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Short paper / *Note*

On global existence and bounds for the blow-up time in a semilinear heat equation involving parametric variable sources

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Abstract. This paper is concerned with the blow-up of the solutions to a semilinear heat equation with a reaction given by parametric variable sources. Some conditions to parameters and exponents of sources are given to obtain lower–upper bounds for the time of blow-up and some global existence results.

Keywords. Parametric variable sources, Global existence, Heat equation, Semilinear parabolic problem, Lower bound, Upper bound.

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1. Introduction

In this paper we study the lower and upper bounds for the blow-up time in a semilinear parabolic problem involving variable sources

$$\begin{cases} u_t - \Delta u = \lambda u^{p(x)} - \mu u^{q(x)}, \quad x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ and λ, μ are two real parameters and $p, q : \Omega \to (1, +\infty)$ functions verifying suitable conditions.

It is well known that the source term causes finite-time blow-up of the solution. The question of blow-up of solutions to semilinear parabolic equations and systems has received considerable attention since the elegant work of Fujita [1]. In practical situations, one would like to know,

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among other things, whether the solutions blow-up, and if so, at what time T blow-up occurs. However, when the solution does blow-up at some finite T, this time can seldom be determined explicitly, and much effort has been devoted to the calculation of bounds for T. Most of the methods used until recently can only yield upper bounds for T, which are of little value in particular situations when blow-up has to be avoided. By using the first-order differential inequality technique, lower bounds for the blow-up time of solutions to semilinear heat equations under different boundary conditions and suitable constraints on the data were obtained by Payne *et al.* [2]. We refer the interested readers to the survey papers [3–5].

Problems related to parabolic equations arise in many mathematical models of applied science, such as nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences etc., and have attracted a great deal of attention in the literature, see [6, 7] and the references therein.

In [8], Pinasco considered a heat equation with a forcing term of variable exponent nonlinearities:

$$\begin{cases} u_t = \Delta u + a(x)u^{p(x)}, \quad (x,t) \in \Omega \times [0,T), \\ u(x,t) = 0, (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, the continuous function $a(x) : \Omega \to \mathbb{R}$ with $0 < c_a \le a(x) \le C_a < +\infty$, and $1 < p^- \le p(x) \le p^+ < +\infty$. Under specific conditions, they proved the local existence of positive solutions and showed that solutions with sufficiently large initial data blow-up in finite time.

From a mathematical point of view, equations of the types (1) and (2) with variable exponent are usually referred to as equations with nonstandard growth conditions. Under certain conditions on the initial data and certain ranges of exponents, the existence, uniqueness and other qualitative properties of solutions for parabolic and hyperbolic equations with variable nonlinearity have been studied by many authors (see [9–17] and references therein).

Particularly, when $a(x) = 1$, Baghaei *et al.* [18], were concerned with the blow-up of the solutions to a semilinear parabolic problem (2), with a reaction given by a variable source. The authors obtained the lower bounds for the blow-up time t^* under some appropriate conditions. Also, fundamental results related with the finite-time blow-up solutions for type (2) problems were obtained by the [19–22].

In this paper, we give a sufficient condition for the lower bounds for the blow-up in L^k $(k > 2)$ and L^2-norms and establish an upper bound for the blow-up in L^1 for a sufficiently large initial datum u_0 for problem (1).

2. Main result and proofs

Let $h : \Omega \to (1, \infty)$ be a measurable function. We introduce h^- and h^+ such that

$$1 < h^- \le h(x) \le h^+ < +\infty, \quad x \in \Omega. \tag{3}$$

2.1. *Global existence*

In this section, we first show the global existence result for $N \ge 1$. The main idea of this section is the comparison principle. Let $\varphi(x)$ satisfy the following elliptic problem:

$$-\Delta\varphi = 1, \quad x \in \Omega; \quad \varphi(x) = 1, \quad x \in \partial\Omega.$$

By using the result in [23], we can see that the above nonlinear problem has a unique solution, and the following inequalities hold:

$$M := \max_{x \in \overline{\Omega}} \varphi(x) < +\infty; \quad \varphi(x) > 1, \quad x \in \Omega,$$

and

$$M_0 := \min_{x\in\overline{\Omega}} \varphi(x) < +\infty; \quad \varphi(x) > 1, \quad x \in \Omega.$$

Theorem 1. *Let u be a solution of (1) and functions p, q satisfy the condition (3).*

(i) *if $p(x) < q(x)$, $x \in \overline{\Omega}$ with $p^+ < q^-$, u remains globally bounded with the parameters $\lambda \geq 0$, $\mu > 0$;*
(ii) *if $p(x) = q(x) > 1$, $x \in \overline{\Omega}$, u remains globally bounded with the parameters $\lambda > \mu \geq 0$;*
(iii) *if $q(x) < p(x)$, $x \in \overline{\Omega}$ with $q^+ < p^-$, u remains globally bounded with the parameters $\lambda > 0$, $\mu \geq 0$.*

Now we state and prove a lemma that is useful to the proof of Theorem 1.

Lemma 2. *If $\omega \in (0,\infty)$ satisfies the following inequality:*

$$\omega^s \leq a\omega^q + b\omega^r,$$

where s, q, r, a, and b are constants with $a, b > 0$ and $s > \max\{q, r\} \geq 0$, then

$$0 < \omega \leq \inf_{\varepsilon\in(0,1)} \max\left\{ \left(\frac{b}{\varepsilon}\right)^{\frac{1}{s-r}}, \left(\frac{a}{1-\varepsilon}\right)^{\frac{1}{s-q}} \right\}.$$

Proof of Lemma 2. Let $\varepsilon \in (0,1)$.

(i_0) if $\varepsilon\omega^s < b\omega^r$, then $\omega < (b/\varepsilon)^{1/(s-r)}$;
(ii_0) if $\varepsilon\omega^s \geq b\omega^r$, then $\omega^s \leq a\omega^q + \varepsilon\omega^s \Rightarrow (1-\varepsilon)\omega^s \leq a\omega^q$, which implies $\omega \leq (a/(1-\varepsilon))^{1/(s-q)}$.

The Lemma 2 is proved. □

Proof of Theorem 1. Let consider the following elementary inequality:

$$a_0\delta^l - b_0\delta^d \leq a_0\left(\frac{a_0}{b_0}\right)^{\frac{l}{d-l}}, \quad \forall \delta > 0, \tag{4}$$

where $a_0 \geq 0$, $b_0 > 0$ and $0 < l < d$.

(i) Let $p(x) < q(x)$, $x \in \overline{\Omega}$ with $p^+ < q^-$. Define $\overline{u} = A\varphi(x)$, where constant $A > 0$ satisfies that

$$A \geq \max\left\{ \lambda\left(\left(\frac{\lambda}{\mu}\right)^{\frac{p^+}{q^- - p^+}} + 1\right), \quad \max_{x\in\overline{\Omega}} u_0(x) \right\}. \tag{5}$$

By (4), we have

$$\lambda\overline{u}^{p(x)} - \mu\overline{u}^{q(x)} \leq \lambda\left(\frac{\lambda}{\mu}\right)^{\frac{p(x)}{q(x)-p(x)}} \leq \lambda\left(\left(\frac{\lambda}{\mu}\right)^{\frac{p^+}{q^- - p^+}} + 1\right).$$

Then we obtain

$$\Delta\overline{u} + \lambda\overline{u}^{p(x)} - \mu\overline{u}^{q(x)} \leq -A + \lambda\left(\left(\frac{\lambda}{\mu}\right)^{\frac{p^+}{q^- - p^+}} + 1\right) \leq \overline{u}_t = 0,$$

where A satisfies condition (5) with the parameters $\lambda \geq 0$, $\mu > 0$.

(ii) Let $q(x) = p(x) > 1$, $x \in \overline{\Omega}$ and constant $A > 0$ satisfies that

$$\max_{x\in\overline{\Omega}} u_0(x) \leq A \leq \left(\frac{1}{(\lambda-\mu)\, M^{p^+}}\right)^{\frac{1}{p^+-1}}. \tag{6}$$

with the parameters $\lambda > \mu \geq 0$. We gain

$$\Delta\overline{u} + \lambda\overline{u}^{p(x)} - \mu\overline{u}^{q(x)} = \Delta\overline{u} + (\lambda-\mu)\overline{u}^{p(x)} \leq -A + (\lambda-\mu)A^{p^+}M^{p^+} \leq \overline{u}_t = 0,$$

where A satisfies condition (6).

(iii) Let $q(x) < p(x)$, $x \in \overline{\Omega}$ with $q^+ < p^-$ and constant $A > 0$ satisfies that

$$\max_{x\in\overline{\Omega}} u_0(x) \le A \le \inf_{\varepsilon\in(0,1)} \max\left\{ \left(\frac{1}{\varepsilon\lambda M^{p^+}}\right)^{\frac{1}{p^+-1}}, \left(\frac{\mu M_0^{q^-}}{\lambda(1-\varepsilon)M^{p^+}}\right)^{\frac{1}{p^+-q^-}} \right\}, \tag{7}$$

with the parameters $\lambda > 0$, $\mu \ge 0$.

By using Lemma 2, we have

$$\Delta\overline{u} + \lambda\overline{u}^{p(x)} - \mu\overline{u}^{q(x)} \le -A + \lambda A^{p^+} M^{p^+} - \mu A^{q^-} M_0^{q^-} \le \overline{u}_t = 0,$$

where A satisfies condition (7). Moreover, $\overline{u} \ge 0$ on $\partial\Omega \times (0, +\infty)$ and $\overline{u}(x, 0) \ge u_0(x)$ in Ω. By the comparison principle, $\overline{u}$ is a globally bounded supersolution of (1). The proof of Theorem 1 is completed. □

2.2. *Blow-up in finite time for any initial data*

In this section, we give our main results which we seek for the lower and upper bounds for the blow-up time T of problem (1) in some appropriate measure and their proofs.

Firstly, we consider the case $N \ge 3$.

Theorem 3. *Let $u(x,t)$ be the nonnegative solution of problem (1) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \ge 3$ and functions p, q satisfy the condition (3). Define*

$$F(t) = \int_\Omega u^k \,\mathrm{d}x. \tag{8}$$

If $q(x) < p(x)$, $x \in \overline{\Omega}$ with $q^+ < p^-$ and λ, μ are two real parameters with $0 < \mu < \lambda$ and k is a parameter restricted by the condition

$$k > \max\left\{\frac{N(p^+-\gamma)}{2}, 2\right\}, \quad 0 < \gamma < 1,$$

then a lower bound for the time of blow-up for any solution which blows up in L^k norm is given by

$$\int_{\|u_0\|_k^k}^{+\infty} \frac{\mathrm{d}\gamma}{K_2\gamma^{\frac{2(p^++k-\gamma)-N(p^+-\gamma)}{2k-N(p^+-\gamma)}} + K_1} \le T,$$

where $\|u_0\|_k^k = \int_\Omega u_0^k \,\mathrm{d}x$ and K_1, K_2 are positive constants which will be determined later.

Now, we deal with the case $N = 1, 2$.

Theorem 4. *Let $u(x,t)$ be the nonnegative solution of problem (1) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, functions p, q satisfy the condition (3) and F given in (8). If $q(x) < p(x)$, $x \in \overline{\Omega}$ with $q^+ < p^-$ and λ, μ are two real parameters with $0 < \mu < \lambda$ and k is a parameter such that*

$$k > \max\{p^+ - \gamma, 2\}, \quad 0 < \gamma < 1,$$

then a lower bound for the time of blow-up for any solution which blows up in L^k norm is given by

$$\int_{\|u_0\|_k^k}^{+\infty} \frac{\mathrm{d}\xi}{K_3\xi^{\frac{k}{k-p^++\gamma}} + K_1} \le T,$$

where K_1, K_3 are positive constants which will be determined later.

Finally, we give a sufficient condition for the solution of problem (1) to blow-up in L^1-norm and establish an upper bound for the blow-up time.

Theorem 5. *Let $u(x,t)$ be the nonnegative solution of problem (1) in a bounded domain $\Omega\subset\mathbb{R}^N$, $N\geq 1$ and functions p,q satisfy the condition (3). Define*

$$W(t)=\int_\Omega u\,\mathrm{d}x. \tag{9}$$

Assume that $q(x)<p(x)$, $x\in\overline{\Omega}$ with $q^+<p^-$ and λ, $\mu>0$ are two real parameters with

$$\mu<\lambda\min\left\{L_0\|u_0\|_{q^+}^{p^--q^+},\frac{L_0p^-}{q^+}\right\},$$

then an upper bound for the time of blow-up for any solution which blows up in L^1 norm is given by

$$\int_{\|u_0\|_1}^{+\infty}\frac{\mathrm{d}\zeta}{L_2\zeta^{p^-}-L_1}\geq T,$$

where $\|u_0\|_1=\int_\Omega u_0\,\mathrm{d}x$ and L_0, L_1, L_2 are positive constants which will be determined later.

Proof of Theorem 3. We compute where we have used successively the differential equation (1), the divergence theorem, the boundary condition (1). Next, by using (8), we obtain

$$\begin{aligned}F'(t)&=k\int_\Omega u^{k-1}u_t\,\mathrm{d}x=k\int_\Omega u^{k-1}(\Delta u+\lambda u^{p(x)}-\mu u^{q(x)})\,\mathrm{d}x\\&=k\int_\Omega u^{k-1}\Delta u+\lambda k\int_\Omega u^{p(x)+k-1}\,\mathrm{d}x-\mu k\int_\Omega u^{q(x)+k-1}\,\mathrm{d}x\\&=-k(k-1)\int_\Omega u^{k-2}|\Delta u|^2+\lambda k\int_\Omega u^{p(x)+k-1}\,\mathrm{d}x-\mu k\int_\Omega u^{q(x)+k-1}\,\mathrm{d}x\\&=-\frac{4(k-1)}{k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2+k\lambda\int_\Omega u^{p(x)+k-1}\,\mathrm{d}x-k\mu\int_\Omega u^{q(x)+k-1}\,\mathrm{d}x.\end{aligned} \tag{10}$$

On the other hand, we obtain

$$\lambda u^{p(x)+k-1}-\mu u^{q(x)+k-1}=(\lambda-\mu)u^{p(x)+k-1}+\mu(u^{p(x)+k-1}-u^{q(x)+k-1}). \tag{11}$$

By the condition $q(x)<p(x)$, $x\in\overline{\Omega}$, we get the following

$$q(x)+k-1<p(x)+k-1<p(x)+k-\gamma,\quad 0<\gamma<1,$$

and we see that

$$u^{q(x)+k-1}+u^{p(x)+k-\gamma}\geq u^{p(x)+k-1}. \tag{12}$$

By (11) and (12), we derive

$$\lambda u^{p(x)+k-1}-\mu u^{q(x)+k-1}\leq(\lambda-\mu)u^{p(x)+k-1}+\mu u^{p(x)+k-\gamma}, \tag{13}$$

with $0<\mu<\lambda$. From (10) and (13) we get

$$\begin{aligned}F'(t)\leq&-\frac{4(k-1)}{k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2\\&+k(\lambda-\mu)\int_\Omega u^{p(x)+k-1}\,\mathrm{d}x+k\mu\int_\Omega u^{p(x)+k-\gamma}\,\mathrm{d}x.\end{aligned} \tag{14}$$

Furthermore, by the condition (3), we derive

$$\begin{aligned}\int_\Omega u^{p(x)+k-\gamma}\,\mathrm{d}x&=\int_{\Omega\cap\{x:u\geq 1\}}u^{p(x)+k-\gamma}\,\mathrm{d}x+\int_{\Omega\cap\{x:u<1\}}u^{p(x)+k-\gamma}\,\mathrm{d}x\\&\leq\int_\Omega u^{p^++k-\gamma}\,\mathrm{d}x+|\Omega|=\|u\|_{p^++k-\gamma}^{p^++k-\gamma}+|\Omega|.\end{aligned} \tag{15}$$

Then (14) and (15), we have

$$\begin{aligned}F'(t)\leq&-\frac{4(k-1)}{k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2\\&+k(\lambda-\mu)\|u\|_{p^++k-1}^{p^++k-1}+k\mu\|u\|_{p^++k-\gamma}^{p^++k-\gamma}+k\mu|\Omega|.\end{aligned} \tag{16}$$

Since $p^+ + k - 1 < p^+ + k - \gamma$, $0 < \gamma < 1$ and $0 < \mu < \lambda$, then by using (4) inequality, we get

$$\begin{aligned} & k(\lambda-\mu)u^{p^+ + k-1} - k\mu u^{p^+ + k-\gamma} \\ & \le k(\lambda-\mu)\left(\frac{\lambda-\mu}{\mu}\right)^{\frac{p^+ + k-1}{1-\gamma}} := K_0(k,\gamma,\lambda,\mu,p^+) := K_0 > 0. \end{aligned} \tag{17}$$

So, from (17), we arrive at

$$k(\lambda-\mu)\|u\|_{p^+ + k-1}^{p^+ + k-1} \le k\mu\|u\|_{p^+ + k-\gamma}^{p^+ + k-\gamma} + K_0|\Omega|. \tag{18}$$

Combining (16) and (18), we have

$$F'(t) \le -\frac{4(k-1)}{k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2 + 2k\mu\|u\|_{p^+ + k-\gamma}^{p^+ + k-\gamma} + K_1, \tag{19}$$

where

$$K_1 := (K_0 + k\mu)|\Omega| > 0.$$

We deduce from the Gagliardo–Nirenberg interpolation inequality (see [24]) that there exists a constant $C := C(N, p^+, \gamma, k) > 0$, independent of u, such that

$$\|u\|_{p^+ + k-\gamma}^{p^+ + k-\gamma} = \left\|u^{\frac{k}{2}}\right\|_{\frac{2(p^+ + k-\gamma)}{k}}^{\frac{2(p^+ + k-\gamma)}{k}} \le C\left\|\nabla u^{\frac{k}{2}}\right\|_2^{\frac{2\theta(p^+ + k-\gamma)}{k}}\left\|u^{\frac{k}{2}}\right\|_2^{\frac{2(1-\theta)(p^+ + k-\gamma)}{k}},$$

where $\theta = N(p^+ - \gamma)/(2(p^+ + k - \gamma)) \in (0,1)$ and $k > \max\{(N-2)(p^+ - \gamma)/2, 2\}$, then

$$\|u\|_{p^+ + k-\gamma}^{p^+ + k-\gamma} \le C\left\|\nabla u^{\frac{k}{2}}\right\|_2^{\frac{N(p^+ - \gamma)}{k}}\left\|u^{\frac{k}{2}}\right\|_2^{\frac{2(p^+ + k-\gamma)-N(p^+ - \gamma)}{k}}.$$

By using Young's inequality, we get

$$\begin{aligned} \|u\|_{p^+ + k-\gamma}^{p^+ + k-\gamma} &\le \frac{CN(p^+ - \gamma)\varepsilon^{\frac{2k}{N(p^+ - \gamma)}}}{2k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2 + \frac{C[2k - N(p^+ - \gamma)]}{2k\varepsilon^{\frac{2k}{2k-N(p^+ - \gamma)}}}\left\|u^{\frac{k}{2}}\right\|_2^{\frac{2[2(p^+ + k-\gamma)-N(p^+ - \gamma)]}{2k-N(p^+ - \gamma)}} \\ &= \frac{CN(p^+ - \gamma)\varepsilon^{\frac{2k}{N(p^+ - \gamma)}}}{2k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2 + \frac{2Ck - CN(p^+ - \gamma)}{2k\varepsilon^{\frac{2k}{2k-N(p^+ - \gamma)}}}F(t)^{\frac{2(p^+ + k-\gamma)-N(p^+ - \gamma)}{2k-N(p^+ - \gamma)}}, \end{aligned} \tag{20}$$

where ε is a positive constant to be determined later. Combining (19) with (20), we obtain

$$\begin{aligned} F'(t) \le & -\frac{4(k-1)}{k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2 + 2k\mu\left(\frac{CN(p^+ - \gamma)\varepsilon^{\frac{2k}{N(p^+ - \gamma)}}}{2k}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2\right. \\ & \left. + \frac{2Ck - CN(p^+ - \gamma)}{2k\varepsilon^{\frac{2k}{2k-N(p^+ - \gamma)}}}F(t)^{\frac{2(p^+ + k-\gamma)-N(p^+ - \gamma)}{2k-N(p^+ - \gamma)}}\right) + K_1. \end{aligned}$$

If we choose $\varepsilon > 0$ such that

$$\varepsilon = \left(\frac{4(k-1)}{k\mu CN(p^+ - \gamma)}\right)^{\frac{N(p^+ - \gamma)}{2k}},$$

then, we obtain the differential inequality

$$F'(t) \le K_2 F(t)^{\frac{2(p^+ + k-\gamma)-N(p^+ - \gamma)}{2k-N(p^+ - \gamma)}} + K_1, \tag{21}$$

where

$$K_2 := \frac{\mu(2Ck - CN(p^+ - \gamma))}{\varepsilon^{\frac{2k}{2k-N(p^+ - \gamma)}}} > 0.$$

An integration of the differential inequality (21) from 0 to t, we obtain the following inequality

$$\int_{F(0)}^{F(t)} \frac{\mathrm{d}\gamma}{K_2\gamma^{\frac{2(p^+ + k-\gamma)-N(p^+ - \gamma)}{2k-N(p^+ - \gamma)}} + K_1} \le t$$

which with $\lim_{t\to T^-} F(t) = +\infty$ implies

$$\int_{F(0)}^{+\infty} \frac{\mathrm{d}\gamma}{K_2\gamma^{\frac{2(p^++k-\gamma)-N(p^+-\gamma)}{2k-N(p^+-\gamma)}} + K_1} \le T,$$

where $F(0) = \int_\Omega u_0^k(x)\,\mathrm{d}x$. Note that $k > (N(p^+-\gamma))/2$, $0<\gamma<1$, hence the right-hand side of the above inequality is finite. The proof of Theorem 3 is completed. □

Proof of Theorem 4. Now, recall the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ which provides the inequality

$$\|u\|_\infty \le B\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega), \tag{22}$$

where B is the best constant of the Sobolev embedding. Using (22) and Hölder's inequality to $\|u\|_{p^++k-\gamma}^{p^++k-\gamma}$ which is in (19), we get

$$\begin{aligned}\|u\|_{p^++k-\gamma}^{p^++k-\gamma} &= \|u\|_k^k \|u\|_\infty^{p^+-\gamma}\\ &= F(t)\left\|u^{\frac{k}{2}}\right\|_\infty^{\frac{2(p^+-\gamma)}{k}} \le B^{\frac{2(p^+-\gamma)}{k}} F(t)\left\|\nabla u^{\frac{k}{2}}\right\|_2^{\frac{2(p^+-\gamma)}{k}}.\end{aligned} \tag{23}$$

Now, by using Young's inequality to (23), we have for all $\varepsilon>0$,

$$\|u\|_{p^++k-\gamma}^{p^++k-\gamma} \le \frac{kB^{\frac{2(p^+-\gamma)}{k}}}{k-p^++\gamma}\varepsilon^{-\frac{k-p^++\gamma}{k}} F(t)^{\frac{k}{k-p^++\gamma}} + \frac{kB^{\frac{2(p^+-\gamma)}{k}}}{p^+-\gamma}\varepsilon^{\frac{p^+-\gamma}{k}}\left\|\nabla u^{\frac{k}{2}}\right\|_2^2. \tag{24}$$

Let us choose $\varepsilon>0$ such that

$$\varepsilon = \left(\frac{2(k-1)p^+-\gamma}{\mu k^3 B^{\frac{2(p^+-\gamma)}{k}}}\right)^{\frac{k}{p^+-\gamma}}.$$

Thus, from the relation between (19) and (24), we have

$$F'(t) \le \frac{2\mu k^2 B^{\frac{2(p^+-\gamma)}{k}}}{k-p^++\gamma}\varepsilon^{-\frac{k-p^++\gamma}{k}} F(t)^{\frac{k}{k-p^++\gamma}} + K_1 = K_3 F(t)^{\frac{k}{k-p^++\gamma}} + K_1, \tag{25}$$

where

$$K_3 = \frac{2\mu k^2 B^{\frac{2(p^+-\gamma)}{k}}}{(k-p^++\gamma)\left(\frac{2(k-1)p^+-\gamma}{\mu k^3 B^{\frac{2(p^+-\gamma)}{k}}}\right)^{\frac{k-p^++\gamma}{p^+-\gamma}}} > 0.$$

Then from (25), we can gain a lower bound for blow-up time T:

$$\int_{F(0)}^{+\infty} \frac{\mathrm{d}\xi}{K_3\xi^{\frac{k}{k-p^++\gamma}} + K_1} \le T.$$

The Theorem 4 is proved. □

Proof of Theorem 5. From (1) and (9), we obtain

$$\begin{aligned}W'(t) &= \int_\Omega u_t\,\mathrm{d}x = \int_\Omega (\Delta u + \lambda u^{p(x)} - \mu u^{q(x)})\,\mathrm{d}x\\ &\ge \lambda\int_\Omega u^{p(x)}\,\mathrm{d}x - \mu\int_\Omega u^{q(x)}\,\mathrm{d}x.\end{aligned} \tag{26}$$

If p and q satisfy the condition (3), we have

$$\int_\Omega u^{p(x)}\,\mathrm{d}x = \int_{\Omega\cap\{x:u\ge1\}} u^{p(x)}\,\mathrm{d}x + \int_{\Omega\cap\{x:u<1\}} u^{p(x)}\,\mathrm{d}x,$$

and

$$\begin{aligned}\int_\Omega u^{p(x)}\,\mathrm{d}x &\geq \int_{\Omega\cap\{x:u\geq 1\}} u^{p^-}\,\mathrm{d}x = \int_\Omega u^{p^-}\,\mathrm{d}x - \int_{\Omega\cap\{x:u<1\}} u^{p^-}\,\mathrm{d}x\\ &\geq \int_\Omega u^{p^-}\,\mathrm{d}x - \int_\Omega \mathrm{d}x = \int_\Omega u^{p^-}\,\mathrm{d}x - |\Omega|.\end{aligned}$$

Hence,

$$\int_\Omega u^{p(x)}\,\mathrm{d}x \geq \int_\Omega u^{p^-}\,\mathrm{d}x - |\Omega|, \tag{27}$$

and

$$\int_\Omega u^{q(x)}\,\mathrm{d}x \leq \|u\|_{q^+}^{q^+} + |\Omega|. \tag{28}$$

From (26), (27), (28) and using inverse Holder's inequality, we get

$$\begin{aligned}W'(t) &\geq \lambda\|u\|_{p^-}^{p^-} - \mu\|u\|_{q^+}^{q^+} - L_1\\ &\geq \lambda L_0\|u\|_{q^+}^{p^-} - \mu\|u\|_{q^+}^{q^+} - L_1\\ &= \mu\|u\|_{q^+}^{q^+}\left(\frac{\lambda L_0}{\mu}\|u\|_{q^+}^{p^- - q^+} - 1\right) - L_1,\end{aligned} \tag{29}$$

where

$$L_0 = |\Omega|^{\frac{q^+ - p^-}{q^+}} > 0, L_1 = (\lambda+\mu)|\Omega| > 0.$$

Obviously, since $1 < q^+ < p^-$, we can get that the function $f(\vartheta) = \vartheta^{p^- - q^+}$ is monotone increasing and if $\lambda, \mu > 0$ and u_0 satisfies

$$\mu < \lambda L_0\|u_0\|_{q^+}^{p^- - q^+},$$

then we can know that the solution of problem (1) blows up in finite time.

Thus, by (26) and (29), we have

$$\begin{aligned}W'(t) &\geq \lambda L_0\|u\|_{q^+}^{p^-} - \mu\|u\|_{q^+}^{q^+} - L_1\\ &= \lambda L_0\|u\|_{q^+}^{p^-} - \mu\left(\delta\|u\|_{q^+}^{q^+}\right)^{\frac{p^-}{q^+}\cdot\frac{q^+}{p^-}} \delta^{-\frac{p^-}{p^- - q^+}\cdot\frac{p^- - q^+}{p^-}} - L_1\\ &\geq \lambda L_0\|u\|_{q^+}^{p^-} - \frac{\mu\delta^{\frac{p^- - q^+}{q^+}} q^+}{p^-}\|u\|_{q^+}^{p^-} - L_1\\ &\geq L_2\|u\|_1^{p^-} - L_1,\end{aligned} \tag{30}$$

where

$$L_2 = |\Omega|^{\frac{(1-q^+)p^-}{q^+}}\left(\lambda L_0 - \frac{\mu\delta^{\frac{p^- - q^+}{q^+}} q^+}{p^-}\right).$$

Choosing δ satisfies $0 < \delta = (\lambda L_0 p^-/2\mu q^+)^{q^+/(p^- - q^+)}$, then from (9) and (30) it can be rewritten as

$$W'(t) \geq L_2 W^{p^-}(t) - L_1. \tag{31}$$

Hence, if u_0 is large enough satisfying

$$W(0) = \|u_0\|_1 > \left(\frac{L_1}{L_2}\right)^{\frac{1}{p^-}},$$

by virtue of (31), we can derive that the blow-up time T satisfies

$$\int_{W(0)}^{+\infty} \frac{\mathrm{d}\zeta}{L_2\zeta^{p^-} - L_1} \geq T.$$

The proof of Theorem 5 is completed. □

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