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Convex Soft Geometries

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Abstract: This paper introduces convex soft geometries. The import of this model is that it extends a successful field of research, namely convex geometries, to account for alternatives characterized by a multiplicity of attributes. Related concepts include soft convex hulls and a notion of extreme elements. Some fundamental results are proven. Among them, we guarantee that an anti-exchange property holds true. Importantly, the existence of extreme elements is guaranteed, hence proving their suitability for applied studies. Both results are natural (but non-trivial) extensions of comparable outputs for convex geometries. We present a detailed research program that might stimulate future investigations on this new research area.

Keywords: soft set; convex soft geometry

1. Introduction

This article lies at the crossroads of two subjects, specifically convex geometry and soft set theory. This blend initiates a new area of research for which many successful parallels exist in the soft computing literature.

Indeed, from the moment Zadeh (1965) launched fuzzy set theory, interest on fuzzy models grew and substantial research on hybrid constructions was conducted. Thus, for example, topological notions were soon combined with fuzzy set theory and its extensions. Chang (1968) defined fuzzy topology, and the notion was reformulated by Goguen (1973), Lowen (1976) (who clarified the links between fuzzy topologies and point-set or set-theoretic topologies), Hutton (1980) (who insisted on "pointless" definitions for both structures and properties), or Šostak (1985). Katsaras and Liu (1977) initiated the study of fuzzy (topological) vector spaces (v., Katsaras (1981, 1984) for deeper investigations on fuzzy topological vector spaces). Also fuzzy group theory (Mordeson et al., 2005) was developed soon after the inception of fuzzy sets.

Many other models of uncertain knowledge were proposed after the advent of fuzzy set theory, for example, rough set theory (Pawlak, 1982) or soft set theory (Molodtsov, 1999). The latter uses parameterized descriptions of the alternatives in terms of a set of attributes. These descriptions are binary in Molodtsov (1999) and multinary in Fatimah et al. (2018). Thus, whereas fuzzy sets allow us to represent alternatives whose unique identifying characteristic can be partially fulfilled, soft sets are designed to represent alternatives with a multiplicity of crisp characteristics, each of which can be either fulfilled or not in the case of Molodtsov (1999), or occur in various grades in Fatimah et al. (2018). Mixed opinions can be taken, for example, with fuzzy soft sets (Maji et al., 2001).

In particular, topology has exerted a special pull among researchers on soft set theory since Shabir and Naz (2011) launched soft topology. It stems from the hybridization of the axioms of topology with soft set theory. Beyond this model, Tanay and Kandemir (2011) defined fuzzy soft topological spaces, which extended the applicability of fuzzy soft sets (Maji et al., 2001) in multi-criteria group decision making (in medical practice, Hassan et al. (2017), for example) to a topological context (cf., Khameneh et al. (2017)). Tradeoffs between both models have been stated recently (Alcantud, 2021).

Admittedly, this paper is inspired by the success of soft topological studies, and it is motivated by the remarkable appeal of this abstract soft structure and its applications to decision-making theory (Al-shami, 2021a, 2021b; Alcantud, 2021). Our main goal, however, is the introduction of a totally new soft structure with an inclination toward abstract convexity. Now the inspirational model is convex geometries (cf., Edelman and Jamison (1985)). This is a combinatorial abstraction of convexity that embraces all standard examples and allows one to make use of techniques from convexity theory, the theory of ordered sets, and graph theory.

It is timely to clarify that already Deli (2013) and afterwards Majeed (2016) studied convex and concave soft sets. But these models require an infinite set of attributes, on which convex combinations can be defined properly. Our approach to "soft convexity" is radically different. To begin with, we shall work in a totally finite environment where both the sets of alternatives and relevant attributes are finite. Importantly, no further requirements are imposed on these sets. Then we consider that a way to incorporate convexity considerations consists of extending the idea of a convex geometry (Edelman & Jamison, 1985) to a soft set context. We can regard this extension from two perspectives. Since convex geometries are defined on a set, we can faithfully assume that this set is defined by a unique attribute (the one that defines "belongingness" to it). From this viewpoint, our model shall extend the ethos of convex geometries to sets whose alternatives are characterized by a multiplicity of attributes. Alternatively, from the perspective of soft set theory we are introducing a novel convexity structure that is free from any constraints on the ground set or the relevant attributes. Another remarkable advantage of this abstract structure is that it builds upon a strong theoretical foundation.

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In addition, it is important to highlight that this mostly theoretical approach has its upside. A rigorous theoretical foundation of abstract convexity in soft set theory should forward this discipline and guarantee its interest for formal studies. This pioneering analysis can also promote the study of relationships with other types of abstract soft structures like soft topologies or others (Akram et al., 2018, 2019), for which a considerable theoretical background already exists.

This research article is divided into the following parts. Section 2 gives necessary notions from soft set theory and soft topology, and convex geometries. We provide some examples too. Section 3 introduces the formal model consisting of convex soft geometries. In addition, we study how we can associate a convex geometry with each convex soft geometry, and conversely. We prove some preliminary results and define concepts for the development of this new theory. These ideas include soft convex hulls and extreme elements for a soft set (in a given convex soft geometry). Section 4 provides two core results. Our first theorem proves that convex soft geometries satisfy an anti-exchange property. With this result, we prove another technical theorem that in particular, ensures the existence of extreme elements for any soft set. The aim of Section 5 is to end this paper. We give a concise summary of our main findings. And especially, we state various lines for future research. To facilitate the reading of our results, a summary of notation is given after Section 5.

2. Preliminary Concepts

Henceforth, X shall denote a fixed nonempty set, and E shall denote a set of characteristics of the alternatives in X. When A is a set, $\mathcal{P}(A)$ shall refer to the set of all the subsets of A, also called the set of parts of A or 2^A. Set complements in A are denoted by the $\setminus$ symbol, thus $A \setminus B$ means the elements in A that are not in B.

Section 2.1 recalls fundamental concepts from soft set theory, with some examples. Then Section 2.2 states the definition of a soft topology. And Section 2.3 recalls the definition of a convex geometry and states some important facts about them.

2.1. Elements of soft set theory

A soft set on X consists of (F, E), E being the characteristics that fully describe the members of X, such that $F : E \longrightarrow \mathcal{P}(X)$. Mathematically, it is a correspondence or multi-valued mapping from the set of attributes to the set of alternatives. We can therefore describe a soft set over X by a collection of subsets of X parameterized by E. The set of all soft sets on X characterized by E shall be represented by $SS_E(X)$ or simply $SS(X)$ if E is common knowledge.

For each $e \in E$, $F(e) \subseteq X$ can be denoted as $(F, E)(e)$ in more accurate notation. It is the set of e-approximate elements of X, also called the subset of X approximated by e. In the particular case $E = \{e\}$, the soft set (F, E) can be naturally identified with the subset $F(e) \subseteq X$. Therefore, $SS_E(X)$ can be identified with $\mathcal{P}(X)$. In the case of a general E, we can regard $SS_E(X)$ as an extension of the concept of $\mathcal{P}(X)$ to the case where membership is multiply (or "softly") defined. And then each soft set $(F, E) \in SS_E(X)$ corresponds to an extended version of the idea of a subset of X that embeds all subsets of X approximated by the attributes in E.

A soft set (F, E) on X is also represented by $\{(a, F(a)) : a \in E\}$. It can be described in tabular form if E and X are finite (Maji et al., 2003). (F, E) is called finite (respectively, countable) when $F(e)$ is finite (respectively, countable) for each $e \in E$ (Das & Samanta, 2013; Nazmul & Samanta, 2014).

The absolute and null soft sets on X are basic examples. The absolute soft set $\widetilde{X} = (\widetilde{X}, E)$ satisfies $\widetilde{X}(e) = X$ for all $e \in E$ [Maji et al., 2003, Definition 8]. The null soft set $\boldsymbol{\Phi} = (\boldsymbol{\Phi}, E)$ satisfies $\boldsymbol{\Phi}(e) = \varnothing$ for all $e \in E$ [Maji et al., 2003, Definition 7].

Special soft sets are "soft points." The next remark explains that some issues arise with this term.

Remark 1. Let us summarize the terminological problem with the utilization of the expression "soft point":

1. In [Das and Samanta, 2013, Definition 1], a "soft point" is (F, E) for which $x \in X$ and $a \in E$ exist such that $F(a) = \{x\}$ and $F(a') = \varnothing$ if $a \neq a'$. However [Nazmul & Samanta, 2014, Definition 1] refers to this definition as a "soft element." This particular soft set shall be denoted $(\{x\}_e, E)$, and it is called a "soft spot" in Alcantud (2021).
2. "Soft points" in [Zorlutuna et al. (2012), Definition 7] are different: they are soft sets (F, E) such that there is $a \in E$ with $F(a) \neq \varnothing$, and $F(a') = \varnothing$ if $a' \neq a$.
3. "Soft points" in Shabir and Naz (2011) and Terepeta (2019) are soft sets (F, E) such that $x \in X$ exists with the property $F(a) = \{x\}$ for all $a \in E$. Here (x, E) shall denote this special soft set.
4. Finally, [7, Definition 11] defined "soft point" with a concept more general than the definition in Shabir and Naz (2011) and Terepeta (2019): for Aygünoğlu and Aygün (2012), (F, E) is a "soft point" when $A \subseteq E$ and $x \in X$ exist, with the properties $F(a) = \{x\}$ when $a \in A$, $F(a') = \varnothing$ when $a' \neq a$.

Unless otherwise stated, in this article the term "soft point" will exclusively refer to the concept defined in Shabir and Naz (2011) and Terepeta (2019). Hence as said above, we shall denote by (x, E) the soft point (F, E) with $F(a) = \{x\}$ for all $a \in E$. This notion has been used to define important axioms in soft topology (Shabir and Naz, 2011; Terepeta, 2019).

Tables 1 and 2 give a visual example to help intuition when $|X| = |E| = 2$. Under these assumptions, $|SS_E(X)| = 16$. Table 1 contains tabular representations of some of the concepts defined in Remark 1. Table 2 completes the list of 16 soft sets on X. We shall use them in subsequent sections.

Soft set theory has incorporated some basic set-theoretic operations. For example, on $SS_E(X)$ one can define intersections, unions, and inclusions in the following way (Maji et al., 2003): for every $(F_1, E), (F_2, E) \in SS_E(X)$,

(1) $(F_1, E) \sqcup (F_2, E)$ is $(F_3, E) \in SS_E(X)$ that satisfies $F_3(a) = F_1(a) \cup F_2(a)$ for each $a \in E$.

Alternatively, we can write $((F_1, E) \sqcup (F_2, E))(a) = F_1(a) \cup F_2(a)$ for each $a \in E$.

Notice that unions of arbitrary collections of soft sets can be defined similarly to the case of two soft sets.

Table 1
The case $\boldsymbol{X} = \{\boldsymbol{x}, \boldsymbol{y}\}, \boldsymbol{E} = \{\boldsymbol{e}, \boldsymbol{e}'\}$: tabular representations of the null soft set; $(\{\boldsymbol{x}\}_{\boldsymbol{e}}, \boldsymbol{E})$ and $(\{\boldsymbol{x}\}_{\boldsymbol{e}'}, \boldsymbol{E})$, which are "soft points" in the sense of Das and Samanta (2013) or "soft spots" in Alcantud (2021); a soft point (x, E); $(\boldsymbol{F_1}, \boldsymbol{E})$, which is "soft point" in the sense of Zorlutuna et al. (2012); and the full soft set $\widetilde{\boldsymbol{X}}$

(Φ, E)	e	e'
x	0	0
y	0	0

$(\{x\}_e, E)$	e	e'
x	1	0
y	0	0

$(\{x\}_{e'}, E)$	e	e'
x	0	1
y	0	0

(x, E)	e	e'
x	1	1
y	0	0

(F_1, E)	e	e'
x	1	0
y	1	0

$(\tilde{X}, E)$	e	e'
x	1	1
y	1	1

Table 2
The case $\boldsymbol{X} = \{\boldsymbol{x}, \boldsymbol{y}\}, \boldsymbol{E} = \{\boldsymbol{e}, \boldsymbol{e}'\}$: tabular representations of other soft sets

$(\{y\}_e, E)$	e	e'
x	0	0
y	1	0

$(\{y\}_{e'}, E)$	e	e'
x	0	0
y	0	1

(y, E)	e	e'
x	0	0
y	1	1

(F_2, E)	e	e'
x	0	1
y	0	1

(F_3, E)	e	e'
x	1	0
y	0	1

(F_4, E)	e	e'
x	0	1
y	1	0

(F_5, E)	e	e'
x	1	1
y	1	0

(F_6, E)	e	e'
x	1	1
y	0	1

(F_7, E)	e	e'
x	1	0
y	1	1

(F_8, E)	e	e'
x	0	1
y	1	1

(2) $(F_1, E) \sqcap (F_2, E)$ is $(F_4, E) \in SS_E(X)$ that satisfies $F_4(a) = F_1(a) \cap F_2(a)$ for each $a \in E$.
Alternatively, we can write $((F_1, E) \sqcap (F_2, E))\ (a) = F_1(a) \cap F_2(a)$ for each $a \in E$.
Notice that intersections of arbitrary collections of soft sets can be defined similarly to the case of two soft sets.
(3) We write $(F_1, E) \sqsubseteq (F_2, E)$ when $F_1(a) \subseteq F_2(a)$ for every $a \in E$.

If $(F, E) \in SS_E(X)$, its complement $(F, E)^c$ is the soft set $(F^c, E) \in SS_E(X)$ defined by $F^c(a) = X \setminus F(a)$ for each $a \in E$ [Maji et al., 2003, Definition 6].

The soft equality $(F, E) = (F', E)$ means $F(a) = F'(a)$ for each $a \in E$ [Maji et al., 2003, Definition 4]. Thus $(F, E) = (F', E)$ is equivalent to $(F, E) \sqsubseteq (F', E)$ and $(F', E) \sqsubseteq (F, E)$. Soft sets that are not soft equal are called different.

Example 1. To illustrate the concepts defined above, consider $X = \{x, y\}$ with attributes $E = \{e_1, e_2, e_3\}$. Define the following two soft sets (F'_1, E) and (F'_2, E):
$F'_1(e_1) = \{y\}$, $F'_1(e_2) = X$, $F'_1(e_3) = \emptyset$,
$F'_2(e_1) = F'_2(e_2) = \{y\}$, and $F'_2(e_3) = X$.
In Table 3, we can see tabular representations for these soft sets, and also for $(F'_1, E) \sqcup (F'_2, E)$, $(F'_1, E) \sqcap (F'_2, E)$, and $((F'_2)^c, E)$.

2.2. Elements of soft topology

Let us fix a set of attributes E. We can now define soft topology on X as any collection of soft sets, $\tau \subseteq SS_E(X)$, such that three suitable axioms hold true for it. These axioms replicate those of point-set (or crisp) topology as follows:

Definition 1. (Çağman et al., 2011; Shabir & Naz, 2011) A *soft topology* τ over X is $\tau \subseteq SS_E(X)$, a collection of soft sets on X, that satisfies:

(1) The null and absolute soft sets belong to τ.
(2) If a soft set is a union of soft sets from τ, then the soft set also belongs to τ.
(3) If a soft set is an intersection of any finite collection of soft sets from τ, then the soft set also belongs to τ.

The members of τ are usually called soft open sets.

For future reference, we state a proposition that produces soft topologies from point-set topologies, and conversely. It summarizes results from various papers:

Proposition 1. (Shabir & Naz, 2011; Terepeta, 2019) *Let τ be a soft topology on X. Then for all $a \in E$, $\Sigma_a = \{F(a) | (F, E) \in \tau\}$ is a point-set topology on X.*

Conversely, when $\Sigma = \{\Sigma_a\}_{a \in E}$ is a collection of point-set topologies over X, the family of soft sets

$$\tau(\Sigma) = \left\{(F, E)\, \textit{for which}\, F(a) \in \Sigma_a \textit{ when ever } a \in E\right\}, \tag{1}$$

is a soft topology over X.

2.3. Elements of convex geometry

This section owes to Edelman and Jamison (1985). Here we recall the concept of convex geometry. We also state some fundamental results in this interesting theory.

Definition 2. (Edelman & Jamison, 1985) A *convex geometry* $\mathbf{G}$ on a nonempty *finite* set X is a family $\mathbf{G} \subseteq \mathcal{P}(X)$ that satisfies:
(1) $\emptyset \in \mathbf{G}$.
(2) The family $\mathbf{G}$ is closed under intersection, that is, $A \cap B \in \mathbf{G}$ whenever $A, B \in \mathbf{G}$.
(3) When $A \in \mathbf{G}$ and $A \neq X$, there is $x \in X \setminus A$ such that $A \cup \{x\} \in \mathbf{G}$.

The members of $\mathbf{G}$ are usually called *convex sets*.

Observe that the application of axiom (3) in Definition 2 to $A = \emptyset$ guarantees the existence of a singleton that is convex. When it is the case that $x \in \mathbf{G}$ for all $x \in X$, we say that $\mathbf{G}$ is *atomic* (Edelman & Jamison, 1985).

Example 2. Obviously, $\mathcal{P}(X)$ is an atomic convex geometry on X, for all X.

When $X = \{x, y\}$, there are only three convex geometries on X, namely $\{\emptyset, \{x\}, X\}$, $\{\emptyset, \{y\}, X\}$, and $\{\emptyset, \{x\}, \{y\}, X\}$. Only the last one is atomic. There is an isomorphism between $\{\emptyset, \{x\}, X\}$ and $\{\emptyset, \{y\}, X\}$. Intuitively, this means that if we permute the labels of the alternatives in X, each of these two convex geometries becomes the other.

When $X = \{x, y, z\}$, there are six non-isomorphic convex geometries on X (cf., [8, Example 3]). For example, we have
$\mathbf{G}_1 = \{\emptyset, \{x\}, \{x, y\}, \{x, z\}, X\}$, which is not atomic, and
$\mathbf{G}_2 = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, X\}$, which is atomic.

Table 3
Soft union, intersection, and complement: their tabular representation for the soft sets defined in Example 1

(F'_1, E)	e_1	e_2	e_3
x	0	1	0
y	1	1	0

(F'_2, E)	e_1	e_2	e_3
x	0	0	1
y	1	1	1

$(F'_1 \sqcup F'_2, E)$	e_1	e_2	e_3
x	0	1	1
y	1	1	1

$(F'_1 \sqcap F'_2, E)$	e_1	e_2	e_3
x	0	0	0
y	1	1	0

$((F'_2)^c, E)$	e_1	e_2	e_3
x	1	1	0
y	0	0	0

Definition 3. (Edelman & Jamison, 1985) The *convex hull* of $Y \subseteq X$ in the convex geometry $\mathbf{G}$ is the smallest convex superset of Y. Formally:

$$\text{conv}(Y) = \bigcap\{G \in \mathbf{G} \text{ such that } Y \subseteq G\}. \quad (2)$$

Any convex geometry $\mathbf{G}$ on X satisfies the following *anti-exchange property* [Edelman & Jamison, 1985, Theorem 1]:

$$\text{Suppose } G \in \mathbf{G}, x, y \notin G, \text{ and } x \neq y. \text{ Then } y \in \text{conv}(G \cup \{x\}) \text{ implies } x \notin \text{conv}(G \cup \{y\}). \quad (3)$$

As Koshevoy puts it, Equation (3) arises as a combinatorial abstraction of one of the geometric properties of the standard convex closure defined in any Euclidean space [Koshevoy (1999), Section 3].

Finally, for any convex geometry $\mathbf{G}$ on X, any nonempty subset of X has at least one extreme element. This concept is defined as follows:

Definition 4. (Edelman & Jamison, 1985) Let us fix a convex geometry $\mathbf{G}$ on X. When $\varnothing \neq Y \subseteq X$, we say that $x \in Y$ is an *extreme element* of Y if $x \notin \text{conv}(Y \setminus \{x\})$. We denote by $\text{ex}(Y)$ the set of extreme elements for Y (in the convex geometry $\mathbf{G}$).

One reason for the existence of extreme elements is the next remarkable property:

Proposition 2. [Monjardet & Raderanirina, 2001, Theorem 2 (2)] *Let us fix a convex geometry* $\mathbf{G}$ *on* X. *If* $Y \subseteq X$, *then* $\text{conv}(\text{ex}(Y)) = \text{conv}(Y)$.

3. Introducing Convex Soft Geometries

This section is dedicated to produce our new model. As argued above, it intends to incorporate abstract convexity considerations into soft set theory on finite sets with a finite collection of attributes. The new model is motivated by Definition 2 and in some sense, it extends it to a finite soft set scenario (Remark 2 below gives the formal expression of this "extension"). The model adopts the following form:

Definition 5. Suppose that E is a nonempty finite set of attributes. A *convex soft geometry* $\mathcal{G}$ on a nonempty *finite* set X is $\mathcal{G} \subseteq SS_E(X)$, a collection of soft sets on X, such that the following properties hold true:

(G.1) $\Phi \in \mathcal{G}$.

(G.2) The intersection of soft sets in $\mathcal{G}$ belongs to $\mathcal{G}$, that is, if $(F, E), (F', E) \in \mathcal{G}$, then $(F, E) \sqcap (F', E) \in \mathcal{G}$.

(G.3) For each $(F, E) \in \mathcal{G} \setminus \widetilde{X}$, there is $x \in X$ such that $(x, E) \sqsubseteq (F, E)$ is false, and $(x, E) \sqcup (F, E) \in \mathcal{G}$.

The members of $\mathcal{G}$ are called *soft convex sets*, or soft $\mathcal{G}$-convex sets for better clarity.

Axioms (G.1) and (G.2) are self-explanatory. Axiom (G.3) requires that any non-absolute soft $\mathcal{G}$-convex set can be extended by adding a soft point, in a way that produces another "strictly larger" soft $\mathcal{G}$-convex set: notice that the requirement $(x, E) \not\sqsubseteq (F, E)$ guarantees $(F, E) \neq (x, E) \sqcup (F, E)$.

As a trivial example, $\mathcal{G} = SS_E(X)$ is the *discrete* convex soft geometry on X. Another simple universal construction is given below in Example 5.

The next Remark explains why and how Definition 5 extends Definition 2:

Remark 2. The case $E = \{e\}$, or $|E| = 1$, produces a convex soft geometry that can be identified with a convex geometry on X in a trivial manner. To make this formal, notice that any $(F, E) \in SS_E(X)$ can be identified with $F(e) \subseteq X$ when $E = \{e\}$, thus $\mathcal{G} \subseteq SS_E(X)$ can be identified with a family $\mathbf{G} \subseteq X$. This collection of subsets of X satisfies Definition 2. Conversely, a convex geometry $\mathbf{G} \subseteq X$ can be identified with a convex soft geometry on X for $E = \{e\}$ (with any arbitrarily chosen attribute e) in a similar manner.

As in the case of (soft) topologies, convex soft geometries can be compared by inclusion. Suppose that $\mathcal{G}$ and $\mathcal{G}'$ are convex soft geometries on X such that $\mathcal{G} \subseteq \mathcal{G}'$. Then we say that $\mathcal{G}'$ is *finer* (or stronger, or larger) than $\mathcal{G}$, and that $\mathcal{G}$ is *coarser* (or weaker, or smaller) than $\mathcal{G}'$. The next example clarifies these concepts:

Example 3. To illustrate Definition 5, we shall take advantage of the notation adopted in Tables 1 and 2, which give names to all soft sets on $X = \{x, y\}$ and the set of attributes is $E = \{e, e'\}$. Taking into account the characteristics explained above, the next collections define convex soft geometries on X:

$\mathcal{G}_x = \{\Phi, (x, E), \widetilde{X}\}$ and $\mathcal{G}_y = \{\Phi, (y, E), \widetilde{X}\}$

$\mathcal{G}_1 = \{\Phi, (x, E), (y, E), \widetilde{X}\}$

$\mathcal{G}_2 = \{\Phi, (\{x\}_e, E), (x, E), \widetilde{X}\}$

and $\mathcal{G}_3 = \{\Phi, (\{x\}_{e'}, E), (x, E), \widetilde{X}\}$

$\mathcal{G}_4 = \{\Phi, (x, E), (F_5, E), (F_6, E), \widetilde{X}\}$

$\mathcal{G}_5 = \{\Phi, (x, E), (F_5, E), \widetilde{X}\}$

and $\mathcal{G}_6 = \{\Phi, (x, E), (F_6, E), \widetilde{X}\}$

$\mathcal{G}_7 = \{\Phi, (\{x\}_e, E), (x, E), (F_7, E), \widetilde{X}\}$

and $\mathcal{G}_8 = \{\Phi, (\{x\}_{e'}, E), (x, E), (F_8, E), \widetilde{X}\}$

All these convex soft geometries contain (x, E), except $\mathcal{G}_y$ that contains (y, E). One can easily design symmetric expressions for convex soft geometries that contain (y, E), by swapping the roles of x and y.

We can observe that $\mathcal{G}_2 \subseteq \mathcal{G}_7$, thus $\mathcal{G}_7$ is finer than $\mathcal{G}_2$, or $\mathcal{G}_2$ is coarser than $\mathcal{G}_7$. Similarly, we notice $\mathcal{G}_3 \subseteq \mathcal{G}_8$, $\mathcal{G}_x \subseteq \mathcal{G}_1$, $\mathcal{G}_y \subseteq \mathcal{G}_1$, $\mathcal{G}_5 \subseteq \mathcal{G}_4$, or $\mathcal{G}_6 \subseteq \mathcal{G}_4$.

Remark 3. It is possible to develop other related notions of convex soft geometry, which would rely on a different choice of "soft points." We refrain from giving more definitions to avoid confusion at this early stage of development.

In the next Section 3.1, we proceed to investigate fundamental properties of convex soft geometries. Subsequent sections will investigate the interplay between convex soft geometries and convex geometries on a given set. Two types of constructions will be discussed in connection with this issue (cf., Sections 3.2 and 3.3). Related concepts are introduced in Section 3.4. Examples illustrate the main ideas.

3.1. Basic properties

We proceed to present some basic properties of the concept of convex soft geometry that are worth mentioning.

Notice first that we do not lose generality if in Definition 5, axiom (G.1) is strengthened to the following requirement:

(G′.1) $\Phi, \widetilde{X} \in \mathcal{G}$.

The reason is that axioms (G.1) and (G.3) guarantee that $\widetilde{X} \in \mathcal{G}$, because X is finite. Hence, $\mathcal{G} \subseteq SS_E(X)$ satisfies axioms (G.1), (G.2), and (G.3) if and only if it satisfies (G′.1), (G.2), and (G.3)

Also observe that for any convex soft geometry $\mathcal{G} \subseteq SS_E(X)$,

$$\text{there must be } x \in X \text{ such that } (x, E) \in \mathcal{G}, \quad (4)$$

that is, in all convex geometries there is a soft point that is convex. To guarantee this property, we just need to apply axiom (G.3) to the null soft set.

When $(x, E) \in \mathcal{G}$ for all $x \in X$, we say that the convex soft geometry $\mathcal{G}$ is *atomic*.

3.2. Relationships between concepts: First approach

In this subsection, we explain one procedure that produces a convex geometry on X from an arbitrary convex soft geometry (cf., Proposition 3 below). The next section revisits this problem from a different perspective, and there we shall produce a second procedure. Example 4 will illustrate the construction proposed in this section. Afterwards we shall construct convex soft geometries from arbitrary convex geometries (cf., Proposition 4 below). And then Example 6 will demonstrate how this procedure works.

Both transitions are done with the aid of the following auxiliary definitions:

Definition 6. The *selection mapping* operates on $SS_E(X)$ and produces subsets of X as follows:

$$\eta: SS_E(X) \to \mathcal{P}(X)$$
$$(F, E) \mapsto \{x \in X \mid (x, E) \sqsubseteq (F, E)\}. \tag{5}$$

Conversely, we define the following *soft-making mapping* that operates on subsets of X and produces soft sets on X:

$$\psi: \mathcal{P}(X) \to SS_E(X)$$
$$Y \mapsto (F, E) \text{ defined by } F(a) = Y \text{ for all } a \in \text{E}. \tag{6}$$

Alternatively, the soft-making mapping can be expressed by the compact expression $\psi(Y) = \sqcup_{y \in Y}(y, E)$ for each $Y \subseteq X$.

Both mappings are monotonic. More precisely:

$$(F, E) \sqsubseteq (F', E) \text{ implies } \eta(F, E) \subseteq \eta(F', E), \tag{7}$$

and

$$Y \subseteq Y' \Leftrightarrow \psi(Y) \sqsubseteq \psi(Y'). \tag{8}$$

In relation with Equation (7), notice that $(F_5, E) \not\sqsubseteq (F_6, E)$ but $\eta(F_5, E) = \eta(F_6, E) = \{x\}$. Hence, the relationship stated by Equation (7) cannot be improved to become an equivalence.

The next two propositions formalize the constructions that we have announced. Their proofs are straightforward.

Proposition 3. *Let $\mathcal{G} \subseteq SS_E(X)$ be a convex soft geometry on X. Then $\mathbf{G} = \eta(\mathcal{G}) = \{\eta(F, E) | (F, E) \in \mathcal{G}\}$ is a convex geometry on X.*

Example 4. Let us place ourselves in the framework of Example 3. Using Proposition 4, the convex soft geometries $\mathcal{G}_x$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ on $X = \{x, y\}$ produce the convex geometry $\{\varnothing, \{x\}, X\}$ on X. Besides, $\mathcal{G}_1$, $\mathcal{G}_7$, and $\mathcal{G}_8$ in Example 3 produce $\{\varnothing, \{x\}, \{y\}, X\}$. And $\mathcal{G}_y$ in Example 3 yields $\{\varnothing, \{y\}, X\}$.

Proposition 4. *Let $\boldsymbol{G} \subseteq X$ be a convex geometry on X. Then $\mathcal{G} = \psi(\boldsymbol{G}) = \{\psi(G) | G \in \boldsymbol{G}\}$ is a convex soft geometry on X.*

Proposition 4 facilitates the construction of another remarkable class of examples:

Example 5. Suppose $X = \{x_1, \ldots, x_n\}$. Then

$$\{\Phi, \psi(\{x_1\}), \psi(\{x_1, x_2\}), \ldots, \psi(\{x_1, x_2, \ldots x_{n-1}\}), \widetilde{X}\},$$

is a convex soft geometry because $\{\varnothing, \{x_1\}, r\{x_1, x_2\}, \ldots, \{x_1, x_2, \ldots x_{n-1}\}, X\}$ is a convex geometry.

Example 6. Using Proposition 4, the convex geometry $\mathbf{G}_1$ in Example 2 produces the following convex soft geometry on $X = \{x, y, z\}$, for each set of attributes E:

$$\mathcal{G}^1 = \psi(\mathbf{G}_1) = \{\Phi, (x, E), (x, E) \sqcup (y, E), (x, E) \sqcup (z, E), \widetilde{X}\},$$

whereas $\mathbf{G}_2$ in Example 2 produces

$$\mathcal{G}^2 = \psi(\mathbf{G}_2) = \{\Phi, (x, E), (y, E), (z, E), (x, E) \sqcup (y, E), (x, E) \sqcup (z, E), \widetilde{X}\}.$$

One should wonder whether the processes defined by Propositions 3 and 4 are inverse to each other. The next result clarifies this issue:

Proposition 5. If $Y \subseteq X$, *then* $\eta(\psi(Y)) = Y$. *Conversely, if* $(F, E) \in SS_E(X)$, *then* $\psi(\eta(F, E)) \sqsubseteq (F, E)$, *but* $\psi(\eta(F, E)) = (F, E)$ *is in general false.*

Proof. Both implications are straightforward. To prove that $\psi(\eta(F, E)) = (F, E)$ is not true in general, a counter example is in order. Consider $X = \{x, y\}$ and select an arbitrary $e \in E$ with $|E| > 1$. Then $\psi(\eta(\{x\}_e, E)) = \psi(\varnothing) = \Phi$ hence $(\{x\}_e, E) \not\sqsubseteq \psi(\eta(\{x\}_e, E))$. This proves that for $(F, E) = (\{x\}_e, E)$, the property $(F, E) = \psi(\eta(F, E))$ is false. □

In words, Proposition 5 states that if we start with a convex geometry, then produce its associated convex soft geometry, and afterwards we generate the convex geometry associated with the later, then we end up with the original convex geometry. For example, in the situation of Example 6, $\eta(\psi(\mathbf{G}_1)) = \eta(\mathcal{G}^1) = \mathbf{G}_1$. But the analogous reasoning when we start with a convex soft geometry fails to hold true.

Next we proceed to discuss a possible alternative to the constructions studied in this section.

3.3. Relationships between concepts: second approach

In this subsection, we present a different procedure that produces a family of convex geometries on X, indexed by E, from an arbitrary convex soft geometry on X. This is done in Proposition 6 below. However, Example 7 will show that the analogous inverse process in general fails to produce convex soft geometries from convex geometries. Admittedly, the attempt in this section is motivated by Proposition 1, a more favorable result in soft topology.

Proposition 6. *Let $\mathcal{G} \subseteq SS_E(X)$ be a convex soft geometry on X. Then $\boldsymbol{G}_e = \{F(e) | (F, E) \in \mathcal{G}\}$ is a convex geometry on X, for each $e \in E$.*

Proof. Let us prove that $\mathbf{G}_e$ satisfies Definition 2, with $e \in E$ arbitrary but fixed. Condition (1) is obvious: $\Phi(e) = \varnothing$ and $\Phi \in \mathcal{G}$. Also (2) follows easily from a routine argument: select $A, B \in \mathbf{G}_e$, then there are $(F, E), (F', E) \in \mathcal{G}$ with $F(e) = A$

and $F'(e) = B$, therefore $(F, E) \sqcap (F', E) \in \mathcal{G}$ satisfies $((F, E) \sqcap (F', E))(e) = F(e) \cap F'(e) = A \cap B$, which justifies $A \cap B \in \mathbf{G}_e$.

In order to prove (3) in Definition 2, let us fix $A \in \mathbf{G}_e$ with $A \neq X$. Select $(F, E) \in \mathcal{G}$ with $F(e) = A$. We apply a recursive argument based on (G.3) and the finiteness assumption. Since $(F, E) \neq \widetilde{X}$, there is $x \in X$ such that $(x, E) \not\sqsubseteq (F, E)$ and $(x, E) \sqcup (F, E) \in \mathcal{G}$ by (G.3). We distinguish two cases.

Case 1: $x \notin F(e)$. Then we conclude because the soft convex set $(F', E) = (x, E) \sqcup (F, E) \in \mathcal{G}$ satisfies $F'(e) = F(e) \cup \{x\} = A \cup \{x\}$, hence $A \cup \{x\} \in \mathbf{G}_e$ by definition.

Case 2: $x \in F(e)$. Then we repeat the argument with $(F', E) = (x, E) \sqcup (F, E) \in \mathcal{G}$, since it still satisfies $(F', E) \neq \widetilde{X}$ and $F'(e) = A \neq X$.

The finiteness assumption assures that this process eventually ends up with an extension that pertains to Case 1, due to $(F, E) \neq (F', E)$ and $(F, E) \sqsubseteq (F', E)$ which produce strict soft inclusion. □

It would be very convenient if we could use an inverse process to define convex soft geometries from an indexed family of convex geometries on the same set X. However, the next simple example shows that this is not a valid procedure for the production of convex soft geometries:

Example 7. Consider $X = \{x, y\}$ and $E = \{e_1, e_2\}$. Define the convex geometries

$\mathbf{G}' = \{\varnothing, \{x\}, X\}$ and $\mathbf{G}'' = \{\varnothing, \{y\}, X\}$.

Now define $\mathcal{G} = \{(F, E) \in SS_E(X) | F(e_1) \in \mathbf{G}', F(e_2) \in \mathbf{G}''\}$. This family of soft sets on X is not a convex soft geometry, because neither $(x, E) \in \mathcal{G}$ nor $(y, E) \in \mathcal{G}$, contradicting Equation (4). In words, no soft point belongs to $\mathcal{G}$ defined above.

This disappointing setback has motivated us to prove Proposition 4 as a correct methodology for the construction of convex soft geometries from known convex geometries. Notice that Proposition 4 uses a process that reverses the operations in Proposition 6 *with a common convex geometry*, that is, we can circumvent the problem posed by Example 7 with the utilization of the same convex geometry for all the parameters.

3.4. Other relevant concepts: Soft convex hulls and extreme elements

Let us fix a convex soft geometry $\mathcal{G}$ on X for the remaining of this section.

By inspiration of Definition 3, soft convex hulls can be defined for every soft set, in the convex soft geometry $\mathcal{G}$, as follows:

Definition 7. The *soft convex hull* of $(F, E) \in SS_E(X)$ in the convex soft geometry $\mathcal{G}$ is the smallest convex soft set that contains (F, E). Formally:

$$\mathrm{conv}_{\mathcal{G}}(F, E) = \sqcap\{(F', E) \in \mathcal{G} \text{ such that } (F, E) \sqsubseteq (F', E)\}. \quad (9)$$

When $\mathcal{G}$ is clear from the context, we may drop the subindex to denote $\mathrm{conv}_{\mathcal{G}}(F, E)$ as $\mathrm{conv}(F, E)$.

The next properties of soft convex hulls are immediate:

1. $(F, E) \sqsubseteq \mathrm{conv}(F, E) \in \mathcal{G}$, for all $(F, E) \in SS_E(X)$.
2. $(F, E) = \mathrm{conv}(F, E)$ if and only if $(F, E) \in \mathcal{G}$.
3. The operator conv is monotonic with respect to soft inclusion: $(F, E) \sqsubseteq (F', E)$ implies $\mathrm{conv}(F, E) \sqsubseteq \mathrm{conv}(F', E)$ for all $(F, E), (F', E) \in SS_E(X)$.
4. Suppose that $\mathcal{G}'$ is another convex soft geometry on X such that $\mathcal{G} \subseteq \mathcal{G}'$. Then for all $(F, E) \in SS_E(X)$: $\mathrm{conv}_{\mathcal{G}'}(F, E) \sqsubseteq \mathrm{conv}_{\mathcal{G}}(F, E)$.

   Example 9 below gives explicit examples of this property which means that finer convex soft geometries produce "smaller" convex hulls (in terms of soft inclusion). □

Remark 4. Property 4 is reminiscent of a property of topological closures: the closure of a subset in a finer topology is a subset of its closure in a coarser topology.

Example 8. Some tedious but straightforward computations allow us to calculate the soft convex hull of various soft sets from Tables 1 and 2, in the convex soft geometries defined in Example 3. The result appears in Table 4.

Example 9. The data in Example 8 help us check property 4 of the soft convex hull against concrete examples. Noticing that $\mathcal{G}_2 \subseteq \mathcal{G}_7$, we confirm that $\mathrm{conv}_{\mathcal{G}_7}(F, E) \sqsubseteq \mathrm{conv}_{\mathcal{G}_2}(F, E)$ for all the soft sets considered in that example. It is also the case that $\mathcal{G}_3 \subseteq \mathcal{G}_8$, and we confirm that $\mathrm{conv}_{\mathcal{G}_8}(F, E) \sqsubseteq \mathrm{conv}_{\mathcal{G}_3}(F, E)$ for all these soft sets. Similarly, we can repeat this exercise with $\mathcal{G}_x \subseteq \mathcal{G}_1$, $\mathcal{G}_y \subseteq \mathcal{G}_1$, $\mathcal{G}_5 \subseteq \mathcal{G}_4$, or $\mathcal{G}_6 \subseteq \mathcal{G}_4$.

It is natural to wonder, how do the selection and soft-making mappings interact with (soft) convex hulls? The next two propositions dissect the behavior of (soft) convex hulls with respect to them:

Table 4
The soft convex hulls of various soft sets in the convex soft geometries defined in Example 3. See Tables 1 and 2 for notation

	$(\{x\}_{e'}, E)$	(F_1, E)	$(\{y\}_{e'}, E)$	(F_3, E)	(F_6, E)	(x, E)	(y, E)
$\mathcal{G}_x$	(x, E)	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	(x, E)	$\tilde{X}$
$\mathcal{G}_y$	$\tilde{X}$	$\tilde{X}$	(y, E)	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	(y, E)
$\mathcal{G}_1$	(x, E)	$\tilde{X}$	(y, E)	$\tilde{X}$	$\tilde{X}$	(x, E)	(y, E)
$\mathcal{G}_2$	(x, E)	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	(x, E)	$\tilde{X}$
$\mathcal{G}_3$	$(\{x\}_{e'}, E)$	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	(x, E)	$\tilde{X}$
$\mathcal{G}_4$	(x, E)	(F_5, E)	(F_6, E)	(F_6, E)	(F_6, E)	(x, E)	$\tilde{X}$
$\mathcal{G}_5$	(x, E)	(F_5, E)	$\tilde{X}$	$\tilde{X}$	$\tilde{X}$	(x, E)	$\tilde{X}$
$\mathcal{G}_6$	(x, E)	$\tilde{X}$	(F_6, E)	$\tilde{X}$	(F_6, E)	(x, E)	$\tilde{X}$
$\mathcal{G}_7$	(x, E)	(F_7, E)	(F_7, E)	(F_7, E)	$\tilde{X}$	(x, E)	(F_7, E)
$\mathcal{G}_8$	$(\{x\}_{e'}, E)$	$\tilde{X}$	(F_8, E)	$\tilde{X}$	$\tilde{X}$	(x, E)	(F_8, E)

Table 5
The extreme elements for three soft sets on $X = \{x, y\}$, in the convex soft geometries defined in Example 3. See Tables 1 and 2 for notation, and Table 4 for intermediate calculations

	(F_2, E)	(F_6, E)	(F_8, E)
$\mathcal{G}_x$	$\{y\}$	$\{y\}$	$\{y\}$
$\mathcal{G}_y$	$\{x\}$	$\{x\}$	$\{x\}$
$\mathcal{G}_1$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$
$\mathcal{G}_2$	$\{y\}$	$\{x, y\}$	$\{y\}$
$\mathcal{G}_3$	$\{y\}$	$\{y\}$	$\{y\}$
$\mathcal{G}_4$	$\{y\}$	$\{y\}$	$\{y\}$
$\mathcal{G}_5$	$\{y\}$	$\{y\}$	$\{y\}$
$\mathcal{G}_6$	$\{y\}$	$\{y\}$	$\{y\}$
$\mathcal{G}_7$	$\{y\}$	$\{x, y\}$	$\{x, y\}$
$\mathcal{G}_8$	$\{y\}$	$\{x, y\}$	$\{x, y\}$

Proposition 7. *Let $\mathcal{G}$ be a convex soft geometry on X. Then* $\mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E) \subseteq \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$, *for each* $(F, E) \in SS_E(X)$. *However,* $\mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E) = \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$ *is not in general true.*

Proof. On one hand, $x \in \mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E)$ if and only if $x \in \eta(F', E)$, for each $(F', E) \in \mathcal{G}$ with $\eta(F, E) \sqsubseteq \eta(F', E)$. Therefore, $x \in \mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E)$ implies $x \in \eta(F', E)$, for each $(F', E) \in \mathcal{G}$ with $(F, E) \sqsubseteq (F', E)$, by virtue of Equation (7).

On the other hand, $x \in \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$ if and only if $(x, E) \sqsubseteq (F', E)$, for each $(F', E) \in \mathcal{G}$ with $(F, E) \sqsubseteq (F', E)$. Therefore, $x \in \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$ if and only if $x \in \eta(F', E)$, for each $(F', E) \in \mathcal{G}$ with $(F, E) \sqsubseteq (F', E)$

It is now apparent that $x \in \mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E)$ implies $x \in \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$, hence $\mathrm{conv}_{\eta(\mathcal{G})}\eta(F, E) \subseteq \eta(\mathrm{conv}_{\mathcal{G}}(F, E))$.

To prove that this inclusion cannot be improved to become an equality, consider the convex soft geometry $\mathcal{G}_1$ defined in Example 3 (the argument remains valid with $\mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6$, or $\mathcal{G}_7$). Notice that $\mathrm{conv}_{\eta(\mathcal{G})}\eta(\{x\}_{e'}, E) = \varnothing$ for each convex soft geometry on X, because $\eta(\{x\}_{e'}, E) = \varnothing$. However, Table 4 shows $\mathrm{conv}_{\mathcal{G}_1}(\{x\}_{e'}, E) = (x, E)$, which ensures $\eta(\mathrm{conv}_{\mathcal{G}_1}(\{x\}_{e'}, E)) = \{x\}$. □

Proposition 8. *Let $\mathbf{G} \subseteq X$ be a convex geometry on X. If $Y \subseteq X$, then* $\mathrm{conv}_{\psi(\mathbf{G})}\psi(Y) = \psi(\mathrm{conv}_{\mathbf{G}}(Y))$.

Proof. By construction, $\mathrm{conv}_{\psi(\mathbf{G})}\psi(\mathrm{Y}) = \sqcap\{\psi(\mathrm{Y}') | \mathrm{Y}' \in \mathbf{G}, \psi(\mathrm{Y}) \sqsubseteq \psi(\mathrm{Y}')\}$. And by definition, $\psi(\mathrm{conv}_{\mathbf{G}}(\mathrm{Y})) = \psi(\cap\{Y' \in G | Y \subseteq Y'\})$.

To prove the soft equality of these soft sets, let us fix $e \in E$. On one hand,

$$\mathrm{conv}_{\psi(\mathbf{G})}\psi(\mathrm{Y})(e) = \cap\{\psi(\mathrm{Y}')(e) | \mathrm{Y}' \in \mathbf{G}, \psi(\mathrm{Y}) \sqsubseteq \psi(\mathrm{Y}')\}.$$

We resort to Equation (8) in order to transform the equality $\psi(Y) \sqsubseteq \psi(Y')$ into $Y \subseteq Y'$, thus

$$\mathrm{conv}_{\psi(\mathbf{G})}\psi(\mathrm{Y})(e) = \cap\{\psi(\mathrm{Y}')(e) | \mathrm{Y}' \in \mathbf{G}, \mathrm{Y} \subseteq \mathrm{Y}'\}.$$

Now the definition of ψ produces

$$\mathrm{conv}_{\psi(\mathbf{G})}\psi(\mathrm{Y})(e) = \cap\{\mathrm{Y}' | \mathrm{Y}' \in \mathbf{G}, \mathrm{Y} \subseteq \mathrm{Y}'\}.$$

On the other hand,

$$\psi(\mathrm{conv}_{\mathbf{G}}(\mathrm{Y}))(e) = \mathrm{conv}_{\mathbf{G}}(\mathrm{Y}),$$

by the definition of ψ. Now Equation (2) proves the coincidence of $\psi(\mathrm{conv}_{\mathbf{G}}(\mathrm{Y}))(e)$ and $\mathrm{conv}_{\psi(\mathbf{G})}\psi(\mathrm{Y})(e)$. □

Now we turn our attention to extreme elements. We shall need the following auxiliary notation in order to adapt the idea from convex geometries (cf., Definition 4) to our soft context: when $(F, E) \in SS_E(X)$ and $x \in X$, the soft set $(F, E) \setminus \{x\}$ is (F', E) such that $F'(e) = F(e) \setminus \{x\}$, for all $e \in E$.

Definition 8. Let us fix a convex soft geometry $\mathcal{G}$ on X. When $\Phi \neq (F, E) \in SS_E(X)$, we say that

$$x \in X \text{ is an } \textit{extreme element} \text{ for } (F, E) \text{ in } \mathcal{G} \text{ when } (x, E) \sqcap (F, E) \neq \Phi \text{ and } (x, E) \not\sqsubseteq \mathrm{conv}((F, E) \setminus \{x\}). \quad (10)$$

We shall denote by $\mathrm{ex}_{\mathcal{G}}(F, E)$, or simply $\mathrm{ex}(F, E)$ if $\mathcal{G}$ is common knowledge, the set of extreme elements for (F, E) in the convex soft geometry $\mathcal{G}$.

Also, $(\mathrm{ex}_{\mathcal{G}}(F), E)$ or simply $(\mathrm{ex}(F), E)$ will denote the soft set associated with the extreme elements for (F, E) in $\mathcal{G}$, that is, $\psi(\mathrm{ex}_{\mathcal{G}}(F), E)$. Thus, it is the soft set such that $\mathrm{ex}(F)(e) = \mathrm{ex}_{\mathcal{G}}(F, E)$, for all $e \in E$.

Example 10. Some tedious but straightforward computations, which use the information in Table 4, allow us to calculate the extreme elements for various soft sets from Tables 1 and 2, in the convex soft geometries defined in Example 3. The result is displayed in Table 5.

The soft sets associated with the extreme elements for the soft sets in Table 5 are of three types: $(\{x\}, E)$, $(\{y\}, E)$ and $(\{x, y\}, E) = \widetilde{X}$. Table 6 displays examples of these three cases. Notice $(\mathrm{ex}_{\mathcal{G}_x}(F_2), E) = (y, E)$, $(\mathrm{ex}_{\mathcal{G}_y}(F_2), E) = (x, E)$, and $(\mathrm{ex}_{\mathcal{G}_1}(F_2), E) = \widetilde{X}$.

Besides, $(\mathrm{ex}_{\mathcal{G}_x}(F_2), E) = (\mathrm{ex}_{\mathcal{G}_x}(F_6), E) = (\mathrm{ex}_{\mathcal{G}_x}(F_8), E)$, $(\mathrm{ex}_{\mathcal{G}_y}(F_2), E) = (\mathrm{ex}_{\mathcal{G}_y}(F_6), E) = (\mathrm{ex}_{\mathcal{G}_y}(F_8), E)$, and $(\mathrm{ex}_{\mathcal{G}_1}(F_2), E) = (\mathrm{ex}_{\mathcal{G}_1}(F_6), E) = (\mathrm{ex}_{\mathcal{G}_1}(F_8), E)$. The reader can easily produce the other instances by reference to these examples.

4. Main Results

The aim of this section is to provide two non-trivial results about convex soft geometries. As in the inspirational case of convex geometries, both are connected. First we prove that an

Table 6
Soft sets associated with the extreme elements for (F_2, E): see Table 5

$(\mathrm{ex}_{\mathcal{G}_x}(F_2), E)$	e	e'	$(\mathrm{ex}_{\mathcal{G}_y}(F_2), E)$	e	e'	$(\mathrm{ex}_{\mathcal{G}_1}(F_2), E)$	e	e'
x	0	0	x	1	1	x	1	1
y	1	1	y	0	0	y	1	1

anti-exchange property holds true for any convex soft geometry. Then we use this feature of convex soft geometries to assure the existence of extreme elements for any soft set. We present both results separately in respective subsections. We emphasize that they are extensions of classical results in the investigation of convex geometries (cf., Section 2.3), which correspond to the case $|E| = 1$ under the natural identification explained in Remark 2.

4.1. The anti-exchange property

Equation (3) defined the anti-exchange property that all convex geometries must satisfy. This section proves that convex soft geometries satisfy the following extended property:

Definition 9. Let us fix a convex soft geometry $\mathcal{G}$ on X. The anti-exchange property of $\mathcal{G}$ states that when $(F, E) \in \mathcal{G}$, and $x, y \in X$ with $x \neq y$ are such that $(x, E) \not\sqsubseteq (F, E)$ and $(y, E) \not\sqsubseteq (F, E)$,

$$(y, E) \sqsubseteq \text{conv}((F, E) \sqcup (x, E)) \Rightarrow (x, E) \not\sqsubseteq \text{conv}((F, E) \sqcup (y, E)). \tag{11}$$

The demonstration of that property uses the following auxiliary result:

Lemma 1. *Let $\mathcal{G} \subseteq SS_E(X)$ be a convex soft geometry on X. Then when $x \in X$ and $(F, E) \in \mathcal{G}$ satisfies $(x, E) \not\sqsubseteq (F, E)$, there exists $(F_x, E) \in \mathcal{G}$ with $(F, E) \sqsubseteq (F_x, E)$, $(x, E) \not\sqsubseteq (F_x, E)$, and $(x, E) \sqcup (F_x, E) \in \mathcal{G}$.*

Proof. With (F, E) and $x \in X$ we associate $(F_x, E) \in \mathcal{G}$, a maximal soft set with respect to $\sqsubseteq$ among those that satisfy $(x, E) \not\sqsubseteq (F', E)$ and $(F, E) \sqsubseteq (F', E) \in \mathcal{G}$. In other words, $(x, E) \not\sqsubseteq (F_x, E)$; $(F, E) \sqsubseteq (F_x, E)$; and if $(x, E) \not\sqsubseteq (F', E)$ and $(F, E) \sqsubseteq (F', E) \in \mathcal{G}$, then $(F', E) \sqsubseteq (F_x, E) \in \mathcal{G}$. Its existence is guaranteed by a recursive argument based on (G.3), and the finiteness assumption.

Consider $(F_x, E) \in \mathcal{G} \setminus \widetilde{X}$. Property (G.3) of a convex soft geometry assures the existence of $x' \in X$ such that $(x', E) \not\sqsubseteq (F_x, E)$ and $(x', E) \sqcup (F_x, E) \in \mathcal{G}$.

It must be the case that $x = x'$ due to the construction of (F_x, E): if we suppose $x \neq x'$, then the soft set $(F', E) = (x', E) \sqcup (F_x, E) \in \mathcal{G}$ would satisfy $(x, E) \not\sqsubseteq (F', E)$ and $(F, E) \sqsubseteq (F_x, E) \sqsubseteq (F', E) \in \mathcal{G}$, which entails $(F', E) \sqsubseteq (F_x, E)$, a contradiction with $(x', E) \not\sqsubseteq (F_x, E)$. In conclusion, $(x, E) \sqcup (F_x, E) \in \mathcal{G}$, which concludes the argument. □

Lemma 1 considers a soft convex set and a soft point not included in it. It is not necessarily true that their soft union produces a soft convex set. However, there must be another soft convex set larger than the original soft convex set, for which these properties are true (i.e., that does not "contain" the soft point but produces a soft convex set when the soft point is "joined" to it).

We are ready to prove our first main theorem: the anti-exchange property is valid for all convex soft geometries.

Theorem 1. *Any convex soft geometry $\mathcal{G}$ on X satisfies Definition 9.*

Proof. Fix a convex soft geometry $\mathcal{G}$ on X. Let us assume $(F, E) \in \mathcal{G}$, and $x, y \in X$ with $y \neq x$ satisfy $(y, E) \not\sqsubseteq (F, E)$, $(x, E) \not\sqsubseteq (F, E)$.

Before proving the claim, we need to produce two auxiliary constructions.

The application of Lemma 1 to (F, E) and $x \in X$ guarantees the existence of $(F_x, E) \in \mathcal{G}$, such that $(x, E) \not\sqsubseteq (F_x, E)$, $(F, E) \sqsubseteq (F_x, E)$, and $(x, E) \sqcup (F_x, E) \in \mathcal{G}$.

We can proceed similarly with (F_y, E) and $y \in X$, which produces $(y, E) \sqcup (F_y, E) \in \mathcal{G}$ for some $(F_y, E) \in \mathcal{G}$ such that $(y, E) \not\sqsubseteq (F_y, E)$ and $(F, E) \sqsubseteq (F_y, E)$.

We are ready to check Equation (11). Therefore, let us assume $(y, E) \sqsubseteq \text{conv}((F, E) \sqcup (x, E))$. Then we deduce $(y, E) \sqsubseteq \text{conv}((F_y, E) \sqcup (x, E))$ from the monotonicity of the conv operator, since $(F, E) \sqsubseteq (F_y, E)$.

We claim that the soft inclusion $(x, E) \sqsubseteq \text{conv}(F_y, E)$ must be false: should $(x, E) \sqsubseteq \text{conv}(\text{F}_\text{y}, \text{E})$ hold true, we would have $(y, E) \sqsubseteq \text{conv}((F_y, E) \sqcup (x, E)) = \text{conv}(F_y, E) = (F_y, E)$ using $(F_y, E) \in \mathcal{G}$, but $(y, E) \sqsubseteq (F_y, E)$ is false.

Now the facts $x \neq y$ and $(x, E) \not\sqsubseteq (F_y, E)$ guarantee $(x, E) \not\sqsubseteq (F_y, E) \sqcup (y, E) \in \mathcal{G}$. This fact means $(x, E) \not\sqsubseteq \text{conv}((F_y, E) \sqcup (y, E))$, and from this we conclude $(x, E) \not\sqsubseteq \text{conv}((F, E) \sqcup (y, E))$ because $(F, E) \sqsubseteq (F_y, E)$ and conv is a monotonic operator. □

4.2. Extreme elements for soft sets: An existence theorem

This section investigates the existence of extreme elements for soft sets.

Let us go back to the inspirational case of convex geometries for motivation. Proposition 2 has assured that any $Y \subseteq X$ has an extreme element (for any fixed convex geometry **G** on *X*). We have explained that this result is a consequence of the property $\text{conv}(\text{ex}(\text{Y})) = \text{conv}(\text{Y})$. In fact, the gist of the argument boils down to the simpler fact $\varnothing \neq \text{conv}(\text{Y}) \subseteq \text{conv}(\text{ex}(\text{Y}))$, since this subsethood necessarily implies $\varnothing \neq \text{ex}(\text{Y})$.

Our main result proves that in the case of convex soft geometries, maximal elements exist for any non-trivial soft set as well. Their existence is guaranteed by a general relationship (see Example 11 below for complementary information):

Theorem 2. *Let us fix a convex soft geometry $\mathcal{G}$ on X. When (F, E) is a soft set that is not the null soft set,*

$$(F, E) \sqsubseteq \text{conv}(F, E) \sqsubseteq \text{conv}(ex(F), E). \tag{12}$$

In particular, $ex(F, E) \neq \varnothing$ for any soft set other than the null soft set.

Proof. We already know $(F, E) \sqsubseteq \text{conv}(F, E)$. Let us prove $\text{conv}(F, E) \sqsubseteq \text{conv}(\text{ex}(F), E)$. We distinguish two cases.

Case 1: $(F, E) \sqsubseteq \text{conv}(\text{ex}(F), E)$. Then because conv is a monotonic operator, and $\text{conv}(\text{conv}(\text{ex}(F), E)) = \text{conv}(\text{ex}(F), E)$ we are done.

Case 2: $(F, E) \not\sqsubseteq \text{conv}(\text{ex}(F), E)$. This obviously implies the existence of $(\{x\}_a, E)$, a soft spot such that $(\{x\}_a, E) \sqsubseteq (F, E)$ but $(\{x\}_a, E) \sqcap \text{conv}(\text{ex}(F), E) = \Phi$. Then $(\{x\}_a, E) \sqcap (\text{ex}(F), E) = \Phi$. By the definition of $(\text{ex}(F), E)$, this forcefully yields that x is not extreme element for (F, E). Since $(x, E) \sqcap (F, E) \neq \Phi$, we deduce from the definition of extreme element that $(x, E) \sqsubseteq \text{conv}((F, E) \setminus \{x\})$.

We claim $\text{conv}(F, E) = \text{conv}((F, E) \setminus \{x\})$. Let us prove $\text{conv}(F, E) \sqsubseteq \text{conv}((F, E) \setminus \{x\})$, since it is obvious that $\text{conv}((F, E) \setminus \{x\}) \sqsubseteq \text{conv}(F, E)$ because conv is monotonic with respect to $\sqsubseteq$.

It is clear that $(F,E)\setminus\{x\}\sqsubseteq \mathrm{conv}((F,E)\setminus\{x\})$. We have deduced $(x,E)\sqsubseteq \mathrm{conv}((F,E)\setminus\{x\})$. Altogether, both facts guarantee $(F,E)\sqsubseteq \mathrm{conv}((F,E)\setminus\{x\})$. Since conv is monotonic with respect to $\sqsubseteq$,

$$\mathrm{conv}(F,E)\sqsubseteq \mathrm{conv}(\mathrm{conv}((F,E)\setminus\{x\}))=\mathrm{conv}((F,E)\setminus\{x\}),$$

hence the claim is proven.

Let us now distinguish two subcases.

Case 2.1. We first suppose that x is the only member of X with the property $(\{x\}_a,E)\sqsubseteq(F,E)$ but $(\{x\}_a,E)\sqcap\mathrm{conv}(\mathrm{ex}(F),E)=\Phi$ for some $a\in E$. This assumption yields $(F,E)\setminus\{x\}\sqsubseteq \mathrm{conv}(\mathrm{ex(F),E})$. We now use the monotonicity of the conv operator to obtain $\mathrm{conv}((F,E)\setminus\{x\})\sqsubseteq\mathrm{conv}(\mathrm{ex}(F),E)$. Since $\mathrm{conv}(F,E)=\mathrm{conv}(\mathrm{conv}((F,E)\setminus\{x\}))$, the desired consequence $\mathrm{conv(F,E)}\sqsubseteq\mathrm{conv(ex(F),E)}$ follows immediately.

Case 2.2. Now we suppose that there exists $y\neq x$, $y\in X$, such that $(\{y\}_{a'},E)\sqsubseteq(F,E)$ but $(\{y\}_{a'},E)\sqcap\mathrm{conv}(\mathrm{ex}(F),E)=\Phi$ for some $a'\in E$. Then $(y,E)\sqcap(F,E)\neq\Phi$, and the argument that we used above for x also applies to y. We deduce $(y,E)\sqsubseteq\mathrm{conv}((F,E)\setminus\{y\})$.

Set $(F',E)=((F,E)\setminus\{x\})\setminus\{y\}=((F,E)\setminus\{y\})\setminus\{x\}$. Clearly, both $(x,E)\sqsubseteq(F',E)$ and $(y,E)\sqsubseteq(F',E)$ are false. A contradiction with Theorem 1 follows from the following facts:

$(y,E)\sqsubseteq\mathrm{conv}((F,E)\setminus\{y\})\sqsubseteq\mathrm{conv}((F',E)\cup(x,E))$, and

$(x,E)\sqsubseteq\mathrm{conv}((F,E)\setminus\{x\})\sqsubseteq\mathrm{conv}((F',E)\cup(y,E))$.

Now the claim $ex(F,E)\neq\varnothing$ follows immediately. □

In order to complete the information that Theorem 2 renders, we point out that the soft set equality $\mathrm{conv}(\mathrm{ex}(F),E)=\mathrm{conv}(F,E)$ is *not* necessarily true. The next counterexample proves this claim:

Example 11. Consider $(F_6,E)\in SS_E(X)$ defined in Table 2, and the convex soft geometries defined in Example 3. Using Tables 4 and 5 for intermediate calculations, the reader can check that $\mathrm{conv}_{\mathcal{G}}(\mathrm{ex(F_6),E})=\widetilde{\mathrm{X}}$ when $\mathcal{G}$ is any of these convex soft geometries. However, $\mathrm{conv}_{\mathcal{G}_4}(F_6,E)=\mathrm{conv}_{\mathcal{G}_6}(F_6,E)=(F_6,E)$. In conclusion, $\mathrm{conv}_{\mathcal{G}_4}(\mathrm{ex}(F_6),E)\not\sqsubseteq\mathrm{conv}_{\mathcal{G}_4}(F_6,E)$, and $\mathrm{conv}_{\mathcal{G}_6}(\mathrm{ex}(F_6),E)\not\sqsubseteq\mathrm{conv}_{\mathcal{G}_6}(F_6,E)$.

Hence, Theorem 2 informs us that our definition of extreme element is quite convenient, as its existence is guaranteed in any non-trivial case. We remind the reader that Table 5 had provided many examples where this property can be observed.

5. Conclusion

This paper produces the first rigorous analysis of abstract convexity in an unrestricted soft set setting. We have shown how this issue gives rise to some fundamental results that ensure its attractiveness for further analyses. The methodology that we have used combines combinatorial arguments with the standard reasoning in soft set theory. In this seminal paper, we have given many examples of the main novel concepts, and we have studied their relationships with comparable notions from convex geometries, our main source of inspiration. In addition, we have proved some results that are rightful extensions of fundamental theorems about convex geometries. Our conclusions ensure that we have produced a working blend of disciplines inclusive of abstract convexity and the theory of vagueness represented by soft sets.

Let us summarize the main contributions of this paper. Consider a set X whose elements are characterized by the attributes E in a binary manner. Both X and E are nonempty and finite.

1. Convex soft geometries on X have been defined.
2. In the case where E is a singleton, any convex soft geometry on X is a convex geometry on X with a simple identification. In the general case, any convex soft geometry on X defines a convex geometry on X.
3. Conversely, any convex geometry on X produces a convex soft geometry on X. We reveal to what extent the two processes defined above are reciprocal.
4. We explore a different mechanism for passing from convex soft geometries to a family of convex geometries (one for each parameter). The converse mechanism, however, does not necessarily produce convex soft geometries from convex geometries.
5. For any fixed convex soft geometry on X, we define associated concepts and properties:
   (5.1) Soft convex hull of a soft set. Its interaction with the two processes defined in items 2. and 3. is disclosed.
   (5.2) The anti-exchange property of the convex soft geometry.
   (5.3) Extreme element for a soft set.
6. We prove that all convex soft geometries on X satisfy the anti-exchange property. The case of a unique attribute returns the classical result that convex geometries satisfy the anti-exchange property.
7. We prove that in all convex soft geometries on X, any soft set has an extreme element. The case of a unique attribute returns the classical result that for each convex geometry, any subset has an extreme element.

Needless to say, we have not intended to exhaust all possible research directions in this innovative study, as explained at some parts of this paper. So to finalize, we present some more lines for future investigation in relation to the model described in this paper.

- We have refrained from studying the standard algebraic manipulations (e.g., unions of convex soft geometries, or their restrictions to smaller subsets). Edelman and Jamison [1985, Section 5] can be used for inspiration. Here we can also find motivation for additional constructions of convex soft geometries.
- What is the right concept of isomorphism for convex soft geometries? Cantone et al. [8, Footnote 2] state the appropriate concept in the standard analysis of convex geometries (a verbal explanation is given in Example 3). This is important for the next item.
- How many non-isomorphic convex soft geometries exist for any fixed X and E? In the inspirational case (which corresponds with $|E|=1$), the number is 2 when $|X|=2$, 6 when $|X|=3$, 34 when $|X|=4$, 672 when $|X|=5$, 199572 when $|X|=6$, and 12884849614 when $|X|=7$ (v. [8, Section 2.1] for a reference to this issue).

  This research pertains to *combinatorial soft set theory*, a possible new branch within soft set theory.
- We can think of the structure that combines the traits of soft topology and convex soft geometry. Basically, one needs to add the property that the union of soft sets in $\mathcal{G}$ is in $\mathcal{G}$ to Definition 5.

  Notice that in case $|E|=1$, this structure has been identified and it is called an ordinal convex geometry (Edelman and Jamison [1985, Theorem 2], Cantone et al. [8, Theorem 5]).
- Soft topologies are point-set topologies on a product space (cf., Matejdes (2021)). Is there a comparable result for convex soft geometries? More precisely, can we find a correspondence between convex soft geometries and convex geometries on a suitable set that allows us to transfer properties from one setting to the other?
- The abstract structure defined on soft sets can be extended to fuzzy soft sets (i.e., the elements are characterized by attributes that can be partially fulfilled (Maji et al. 2001)), N-soft sets (a multinary, ordinal evaluation by the attributes can be applied (Fatimah et al. 2018)), or other more general contexts.

- Conversely, we can think of a weaker abstract structure defined on soft sets (and the models mentioned above) that corresponds with the extension of alignments Edelman and Jamison [1985, Section 2] to a setting with a multiplicity of attributes.
- One of the most striking results in the analysis of convex geometries is Koshevoy's theorem (cf., Koshevoy (1999), also [8, Theorem 2.16]). It gives an excellent connection with abstract choice theory: convex geometries and path-independent choice spaces can be put in one-to-one correspondence. The notion of extreme element is crucial in this correspondence; hence, Koshevoy's theorem has stimulated the analysis of extremes. Its extension to our context would yield a beautiful motivation for soft-set-supported abstract choice theory.

Notation and conventions

$\sqcap$, $\sqcup$, $\sqsubseteq$ denote the soft intersection, union, and inclusion.
(F, E), (F', E), (F_1, E), ... denote soft sets.
(x, E) denotes the soft point with $x(e) = \{x\}$ for all $e \in E$.
$(\{x\}_a, E)$ denotes the soft spot with $\{x\}_a(a') = \varnothing$ when $a' \neq a$, $\{x\}_a(a) = \{x\}$.
Φ and $\tilde{X}$ denote the null and absolute soft sets, respectively.
$\mathbf{G}$, $\mathbf{G}_1$, $\mathbf{G}_2$, ... denote convex geometries.
$\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_x$, $\mathcal{G}_y$, ... denote convex soft geometries.
η is the selection mapping.
ψ is the soft-making mapping.
$\mathrm{conv}_{\mathbf{G}}(Y)$ stands for the convex hull of Y in $\mathbf{G}$, a convex geometry.
$\mathrm{conv}_{\mathcal{G}}(F, E)$ stands for the soft convex hull of (F, E) in $\mathcal{G}$, a convex soft geometry.
$\mathrm{ex}_{\mathcal{G}}(F, E)$ denotes the set of extreme elements for (F, E) in the convex soft geometry $\mathcal{G}$.

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