Gravitational partition function modified by superlight braneworld perturbative modes

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In this paper, we will analyze the effects of superlight braneworld perturbative modes on clustering of galaxies. In the present manuscript, we use the Boltzmann and Tsallis statistical approaches to study the large distance modification of gravity in the braneworld. The impact of modified potential on clustering of galaxies is analyzed in both the approaches. The infinities associated with Newtonian point mass approximation of gravity models is removed using the analytical extensions. The regularized and finite partition function is obtained for the system of galaxies in the braneworld model and is used to evaluate the regularized thermodynamic properties of the system in the form of equations of state. Hence, we study thermodynamic quantities and discuss about the thermodynamic stability of the model. We find that large number of galaxies may lead to the thermodynamic instability. The appealing feature of the paper revolve around the generalization of the dimensional regularization (GDR) of Bollini and Giambiagi supplemented with the statistical adequacy of Boltzmann and Tsallis.

I. INTRODUCTION

In this paper, we will analyze a large distance modification to the gravitational partition function. This gravitational partition function has been used in analyzing the large scale structure of our universe, and hence any large modification of the gravitational potential would modify it.

Clustering of galaxies and the structure formation in the expanding universe using Newtonian potential as a source of attraction and by approximating the galaxies as point masses has been extensively analyzed [1]; [2]; [3]. The study has been followed and improved to analyze the structure formation for extended objects [4,5].

An empirical modification of Newtonian dynamics (MOND), has been proposed and discussed at low accelerations [6–8]. Gravitational clustering with modified potentials have been studied in various modifications of Newtonian potential to explain clustering [9–14]. The modified theories like the nonlocal extensions of GR can

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be a possible treatment for the singularities like black hole, big bang and the cosmic expansion [15–19].

In most of the studies the standard techniques of statistical mechanics have been followed [20,21]. In our early studies we have analyzed the clustering of galaxies using a Newtonian potential modified by superlight modes of a braneworld model and obtained considerable effects on the clustering parameter and have elaborated the effects on cosmic energy equations [9]. The large distance corrections to the Newtonian potential from superlight modes in braneworld models have been studied by considering the universe a brane in higher dimensions [22]. The string theory due to the inherent characteristics of possessing extra dimensions thus, become the motivation to discuss these models [23]. Due to the lack of testing of general relativity and the corresponding Newtonian approximation at very large or very small distances and due to the propagation of gravity in to higher dimensional bulk, the Newtonian potential gets brane corrections [24]; [25]; [26]. Keeping in consideration the importance of models with superlight perturbation modes and the modification of gravitational interaction at astronomical scales, the form of Newtonian potential used in this paper to study the clustering can be a promising candidate to resolve the issue of dark matter in galaxies [22].

It may be noted that most of the work done on clustering of galaxies, using this formalism, has been done using the Boltzmann-Gibbs entropy. However, it can be argued that as the Boltzmann-Gibbs entropy is based on the extensive property of the system, it might not explain the gravitational systems, as these systems can violate the extensive property [27]. Such a violation occurs because of breakdown of ergodicity for such gravitational systems [28,29]. Thus, it has been suggested that such system can be better described by Tsallis statistical mechanics [30,31]. In fact, it has been demonstrated that the Tsallis statistical mechanics can be used to study self-interacting gravitational particles, even though such systems can violate the extensive property [32]. So, we will also analyze the Tsallis statistical mechanics for this system of galaxies corrected by superlight braneworld perturbative modes.

Though in our early studies we have analyzed the braneworld modified Newtonian potential to galactic clustering using the techniques of statistical mechanics and have further modified the potential using the softening parameter to eliminate the mathematical divergences. In the present study we use the Boltzmann statistical as well as Tsallis statistical approaches to study the gravitating gas interacting through brane potential. Dimensional regularization of partition function to get the divergence-free results is the main thrust involved in the paper. The paper involves rigorous work on dimensional regularization to regularize the thermodynamic properties in both the approaches. Besides giving physical impact of the large distance modification of gravity in the braneworld, the mathematics of the paper is quite appealing due to the generalization of the dimensional regularization of Bollini and Giambiagi [33–35]. This generalization was based on the general quantification method of QFT's [36–39] using ultradistributions of Sebastiao e Silva, also known as ultrahyperfunctions [40–42].

Our analysis is valid if the following assumptions are satisfied:

(1) The graviton is massless, as suggested by the LIGO experiment. In that case the graviton does not present polarizations that produce strong corrections to the gravitational potential, given for example by positive powers of r as shown in Ref. [43]. This was proved in Refs. [44,45], in which gravity is quantized following the paths suggested by Suraj N. Gupta and Richard P. Feynman, through the theory of ultrahyperfunctions [36–39]. This theory allows to quantize nonrenormalizable quantum field theories.

(2) The graviton is nonmassless. In that case the result of Ref. [43] shows that there is a polarization of the graviton that produces a strong correction that is a positive power of r. So, for our result to be valid, the sources must be beyond the radius below which the additional polarizations of the graviton are strongly interacting.

II. PARTITION FUNCTION

Let us consider the distribution $\frac{1}{r} = PV\frac{1}{r}$. Then $\frac{1}{r}|_{r=0} = 0$. The general partition function of a system of N particles of mass m interacting through the modified gravitational potential with potential energy is Φ, can be written as

$$\mathcal{Z}_\nu = \frac{1}{N!} \int d^\nu p \, d^\nu r$$
$$\times \exp\left(-\left[\sum_{i=1}^N \frac{p_i^2}{2m} + \Phi(r_1, r_2, r_3, \ldots, r_N)\right]T^{-1}\right), \quad (2.1)$$

The interaction potential energy between galaxies in braneworld model with large extra dimensions can be written as [23,46]

$$\phi_{i,j} = -\frac{Gm^2}{r_{ij}}\left(1 + \frac{k_m}{r_{ij}^2}\right), \quad (2.2)$$

For N galaxies the potential energy can be expressed as

$$\phi_{i,j} = -\frac{N(N-1)Gm^2}{2r_{ij}}\left(1 + \frac{k_m}{r_{ij}^2}\right), \quad (2.3)$$

The partition function $\mathcal{Z}$ in ν dimensions.

$$\mathcal{Z}_\nu = \frac{1}{N!} \int_{-\infty}^{\infty} d^\nu x$$
$$\times \int_{-\infty}^{\infty} d^\nu p \exp\left[\beta\left(\frac{N(N-1)Gm^2}{2r}\left(1 + \frac{k_m}{r^2}\right) - \frac{Np^2}{2m}\right)\right]$$
$$(2.4)$$

Evaluating the integral over the angles we have:

$$\mathcal{Z}_\nu = \frac{1}{N!}\left(\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right)^2$$
$$\times \int_0^\infty r^{\nu-1} dr \exp\left(\beta\frac{N(N-1)Gm^2}{2r}\left(1 + \frac{k_m}{r^2}\right)\right)$$
$$\times \int_0^\infty p^{\nu-1} dp \exp\left(-\frac{Np^2}{2m}\right) \quad (2.5)$$

For the integral over the momentum we obtain:

$$\int_0^\infty p^{\nu-1} dr \exp\left(-\beta\left(\frac{Np^2}{2m}\right)\right) = \frac{\Gamma(\frac{\nu}{2})(2m)^{\frac{\nu}{2}}}{2(\beta N)^{\frac{\nu}{2}}} \quad (2.6)$$

To solve the integral over r

$$\int_0^\infty r^{\nu-1} dr \exp\left(\beta\frac{N(N-1)Gm^2}{2r}\left(1 + \frac{k_m}{r^2}\right)\right)$$

we make use of the approximation:

$$\int_0^\infty x^{-\nu-1} dx \exp(\beta_1 x + \beta_2 x^3)$$
$$= \int_0^\infty x^{-\nu-1} dx \exp(\beta_1 x)(1 + \beta_2 x^3).$$

Thus we get:

$$\int_0^\infty x^{-\nu-1} dx \exp(\beta_1 x) + \beta_2 \int_0^\infty x^{-\nu+2} dx (\exp(\beta_1 x))$$
$$= \beta_1^\nu \left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \frac{\beta_2}{\beta_1^3} \right] \Gamma(3-\nu).$$

Using this result, we have for $\mathcal{Z}$ the expression:

$$\mathcal{Z}_\nu = \frac{1}{N!} \frac{2\pi^\nu}{\Gamma(\frac{\nu}{2})} \left(\frac{2m}{\beta N} \right)^{\frac{\nu}{2}} \left(\beta \frac{N(N-1)Gm^2}{2} \right)^\nu$$
$$\times \left[\frac{1}{-\nu(1-\nu)(2-\nu)} + k_m \left(\beta \frac{N(N-1)Gm^2}{2} \right)^{-2} \right] \Gamma(3-\nu)$$

$$(2.7)$$

Now, we proceed to do Laurent series development around $\nu = 3$. We write:

$$\mathcal{Z}_\nu = f(\nu) \Gamma(3-\nu) \qquad (2.8)$$

where

$$f(\nu) = \frac{1}{N!} \frac{2\pi^\nu}{\Gamma(\frac{\nu}{2})} \left(\frac{2m}{\beta N} \right)^{\frac{\nu}{2}} \left(\beta \frac{N(N-1)Gm^2}{2} \right)^\nu$$
$$\times \left[\frac{1}{-\nu(1-\nu)(2-\nu)} + k_m \left(\beta \frac{N(N-1)Gm^2}{2} \right)^{-2} \right]$$

$$(2.9)$$

In three dimensions it is

$$f(3) = \frac{1}{N!} 4\pi^{\frac{5}{2}} \left(\frac{2m}{\beta N} \right)^{\frac{3}{2}} \left[-\frac{1}{6}\alpha + k_m \alpha \right], \qquad (2.10)$$

where:

$$\alpha = \beta \frac{N(N-1)Gm^2}{2}. \qquad (2.11)$$

We get:

$$\frac{f'(\nu)}{f(\nu)} = \frac{\nu}{2} \psi\left(\frac{\nu}{2}\right) + \frac{1}{2} \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2} \right)$$
$$+ \frac{-2 + 6\nu - 3\nu^2}{-\nu(1-\nu)(2-\nu) + k_m \alpha^{-2} \nu^2 (1-\nu)^2 (2-\nu)^2}.$$

$$(2.12)$$

In three dimensions (2.12) takes the form:

$$\frac{f'(3)}{f(3)} = \frac{3}{2} \psi\left(\frac{3}{2}\right) + \frac{1}{2} \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2} \right)$$
$$+ \frac{11}{6} \left(\frac{\alpha^2}{\alpha^2 - 6k_m} \right). \qquad (2.13)$$

For the Γ function we have

$$\Gamma(3-\nu) = \frac{1}{3-\nu} - C + \sum_{k=1}^\infty c_k (\nu - 3)^k. \qquad (2.14)$$

As a consequence the partition function in ν dimensions is

$$\mathcal{Z}_\nu = \frac{f(3)}{3-\nu} - f(3)C - f'(3) + \sum_{k=1}^\infty a_k (\nu - 3)^k. \qquad (2.15)$$

Thus:

$$\mathcal{Z} = -f(3)\left(C + \frac{f'(3)}{f(3)} \right). \qquad (2.16)$$
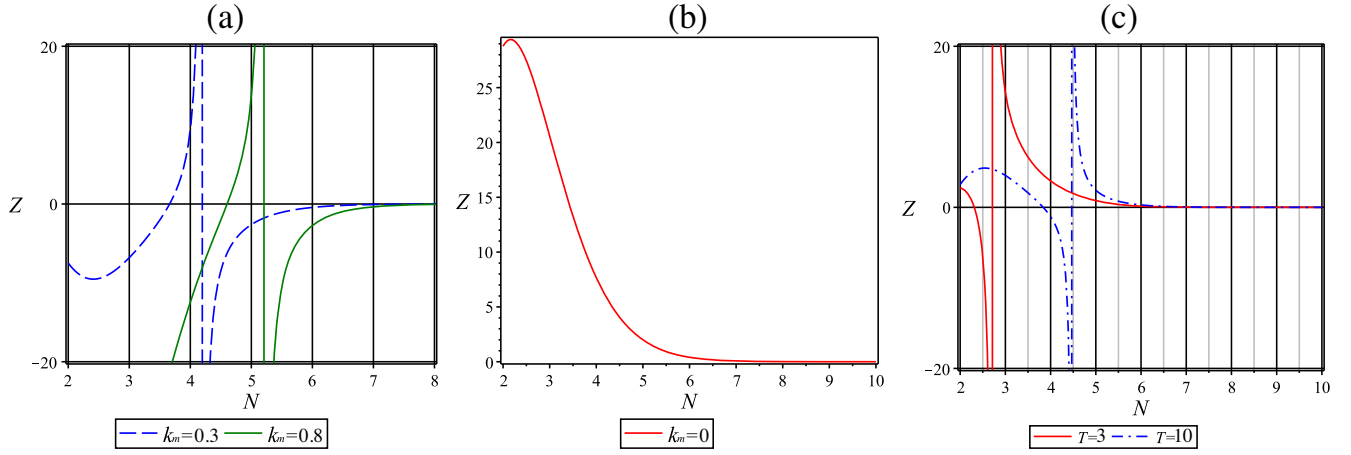
For three dimensions it is explicitly expressed as:

$$\mathcal{Z} = -\frac{4}{N!} \pi^{\frac{5}{2}} \left(\frac{2m}{\beta N} \right)^{\frac{3}{2}} \left[-\frac{1}{6}\alpha + k_m \alpha \right]$$
$$\times \left(C + \frac{3}{2}\psi\left(\frac{3}{2}\right) + \frac{1}{2}\ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2} \right) \right.$$
$$\left. + \frac{11}{6}\left(\frac{\alpha^2}{\alpha^2 - 6k_m} \right) \right). \qquad (2.17)$$

Finally after simplification we arrive at the expression of finite partition function:

$$\mathcal{Z} = \left(6 - C + \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{128} \right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2 - 6k_m} \right) \right)$$
$$\times \frac{2\alpha}{N!} \pi^{\frac{5}{2}} \left(\frac{2m}{\beta N} \right)^{\frac{3}{2}} \left(\frac{1}{6} - k_m \right). \qquad (2.18)$$

In Fig. 1 we can see behavior of the partition function for various values of the model parameters. The most important point here is to have positive partition function which is corresponding to the physical system. It help us to constrain model parameters. We can use this fact to restrict values of N and k_m. First of all we should note that interacting system

FIG. 1.   Partition function in the braneworld model with $G = m = 1$. (a) $T = 5$; (b) $T = 3$; (c) $k_m = 0.1$.

need $N \geq 2$. Hence, in the case of $k_m > \frac{1}{6}$, partition function is mostly negative except for special number of galaxies like $N = 4$ or $N = 5$ [see Fig. 1(a)]. On the other hand, for the case of $k_m < \frac{1}{6}$ we can obtain a lower bound for the number of galaxies. In the special case of $k_m = 0$ the partition function is completely positive for $N \geq 2$ (see Fig. 1(b) where drawn for $T = 3$ however other temperatures yield to the similar result). Finally, in Fig. 1(c) we can see $0 < k_m < \frac{1}{6}$ yields to positive partition function if $N > N_{\min}$. Value of $N_{\min}$ is depend on the temperature. Increasing temperature, increased value of $N_{\min}$.

III. THERMODYNAMIC PROPERTIES FROM BOLTZMANN STATISTICS

We know from Boltzmann Gibbs statistical mechanics and thermodynamics, that there are several physical quantities like average energy U, Helmholtz free energy F, entropy S and chemical potential μ, that can be of interest to a physical system. Therefore, we dedicate this section to demonstrate how we can calculate various thermodynamic properties of the gravitational system in question. These quantities may not have the same meaning as in usual thermodynamics since these are for a gravitational system, nonetheless, their physical significance is intact. As we shall see, these quantities will be crucial in further sections.

We start with calculating the mean energy $\langle \mathcal{U} \rangle$ of a system at constant N, V by using the formula:

$$\langle \mathcal{U} \rangle = -\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} \tag{3.1}$$

Then, we get for $\langle \mathcal{U} \rangle$:

$$\langle \mathcal{U} \rangle = \frac{1}{\beta} - \frac{1}{\beta} \left(\frac{1 - 44\left(\frac{k_m \alpha^2}{(\alpha^2 - 6k_m)^2}\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2 - 6k_m}\right)} \right). \tag{3.2}$$

It help us to obtain the specific heat via

$$C_v = \left(\frac{\partial U}{\partial T}\right)_V, \tag{3.3}$$

which yields to the following expression,

$$C_v = k + \frac{k\left(\frac{176\alpha^2 k_m}{(\alpha^2 - 6k_m)^3} - \frac{44\alpha^2 k_m}{(\alpha^2 - 6k_m)^2} - 1\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2 - 6k_m}\right)} - \frac{k\left(1 - \left(\frac{44 k_m \alpha^2}{(\alpha^2 - 6k_m)^2}\right)^2\right)}{\left(-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2 \beta N (N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2 - 6k_m}\right)\right)^2} \tag{3.4}$$

Figure 2 shows mean energy function in the braneworld model. In Fig. 3 we see that the specific heat is negative. Thus we see that the system is self-gravitating, as established by Lynden Bell in [47].
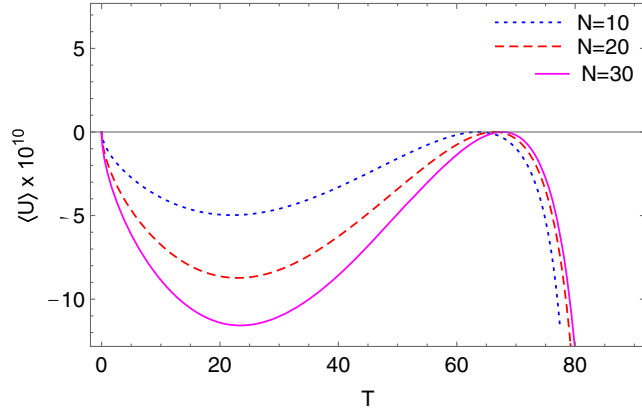
FIG. 2.   Mean energy function in the braneworld model with $G = m = 1$, $k_m = 1/6$.
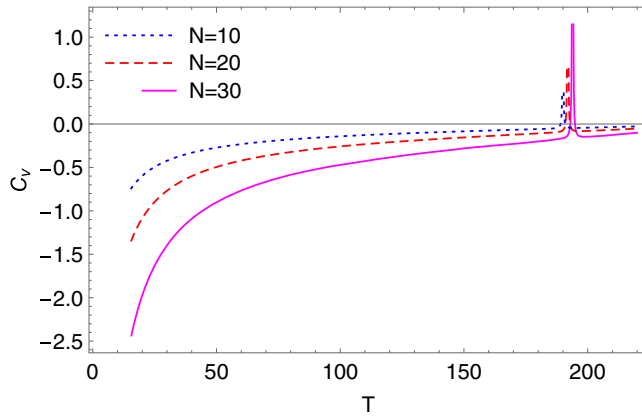

FIG. 4.   Free energy with $G = m = 1$, $k_m = 1/6$.

FIG. 3.   Specific heat of Boltzmann statistics with $G = m = 1$, $k_m = 1/6$.

FIG. 5.   Entropy of Boltzmann statistics with $G = m = 1$, $k_m = 1/6$.

Figure 4 shows free energy function and Fig. 5 entropy, in the braneworld model. Helmholtz free energy F is obtained using

$$F = -\frac{1}{\beta}\ln \mathscr{Z} \tag{3.5}$$

Therefore we get:

$$F = -\frac{1}{\beta}\left[\ln \pi + N - N \ln N + \ln(\alpha - 6k_m\alpha)\right.$$
$$+ \ln\left(18 - 3C - 18\ln 2\right.$$
$$\left.+ 3\ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11\alpha^2}{\alpha^2 - 6k_m}\right)\right] \tag{3.6}$$

The entropy S is obtained from
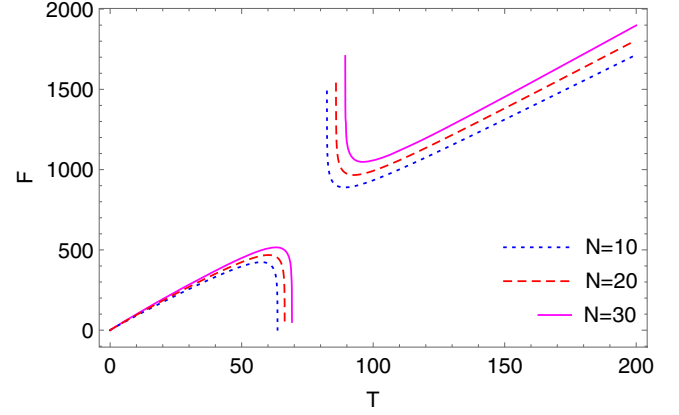
$$TS = U - F. \tag{3.7}$$

Its explicit expression is

$$S = k - k\left(\frac{1 - 44\left(\frac{k_m\alpha^2}{(\alpha^2 - 6k_m)^2}\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2 - 6k_m}\right)}\right)$$
$$+ k\left[\ln\pi + N - N\ln N + \ln(\alpha - 6k_m\alpha)\right.$$
$$+ \ln\left(18 - 3C - 18\ln 2\right.$$
$$\left.+ 3\ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11\alpha^2}{\alpha^2 - 6k_m}\right)\right]. \tag{3.8}$$

Analyzing the entropy shows positive region for $N_{\min} \leq N \leq N_{\max}$ in the case of $0 < k_m < \frac{1}{6}$, hence we find upper limit for the number of galaxies which is strongly depend on the temperature.

The chemical potential is defined as:

$$\mu = \left(\frac{\partial F}{\partial N}\right)_T \tag{3.9}$$

Accordingly:

$$\mu = -\frac{1}{\beta}\left[-\ln N + \frac{2N-1}{N(N-1)} + \frac{\frac{3(3N-1)}{N(N-1)} + \frac{2N-1}{N(N-1)}\left(\frac{22\alpha^2}{\alpha^2-6k_m} - \frac{22\alpha^4}{(\alpha^2-6k_m)^2}\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2-6k_m}\right)}\right]. \tag{3.10}$$

In addition, we have the following formula, which we will use later:

$$\exp\beta N\mu = N^N \exp\left[\frac{2N-1}{(N-1)} + \frac{\frac{3(3N-1)}{(N-1)} + \frac{2N-1}{(N-1)}\left(\frac{22\alpha^2}{\alpha^2-6k_m} - \frac{22\alpha^4}{(\alpha^2-6k_m)^2}\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2-6k_m}\right)}\right]. \tag{3.11}$$

Pressure P is obtained using

$$PV = \frac{2N}{3}\langle\mathcal{U}\rangle. \tag{3.12}$$

Using (3.2) we arrive to the state equation:

$$\beta PV = \frac{2N}{3} - \frac{2N}{3}\left(\frac{1 - 44\left(\frac{k_m\alpha^2}{(\alpha^2-6k_m)^2}\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2-6k_m}\right)}\right). \tag{3.13}$$

In Fig. 7 we can see equation of state (EoS) for various interaction strength. In the case of $k_m = 0$ we can see linear behavior. For the large N limit, there is no important interaction effect.

A. Probability distribution function

Up to this point, we have dealt with the microcanonical ensemble. However, now we shall define the grand canonical partition function Z_G to calculate the probability distribution of the system. We define Z_G as follows

$$\ln Z_G = \beta PV \tag{3.14}$$

From chemical potential μ we find the fugacity z as

$$z^N = e^{N\beta\mu}. \tag{3.15}$$

The distribution function follows as:

$$F(N) = \frac{z^N \mathcal{Z}}{Z_G}. \tag{3.16}$$

Using (3.11), the distribution function can be written as:

$$F(N) = \frac{2N^N\pi^{\frac{5}{2}}}{N!}\left(\frac{2m}{\beta N}\right)^{\frac{3}{2}}\left[\frac{1}{6}\alpha - k_m\alpha\right]\exp\left[\frac{2N-1}{(N-1)} - \frac{2N}{3} + \frac{\frac{3(3N-1)}{(N-1)} + \frac{2N-1}{(N-1)}\left(\frac{22\alpha^2}{\alpha^2-6k_m} - \frac{22\alpha^4}{(\alpha^2-6k_m)^2}\right) + \frac{2N}{3}\left(1 - 44\left(\frac{k_m\alpha^2}{(\alpha^2-6k_m)^2}\right)\right)}{-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2-6k_m}\right)}\right]$$

$$\times\left(-C - 6\ln 2 + 6 + \ln\left(\frac{\pi^2\beta N(N-1)^2 G^2 m^5}{2}\right) + \frac{11}{3}\left(\frac{\alpha^2}{\alpha^2-6k_m}\right)\right). \tag{3.17}$$

IV. TSALLIS PARTITION FUNCTION OF THE SYSTEM IN THE BRANEWORLD MODEL

In 1988, C. Tsallis proposed a nonextensive generalization of statistical mechanics (widely called as Tsallis statistical mechanics) [27]. Based on the motivations laid out in the introduction, we now try to investigate the many-body gravitational system using Tsallis's formalism. Tsallis q-exponential is defined as the distribution:

$$e_q(x) = [1 + (q-1)x]_+^{\frac{1}{q-1}} \tag{4.1}$$

Or equivalently

$$e_q(x) = \begin{cases} 1 + (q-1)x]^{\frac{1}{q-1}}; & 1 + (q-1)x > 0 \\ 0; & 1 + (q-1)x < 0. \end{cases} \tag{4.2}$$

Let us consider again the distribution: $\frac{1}{r} = PV\frac{1}{r}$. First we do the calculation in ν dimensions and assume $q > 1$. Thus

$$\mathcal{Z}_\nu = \int_{-\infty}^{\infty} d^\nu x \int_{-\infty}^{\infty} d^\nu p \left[1 + (q-1)\beta \left(\frac{N(N-1)Gm^2}{2r} + \frac{kN(N-1)Gm^2}{2r^3} - \frac{Np^2}{2m} \right) \right]_+^{\frac{1}{q-1}}. \tag{4.3}$$

Here we put $q = 4/3$. The choice of that value of q is not arbitrary. It is the value of q where Verlinde's conjecture of emergent gravity could be proved, in the nonrelativistic case [48]. Accordingly:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \right]^2 \int_0^{\infty} r^{\nu-1} dr \int_0^{\infty} p^{\nu-1} dp \left[1 + \beta \left(\frac{N(N-1)Gm^2}{6r} + \frac{N(N-1)Gm^2}{6r^3} - \frac{Np^2}{6m} \right) \right]_+^3 \tag{4.4}$$

The expression for $\mathcal{Z}$ in (4.4) can be rewritten in the form:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \right]^2 \left(\frac{6m}{\beta N} \right)^{\frac{\nu}{2}} B\left(4, \frac{\nu}{2} \right) \int_0^{\infty} r^{-\frac{\nu}{2}-10} dr (r^3 + \alpha_t r + k\alpha_t)^{\frac{\nu}{2}+3} \tag{4.5}$$

where

$$\alpha_t = \frac{\beta N(N-1)Gm^2}{6}$$

Hence we can write

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \right]^2 \left(\frac{6m}{\beta N} \right)^{\frac{\nu}{2}} B\left(4, \frac{\nu}{2} \right) \int_0^{\infty} r^{-\frac{\nu}{2}-10} dr (r+b)^{\frac{\nu}{2}+3} (r-r_1)^{\frac{\nu}{2}+3} (r-r_3)^{\frac{\nu}{2}+3} \tag{4.6}$$

where r_1, $r_2 = -b$, and r_3 are the cube roots of the integrand. Also,

$$r_1 = \gamma_1 + e^{\frac{i\pi}{3}} \gamma_2$$

$$r_3 = \gamma_1 + e^{-\frac{i\pi}{3}} \gamma_2$$

$$b = \gamma_2 - \gamma_1 - \frac{\alpha}{3}$$

$$\gamma_1 = \left(\sqrt{\frac{1}{4} \left(\frac{2\alpha_t^3}{27} + \alpha_t k \right)^2 + \frac{1}{27} \left(\frac{\alpha}{3} (\alpha_t - 6) \right)^2} - \frac{1}{2} \left(\frac{2\alpha_t^3}{27} + \alpha_t k \right) \right)^{\frac{1}{3}}$$

$$\gamma_2 = \left(\sqrt{\frac{1}{4} \left(\frac{2\alpha_t^3}{27} + \alpha_t k \right)^2 + \frac{1}{27} \left(\frac{\alpha}{3} (\alpha_t - 6) \right)^2} + \frac{1}{2} \left(\frac{2\alpha_t^3}{27} + \alpha_t k \right) \right)^{\frac{1}{3}}$$

After solving the integral we have

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \right]^2 \left(\frac{6m}{\beta N} \right)^{\frac{\nu}{2}} B\left(4, \frac{\nu}{2} \right) b^\nu \frac{\Gamma(-\frac{\nu}{2}-9)}{\Gamma(-\frac{3\nu}{2}-9)} \Gamma(-\nu) F\left(-\nu, -\frac{\nu}{2}-3, -\frac{\nu}{2}-3, -\frac{3\nu}{2}-9; 1+\frac{r_1}{b}; 1+\frac{r_3}{b} \right). \tag{4.7}$$

Here, F is the hypergeometric function of two variables. After a long calculation we get (See Appendix):

$$\mathcal{Z}_\nu = \frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(-\frac{\nu}{2}-9)}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(-\frac{3\nu}{2}-9)}$$
$$\times (\Gamma(3-\nu)h_\nu + \phi_\nu(z_1,z_2)) \qquad (4.8)$$

A. Evaluation of partition function

Now we do the Laurent development explicitly, taking into account that $\phi_\nu(z_1,z_2)$ does not have a pole at $\nu = 3$ and take the expression for ϕ_3 at the end of the development. Thus the Laurent development goes as:

We can write

$$\mathcal{Z}_\nu = \Gamma(3-\nu)f_\nu + g_\nu\phi_\nu(z_1,z_2) \qquad (4.9)$$

where

$$f_\nu = \frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(-\frac{\nu}{2}-9)}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(-\frac{3\nu}{2}-9)}h_\nu$$
$$(4.10)$$

$$f_\nu = \frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(\frac{3\nu}{2}+10)\sin\frac{3\nu\pi}{2}}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(\frac{\nu}{2}+10)\sin\frac{\pi\nu}{2}}h_\nu$$
$$(4.11)$$

$$g_\nu = \frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(\frac{3\nu}{2}+10)\sin\frac{3\nu\pi}{2}}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(\frac{\nu}{2}+10)\sin\frac{\pi\nu}{2}}$$
$$(4.12)$$

$$f_\nu = f_3 + f_3'(\nu-3) + \sum_{k=2}^{\infty} b_k(\nu-3)^k. \qquad (4.13)$$

For the Γ function we have

$$\Gamma(3-\nu) = \frac{1}{3-\nu} - C + \sum_{k=1}^{\infty} c_k(\nu-3)^k. \qquad (4.14)$$

As a consequence:

$$\mathcal{Z}_\nu = \frac{f_3}{3-\nu} - f_3 C - f_3'$$
$$+ \sum_{k=1}^{\infty} a_k(\nu-3)^k + g(3)\phi_3(s). \qquad (4.15)$$

Now we write partition function as:

$$\mathcal{Z} = -f_3\left(C + \frac{f_3'}{f_3}\right) + g_3\phi_3(z_1,z_2) \qquad (4.16)$$

where:

$$f_3 = -\frac{256\pi^2}{105}\left(\frac{6m}{\beta N}\right)^{\frac{3}{2}} b^3 \frac{\Gamma(\frac{29}{2})}{\Gamma(\frac{23}{2})}h_3 \qquad (4.17)$$

and

$$g_3 = -\frac{256\pi^2}{105}\left(\frac{6m}{\beta N}\right)^{\frac{3}{2}} b^3 \frac{\Gamma(\frac{29}{2})}{\Gamma(\frac{23}{2})}. \qquad (4.18)$$

We get:

$$\frac{f_3'}{f_3} = \frac{1}{2}\ln\frac{6m\pi^2 b^2}{\beta N} - 2\psi\left(\frac{3}{2}\right) + \psi\left(\frac{29}{2}\right) - \psi\left(\frac{23}{2}\right)$$
$$-\frac{143}{315} + \frac{h_3'}{h_3} \qquad (4.19)$$

and also:

$$h_3 = \left(-\frac{1}{6} + \frac{1}{6}(z_1 + z_2) - \frac{1}{75}(18z_1 z_2 + 7z_1^2 + 7z_2^2) + \frac{21}{31050}(5z_1^3 + 5z_2^3 - 27z_1^2 z_2 - 27z_1 z_2^2)\right) \qquad (4.20)$$

$$h_3' = \left(\frac{11}{36} - \frac{1}{4}(z_1 + z_2) + \frac{454}{1875}(z_1 z_2) + \frac{171}{1875}(z_1^2 + z_2^2) + \frac{62}{198375}(z_1^3 + z_2^3) + \frac{3088}{991875}(z_1^2 z_2 + z_1 z_2^2)\right). \qquad (4.21)$$

Finally we obtain the finite expression for partition function which is divergence free and write it as:

$$\mathcal{Z} = -g_3\left[h_3\left(C + \frac{f_3'}{f_3}\right) - \phi_3(z_1,z_2)\right]. \qquad (4.22)$$

B. Mean energy in ν dimensions

We now proceed to evaluate the average energy of the system. For it is

$$\mathcal{Z}\langle\mathcal{U}\rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \int_0^\infty r^{\nu-1}dr \int_0^{P_0} p^{\nu-1}dp \left(-\frac{N(N-1)Gm^2}{2r} - \frac{kN(N-1)Gm^2}{2r^2} + \frac{Np^2}{2m}\right)$$
$$\times \left[1 + (q-1)\beta\left(\frac{N(N-1)Gm^2}{2r} + \frac{kN(N-1)Gm^2}{2r^3} - \frac{Np^2}{2m}\right)\right]^{\frac{1}{q-1}} \tag{4.23}$$

Or, equivalently:

$$\mathcal{Z}\langle U\rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}}\right]^2 \left[\frac{\beta N(q-1)}{2m}\right]^{\frac{1}{q-1}}$$
$$\times \left(\int_0^\infty r^{\nu-1}dr \int_0^{P_0} \frac{Np^{\nu+1}}{2m}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}}dp\right.$$
$$-\frac{N(N-1)Gm^2}{2}\int_0^\infty r^{\nu-2}dr \int_0^{P_0} p^{\nu-1}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}}dp$$
$$\left.-\frac{kN(N-1)Gm^2}{2}\int_0^\infty r^{\nu-3}dr \int_0^{P_0} p^{\nu-1}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}}dp\right). \tag{4.24}$$

For $q = 4/3$ we obtain:

$$\mathcal{Z}\langle U\rangle_\nu = \frac{3}{2\beta}\left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \left(\frac{6m}{N\beta}\right)^{\frac{\nu}{2}}\left[-\alpha B\left(4,\frac{\nu}{2}\right)I_1 - k\alpha B\left(4,\frac{\nu}{2}\right)I_2 + B\left(4,\frac{\nu}{2}+1\right)I_3\right]. \tag{4.25}$$

Again, after a long calculation we obtain (See Appendix):

$$\mathcal{Z}\langle U\rangle_\nu = G_\nu[\Gamma(3-\nu)F_\nu + \Phi_\nu(z_1,z_2)], \tag{4.26}$$

where we have defined:

$$\Phi_\nu(z_1,z_2) = \phi_1 + \phi_2 + \phi_3$$

with:

$$G_\nu = \frac{3}{2\beta}\frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}}b^\nu\frac{\Gamma(\frac{3\nu}{2}+11)\sin\frac{3\nu\pi}{2}}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(\frac{\nu}{2}+11)\sin\frac{\pi\nu}{2}} \tag{4.27}$$

and:

$$F_\nu = \left[-\alpha h_1(\nu) - k\alpha h_2(\nu)\frac{(\frac{3\nu}{2}+12)(\frac{3\nu}{2}+11)}{(\frac{\nu}{2}+12)(\frac{\nu}{2}+11)}\right.$$
$$\left.+ h_3(\nu)\frac{(\frac{3\nu}{2}+12)(\frac{3\nu}{2}+11)}{(\frac{\nu}{2}+12)(\frac{\nu}{2}+11)}\right]. \tag{4.28}$$

C. Calculation of mean energy

We follow the Laurent development as:

$$G_\nu F_\nu = G_3 F_3 + (G_3 F_3)'(\nu-3) + \sum_{k=2}^\infty b_k(\nu-3)^k. \tag{4.29}$$

For the Γ function we have

$$\Gamma(3-\nu) = \frac{1}{3-\nu} - C + \sum_{k=1}^\infty c_k(\nu-3)^k. \tag{4.30}$$

As a consequence:

$$\mathcal{Z}\langle U\rangle_\nu = \frac{G_3 F_3}{3-\nu} - G_3 F_3 C - (G_3 F_3)'$$
$$+ \sum_{k=1}^\infty a_k(\nu-3)^k + G_3\Phi(3,s) \tag{4.31}$$

and

$$\mathcal{Z}\langle U\rangle = -G_3 F_3\left(C + \frac{(G_3 F_3)'}{G_3 F_3}\right) + G_3\Phi(3,z_1,z_2). \tag{4.32}$$

Equivalently:

$$\mathcal{Z}\langle U \rangle = -G_3 F_3 \left(C + \frac{G_3'}{G_3} + \frac{F_3'}{F_3} \right) + G_3 \Phi(3, z_1, z_2). \quad (4.33)$$

We evaluate F_3, F_3', G_3 and G_3' in three dimensions:. The result is

$$G_3 = -\frac{128\pi^2}{35\beta} \left(\frac{6m}{\beta N} \right)^{\frac{3}{2}} b^3 \frac{\Gamma(\frac{31}{2})}{\Gamma(\frac{25}{2})} \quad (4.34)$$

$$F_3 = \left[-\alpha h_1(3) - \frac{1023}{675} k\alpha h_2(3) + \frac{1023}{675} h_3(3) \right] \quad (4.35)$$

$$\frac{G_3'}{G_3} = \frac{1}{2} \ln\left(\frac{6m}{\beta N} \pi^2 b^2 \right) + \psi\left(\frac{31}{2} \right) - \psi\left(\frac{25}{2} \right)$$
$$- 2\psi\left(\frac{3}{2} \right) - \frac{143}{315} \quad (4.36)$$

$$F_3' = \left[-\alpha h_1'(3) - \frac{1023}{675} k\alpha h_2'(3) + \frac{1023}{675} h_3'(3) \right.$$
$$\left. + \frac{3332}{151875} k\alpha h_2(3) - \frac{3332}{151875} h_3(3) \right] \quad (4.37)$$

We get then:

$$\langle U \rangle = -\frac{G_3 F_3}{\mathcal{Z}} \left(C + \frac{G_3'}{G_3} + \frac{F_3'}{F_3} \right) + G_3 \Phi(3, z_1, z_2).$$
$$(4.38)$$

We can see the behavior of the above energy in terms of temperature and number by plots of Fig. 6. From Fig. 6(a) we can see that $\langle U \rangle$ is increasing function of N. However, Fig. 6(b) shows that behavior of system is different for small and large N. For the large N limit we can see energy is decreased by temperature which may be sign of an instability. In order to study thermodynamics stability we need to investigate about the specific heat.
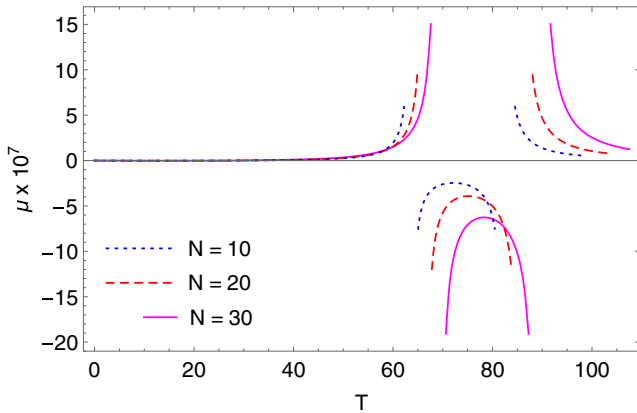
Figure 8 shows distribution function of Boltzmann statistics. Finally, as is usual, the specific heat is obtained by:

$$C_v = \frac{\partial \langle U \rangle}{\partial T}. \quad (4.39)$$

Our numerical analysis indicated that specific heat diverges at special N, which is something like phase transition, hence we can suggest a maximum number for N where the system is in thermodynamics stability. Also we can see that specific heat has a maximum in the stable region which is like Schottky anomaly. These are illustrated by Fig. 7(a). On the other hand, from Fig. 7(b) we can see behavior of specific heat in terms of the temperature. Similar to the energy, we can see different behavior for small and large N. For the small numbers of galaxies we see the specific heat is increasing function of
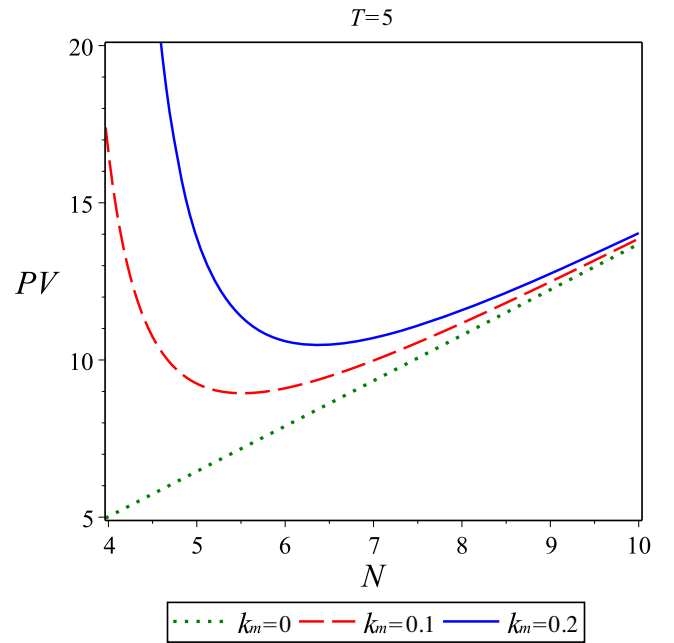

FIG. 7. EoS of Boltzmann statistics with $G = m = 1$.

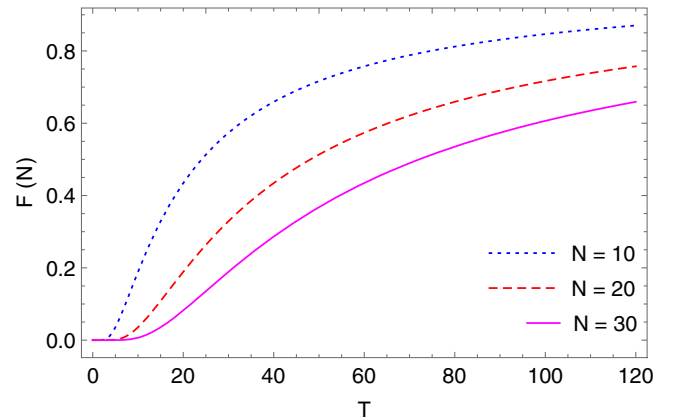FIG. 6. Chemical potential with $G = m = 1$, $k_m = 1/6$.

FIG. 8. Distribution function of Boltzmann statistics with $G = m = 1$, $k_m = 1/6$.
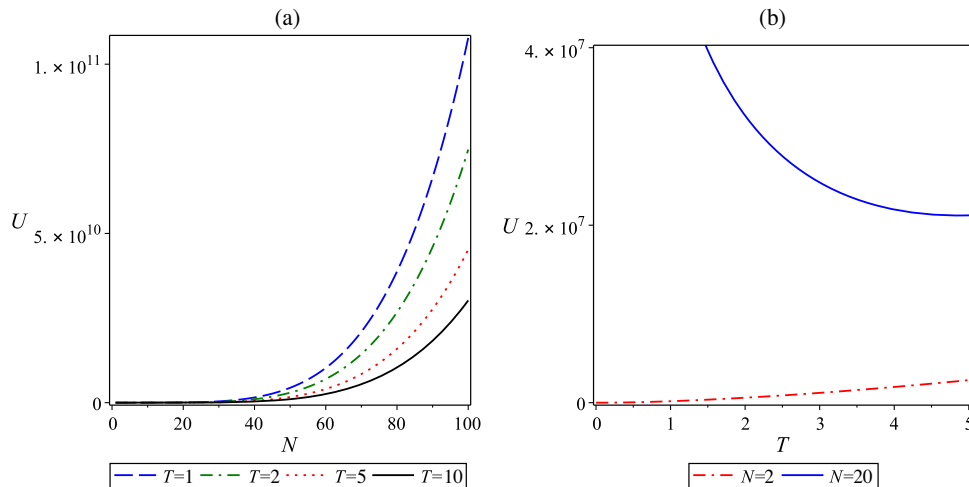
FIG. 9.   Energy of Tsallis statistics with $G = m = k = 1$. (a) in terms of N; (b) in terms of T.
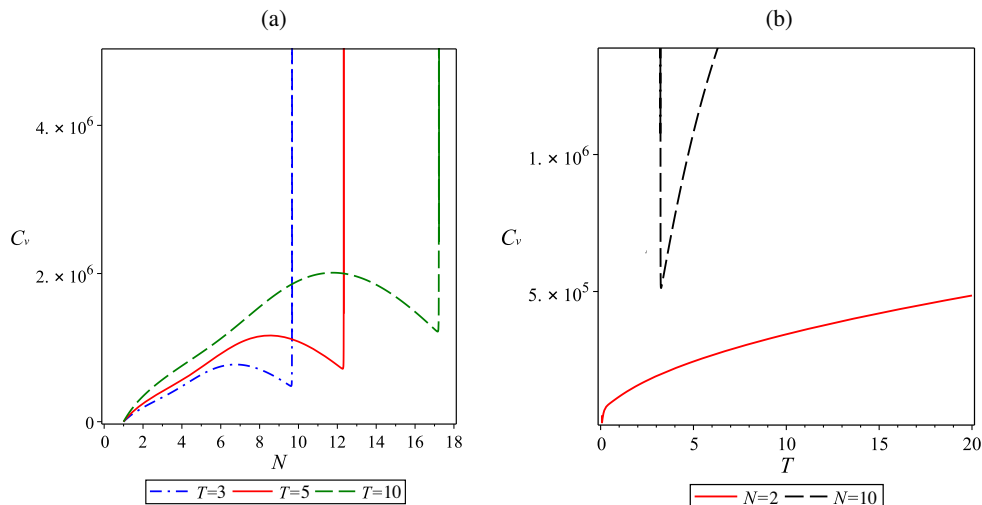

FIG. 10.   Specific heat of Tsallis statistics with $G = m = k = 1$. (a) in terms of N; (b) in terms of T.

temperature as expected. Figure 9 shows mean energy and Fig. 10 specic heat of Tsallis' statistics.

V. CONCLUSION

The main objective of the paper is to obtain divergence-free partition function for the system of galaxies clustering in the braneworld model. Statistically adequate in content, the paper follows Boltzmann statistical and the Tsallis statistical treatment to evaluate the partition function and the corresponding thermodynamics in the form of equations of state. The speciality of paper lies in the successful elimination of divergences arising due to the point mass approximations, which otherwise is mathematically a challenging issue for cosmologists. In the present work the main conclusion of the paper is in the form of finite and singularity free partition function and thus the regularized thermodynamics. This task is accomplished by using the generalization

of the dimensional regularization of Bollini and Giambiagi [33–35]. This generalization was based on the general quantification method of QFT's [36–39] using ultra-distributions of Sebastiao e Silva, also known as ultra-hyperfunctions [40–42]. In 2018 the problem of exponentially divergent nature of gravitational partition function was solved by Plastino and Rocca [48–50], thus deserves a special mention in the conclusive section of this paper.

General relativity has been thoroughly tested at solar system scale [51], and it is known to be valid at that scale. Thus, at that scale the nonlocality is strongly constrained as general relativity is a local theory [52,53]. However, it is still possible for nonlocality to occur at very large distance scales. In fact, it has been suggested that quantum gravitational effects would modify the gravitational potential at large distances, and this can explain the accelerated expansion of the universe without dark energy [54,55].

So, nonlocalities can have important consequences for large scale structure formation in our universe [56]. Such nonlocal effects of gravity could be detected using gravitational waves [57]. It would be interesting to construct a nonlocal gravitational partition function, using the methods used in this paper.

APPENDIX: AUXILIARY CALCULATIONS

1. Calculations of: Tsallis partition function of the system in the braneworld model

In 1988, C. Tsallis proposed a nonextensive generalization of statistical mechanics (widely called as Tsallis statistical mechanics) [27]. Based on the motivations laid

out in the introduction, we now try to investigate the many-body gravitational system using Tsallis's formalism. Tsallis q-exponential is defined as the distribution:

$$e_q(x) = [1 + (q-1)x]_+^{\frac{1}{q-1}}. \tag{A1}$$

Or equivalently

$$e_q(x) = \begin{cases} 1 + (q-1)x]^{\frac{1}{q-1}}; & 1 + (q-1)x > 0 \\ 0; & 1 + (q-1)x < 0. \end{cases} \tag{A2}$$

Let us consider again the distribution: $\frac{1}{r} = PV\frac{1}{r}$. First we do the calculation in ν dimensions and assume $q > 1$. Thus

$$\mathcal{Z}_\nu = \int_{-\infty}^{\infty} d^\nu x \int_{-\infty}^{\infty} d^\nu p \left[1 + (q-1)\beta\left(\frac{N(N-1)Gm^2}{2r} + \frac{kN(N-1)Gm^2}{2r^3} - \frac{Np^2}{2m}\right)\right]_+^{\frac{1}{q-1}} \tag{A3}$$

Here we put $q = 4/3$. The choice of that value of q is not arbitrary. It is the value of q where Verlinde's conjecture of emergent gravity could be proved, in the nonrelativistic case [48]. Accordingly:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \int_0^\infty r^{\nu-1}dr \int_0^\infty p^{\nu-1}dp \left[1 + \beta\left(\frac{N(N-1)Gm^2}{6r} + \frac{N(N-1)Gm^2}{6r^3} - \frac{Np^2}{6m}\right)\right]_+^3 \tag{A4}$$

The expression for $\mathcal{Z}$ in (4.4) can be rewritten in the form:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \left(\frac{6m}{\beta N}\right)^{\frac{k}{2}} B\left(4, \frac{\nu}{2}\right) \int_0^\infty r^{-\frac{k}{2}-10}dr(r^3 + \alpha_t r + k\alpha_t)^{\frac{k}{2}+3} \tag{A5}$$

where

$$\alpha_t = \frac{\beta N(N-1)Gm^2}{6}.$$

Hence we can write

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \left(\frac{6m}{\beta N}\right)^{\frac{k}{2}} B\left(4, \frac{\nu}{2}\right) \int_0^\infty r^{-\frac{k}{2}-10}dr(r+b)^{\frac{k}{2}+3}(r-r_1)^{\frac{k}{2}+3}(r-r_3)^{\frac{k}{2}+3} \tag{A6}$$

where r_1, $r_2 = -b$, and r_3 are the cube roots of the integrand. Also,

$$r_1 = \gamma_1 + e^{\frac{i\pi}{3}}\gamma_2$$

$$r_3 = \gamma_1 + e^{-\frac{i\pi}{3}}\gamma_2$$

$$b = \gamma_2 - \gamma_1 - \frac{\alpha}{3}$$

$$\gamma_1 = \left(\sqrt{\frac{1}{4}\left(\frac{2\alpha_t^3}{27} + \alpha_t k\right)^2 + \frac{1}{27}\left(\frac{\alpha}{3}(\alpha_t - 6)\right)^2} - \frac{1}{2}\left(\frac{2\alpha_t^3}{27} + \alpha_t k\right)\right)^{\frac{1}{3}}$$

$$\gamma_2 = \left(\sqrt{\frac{1}{4}\left(\frac{2\alpha_t^3}{27} + \alpha_t k\right)^2 + \frac{1}{27}\left(\frac{\alpha}{3}(\alpha_t - 6)\right)^2} + \frac{1}{2}\left(\frac{2\alpha_t^3}{27} + \alpha_t k\right)\right)^{\frac{1}{3}}$$

After solving the integral we have

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\right]^2 \left(\frac{6m}{\beta N}\right)^{\frac{k}{2}} B\left(4, \frac{\nu}{2}\right) b^\nu \frac{\Gamma(-\frac{k}{2}-9)}{\Gamma(-\frac{3\nu}{2}-9)} \Gamma(-\nu) F\left(-\nu, -\frac{\nu}{2}-3, -\frac{\nu}{2}-3, -\frac{3\nu}{2}-9; 1+\frac{r_1}{b}; 1+\frac{r_3}{b}\right). \tag{A7}$$

Here, F is the hypergeometric function of two variables. Using transformation formula we put $z_1 = 1+\frac{r_1}{b}$, $z_2 = 1+\frac{r_3}{b}$ and get:

$$\Gamma(-\nu) F\left(-\nu, -\frac{\nu}{2}-3, -\frac{\nu}{2}-3, -\frac{3\nu}{2}-9; 1+\frac{r_1}{b}; 1+\frac{r_3}{b}\right)$$

$$= \Gamma(-\nu)\left[1 - \nu\left(\frac{-\frac{\nu}{2}-3}{-\frac{3\nu}{2}-9} z_1 + \frac{-\frac{\nu}{2}-3}{-\frac{3\nu}{2}-9} z_2\right)\right.$$

$$- \nu(1-\nu)\left(\frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)} z_1 z_2 + \frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{2(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)} z_1^2 + \frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{2(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)} z_2^2\right)$$

$$- \nu(1-\nu)(2-\nu)\left(\frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)(-\frac{\nu}{2}-1)}{6(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)(-\frac{3\nu}{2}-7)} z_1^3 + \frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)(-\frac{\nu}{2}-1)}{6(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)(-\frac{3\nu}{2}-7)} z_2^3\right.$$

$$\left.\left. + \frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)(-\frac{\nu}{2}-3)}{2(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)(-\frac{3\nu}{2}-7)} z_1^2 z_1 + \frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)(-\frac{\nu}{2}-3)}{2(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)(-\frac{3\nu}{2}-7)} z_1 z_2^2\right)\right] + \phi(z_1, z_2) \tag{A8}$$

where:

$$\phi_\nu(z_1, z_2) = \sum_{m+n=4}^{\infty} \frac{\Gamma(m+n-\nu)(-\frac{\nu}{2}-3)_m(-\frac{\nu}{2}-3)_n}{(-\frac{3\nu}{2}-9)_{m+n} m! n!} z_1^m z_2^n. \tag{A9}$$

Equation (A8) can be simplified and translated into the form:

$$\Gamma(-\nu) F\left(-\nu, -\frac{\nu}{2}-3, -\frac{\nu}{2}-3, -\frac{3\nu}{2}-9; 1+\frac{r_1}{b}; 1+\frac{r_3}{b}\right)$$

$$= \Gamma(-\nu)\left[1 - \nu\left(\frac{-\frac{\nu}{2}-3}{-\frac{3\nu}{2}-9}\right)(z_1 + z_2) - \nu(1-\nu)\left(\frac{(-\frac{\nu}{2}-3)}{(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)}\right)\right.$$

$$\times \left(2\left(-\frac{\nu}{2}-3\right) z_1 z_2 + \left(-\frac{\nu}{2}-2\right) z_1^2 + \left(-\frac{\nu}{2}-2\right) z_2^2\right)$$

$$- \nu(1-\nu)(2-\nu)\left(\frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{6(-\frac{3\nu}{2}-9)(-3\frac{3\nu}{2}-8)(-\frac{3\nu}{2}-7)}\right)$$

$$\times \left.\left(\left(-\frac{\nu}{2}-1\right) z_1^3 + \left(-\frac{\nu}{2}-1\right) z_2^3 + 3\left(-\frac{\nu}{2}-3\right) z_1^2 z_1 + 3\left(-\frac{\nu}{2}-3\right) z_1 z_2^2\right)\right] + \phi_\nu(z_1, z_2). \tag{A10}$$

Then, we have for $\nu = 3$, the expression:

$$\phi_3(z_1, z_2) = \sum_{m+n=4}^{\infty} \frac{\Gamma(m+n-3)(-\frac{9}{2})_m(-\frac{9}{2})_n}{(-\frac{27}{2})_{m+n} m! n!} z_1^m z_2^n \tag{A11}$$

$$\Gamma(-\nu)F\left(-\nu, -\frac{\nu}{2} - 3, -\frac{\nu}{2} - 3, -\frac{3\nu}{2} - 9; 1 + \frac{r_1}{b}; 1 + \frac{r_3}{b}\right)$$

$$= \Gamma(3-\nu)\left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \left(\frac{-\frac{\nu}{2} - 3}{(1-\nu)(2-\nu)(-\frac{3\nu}{2} - 9)}\right)(z_1 + z_2) + \left(\frac{(-\frac{\nu}{2} - 3)}{(2-\nu)(-\frac{3\nu}{2} - 9)(-\frac{3\nu}{2} - 8)}\right)\right.$$

$$\times \left(2\left(-\frac{\nu}{2} - 3\right)z_1 z_2 + \left(-\frac{\nu}{2} - 2\right)z_1^2 + \left(-\frac{\nu}{2} - 2\right)z_2^2\right) + \left(\frac{(-\frac{\nu}{2} - 3)(-\frac{\nu}{2} - 2)}{6(-\frac{3\nu}{2} - 9)(-\frac{3\nu}{2} - 8)(-\frac{3\nu}{2} - 7)}\right)$$

$$\times \left.\left(\left(-\frac{\nu}{2} - 1\right)z_1^3 + \left(-\frac{\nu}{2} - 1\right)z_2^3 + 3\left(-\frac{\nu}{2} - 3\right)z_1^2 z_1 + 3\left(-\frac{\nu}{2} - 3\right)z_1 z_2^2\right)\right] + \phi_\nu(z_1, z_2). \tag{A12}$$

Let us define h_ν such that

$$h_\nu = \left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \left(\frac{-\frac{\nu}{2} - 3}{(1-\nu)(2-\nu)(-\frac{3\nu}{2} - 9)}\right)(z_1 + z_2) + \left(\frac{(-\frac{\nu}{2} - 3)}{(2-\nu)(-\frac{3\nu}{2} - 9)(-\frac{3\nu}{2} - 8)}\right)\right.$$

$$\times \left(2\left(-\frac{\nu}{2} - 3\right)z_1 z_2 + \left(-\frac{\nu}{2} - 2\right)z_1^2 + \left(-\frac{\nu}{2} - 2\right)z_2^2\right) + \left(\frac{(-\frac{\nu}{2} - 3)(-\frac{\nu}{2} - 2)}{6(-\frac{3\nu}{2} - 9)(-\frac{3\nu}{2} - 8)(-\frac{3\nu}{2} - 7)}\right)$$

$$\times \left.\left(\left(-\frac{\nu}{2} - 1\right)z_1^3 + \left(-\frac{\nu}{2} - 1\right)z_2^3 + 3\left(-\frac{\nu}{2} - 3\right)z_1^2 z_1 + 3\left(-\frac{\nu}{2} - 3\right)z_1 z_2^2\right)\right]. \tag{A13}$$

Thus we can write the partition function in ν dimensions as

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} B\left(4, \frac{\nu}{2}\right) b^\nu \frac{\Gamma(-\frac{\nu}{2} - 9)}{\Gamma(-\frac{3\nu}{2} - 9)}(\Gamma(3-\nu)h_\nu + \phi_\nu(z_1, z_2)) \tag{A14}$$

$$\mathcal{Z}_\nu = \frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2} \left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(-\frac{\nu}{2} - 9)}{(3 + \frac{\nu}{2})(2 + \frac{\nu}{2})(1 + \frac{\nu}{2})\Gamma(-\frac{3\nu}{2} - 9)}(\Gamma(3-\nu)h_\nu + \phi_\nu(z_1, z_2)). \tag{A15}$$

2. Calculations of: Mean energy in ν dimensions

We now proceed to evaluate the average energy of the system. For it is

$$\mathcal{Z}\langle\mathcal{U}\rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2 \int_0^\infty r^{\nu-1} dr \int_0^{P_0} p^{\nu-1} dp \left(-\frac{N(N-1)Gm^2}{2r} - \frac{kN(N-1)Gm^2}{2r^2} + \frac{Np^2}{2m}\right)$$

$$\times \left[1 + (q-1)\beta\left(\frac{N(N-1)Gm^2}{2r} + \frac{kN(N-1)Gm^2}{2r^3} - \frac{Np^2}{2m}\right)\right]^{\frac{1}{q-1}} \tag{A16}$$

Or, equivalently:

$$\mathcal{Z}\langle U\rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}}\right]^2 \left[\frac{\beta N(q-1)}{2m}\right]^{\frac{1}{q-1}}$$

$$\times \left(\int_0^\infty r^{\nu-1} dr \int_0^{P_0} \frac{Np^{\nu+1}}{2m}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}} dp\right.$$

$$-\frac{N(N-1)Gm^2}{2} \int_0^\infty r^{\nu-2} dr \int_0^{P_0} p^{\nu-1}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}} dp$$

$$\left.-\frac{kN(N-1)Gm^2}{2} \int_0^\infty r^{\nu-3} dr \int_0^{P_0} p^{\nu-1}\left[\frac{1}{\beta N(q-1)} + \frac{(N-1)Gm^2}{2r} + \frac{k(N-1)Gm^2}{2r^3} - \frac{p^2}{2m}\right]^{\frac{1}{q-1}} dp\right) \tag{A17}$$

For $q = 4/3$ we obtain:

$$\mathcal{Z}\langle U\rangle_\nu = \frac{3}{2\beta}\left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}\right]^2\left(\frac{6m}{N\beta}\right)^{\frac{\nu}{2}}\left[-\alpha B\left(4,\frac{\nu}{2}\right)I_1 - k\alpha B\left(4,\frac{\nu}{2}\right)I_2 + B\left(4,\frac{\nu}{2}+1\right)I_3\right] \tag{A18}$$

where

$$I_1 = \int_0^\infty r^{-\frac{\nu}{2}-11}dr(r^3 + \alpha r^2 + k\alpha)^{3+\frac{\nu}{2}} \tag{A19}$$

$$I_2 = \int_0^\infty r^{-\frac{\nu}{2}-13}dr(r^3 + \alpha r^2 + k\alpha)^{3+\frac{\nu}{2}} \tag{A20}$$

$$I_3 = \int_0^\infty r^{-\frac{\nu}{2}-13}dr(r^3 + \alpha r^2 + k\alpha)^{4+\frac{\nu}{2}} \tag{A21}$$

we can further write the integrals as

$$I_1 = \int_0^\infty r^{-\frac{\nu}{2}-11}dr(r+b)^{\frac{\nu}{2}+3}(r-r_1)^{\frac{\nu}{2}+3}(r-r_3)^{\frac{\nu}{2}+3} \tag{A22}$$

$$I_2 = \int_0^\infty r^{-\frac{\nu}{2}-13}dr(r+b)^{\frac{\nu}{2}+3}(r-r_1)^{\frac{\nu}{2}+3}(r-r_3)^{\frac{\nu}{2}+3} \tag{A23}$$

$$I_3 = \int_0^\infty r^{-\frac{\nu}{2}-13}dr(r+b)^{\frac{\nu}{2}+4}(r-r_1)^{\frac{\nu}{2}+4}(r-r_3)^{\frac{\nu}{2}+4}. \tag{A24}$$

Thus after using the formula given below we evaluate the integrals

$$\int_0^\infty r^{\gamma-\alpha-1}dr(r+b)^{B+B'-\gamma}(r-r_1)^B(r-r_3)^{B'} = b^{-\alpha}\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\Gamma(\alpha)F\left(\alpha,B,B',\gamma;1+\frac{r_1}{b};1+\frac{r_3}{b}\right). \tag{A25}$$

Thus we get:

$$I_1 = b^\nu\frac{\Gamma(-\frac{\nu}{2}-10)}{\Gamma(-\frac{3\nu}{2}-10)}\Gamma(-\nu)F\left(-\nu,-\frac{\nu}{2}-3,-\frac{\nu}{2}-3,-\frac{3\nu}{2}-10;1+\frac{r_1}{b};1+\frac{r_3}{b}\right) \tag{A26}$$

$$I_2 = b^\nu\frac{\Gamma(-\frac{\nu}{2}-12)}{\Gamma(-\frac{3\nu}{2}-12)}\Gamma(-\nu)F\left(-\nu,-\frac{\nu}{2}-3,-\frac{\nu}{2}-3,-\frac{3\nu}{2}-12;1+\frac{r_1}{b};1+\frac{r_3}{b}\right) \tag{A27}$$

$$I_3 = b^\nu\frac{\Gamma(-\frac{\nu}{2}-12)}{\Gamma(-\frac{3\nu}{2}-12)}\Gamma(-\nu)F\left(-\nu,-\frac{\nu}{2}-4,-\frac{\nu}{2}-4,-\frac{3\nu}{2}-12;1+\frac{r_1}{b};1+\frac{r_3}{b}\right). \tag{A28}$$

Using transformation formula

$$\Gamma(-\nu)F\left(-\nu,-\frac{\nu}{2}-3,-\frac{\nu}{2}-3,-\frac{3\nu}{2}-10;1+\frac{r_1}{b};1+\frac{r_3}{b}\right) = \Gamma(3-\nu)h_1 + \phi_1(\nu,z_1,z_2) \tag{A29}$$

where we put

$$
\begin{aligned}
h_1(\nu) = &\left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \left(\frac{-\frac{\nu}{2}-3}{(1-\nu)(2-\nu)(-\frac{3\nu}{2}-10)}\right)(z_1+z_2) + \left(\frac{(-\frac{\nu}{2}-3)}{(2-\nu)(-\frac{3\nu}{2}-10)(-\frac{3\nu}{2}-9)}\right)\right. \\
&\times \left(2\left(-\frac{\nu}{2}-3\right)z_1 z_2 + \left(-\frac{\nu}{2}-2\right)z_1^2 + \left(-\frac{\nu}{2}-2\right)z_2^2\right) + \left(\frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{6(-\frac{3\nu}{2}-10)(-\frac{3\nu}{2}-9)(-\frac{3\nu}{2}-8)}\right) \\
&\times \left.\left(\left(-\frac{\nu}{2}-1\right)z_1^3 + \left(-\frac{\nu}{2}-1\right)z_2^3 + 3\left(-\frac{\nu}{2}-3\right)z_1^2 z_2 + 3\left(-\frac{\nu}{2}-3\right)z_1 z_2^2\right)\right].
\end{aligned}
\tag{A30}
$$

In three dimensions:

$$
h_1(3) = \left[-\frac{1}{6} + \frac{9}{58}(z_1+z_2) - \frac{18}{87}z_1 z_2 + \frac{7}{87}(z_1^2+z_2^2) + \frac{7}{2610}(z_1^3+z_2^3) + \frac{7}{50}(z_1^2 z_2 + z_1 z_2^2)\right]
\tag{A31}
$$

and

$$
[h_1'(3) = \left[\frac{11}{36} - \frac{779}{3364}(z_1+z_2) + \frac{518}{2523}z_1 z_2 + \frac{65}{841}(z_1^2+z_2^2) + \frac{101}{315375}(z_1^3+z_2^3) + \frac{697}{1576875}(z_1^2 z_2 + z_1 z_2^2)\right].
\tag{A32}
$$

In the same way:

$$
\Gamma(-\nu)F\left(-\nu, -\frac{\nu}{2}-3, -\frac{\nu}{2}-3, -\frac{3\nu}{2}-12; 1+\frac{r_1}{b}; 1+\frac{r_3}{b}\right) = \Gamma(3-\nu)h_2 + \phi_2(\nu, z_1, z_2)
\tag{A33}
$$

where:

$$
\begin{aligned}
h_2(\nu) = &\left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \left(\frac{-\frac{\nu}{2}-3}{(1-\nu)(2-\nu)(-\frac{3\nu}{2}-12)}\right)(z_1+z_2) + \left(\frac{(-\frac{\nu}{2}-3)}{(2-\nu)(-\frac{3\nu}{2}-12)(-\frac{3\nu}{2}-11)}\right)\right. \\
&\times \left(2\left(-\frac{\nu}{2}-3\right)z_1 z_2 + \left(-\frac{\nu}{2}-2\right)z_1^2 + \left(-\frac{\nu}{2}-2\right)z_2^2\right) + \left(\frac{(-\frac{\nu}{2}-3)(-\frac{\nu}{2}-2)}{6(-\frac{3\nu}{2}-12)(-\frac{3\nu}{2}-11)(-\frac{3\nu}{2}-10)}\right) \\
&\times \left.\left(\left(-\frac{\nu}{2}-1\right)z_1^3 + \left(-\frac{\nu}{2}-1\right)z_2^3 + 3\left(-\frac{\nu}{2}-3\right)z_1^2 z_1 + 3\left(-\frac{\nu}{2}-3\right)z_1 z_2^2\right)\right].
\end{aligned}
\tag{A34}
$$

Again, in three dimensions:

$$
h_2(3) = \left[-\frac{1}{6} + \frac{3}{22}(z_1+z_2) - \frac{54}{341}z_1 z_2 + \frac{21}{341}(z_1^2+z_2^2) + \frac{35}{1798}(z_1^3+z_2^3) + \frac{189}{1798}(z_1^2 z_2 + z_1 z_2^2)\right]
\tag{A35}
$$

and

$$
h_2'(3) = \left[\frac{11}{36} + \frac{293}{1452}(z_1+z_2) - \frac{17778}{116281}z_1 z_2 + \frac{20059}{348843}(z_1^2+z_2^2) + \frac{461521}{880130889}(z_1^3+z_2^3) + \frac{69105}{97792321}(z_1^2 z_2 + z_1 z_2^2)\right].
\tag{A36}
$$

The last equality is

$$
\Gamma(-\nu)F\left(-\nu, -\frac{\nu}{2}-4, -\frac{\nu}{2}-4, -\frac{3\nu}{2}-12; 1+\frac{r_1}{b}; 1+\frac{r_3}{b}\right) = \Gamma(3-\nu)h_3 + \phi_3(\nu, z_1, z_2)
\tag{A37}
$$

with:

$$h_3(\nu) = \left[\frac{1}{-\nu(1-\nu)(2-\nu)} + \left(\frac{-\frac{\nu}{2}-4}{(1-\nu)(2-\nu)(-\frac{3\nu}{2}-12)}\right)(z_1+z_2) + \left(\frac{(-\frac{\nu}{2}-4)}{(2-\nu)(-\frac{3\nu}{2}-12)(-\frac{3\nu}{2}-11)}\times\right)\right.$$

$$\times \left(2\left(-\frac{\nu}{2}-4\right)z_1 z_2 + \left(-\frac{\nu}{2}-3\right)z_1^2 + \left(-\frac{\nu}{2}-3\right)z_2^2\right) + \left(\frac{(-\frac{\nu}{2}-4)(-\frac{\nu}{2}-3)}{6(-\frac{3\nu}{2}-12)(-\frac{3\nu}{2}-11)(-\frac{3\nu}{2}-10)}\right)$$

$$\left.\times \left(\left(-\frac{\nu}{2}-2\right)z_1^3 + \left(-\frac{\nu}{2}-2\right)z_2^3 + 3\left(-\frac{\nu}{2}-4\right)z_1^2 z_1 + 3\left(-\frac{\nu}{2}-4\right)z_1 z_2^2\right)\right]. \tag{A38}$$

In three dimensions:

$$h_3(3) = \left[-\frac{1}{6} + \frac{1}{6}(z_1+z_2) - \frac{22}{93}z_1 z_2 - \frac{3}{31}(z_1^2+z_2^2) + \frac{7}{1798}(z_1^3+z_2^3) + \frac{33}{1798}(z_1^2 z_2 + z_1 z_2^2)\right] \tag{A39}$$

$$h_3'(3) = \left[\frac{11}{36} - \frac{1}{4}(z_1+z_2) + \frac{686}{2883}z_1 z_2 + \frac{275}{2883}(z_1^2+z_2^2) + \frac{1522}{7273809}(z_1^3+z_2^3) + \frac{80}{2406623}(z_1^2 z_2 + z_1 z_2^2)\right]. \tag{A40}$$

We then obtain:

$$\mathcal{Z}\langle U\rangle_\nu = G_\nu[\Gamma(3-\nu)F_\nu + \Phi_\nu(z_1,z_2)] \tag{A41}$$

where we have defined:

$$\Phi_\nu(z_1,z_2) = \phi_1 + \phi_2 + \phi_3$$

with:

$$G_\nu = \frac{3}{2\beta}\frac{24\pi^\nu}{\Gamma(\frac{\nu}{2})^2}\left(\frac{6m}{\beta N}\right)^{\frac{\nu}{2}} b^\nu \frac{\Gamma(\frac{3\nu}{2}+11)\sin\frac{3\nu\pi}{2}}{(3+\frac{\nu}{2})(2+\frac{\nu}{2})(1+\frac{\nu}{2})\Gamma(\frac{\nu}{2}+11)\sin\frac{\pi\nu}{2}} \tag{A42}$$

and:

$$F_\nu = \left[-\alpha h_1(\nu) - k\alpha h_2(\nu)\frac{(\frac{3\nu}{2}+12)(\frac{3\nu}{2}+11)}{(\frac{\nu}{2}+12)(\frac{\nu}{2}+11)} + h_3(\nu)\frac{(\frac{3\nu}{2}+12)(\frac{3\nu}{2}+11)}{(\frac{\nu}{2}+12)(\frac{\nu}{2}+11)}\right]. \tag{A43}$$

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