variable p by formula $z = t(2p - 1)$ we obtain

$$u_1 = \int_0^1 \frac{\Phi(x + t(2p - 1))}{(p(1 - p))^{1-\frac{\mu}{2}}}\, dp,$$

where

$$\Phi(x+z)=[F(x+z)+F(x-z)+G(x+z)+G(x-z)]$$

is an arbitrary function.

It is easy to see that if $u(x,t;\mu)$ is a solution of (A.20) then $t^{1-\mu}u(x,t;2-\mu)$ is also a solution. Therefore the second solution to (A.20) is

$$u_2 = t^{1-\mu}\int_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\mu/2}}\,dp,$$

where Ψ is an arbitrary function, not coinciding with Φ. Summing u_1 and u_2 we obtain general solution to (A.20) of the form (A.21). From the (A.21) we can see that for summable functions Φ and Ψ such a solution exists for $0<\mu<1$. □

Now we derive a general solution to GEPD type equation by transmutation method.

Proposition A.4 *A general solution to the equation*

$$(B_\nu)_x u = (B_\mu)_t u, \quad u = u(x,t;\nu,\mu) \tag{A.24}$$

for $0<\mu<1$, $0<\nu<1$ *is*

$$\begin{aligned} u = \frac{2\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{\nu}{2})}\Big(x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy\int_0^1 \frac{\Phi(y+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}\,dp \\ + t^{1-\mu}x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy\int_0^1 \frac{\Psi(y+t(2p-1))}{(p(1-p))^{\mu/2}}\,dp.\Big) \end{aligned} \tag{A.25}$$

Proof Applying the Poisson operator (A.10) (again obtained by ITCM) with index ν by variable x to the (A.21) we derive general solution (A.25) to the Eq. (A.24). □

Now let apply transmutations for finding general solution to GEPD type equation with spectral parameter.

Proposition A.5 *A general solution to the equation*

$$(B_\nu)_x u = (B_\mu)_t u + b^2 u, \quad u = u(x,t;\nu,\mu) \tag{A.26}$$

for $0 < \mu < 1$, $0 < \nu < 1$ *is*

$$\begin{aligned} u = \frac{2\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\Big(x^{1-\nu} \int_0^x (x^2 - y^2)^{\frac{\nu}{2}-1} dy \\ \times \int_0^1 \frac{\Phi(y + t(2p - 1))}{(p(1-p))^{1-\frac{\mu}{2}}} j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\, dp \\ + t^{1-\mu} x^{1-\nu} \int_0^x (x^2 - y^2)^{\frac{\nu}{2}-1} dy \\ \times \int_0^1 \frac{\Psi(y + t(2p - 1))}{(p(1-p))^{\mu/2}} j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\, dp.\Big) \end{aligned} \tag{A.27}$$

Proof A general solution to the equation

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u + b^2 u, \quad u = u(x, t; \mu), \quad 0 < \mu < 1$$

is (see [24, p. 328])

$$\begin{aligned} u = \int_0^1 \frac{\Phi(x + t(2p - 1))}{(p(1-p))^{1-\frac{\mu}{2}}} j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\, dp \\ + t^{1-\mu} \int_0^1 \frac{\Psi(x + t(2p - 1))}{(p(1-p))^{\mu/2}} j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\, dp. \end{aligned}$$

Applying Poisson operator (A.10) (again obtained by ITCM) with index ν by variable x to the (A.21) we derive general solution (A.25) to the Eq. (A.24). □

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