

QCD condensates and holographic Wilson loops for asymptotically AdS spaces

R. Carcasses Quevedo,^{1,2,*} J. L. Goity,^{3,4,†} and R. C. Trincherro^{1,2,‡}

¹*Instituto Balseiro, Centro Atómico Bariloche, 8400 San Carlos de Bariloche, Argentina*

²*CONICET, Rivadavia 1917, 1033 Buenos Aires, Argentina*

³*Department of Physics, Hampton University, Hampton, Virginia 23668, USA*

⁴*Thomas Jefferson National Accelerator Facility, Newport News, Virginia 23606, USA*

(Received 5 November 2013; published 14 February 2014)

The minimization of the Nambu-Goto action for a surface whose contour defines a circular Wilson loop of radius a placed at a finite value of the coordinate orthogonal to the boundary is considered. This is done for asymptotically anti-de Sitter (AdS) spaces. The condensates of even dimension $n = 2$ through 10 are calculated in terms of the coefficient of a^n in the expansion of the on-shell subtracted Nambu-Goto action for small a . The subtraction employed is such that it presents no conflict with conformal invariance in the AdS case and need not introduce an additional infrared scale for the case of confining geometries. It is shown that the UV value of the condensates is universal in the sense that they only depend on the first coefficients of the difference with the AdS case.

DOI: [10.1103/PhysRevD.89.036004](https://doi.org/10.1103/PhysRevD.89.036004)

PACS numbers: 11.15.-q, 11.25.Tq, 12.38.Aw, 12.38.Lg

I. INTRODUCTION

The relation between large N gauge theories and string theory [1] together with the AdS/CFT correspondence [2–5] has opened new insights into strongly interacting gauge theories. The application of these ideas to QCD has received significant attention since those breakthroughs. From the phenomenological point of view, the so-called anti-de Sitter/QCD approach has produced very interesting results in spite of the strong assumptions involved in its formulation [6–11]. It seems important to further proceed investigating these ideas and refining the current understanding of a possible QCD gravity dual.

As is well known the vacuum of pure gauge QCD is the simplest setting that presents key nonperturbative effects of QCD. In this regard, the gluon condensate $G_2 \equiv \frac{g^2}{4\pi^2} \langle F_{\mu\nu}^a F_a^{\mu\nu} \rangle$ plays an important role. The existence of a nonvanishing G_2 was early on identified [12]. It has important manifestations in hadron phenomenology [12,13], and there are indications of its nonvanishing from lattice QCD [14–16]. The gluon condensate can be obtained from the vacuum expectation value of a small Wilson loop. In the holographic approach, such an expectation value is obtained by minimizing the Nambu-Goto (NG) action for a loop lying in the boundary space [17,18]. This is known to work in the strictly anti-de Sitter (AdS) case, i.e., for a conformal boundary field theory. In this work we assume that this procedure also works in the nonconformal-QCD case provided an adequate 5-dimensional background metric is chosen.

The features and results of this work are summarized as follows:

- (i) The NG action for a circular loop of radius a lying at a given value of the coordinate orthogonal to the boundary of an asymptotically AdS space is considered.
- (ii) The minimization of this action leads to an equation of motion, whose solution is approximated by a power series in a .
- (iii) The on-shell NG action is subtracted following the procedure in Ref. [17]. More precisely, an extension of this procedure is proposed for the case under consideration, where the base of the loop is at a finite value of the radial coordinate and a natural infrared limit is considered for the case of confining theories.
- (iv) The gluon condensates of even dimension $n = 2$ through 10 are obtained from the coefficients of the expansion in powers of a of the subtracted on-shell NG action $S_{\text{NG}}^{\text{sub}}$, the last four ones assuming the absence of the condensate of dimension 2.
- (v) It is shown that the UV value of these condensates is universal in the sense that for a condensate of a given dimension, its value does not depend on the value of the warp factor's higher order coefficients.

The paper is organized as follows. Section II defines the problem to be considered, including the NG action for the circular loop and the asymptotically AdS background metric. Section III deals with the subtraction of the on-shell NG action. Section IV gives some model independent results, which clarify the relation between condensates and the expansion coefficients of the warp factor. Section V deals with the approximate solution of the equations of motion and the evaluation of the on-shell NG action as a

*robert.carcasses-quevedo@ib.edu.ar

†goity@jlab.org

‡trincher@cab.cnea.gov.ar

power series in a . Section VI gives the results for the gluon condensates, showing the above mentioned universality. Section VII includes some concluding remarks. In addition four appendices are included.

II. NAMBU-GOTO ACTION FOR A CIRCULAR LOOP IN AN ASYMPTOTICALLY AdS SPACE

The distance to be considered has the following general form:

$$ds^2 = e^{2A(z)}(dz^2 + \eta_{ij}dx^i dx^j) = G_{\mu\nu}dx^\mu dx^\nu \quad \mu, \nu = 1, \dots, d+1. \quad (1)$$

It is defined by a metric with no dependence on the boundary coordinates and preserves the boundary space Poincaré invariance. This should be the case if only vacuum properties are considered. The form of the warp factor $A(z)$ to be considered is

$$A(z) = -\ln\left(\frac{z}{L}\right) + f(z), \quad (2)$$

where $f(z)$ is a dimensionless function. In this work $f(z)$ is taken to be a power series in z ,¹ i.e.,

$$f(z) = \sum_{k=1} \alpha_k z^k. \quad (3)$$

The case $f(z) = 0$ corresponds to the AdS metric. This deviation from the AdS case could be produced by a bulk gravity theory including matter fields [19]. Possible candidates for these bulk gravity theories have been considered in [20,21].

The area of a surface embedded in this space is given by the NG action,

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}, \quad (4)$$

where g is the determinant of the induced metric on the surface, which is given by

$$g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu,$$

where X^μ are the coordinates of the surface embedded in the ambient $d+1$ dimensional space. The indices a, b refer to coordinates on the surface. The case to be considered is a circular loop whose contour lies at a constant value z_1 of the

coordinate z and in the $i-j$ spatial plane. The coordinates on the surface are then taken to be r and ϕ , the polar coordinates. Therefore the embedding can be described by the following:

$$X^k = 0 \quad (5)$$

$$X^5 = z(r) \quad (6)$$

$$X^i = r \cos \phi, \quad X^j = r \sin \phi \quad (\forall k \neq i \neq j), \quad (7)$$

with the boundary conditions,

$$z(a) = z_1, \quad z'(0) = 0, \quad (8)$$

which state that the contour of the circular loop of radius a is located at z_1 and that no cusps are admitted. Replacing the embedding (5) in the action (4), after a trivial integration in the angular variable, leads to the following expression:

$$S_{\text{NG}} = \frac{1}{\alpha'} \int_0^a e^{2A(z)} r \sqrt{1+z'^2} dr, \quad (9)$$

where the prime denotes derivative with respect to r . The minimal surface is given by the solution of the following equations of motion with the above mentioned boundary conditions:

$$r \frac{z''(r)}{1+z'(r)^2} + z'(r) - 2r \frac{dA(z)}{dz} = 0. \quad (10)$$

For the AdS case $A(z) = -\ln \frac{z}{L}$ the solution is

$$z(r) = \sqrt{a^2 + z_1^2 - r^2}, \quad (11)$$

which upon replacing in (4) leads to the following expression for the on-shell NG action:

$$S_{\text{NG,AdS}}^{\text{o.s.}} = \frac{L^2}{\alpha'} \left(\sqrt{1 + \frac{a^2}{z_1^2}} - 1 \right). \quad (12)$$

III. THE SUBTRACTED ON-SHELL NAMBU-GOTO ACTION

Replacing the solution of the previous section in the NG action leads to a divergent expression when $z_1 \rightarrow 0$, i.e., near the UV boundary of the space. This happens in the AdS case and also when $f(z) \neq 0$. Therefore, a subtraction procedure should be employed. The action requires regularization, where the most obvious procedure is to choose $z_1 \neq 0$, and a renormalized action is obtained by implementing a subtraction, as it is discussed in detail in this section. A procedure of minimal subtraction, defined by

¹Recalling that near the UV boundary the relation between the conformal coordinate z and Fefferman-Graham [22] coordinate ρ is $\rho = z^2$, then a polynomial in z , as considered in this work, corresponds to an expression involving integer and half-integer powers of ρ . However see Sec. IV where it is shown that half-integers powers of ρ cannot appear for a theory describing QCD.

disregarding the $1/z_1$ term in S_{NG} , was implemented in [23].

As shown in [17] for the rectangular loop, a physically motivated procedure is to subtract the contribution of the ‘‘heavy quark’’ mass to the action. This contribution corresponds to the area of a cylinder with axis parallel to z , extending from $z = \infty$ to $z = 0$, for the case of the base of the loop located at $z = 0$. It could be thought that in the case considered in this paper, the base of the loop located at z_1 , the area of a cylinder of radius a with axis parallel to z and extending from $z = \infty$ to z_1 should be subtracted. However, such a procedure should be modified in two aspects, namely:

- (i) First, in the AdS case, it leads to a loss of conformal invariance, more precisely the value of $S_{\text{NG}}^{\text{sub}}$ would depend on the radius of the loop. Requiring independence of the value of $S_{\text{NG}}^{\text{sub}}$ on the radius a , leads to the following definition of the subtracted action:

$$S_{\text{NG}}^{\text{sub}} = S_{\text{NG}} - \frac{r_0(a, z_1)}{\alpha'} \int_{z_1}^{z_{\text{IR}}} dz e^{2A(z)}, \quad (13)$$

where z_{IR} is an infrared scale whose motivation and definition is explained below. For the AdS case the function $r_0(a, z_1)$ is fixed by conformal invariance and given by

$$r_0^{\text{AdS}}(a, z_1) = \sqrt{a^2 + z_1^2}, \quad (14)$$

leading to

$$S_{\text{NG}}^{\text{sub,AdS}} = -\frac{L^2}{\alpha'}. \quad (15)$$

The radius $r_0(a, z_1)$ corresponds to the radius of a loop located at the boundary whose minimal surface would intersect the plane $z = z_1$ with a circle of radius a . Therefore the following holds:

$$\lim_{z_1 \rightarrow 0} r_0(a, z_1) = a. \quad (16)$$

For non-AdS cases the same procedure could be employed, using the corresponding value $r_0^{\text{Non-AdS}}(a, z_1)$. However it should be noted that in the non-AdS case using the AdS r_0 given in (14) also leads to a finite value for the $S_{\text{NG}}^{\text{sub}}$ and presents no conflict with conformal invariance. This last procedure will be employed below.

- (ii) Second, confining warp factors are such that $e^{A(z)}$ presents a global minimum for a finite value for this factor [24,25]. Let z_m denote the location of this minimum in the coordinate z . In these conditions, integrating $e^{2A(z)}$ between z_1 and ∞ would lead to a divergent result. Introducing an infrared integration limit z_{IR} as in (13) eliminates this divergence at the cost of introducing this *ad hoc* infrared cutoff. In this respect the following remarks are important:

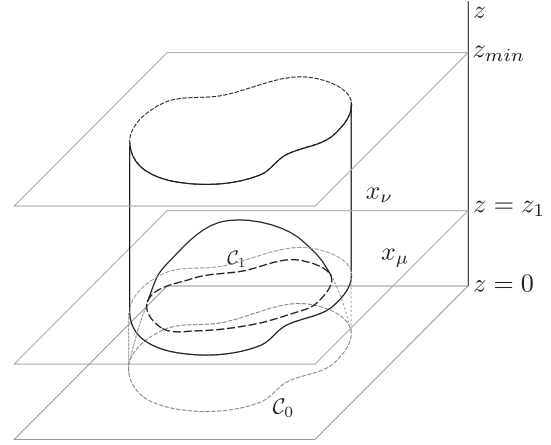


FIG. 1. Subtraction scheme.

- (a) The result for the coefficients of a^n , $n > 1$ in $S_{\text{NG}}^{\text{sub}}$ do not depend on z_{IR} . This fact is shown in Sec. VI, and is due to property (16).
- (b) On the other hand these confining warp factors already have a natural infrared scale. This is given by the location z_m of the global minimum. This is a natural candidate to be identified with z_{IR} . In this respect it is worth noting that for $z_1 < z_m$ the minimal surface could never exceed the value z_m , otherwise it would not be minimal.² In what follows the choice $z_{\text{IR}} = z_m$ is made. It is emphasized that other choices are by no means excluded. Different choices produce different coefficients for the perimeter in $S_{\text{NG}}^{\text{sub}}$.

It is noted that this subtraction is nonvanishing even if the loop is located at a finite value of the coordinate z . Figure 1 illustrates the proposed subtraction procedure, for the case of a minimal surface bounded by a closed contour C_1 .

A. Convergence of the subtracted Nambu-Goto action in the UV limit

The considered warp factors diverge in the UV, the leading singularity is

$$A(z) \sim -\ln\left(\frac{z}{L}\right) \rightarrow A'(z) \sim -\frac{1}{z}, \quad (17)$$

this makes the integrand appearing in the NG action diverge at $z = 0$.

In order to analyze the behavior of the solution near the boundary, the approach in [26] is employed. The NG action is written in terms of r as a function of z , $r(z)$; this leads to

²Indeed, suppose there were a minimal surface with boundary at $z_1 < z_m$ that extends to values of $z > z_m$; then, since the warp factor necessarily grows for these values (recall that z_m is a minimum), a surface stopping at z_m will have less area than the one originally supposed to be minimal, which is a contradiction.

$$S_{\text{NG}} = \frac{1}{\alpha'} \int_{z_1}^{z_0} e^{2A(z)} r(z) \sqrt{1 + r'(z)^2} dz, \quad (18)$$

where $r(0) = r_0$ and $r(z_1) = a$. The equation of motion is

$$(-1 + 2rA'(z)r')(1 + r'^2) + rr'' = 0. \quad (19)$$

Near the boundary (19) implies

$$-\lim_{z \rightarrow 0} 2r \frac{1}{z} r'(1 + r'^2) + \lim_{z \rightarrow 0} r r'' - \lim_{z \rightarrow 0} (1 + r'^2) = 0. \quad (20)$$

This last equation shows that if $\lim_{z_1 \rightarrow 0} r''(z)$ is assumed to be finite, then $\lim_{z_1 \rightarrow 0} r'(z)$ cannot be infinite since in that case it will be impossible to cancel the terms containing r' . Furthermore the cancellation of the terms involving $r'(z)$ require that $\lim_{z_1 \rightarrow 0} r'(z) = 0$ as $z^{1+\epsilon}$, $\epsilon > 0$. In addition the cancellation of the constant term in (20) requires $\epsilon = 0$.

On the contrary, if $\lim_{z \rightarrow 0} r''(z)$ is assumed to be infinite, then $\lim_{z_1 \rightarrow 0} r'(z) \rightarrow \infty$, which can be proved by integrating the former, and again it is not possible to cancel all the divergent terms due to their different degrees of divergence. Therefore,

$$\lim_{z \rightarrow 0} r'(z) = 0 \quad (21)$$

$$r'(z) = -\frac{z}{a} + \dots (z \gg a). \quad (22)$$

Next, this asymptotics is plugged in the $S_{\text{NG}}^{\text{sub}}$ (13). In this respect it is convenient to rewrite it in the form,

$$S_{\text{NG}}^{\text{sub}} = \frac{1}{\alpha'} \int_{z_1}^{z_0} e^{2A(z)} (r(z) \sqrt{1 + r'(z)^2} - r_0(a, z_1)) - \frac{r_0(a, z_1)}{\alpha'} \int_{z_0}^{z_m} dz e^{2A(z)}.$$

For the considered cases of $A(z)$, the second term is convergent because the integrand has no poles in the finite integration interval. The first term is also finite, indeed:

$$\begin{aligned} & \lim_{z \rightarrow 0} e^{2A(z)} \left(r(z) \sqrt{1 + r'(z)^2} - a \right) \\ &= \lim_{z \rightarrow 0} \frac{1}{z^2} \left(a \left(1 + \frac{1}{2} c_0^2 z^2 + \dots \right) - a \right) = \text{finite}. \end{aligned}$$

Thus, the integrand is finite everywhere inside the finite integration region and therefore the integral is finite. Furthermore this last equation shows that the divergent term in the UV of S_{NG} is proportional to the perimeter of the loop, which is consistent with the fact that also the subtraction is proportional to the perimeter of the loop in the UV, as shown by (16). In this respect, it is important to note that if only subtractions proportional to the perimeter of

the loop when $z_1 \rightarrow 0$ are considered, then the coefficients of a^n with $n > 1$ will be independent of the precise subtraction considered. An example of this independence is given by changing z_{IR} , where such a change only affects the coefficient proportional to the perimeter. In fact the role of the radius a in this calculation is similar to that of an the external momentum in a standard correction to a Green function in quantum field theory. Derivatives with respect to the external momenta of order higher than the degree of divergence of such a diagram are not admitted as counterterms. In the calculation considered in this work subtractions proportional to a^n with $n > 1$ are therefore excluded.

IV. MODEL INDEPENDENT RESULTS

In this section some results that follow from the general setting described in the previous sections are considered. No approximation is involved in the derivation of these properties.

A. In QCD $f(z)$ is even

It is recalled that $f(z)$ is the function appearing in the warp factor (2). The title of this subsection means the following: the basic hypothesis underlying this work is that the vacuum expectation value of the Wilson loop in QCD is given by $S_{\text{NG}}^{\text{sub}}$. It will be shown below that under this assumption, the fact that there are no odd-dimensional condensates³ in QCD implies that $f(z) = f(-z)$. The proof of this assertion is based on the following intermediate result.

If the expansion of $\frac{d}{dz} S_{\text{NG}}[z](a)$ as a power series in a only involves even powers of a then,

$$f(z) - f(-z) = \text{const}. \quad (23)$$

Proof: Denoting by $S_{\text{NG}}[z](a)$ the NG action with parameter a , the hypothesis is

$$S_{\text{NG}}[z](a) = S_{\text{NG}}[z](-a).$$

Noting that the change $a \rightarrow -a$ is, at the level of the NG action, the same as changing $z \rightarrow -z$ implies

$$S_{\text{NG}}[z](-a) = S_{\text{NG}}[-z](a)$$

$$\Downarrow (23)$$

$$S_{\text{NG}}[z](a) = S_{\text{NG}}[-z](a).$$

Due to this last equality if $z(r)$ extremizes the NG action so does $-z(r)$. Therefore $-z(r)$ must also be a solution

³By definition a condensate of dimension n is the coefficient of a^n in the expansion of $\frac{d}{dz} S_{\text{NG}}[z](a)$ in powers of a .

of the equation of motion. The equation of motion for $z(r)$ is

$$r \frac{z''(r)}{1+z'(r)^2} + z'(r) - 2r \left(-\frac{1}{z} + \frac{df}{dz}(z) \right) = 0.$$

On the other hand, the equation of motion for $-z(r)$ is

$$-\left(r \frac{z''(r)}{1+z'(r)^2} + z'(r) - 2r \left(-\frac{1}{z} - \frac{df}{dz}(-z) \right) \right) = 0.$$

Adding these two equations leads to

$$\begin{aligned} \frac{df}{dz}(-z) - \frac{df}{dz}(z) = 0 &\Rightarrow \frac{d}{dz} [f(-z) - f(z)] = 0 \\ &\Downarrow \\ f(-z) - f(z) &= \text{const.} \end{aligned}$$

as claimed. For arbitrary z the only solution to the last equation is

$$f_{\text{odd}}(z) = f(z) - f(-z) = 0,$$

which shows that only even functions $f(z)$ are relevant for QCD. In particular for the warp factors considered in the present work, the above general result implies that if the only nonvanishing condensates are even dimensional, the coefficients α_n must vanish if $n = \text{odd}$.

B. Condensates of dimension $n > 1$ are independent of α_m for $m > n$ and $z_1 \rightarrow 0$

First it is noted that $\frac{\alpha'}{L^2} S_{\text{NG}}[z](a, \alpha)$ is dimensionless and that α_n has dimension of length to the $-n$. Therefore if α_m would contribute to the condensate of dimension $n < m$ then inverse powers of α_k should appear for some $k > n$. Therefore in that case $\frac{\alpha'}{L^2} S_{\text{NG}}[z](a, \alpha)$ would diverge when $\alpha_k \rightarrow 0$. However the integrand in $\frac{\alpha'}{L^2} S_{\text{NG}}[z](a, \alpha)$ is well defined when any or all of the α 's vanish. Indeed the only divergence in $\frac{\alpha'}{L^2} S_{\text{NG}}[z](a, \alpha)$ is proportional to a , and appears when all the α_n vanish, but something proportional to a does not contribute to the condensates with $n > 1$. Therefore only positive powers of the α 's can appear and the result follows from dimensional reasons. It should be noted that this result holds for $z_1 \rightarrow 0$, otherwise since z_1 has dimensions of length all the dimensional arguments made above are not valid. In conclusion, the general expression for the expansion in powers of a of the NG action is

$$\begin{aligned} \frac{\alpha'}{L^2} S_{\text{NG}}[z](a, \alpha) &= s^{(0)} + s^{(2)} \alpha_2 a^2 + (s_2^{(4)} \alpha_2^2 + s_4^{(4)} \alpha_4) a^4 \\ &\quad + (s_2^{(6)} \alpha_2^3 + s_{2,4}^{(6)} \alpha_2 \alpha_4 + s_6^{(6)} \alpha_6) a^6 + \dots, \end{aligned}$$

where the coefficients $s^{(n)}$ are dimensionless.

V. ON-SHELL NAMBU-GOTO ACTION EXPANDED IN POWERS OF THE RADIUS a

A. Condensates of dimension 2 and 4

The approach employed in this section is basically the same as in [23]. That is, expand the solution of the equation of motion as a power series in a^2 , replace in the Lagrangian, expand it in powers of a^2 and then integrate. However they differ in some aspects. An important difference is that in this work more general curved backgrounds are considered. More precisely, the warp factors given in (2)–(3) are considered for $n = 1$ and for both α_2 and α_4 nonvanishing. The consideration of $\alpha_4 \neq 0$ is particularly relevant from the phenomenological point of view. This is so because $\alpha_4 \neq 0$ allows for a nonvanishing gluon condensate of dimension 4 without having at the same time one of dimension 2 which is not allowed in QCD.⁴ The other difference concerns the subtraction procedure which in this work is done as described in Sec. III. According to this procedure the NG action should be calculated for a loop lying at a value z_1 of the coordinate orthogonal to the boundary. In this respect it is convenient to define the variable,

$$t = \sqrt{1 + w_1^2 - \rho^2}, \quad w_1 = \frac{z_1}{a}, \quad \rho = \frac{r}{a}.$$

In this variable the AdS solution (11) is written as

$$w(t) = t, \quad w = \frac{z}{a}.$$

In terms of the variable $\psi(t) = w^2(t)$ the NG action is given by

$$S_{\text{NG}} = \frac{L^2}{\alpha'} \int_0^1 \frac{e^{2(a^2 \alpha_2 \psi + a^4 \alpha_4 \psi^2)} t \sqrt{4 + \frac{(1+w_1^2-t^2)\psi'(t)^2}{t^2 \psi(t)}}}{2\psi(t)} dt. \quad (24)$$

The equation of motion for this action reads

$$\begin{aligned} 64a^4 t^3 \alpha_4 \psi'(t)^3 - (1 - t^2 + w_1^2)(2t - \psi'(t))\psi'(t)^2 \\ + 16\psi(t)^2 (2a^2 t^3 \alpha_2 + a^4 t(1 - t^2 + w_1^2) \alpha_4 \psi'(t)^2) \\ - 4\psi(t)(4t^3 - (1 + t^2 + w_1^2)\psi'(t)) \\ - 4\psi(t)(-2a^2 t(1 - t^2 + w_1^2) \alpha_2 \psi'(t)^2 \\ + t(1 - t^2 + w_1^2)\psi''(t)) = 0. \end{aligned} \quad (25)$$

As explained earlier, the boundary conditions to be required are the following ones:

⁴In this assertion the effect of renormalons is neglected. This assumption is supported by the results in [15,27].

$$\psi(w_1) = w_1^2, \quad \psi'(\sqrt{1+w_1^2}) = \text{finite},$$

which correspond to the loop located in the plane $z = z_1$ and the surface with no cusp at $r = 0$.

Next, a power series expansion ansatz for the solution is considered, namely,

$$\psi(t) = \sum_{n=0}^{\infty} a^{2n} \psi_n(t). \quad (26)$$

Replacing in (25) and requiring the vanishing of the coefficient in front of a^{2n} , for $n = 0$ this leads to

$$\begin{aligned} &(-1 + t^2 - w_1^2)(2t - \psi'_0(t))\psi'_0(t)^2 \\ &+ 4\psi_0(t)((1 + t^2 + w_1^2)\psi'_0(t) \\ &+ t(-4t^2 + (-1 + t^2 - w_1^2)\psi''_0(t))) = 0, \end{aligned}$$

$$\begin{aligned} S_{\text{NG}} = \frac{L^2}{\alpha'} &\left(\frac{\sqrt{1+w_1^2}}{w_1} - 1 + a^2 \alpha_2 \left(\frac{10}{3} - 3w_1 + \frac{14}{5} w_1^2 \right) \right. \\ &\left. + a^4 \left\{ \alpha_2^2 \left(\frac{14}{9} (17 - 24 \log 2) - \frac{8}{3} w_1 + w_1^2 \frac{4}{45} (599 - 744 \log 2) \right) + \alpha_4 \left(\frac{14}{9} + \frac{124}{45} w_1^2 \right) \right\} + \dots \right). \quad (27) \end{aligned}$$

The first term in the parenthesis is divergent in the UV limit $w_1 \rightarrow 0$. This divergence, as will be seen in the next section, is canceled by the subtraction S_{CT} . The other terms are finite in this limit. Also, in this limit the result for the coefficients of a^2 and a^4 coincide with the ones in [23].

B. Condensates of dimension 6, 8 and 10

It is recalled that as in Sec. IV A a condensate of dimension n is by definition the coefficient of a^n in the expansion of $\frac{\alpha'}{L^2} S_{\text{NG}}[z](a)$ in powers of a . The calculation of these condensates is done in the UV limit, $z_1 \rightarrow 0$. This procedure is valid since, according to the analysis in the subsection of Sec. III A, the only coefficient that diverges in this limit, is the one corresponding to the perimeter of the loop, i.e., the coefficient of a^1 . Taking into account this remark, the calculation of these condensates follows the same technique as in the previous subsection except that $z_1 = 0$ is taken from the start. Their computation is possible under the assumption $\alpha_2 = 0$, i.e., no dimension 2 condensate. As an example the condensate of dimension 6 is considered. That condensate must be proportional to α_6 . This follows from the dimensional arguments which are considered in Appendix B. There it is shown that for the warp factor of the form (2), $\alpha' S_{\text{NG}}/L^2$ should be dimensionless; thus, the coefficient of a^6 in this quantity should

whose solution is the AdS one $\psi_0(t) = t^2$. For $n = 1$,

$$2(1 + w_1^2)(4t^3 \alpha_2 + \psi'_1(t)) + t(-1 + t^2 - w_1^2)\psi''_1(t) = 0,$$

whose solution up to order $\mathcal{O}(w_1^2)$ is

$$\begin{aligned} \psi_1(t) = &-\frac{1}{1+t} 4\{t(-2-t+t^2 + (-4+(-2+t)t)w_1^2) \\ &+ 2(1+t)(1+2w_1^2)\text{arctanh}(t) \\ &+ (1+t)(1+2w_1^2)\log(1-t^2)\} \alpha_2. \end{aligned}$$

In a similar fashion the equation and its solution for $\psi_2(t)$ are obtained.

Next the NG action expansion in powers of a is computed. Replacing the solution (26) in the integrand of (24), expanding in powers of a and w_1 and integrating leads to

have dimension of length to the -6 . Next recalling that the dimension of α_n is length to the $-n$, then the only way of getting such a dimension in terms of positive⁵ powers of the α 's is by means of $\alpha_2 \alpha_4$ or α_6 ; thus, the assumption $\alpha_2 = 0$ leaves only α_6 . In terms of the variables t and ψ the action to be considered is therefore

$$S_{\text{NG}}^{(6)} = \frac{L^2}{\alpha'} \int_0^1 \frac{e^{2a^6 \alpha_6 \psi^3} t \sqrt{4 + \frac{(1+w_1^2-t^2)\psi'(t)^2}{t^2 \psi(t)}}}{2\psi(t)} dt. \quad (28)$$

Next, an expansion in powers of a^2 of the solution is considered as in (26); replacing this in the equation of motion determines the coefficients $\psi_n(t)$, giving

$$\psi^{(6)}(t) = t^2 + a^6 \frac{\alpha_6}{10} (24t - 12t^2 - 6t^4 - 4t^6 - 24 \log(1+t)). \quad (29)$$

Replacing in (28) gives the following contribution proportional to a^6 :

⁵See IV B.

$$\frac{\alpha'}{L^2} S_{\text{NG}}|_{a^6} = \frac{3}{5} \alpha_6 a^6.$$

For the case of the dimension 8 condensate, for dimensional reasons, only α_4 and α_8 are relevant. The action to be considered thus involves only these two coefficients in the warp factor. The solution can be obtained as a power series in a up to order a^8 , having an expression considerably more lengthy than (29), which is given in Appendix C. The contribution proportional to a^8 to the NG action is given by

$$\frac{\alpha'}{L^2} S_{\text{NG}}|_{a^8} = -\frac{11}{5670} ((-2111 + 3360 \log 2) \alpha_4^2 - 270 \alpha_8) a^8.$$

For the case of the dimension 10 condensate, for dimensional reasons, only α_4 , α_6 and α_{10} are relevant. The action to be considered thus involves only these coefficients in the warp factor. The solution can be obtained as a power series in a up to order a^{10} , having an expression considerably more lengthy than (29), which is given in Appendix C. The contribution proportional to a^{10} to the NG action is given by

$$\frac{\alpha'}{L^2} S_{\text{NG}}|_{a^{10}} = \frac{13}{4725} ((2999 - 5040 \log 2) \alpha_4 \alpha_6 + 175 \alpha_{10}) a^{10}.$$

Proceeding in this way leads to

$$\begin{aligned} \frac{\alpha'}{L^2} S_{\text{CT}} &= \sqrt{a^2 + z_1^2} \int_{z_1}^{z_m} dz \frac{e^{2\alpha_2 z^2 + 2\alpha_4 z^4}}{z^2} \\ &= \sqrt{a^2 + z_1^2} \int_{z_1}^{z_m} dz \left(\frac{1}{z^2} + 2\alpha_2 + (2\alpha_2^2 + 2\alpha_4)z^2 + \left(\frac{4\alpha_2^3}{3} + 4\alpha_2\alpha_4 \right)z^4 + \dots \right) \\ &= \sqrt{a^2 + z_1^2} \left(-\frac{1}{z} + 2\alpha_2 z + \frac{2}{3}(\alpha_2^2 + \alpha_4)z^3 + \frac{4}{15}\alpha_2(\alpha_2^2 + 3\alpha_4)z^5 + \dots \right) \Big|_{z_1}^{z_m}. \end{aligned} \quad (32)$$

Note that the $-1/z$ appearing in the last equality, when evaluated at $z = z_1$ and multiplied by the factor $\sqrt{a^2 + z_1^2}$, cancels the divergent term, when $z_1 \rightarrow 0$, appearing in the on-shell NG action in (24).

⁶For the case considered, $\alpha_n = \delta_{n4}\alpha_4$, the integral in (30) can be explicitly calculated as

$$\int_{z_1}^{z_m} dz \frac{e^{2\alpha_4 z^4}}{z^2} = \frac{1}{4} \left(\frac{E_{\frac{3}{4}}(-2z_1^4 \alpha_4)}{z_1} - \frac{E_{\frac{3}{4}}(-2z_m^4 \alpha_4)}{z_m} \right) = \frac{1}{z_1} + \left(-\frac{E_{\frac{3}{4}}(-2z_m^4 \alpha_4)}{4z_1} + \frac{\Gamma(-\frac{1}{4})(-\alpha_4)^{1/4}}{2 \cdot 2^{3/4}} \right) \cong -\frac{2\alpha_4 z_1^3}{3} - \frac{2}{7} \alpha_4^2 z_1^7 + O(z_1^8),$$

where $E_\nu(z)$ denote the exponential integral and the last approximate equality is an expansion in powers of z_1 . From this expression it is clear that the coefficients of positive powers of z_1 are the same as the ones obtained expanding the integrand in (30).

VI. THE GLUON CONDENSATE, UV UNIVERSALITY

A. The computation of the subtraction

The subtracted NG action is

$$S_{\text{NG}}^{\text{sub}} = S_{\text{NG}} - S_{\text{CT}},$$

where

$$\begin{aligned} S_{\text{CT}} &= \frac{\sqrt{a^2 + z_1^2}}{\alpha'} \int_{z_1}^{z_m} dz e^{2A(z)} \\ &= L^2 \frac{\sqrt{a^2 + z_1^2}}{\alpha'} \int_{z_1}^{z_m} dz \frac{e^{2\sum_{n=1}^{\infty} \alpha_n z^n}}{z^2}, \end{aligned} \quad (30)$$

and z_m denotes the minimum of $e^{2A(z)}$. Here the computation is done for the case where only α_2 and α_4 are different from 0. In this case,

$$z_m = \frac{1}{2} \sqrt{\frac{\sqrt{\alpha_2^2 + 4\alpha_4}}{\alpha_4} - \frac{\alpha_2}{\alpha_4}}. \quad (31)$$

Because the integrand in (30) is well behaved in the integration region, the exponential in the integrand can be expanded before performing the integral.⁶

B. The subtracted on-shell NG action

$S_{\text{NG}}^{\text{sub}}$ is defined in (13). Using (24), (32) and keeping terms up to order z_1 , explicitly, leads to,

$$\begin{aligned} \frac{\alpha'}{L^2} S_{\text{NG}}^{\text{sub}} &= -1 + a \left(\frac{1}{z_m} - 2z_m \alpha_2 \right) + \frac{10}{3} a^2 \alpha_2 \\ &+ a^4 \left(\alpha_2^2 \frac{14}{9} (17 - 24 \log 2) + \alpha_4 \frac{14}{9} \right) \\ &- \frac{8}{3} a^3 z_1 \alpha_2^2 + \mathcal{O}(z_1^2). \end{aligned} \quad (33)$$

The first two lines correspond to the terms that survive in the UV limit $z_1 \rightarrow 0$. The IR scale z_m should be replaced by its expression (31). In this respect it is worth noticing that the contribution of that scale is proportional to a , therefore a change in that scale only changes the coefficient of the perimeter. For $\alpha_4 = 0$ the results for the coefficients of a^2 and a^4 coincide in the UV limit with the ones in [23].

Next the expression of $S_{\text{NG}}^{\text{sub}}$ as a power series in a^2 in the UV limit $z_1 \rightarrow 0$, is given for the case $\alpha_2 = 0$,

$$\begin{aligned} \frac{\alpha'}{L^2} S_{\text{NG}}^{\text{sub}} \Big|_{z_1=0} &= -1 + \frac{1}{z_m} a + \alpha_4 \frac{14}{9} a^4 + \frac{3}{5} \alpha_6 a^6 \\ &- \frac{11}{5670} ((-2111 + 3360 \log 2) \alpha_4^2 - 270 \alpha_8) a^8 \\ &+ \frac{13}{4725} ((2999 - 5040 \log 2) \alpha_4 \alpha_6 + 175 \alpha_{10}) a^{10}. \end{aligned} \quad (34)$$

Due to the proof in IV B these results are unchanged by considering additional α 's in the expression (3). It means that the coefficients of a^n in this expression are exact, the inclusion of additional terms in the expansion of the warp exponent do not change their value. This is a strictly UV result, it is only valid in the limit $z_1 \rightarrow 0$. It shows that if the expectation value of Wilson loops are related to minimal areas in the dual theory, as assumed, then the expectation values of gauge invariant operators in QCD can be used to systematically build the QCD dual background. In particular, since there is no dimension 2 gauge invariant operator in pure QCD then the coefficient of a^2 should be zero.⁷ Thus, under these conditions, this absence implies $\alpha_2 = 0$. The case of the coefficient of a^4 is different since in QCD there is a gauge invariant quantity of dimension 4, which is the expectation value of $\langle F_{\mu\nu} F^{\mu\nu} \rangle$. This coefficient is related to the gluon condensate, and its value fixes the value of α_4 . This procedure can be continued for higher order terms in the expansion. Higher dimensional condensates fix the values of higher index α_i coefficients, once the ones with lower indices are known. This is clearly exemplified by expression (34).

⁷See footnote 4.

C. Computation of the gluon condensates

For the soft wall case $\alpha_i = \delta_{i2} \alpha_2$ the results are the same as in [23].

For the z^4 case, $\alpha_i = \delta_{i4} \alpha_4$. Equation (33) shows that in this case the coefficient of a^4 in $S_{\text{NG}}^{\text{sub}}$ is $\frac{14}{9} \frac{L^2}{a'} \alpha_4$. Using the expression of G_2 in terms of this coefficient appearing in [23] leads to the expression for α_4 :

$$\alpha_4 = \frac{1}{56} \frac{\pi^4}{L^2 a'} G_2, \quad (35)$$

which according to the value of $G_2 = 0.028 \text{ GeV}^4$ in [27] gives $\frac{L^2}{a'} \alpha_4 = 4.9 \times 10^{-3} \text{ GeV}^4$.

In Appendix D it is shown how an additional relation between $\frac{L^2}{a'}$ and α_4 can be obtained by means of computing the string tension for the linear potential between static quarks. This argument is based on the important fact that the coefficient in front of the NG action, i.e., $\frac{L^2}{a'}$, is independent of the loop's shape. In particular it is the same for the circular and for the rectangular loops. In the case $\alpha_i = \delta_{i4} \alpha_4$ this relation is $\sigma = \frac{L^2}{a'} 2\sqrt{e} \sqrt{\alpha_4}$ where σ is the string tension mentioned above. Taking for it the slope in the linear term of the Cornell potential [28], i.e., $\sigma = 0.186 \text{ GeV}^2$ leads to

$$\frac{L^2}{a'} = 0.65, \quad \text{and} \quad \alpha_4 = 7.5 \times 10^{-3} \text{ GeV}^4. \quad (36)$$

For the case $\alpha_i = \delta_{i6} \alpha_6$ the coefficient of a^6 in $S_{\text{NG}}^{\text{sub}}$ is equal to

$$\frac{L^2}{a'} \frac{3}{5} \alpha_6. \quad (37)$$

In [29] a relation between this expectation value and the coefficient of a^6 in the expansion of a circular loop on its radius a is derived, namely,

$$\langle W(C) \rangle \Big|_{a^6} = \frac{\pi^2}{192 N_c} \langle g^3 f_{abc} F_{a\beta}^a F_{\beta\delta}^b F_{\delta\alpha}^c \rangle, \quad (38)$$

which should be equal to the coefficient (37). Furthermore as shown in Appendix D the string tension for this case is given by $\sigma = \frac{L^2}{a'} (6\alpha_6 e)^{1/3}$. Using the value of σ already quoted and the one of $\frac{L^2}{a'}$ obtained in (36), one obtains $\alpha_6 = 1.4 \times 10^{-3} \text{ GeV}^6$, leading to

$$\begin{aligned} \langle g^3 f_{abc} F_{a\beta}^a F_{\beta\delta}^b F_{\delta\alpha}^c \rangle &= \frac{192 N_c}{\pi^2} \frac{L^2}{a'} \frac{3}{5} \alpha_6 \\ &= 3.18 \times 10^{-2} \text{ GeV}^6, \end{aligned}$$

which is compatible with the estimation $\simeq 4.5 \times 10^{-2} \text{ GeV}^6$ given in [12] for this condensate.

VII. CONCLUDING REMARKS

In this work the calculation of the minimal area bounded by a circular loop lying at a certain value z_1 of the radial coordinate z has been considered. This surface is embedded in a 5-dimensional space with a global metric which in conformal coordinates depends only on the warp factor $e^{2A(z)}$. The connection of this calculation with QCD observables is shown in the following scheme:

$$\text{Global metric} \xleftrightarrow{\text{NG}} \text{Min area} \xleftrightarrow{a\text{-exp}} \text{QCD condensates,}$$

where a -exp goes for the expansion of the minimal area in powers of the loop radius a . The continuation of this scheme to the left would require the knowledge of a gravity-string theory from which the warp factor could be obtained. In this respect it is worth remarking that if such a theory would include a dilaton field then the warp factor $e^{2A(z)}$ considered in this work corresponds to the string frame warp factor [25]. The arrows in the above scheme go in both directions, trying to indicate that these connections could be employed in both ways. That is, knowledge of QCD condensates could be employed to obtain warp factors as in (35) and, in the other direction, details of a higher dimensional theory would give information about QCD.

Regarding the connection between the minimal area and the condensates, it is emphasized that an important ingredient for this connection is the subtraction employed. This subtraction involves both UV and IR divergences, the first already present in the AdS case are treated as in [17] and maintaining conformal invariance, the second coming from the consideration of confining warp factors, require an IR scale which is argued to be given naturally by the location of the minimum of these warp factors. In this respect, it is important to realize that the approximations employed are well suited for the calculation of the first coefficients in the expansion in powers of the radius a for the *subtracted* NG action.

Finally it is noted that the techniques employed in this work are not restricted to the particular family of warp factors (2). Any other choice that can be made convergent by the subtractions appearing in Sec. III would work.⁸ If this is not the case other subtractions should be considered.

ACKNOWLEDGMENTS

This work was supported by DOE Contract No. DE-AC05-06OR23177 under which JSA operates the Thomas Jefferson National Accelerator Facility, and by the National Science Foundation (U.S.) through Grants No. PHY-0855789 and No. PHY-1307413 (J.L.G.), and by CONICET (Argentina) PIP No. 11220090101018 (R.C.T.). J.L.G. thanks the Instituto Balseiro and the Centro

Atómico Bariloche for hospitality and support during the early stages of this project.

APPENDIX A: THE SUBTRACTION FOR Dp -BRANES INSPIRED WARP FACTORS

As an example of other warp factors of interest, the following are considered:

$$A(z) = -n \log\left(\frac{z}{L}\right) + f(z). \tag{A1}$$

The case $n = 1$ is the one already studied in III A. For backgrounds generated by a stack of Dp -branes, one often arrives to metrics with $n \leq 1$. These warp factors diverge in the UV; the leading singularity is

$$A(z) \sim -n \log\left(\frac{z}{L}\right) \Rightarrow A'(z) \sim -n \frac{L}{z}. \tag{A2}$$

The equation of motion near the boundary implies

$$-\lim_{z \rightarrow 0} 2rn \frac{1}{z} r'(1+r'^2) + \lim_{z \rightarrow 0} r r'' - \lim_{z \rightarrow 0} (1+r'^2) = 0,$$

which in a similar way as in III A leads to the following asymptotic behavior:

$$\lim_{z \rightarrow 0} r'(z) = 0r'(z) = \frac{1}{a(1-2n)} z + \dots (z \ll 1).$$

Inserting this in $S_{\text{NG}}^{\text{sub}}$, shows that the leading behavior of the integrand in the NG action is governed by

$$\begin{aligned} & \lim_{z \rightarrow 0} e^{2A(z)} (r(z) \sqrt{1+r'(z)^2} - a) \\ &= \lim_{z \rightarrow 0} \frac{1}{z^{2n}} \left(a \left(1 + \frac{1}{2} c_0^2 z^2 + \dots \right) - a \right) \\ &= a \lim_{z \rightarrow 0} \frac{1}{z^{2n}} \left(\frac{1}{2} c_0^2 z^2 + \dots \right) = 0. \end{aligned}$$

This implies that for $n > 1$ the regularization procedure does not work since the expression diverges. However $n \leq 1$ corresponds to the metrics obtained from top-bottom approaches with stacks of Dp -branes. A concrete example can be found in [30], where the area of the circular loop is found to be

$$S_{\text{NG}}|_{Dp} = \frac{1}{2\pi\alpha'} \int_0^a \left(\frac{5-p}{2} \frac{1}{z} \right)^{\frac{7-p}{5-p}} r \sqrt{1+z'^2} dr. \tag{A3}$$

The regularization procedure ensures the convergence of the subtracted area except for the cases $p = 4$ and $p = 5$.

⁸See Appendix A for an example.

APPENDIX B: ADMISSIBLE MONOMIALS IN THE EXPANSION OF THE NAMBU-GOTO ACTION SOLUTION

The NG action times α' is an area and therefore has dimension of length squared. Making explicit the first term in (A1) it is written as follows:

$$\alpha' S_{\text{NG}} = L^{2n} \int_0^a \frac{e^{2\sum_{k=1}^n \alpha_k z^k}}{z^{2n}} r \sqrt{1 + z'^2} dr.$$

Therefore, the integral in this last equation should have dimensions of length to the power $2 - 2n$. In particular for the case $n = 1$ (deformation of AdS), it should be dimensionless. This integral depends on a, z_1 which have dimension of length, and the α 's. In this respect it is useful to note that the α_k has dimensions of length to the power $-k$. Any monomial contributing to $\frac{\alpha'}{L^{2n}} S_{\text{NG}}$ of the general form,

$$a^j z_1^m \alpha_k^l,$$

will have vanishing coefficient unless

$$j + m - kl = 2 - 2n.$$

The same general conclusions are valid for $\frac{\alpha'}{L^{2n}} S_{\text{CT}}$.

APPENDIX C: SOLUTIONS NEEDED TO OBTAIN THE CONDENSATES OF DIMENSIONS 8 AND 10

The solution expanded in powers of a reads

$$\psi(t) = t^2 + \sum_{n=1}^{\infty} a^{2n} \psi_n(t). \quad (\text{C1})$$

To obtain the condensates up to dimension 10 one needs to include terms up to $n = 5$. For the case where $\alpha_2 = 0$, the functions ψ_n are found to be

$$\begin{aligned} \psi_1(t) &= 0 \\ \psi_2(t) &= -\frac{2}{3} \alpha_4 (t(t^3 + 2t - 4) + 4 \log(t + 1)) \\ \psi_3(t) &= -\frac{1}{5} \alpha_6 (t(2t^5 + 3t^3 + 6t - 12) + 12 \log(t + 1)) \\ \psi_4(t) &= -\frac{2}{945} \alpha_4^2 \left(-6720 \text{Li}_2\left(\frac{t+1}{2}\right) \right. \\ &\quad - 8 \log(t + 1)(105t(t^3 + 6t + 6) + 210 \log(t + 1) - 478) \\ &\quad + (t(t(135t^5 - 156t^3 - 304t + 1260) + 5272) - 3824)t \\ &\quad + 13440t \log(2) + 6720 \log(1 - t) \log\left(\frac{2}{t+1}\right) + 560\pi^2 - 3360 \log^2(2) \Big) \\ &\quad \left. - \frac{2}{21} \alpha_8 (t(3t^7 + 4t^5 + 6t^3 + 12t - 24) + 24 \log(t + 1)) \right) \\ \psi_5(t) &= -\frac{1}{1890} \left\{ 2\alpha_4 \alpha_6 \left(-30240 \text{Li}_2\left(\frac{t+1}{2}\right) \right. \right. \\ &\quad + t(t(t(t(420t^5 - 321t^3 - 1772t + 2772) + 240) + 2604) + 26184) - 22128) \\ &\quad - 24 \log(t + 1)(21t(2t^5 + 15t^3 + 42t + 48) + 252 \log(t + 1) - 922) \\ &\quad + 60480t \log(2) + 30240 \log(1 - t) \log\left(\frac{2}{t+1}\right) + 2520\pi^2 - 15120 \log^2(2) \Big) \\ &\quad \left. + 35\alpha_{10} (t(12t^9 + 15t^7 + 20t^5 + 30t^3 + 60t - 120) + 120 \log(t + 1)) \right\}, \quad (\text{C2}) \end{aligned}$$

where Li_2 denotes the dilogarithm function.

APPENDIX D: LINEAR POTENTIAL BETWEEN STATIC QUARKS

The string tension is given by the value at its minimum of the function [24],

$$f(z) = \frac{\alpha'}{L^2} e^{2A(z)}.$$

For the case $\alpha_n = \alpha_4 \delta_{n4}$ it is given by

$$f(z) = \frac{\alpha'}{L^2} e^{2(-\log z + \alpha_4 z^4)}.$$

Its minimum and the corresponding string tension are respectively given by

$$z_0 = \frac{1}{\sqrt{2\alpha_4}^{1/4}}, \quad \sigma = f(z_0) = \frac{2\alpha'}{L^2} \sqrt{e\alpha_4},$$

which has the right units since α_4 has units of length to the minus 4, thus $\sqrt{\alpha_4}$ has units of energy squared as it corresponds to a string tension.

For the case $\alpha_n = \alpha_6 \delta_{n6}$ the minimum and string tension are respectively given by

$$z_0 = \frac{1}{(6\alpha_6)^{1/6}}, \quad f(z_0) = \frac{\alpha'}{L^2} (6\alpha_6 e)^{1/3}.$$

-
- [1] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).
 [2] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
 [3] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
 [4] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Phys. Lett. B* **428**, 105 (1998).
 [5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Phys. Rep.* **323**, 183 (2000).
 [6] S. S. Gubser (unpublished).
 [7] T. Sakai and S. Sugimoto, *Prog. Theor. Phys.* **113**, 843 (2005).
 [8] L. Da Rold and A. Pomarol, *Nucl. Phys.* **B721**, 79 (2005).
 [9] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, *Phys. Rev. Lett.* **95**, 261602 (2005).
 [10] J. Polchinski and M. J. Strassler (unpublished).
 [11] C. Csaki and M. Reece, *J. High Energy Phys.* **05** (2007) 062.
 [12] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys.* **B147**, 385 (1979).
 [13] P. Colangelo and A. Khodjamirian (unpublished).
 [14] A. Di Giacomo and G. C. Rossi, *Phys. Lett.* **100B**, 481 (1981).
 [15] P. E. L. Rakow, *Proc. Sci. LAT2005* (2006) 284.
 [16] T. Banks, R. Horsley, H. R. Rubinstein, and U. Wolff, *Nucl. Phys.* **B190**, 692 (1981).
 [17] J. M. Maldacena, *Phys. Rev. Lett.* **80**, 4859 (1998).
 [18] S. Rey and J. Yee, *Eur. Phys. J. C* **22**, 379 (2001).
 [19] L. Hung, R. C. Myers, and M. Smolkin, *J. High Energy Phys.* **11** (2011) 001.
 [20] U. Gursoy and E. Kiritsis, *J. High Energy Phys.* **02** (2008) 032.
 [21] J. L. Goity and R. C. Trinchero, *Phys. Rev. D* **86**, 034033 (2012).
 [22] C. Fefferman and C. R. Graham, in *Astérisque* 1985, Numero Hors Serie (1984), pp. 95–116.
 [23] O. Andreev and V. I. Zakharov, *Phys. Rev. D* **76**, 047705 (2007).
 [24] Y. Kinar, E. Schreiber, and J. Sonnenschein, *Nucl. Phys.* **B566**, 103 (2000).
 [25] U. Gursoy, E. Kiritsis, and F. Nitti, *J. High Energy Phys.* **02** (2008) 019.
 [26] C. R. Graham and E. Witten, *Nucl. Phys.* **B546**, 52 (1999).
 [27] E. M. Ilgenfritz, *Proc. Sci. ConfinementX* (2012) 301.
 [28] E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. Lane, and T. Yan, *Phys. Rev. Lett.* **34**, 369 (1975).
 [29] M. A. Shifman, *Nucl. Phys.* **B173**, 13 (1980).
 [30] P. N. Kopnin and A. Krikun, *Phys. Rev. D* **84**, 066002 (2011).