INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mécanique

Kuanysh A. Bekmaganbetov, Gregory A. Chechkin and Vladimir V. Chepyzhov

Attractors and a "strange term" in homogenized equation

Volume 348, issue 5 (2020), p. 351-359.

<https://doi.org/10.5802/crmeca.1>

Comptes Rendus
Mécanique
2020, 348, nº 5, p. 351-359
https://doi.org/10.5802/crmeca.1

Attractors and a "strange term" in homogenized equation

Attracteurs et un « terme étrange » dans les équations homogénéisées

Kuanysh A. Bekmaganbetov[a, b], **Gregory A. Chechkin**[*, c, d] **and Vladimir V. Chepyzhov**[e, f]

[a] M. V. Lomonosov Moscow State University, Kazakhstan Branch, Kazhymukan st. 11, Nur-Sultan, 010010, Kazakhstan

[b] Institute of Mathematics and Mathematical Modeling, Pushkin st. 125, Almaty, 050010, Kazakhstan

[c] M. V. Lomonosov Moscow State University, Leninskie Gory, 1, Moscow, 119991, Russia

[d] Institute of Mathematics with Computing Center - Subdivision of the Ufa Federal Research Center of Russian Academy of Sciences, Chernyshevskogo st., 112, Ufa, 450008, Russia

[e] Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetniy 19, Moscow 127994, Russia

[f] National Research University Higher School of Economics, Moscow 101000, Russia

E-mails: bekmaganbetov-ka@yandex.kz (K. A. Bekmaganbetov), chechkin@mech.math.msu.su (G. A. Chechkin), chep@iitp.ru (V. V. Chepyzhov)

Abstract. We study the behavior of attractors of the reaction–diffusion equation in a perforated domain as the small parameter characterizing the perforation tends to zero.

Résumé. Nous étudions le comportement des attracteurs de l'équation de réaction–diffusion dans le domaine perforé car le petit paramètre caractérisant la perforation tend vers zéro.

Keywords. Homogenization, Attractors, Reaction–diffusion equation, Boundary value problem, Perforated domain.

Mots-clés. Homogénéisation, Attracteurs, Équation de réaction–diffusion, Problème de valeur limite, Domaine perforé.

2020 Mathematics Subject Classification. 35B30, 35B40, 35B45, 35B60, 35Q35, 76A05, 76D10.

Manuscript received and accepted 3rd February 2020.

[*] Corresponding author.

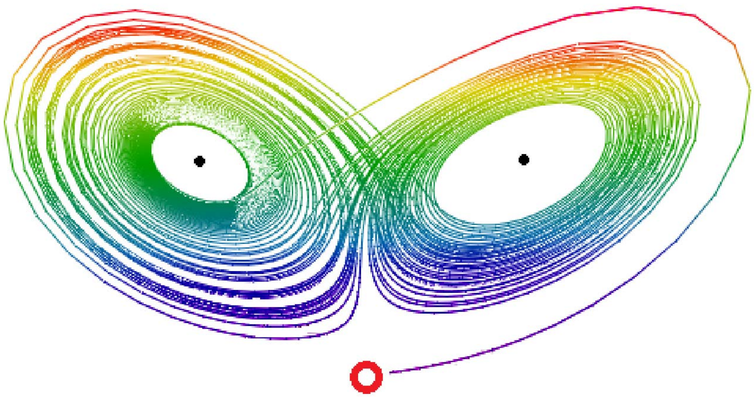

Figure 1. Attractor.

1. Introduction

Homogenization in a perforated domain in critical cases leads to the appearance of an additional potential ("strange term") in the limit (homogenized) equation (see [1–6]). We discovered the same phenomenon in the homogenization of attractors (see Figure 1[1] for example) for the reaction–diffusion equation.

2. Notation and settings

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a piecewise smooth boundary $\partial\Omega$. Let G_0 be a domain in $Y = (-1/2, 1/2)^n$ such that $\overline{G}_0$ is a compact set diffeomorphic to a ball.

For $\delta > 0$ and B, we denote $\delta B = \{x : \delta^{-1} x \in B\}$. Assume that ε is small enough so that

$$\varepsilon^{n/(n-2)} G_0 \subset \varepsilon Y.$$

For $j \in \mathbb{Z}^n$, we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + \varepsilon^{n/(n-2)} G_0.$$

We define the domain $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > \sqrt{n}\,\varepsilon\}$ and the set of admissible indexes as

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}.$$

Note that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, where $d > 0$ is a constant. Consider the following domain:

$$\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon, \quad \text{where } G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j.$$

Denote

$$Q_\varepsilon = \Omega_\varepsilon \times (0, +\infty), \quad Q = \Omega \times (0, +\infty).$$

We study the asymptotic behavior of attractors of the problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon - f(u_\varepsilon) + g(x), & x \in \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^{n/(2-n)} b_\varepsilon^j(x) u_\varepsilon = 0, & x \in \partial G_\varepsilon^j, \ j \in \Upsilon_\varepsilon, \ t \in (0, +\infty), \\ u_\varepsilon = 0, & x \in \partial\Omega, \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, \ t = 0. \end{cases} \tag{1}$$

[1] http://docplayer.ru/32107834-Lekciya-5-haoticheskoe-povedenie-dinamicheskih-sistem-sistema-lorenca.html.

Here, ν is the outward unit vector to the boundary, $g(x) \in L_2(\Omega)$,

$$b_\varepsilon^j(x) = b\left(x, \frac{x - P_\varepsilon^j}{\varepsilon^{n/(n-2)}}\right),$$

where $b(x, y) \in C(\Omega \times \mathbb{R}^n)$, such that $0 < b_0 \le b(x, y) \le B_0$ for some constants b_0 and B_0, $b(x, y)$ is one-periodic in y, and $f(v) \in C(\mathbb{R})$ satisfies the following inequalities:

$$f(v)\cdot v \ge K|v|^p - C, \quad |f(v)| \le C_1(|v|^{p-1} + 1), \quad p \ge 2. \tag{2}$$

Note that we *do not assume* that the nonlinear function $f(v)$ satisfies the Lipschitz condition with respect to v.

We denote the spaces $\mathbf{H} := L_2(\Omega)$, $\mathbf{H}_\varepsilon := L_2(\Omega_\varepsilon)$, $\mathbf{V} := H_0^1(\Omega)$, and $\mathbf{V}_\varepsilon := H^1(\Omega_\varepsilon; \partial\Omega)$—set of functions from $H^1(\Omega_\varepsilon)$ with zero trace on $\partial\Omega$—and $\mathbf{L}_p := L_p(\Omega)$ and $\mathbf{L}_{p,\varepsilon} := L_p(\Omega_\varepsilon)$. The norms in these spaces are denoted, respectively, by

$$\|v\|^2 := \int_\Omega |v(x)|^2\,\mathrm{d}x, \quad \|v\|_\varepsilon^2 := \int_{\Omega_\varepsilon} |v(x)|^2\,\mathrm{d}x, \quad \|v\|_1^2 := \int_\Omega |\nabla v(x)|^2\,\mathrm{d}x,$$
$$\|v\|_{1\varepsilon}^2 := \int_{\Omega_\varepsilon} |\nabla v(x)|^2\,\mathrm{d}x, \quad \|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p\,\mathrm{d}x, \quad \|v\|_{\mathbf{L}_p\,\varepsilon}^p := \int_{\Omega_\varepsilon} |v(x)|^p\,\mathrm{d}x.$$

Recall that $\mathbf{V}' := H^{-1}(\Omega)$ and $\mathbf{L}_q$ are the dual spaces of $\mathbf{V}$ and $\mathbf{L}_p$, respectively, where $q = p/(p-1)$. Moreover, $\mathbf{V}_\varepsilon'$ is the dual space for $\mathbf{V}_\varepsilon$.

As in [7, 8], we study weak solutions of the initial boundary value problem (1), that is, the functions

$$u_\varepsilon(x, s) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap L_p^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_{p,\varepsilon}),$$

which satisfy problem (1) in the distributional sense, that is,

$$\begin{aligned} &-\int_{Q_\varepsilon} u_\varepsilon \frac{\partial \psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla \psi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\varepsilon} f(u_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t \\ &+ \varepsilon^{n/(2-n)} \sum_{j \in \Upsilon_\varepsilon} \int_0^{+\infty} \int_{\partial G_\varepsilon^j} b_\varepsilon^j u_\varepsilon \psi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\varepsilon} g(x)\psi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3}$$

for any $\psi \in C_0^\infty(\mathbb{R}_+; \mathbf{H}_\varepsilon)$.

If $u_\varepsilon(x, t) \in L_p(0, M; \mathbf{L}_{p,\varepsilon})$, then it follows from condition (2) that $f(u(x, t)) \in L_q(0, M; \mathbf{L}_{q,\varepsilon})$. At the same time, if $u_\varepsilon(x, t) \in L_2(0, M; \mathbf{V}_\varepsilon)$, then $\Delta u_\varepsilon(x, t) + g(x) \in L_2(0, M; \mathbf{V}_\varepsilon')$. Therefore, for an arbitrary weak solution $u_\varepsilon(x, s)$ of problem (1), we have

$$\frac{\partial u_\varepsilon(x, t)}{\partial t} \in L_q(0, M; \mathbf{L}_{q,\varepsilon}) + L_2(0, M; \mathbf{V}_\varepsilon').$$

The Sobolev embedding theorem implies that

$$L_q(0, M; \mathbf{L}_{q,\varepsilon}) + L_2(0, M; \mathbf{V}_\varepsilon') \subset L_q(0, M; \mathbf{H}_\varepsilon^{-r}),$$

where the space $\mathbf{H}_\varepsilon^{-r} := H^{-r}(\Omega_\varepsilon)$ and $r = \max\{1, n(1/2 - 1/p)\}$. Hence, for any weak solution $u_\varepsilon(x, t)$ of (1), we have $\partial u_\varepsilon(x, t)/\partial t \in L_q(0, M; \mathbf{H}_\varepsilon^{-r})$.

Remark 1. The existence of a weak solution $u(x, s)$ to problem (1) for every $U \in \mathbf{H}_\varepsilon$ and fixed ε such that $u(x, 0) = U(x)$ can be proved by the standard approach (see for instance [7, 9]). This solution is not necessarily unique because we do not assume the Lipschitz condition for $f(v)$ with respect to v.

The following lemma can be proved similarly to Proposition XV.3.1 from [8].

Lemma 2.1. *Let $u_\varepsilon(x, t) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap L_p^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_{p,\varepsilon})$ be a weak solution of problem* (1)*. Then*

(i) $u \in C(\mathbb{R}_+; \mathbf{H}_\varepsilon)$;

(ii) *the function* $\|u_\varepsilon(\cdot,t)\|_\varepsilon^2$ *is absolutely continuous on* $\mathbb{R}_+$ *and, moreover,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(\cdot,t)\|_\varepsilon^2+\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x,t)|^2\,\mathrm{d}x+\int_{\Omega_\varepsilon}f(u_\varepsilon)u_\varepsilon\,\mathrm{d}x$$
$$+\varepsilon^{\frac{n}{2-n}}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}b_\varepsilon^j|u_\varepsilon(x,t)|^2\,\mathrm{d}x=\int_{\Omega_\varepsilon}g(x)u_\varepsilon\,\mathrm{d}x$$

for almost every $t\in\mathbb{R}_+$.

In further analysis, we shall omit the index ε in the notation of spaces, where it is natural. We now apply the scheme described in [10] to construct the trajectory attractor for problem (1).

To describe the trajectory space $\mathscr{K}_\varepsilon^+$ for problem (1), we follow the general framework of Section 3 from [10] and define the Banach spaces for every $[t_1,t_2]\in\mathbb{R}$,

$$\mathscr{F}_{t_1,t_2}:=L_p(t_1,t_2;\mathbf{L}_p)\cap L_2(t_1,t_2;\mathbf{V})\cap L_\infty(t_1,t_2;\mathbf{H})\cap\left\{v\,\middle|\,\frac{\partial v}{\partial t}\in L_q(t_1,t_2;\mathbf{H}^{-r})\right\},\tag{4}$$

with norm

$$\|v\|_{\mathscr{F}_{t_1,t_2}}:=\|v\|_{L_p(t_1,t_2;\mathbf{L}_p)}+\|v\|_{L_2(t_1,t_2;\mathbf{V})}+\|v\|_{L_\infty(0,M;\mathbf{H})}+\left\|\frac{\partial v}{\partial t}\right\|_{L_q(t_1,t_2;\mathbf{H}^{-r})}.\tag{5}$$

It is clear that the condition

$$\|\Pi_{t_1,t_2}f\|_{\mathscr{F}_{t_1,t_2}}\le C(t_1,t_2,\tau_1,\tau_2)\|f\|_{\mathscr{F}_{\tau_1,\tau_2}},\quad\forall f\in\mathscr{F}_{\tau_1,\tau_2},\tag{6}$$

where $[t_1,t_2]\subseteq[\tau_1,\tau_2]$, Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1,t_2]$, and the constant $C(t_1,t_2,\tau_1,\tau_2)$ is independent of f, holds for norm (5) and the translation semigroup $\{S(h)\}$ satisfies

$$\|S(h)f\|_{\mathscr{F}_{t_1-h,t_2-h}}=\|f\|_{\mathscr{F}\,t_1,t_2},\quad\forall f\in\mathscr{F}_{t_1,t_2}.\tag{7}$$

The space $\mathscr{F}_{t_1,t_2}$ consists of functions $f(s), s\in[t_1,t_2]$ such that $f(s)\in E$ for almost all $s\in[t_1,t_2]$, where E is a Banach space.

Setting $\mathscr{D}_{t_1,t_2}=L_q(t_1,t_2;\mathbf{H}^{-r})$, we have that $\mathscr{F}_{t_1,t_2}\subseteq\mathscr{D}_{t_1,t_2}$, and if $u(s)\in\mathscr{F}_{t_1,t_2}$, then $A(u(s))\in\mathscr{D}_{t_1,t_2}$. We can consider a weak solution of problem (1) as a solution of the equation in the general scheme of Section 3 from [10].

Define the spaces

$$\mathscr{F}_+^{\mathrm{loc}}=L_p^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_p)\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{V})\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H})\cap\left\{v\,\middle|\,\frac{\partial v}{\partial t}\in L_q^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^{-r})\right\},$$
$$\mathscr{F}_{\varepsilon,+}^{\mathrm{loc}}=L_p^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_{p,\varepsilon})\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{V}_\varepsilon)\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}_\varepsilon)\cap\left\{v\,\middle|\,\frac{\partial v}{\partial t}\in L_q^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-r})\right\}.$$

We denote by $\mathscr{K}_\varepsilon^+$ the set of all weak solutions of problem (1). Recall that for any $U\in\mathbf{H}$, there exists at least one trajectory $u(\cdot)\in\mathscr{K}_\varepsilon^+$ such that $u(0)=U(x)$. Therefore, the trajectory space $\mathscr{K}_\varepsilon^+$ of problem (1) is not empty and is sufficiently large.

It is clear that $\mathscr{K}_\varepsilon^+\subset\mathscr{F}_+^{\mathrm{loc}}$ and the trajectory space $\mathscr{K}_\varepsilon^+$ is translation-invariant; that is, if $u(s)\in\mathscr{K}_\varepsilon^+$, then $u(h+s)\in\mathscr{K}_\varepsilon^+$ for all $h\ge 0$. Therefore,

$$S(h)\mathscr{K}_\varepsilon^+\subseteq\mathscr{K}_\varepsilon^+,\quad\forall h\ge 0.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathscr{F}_{t_1,t_2}$ using the norms of the spaces $L_2(t_1,t_2;\mathbf{H})$:

$$\rho_{0,M}(u,v)=\left(\int_0^M\|u(s)-v(s)\|^2\,\mathrm{d}s\right)^{1/2},\quad\forall u(\cdot),v(\cdot)\in\mathscr{F}_{0,M}.$$

These metrics generate the topology Θ_+^{loc} in $\mathscr{F}_+^{\text{loc}}$ (respectively, $\Theta_{\varepsilon,+}^{\text{loc}}$ in $\mathscr{F}_{\varepsilon,+}^{\text{loc}}$). Recall that a sequence $\{v_k\} \subset \mathscr{F}_+^{\text{loc}}$ converges to $v \in \mathscr{F}_+^{\text{loc}}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for each $M > 0$. The topology Θ_+^{loc} is metrizable using, for example, the Frechet metric

$$\rho_+(f_1, f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1, f_2)}{1 + \rho_{0,m}(f_1, f_2)}, \tag{8}$$

and the corresponding metric space is complete. We consider this topology in the trajectory space $\mathscr{K}_\varepsilon^+$ of (1). The translation semigroup $\{S(t)\}$ acting on $\mathscr{K}_\varepsilon^+$ is continuous in the considered topology Θ_+^{loc}.

Following the general scheme, we define bounded sets in $\mathscr{K}_\varepsilon^+$ using the Banach space $\mathscr{F}_+^b := \{f(s) \in \mathscr{F}_+^{\text{loc}} \mid \|f\|_{\mathscr{F}_+^b} < +\infty\}$. We clearly have

$$\mathscr{F}_+^b = L_p^b(\mathbb{R}_+; \mathbf{L}_p) \cap L_2^b(\mathbb{R}_+; \mathbf{V}) \cap L_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \,\middle|\, \frac{\partial v}{\partial t} \in L_q^b(\mathbb{R}_+; \mathbf{H}^{-r}) \right\}, \tag{9}$$

and $\mathscr{F}_+^b$ is a subspace of $\mathscr{F}_+^{\text{loc}}$.

Consider the translation semigroup $\{S(t)\}$ on $\mathscr{K}_\varepsilon^+$, $S(t) : \mathscr{K}_\varepsilon^+ \to \mathscr{K}_\varepsilon^+$, $t \geq 0$.

Let $\mathscr{K}_\varepsilon$ be the kernel of problem (1), which consists of all weak complete solutions $u(s)$, $s \in \mathbb{R}$, of the equation bounded in the space

$$\mathscr{F}^b = L_p^b(\mathbb{R}; \mathbf{L}_p) \cap L_2^b(\mathbb{R}; \mathbf{V}) \cap L_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \,\middle|\, \frac{\partial v}{\partial t} \in L_q^b(\mathbb{R}; \mathbf{H}^{-r}) \right\}.$$

Definition 2.1 ([8]). *A set $\mathfrak{A} \subseteq \mathscr{K}^+$ is called the* TRAJECTORY ATTRACTOR *of the translation semigroup $\{S(t)\}$ on $\mathscr{K}^+$ in the topology Θ_+^{loc} if (i) $\mathfrak{A}$ is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}, (ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and (iii) $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathscr{K}^+$ in the topology Θ_+^{loc}; that is, for each $M > 0$,*

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Here, we assume that $\Theta_{0,M} = L_2(0, M; \mathbf{H})$.

Proposition 2.2. *Under hypotheses* (2)*, problem* (1) *has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathscr{K}_\varepsilon,$$

where the kernel $\mathscr{K}_\varepsilon$ is non-empty and is uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathscr{F}^b$. Recall that the spaces $\mathscr{F}_+^b$ and Θ_+^{loc} depend on ε.

The proof of this proposition almost coincides with the proof given in [8] for a particular case. The existence of an absorbing set that is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc} is proved using Lemma 2.1 similarly to [8].

We note that

$$\mathfrak{A}_\varepsilon \subset \mathscr{B}_0(R), \quad \forall \varepsilon \in (0,1),$$

where $\mathscr{B}_0(R)$ is a ball in $\mathscr{F}_+^b$ with a sufficiently large radius R. The Aubin–Lions–Simon lemma (see [11]) implies that

$$\mathscr{B}_0(R) \Subset L_2^{\text{loc}}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \quad \mathscr{B}_0(R) \Subset C^{\text{loc}}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{10}$$

Using compact inclusions (10), we strengthen the attraction to the constructed trajectory attractor.

Corollary 2.2. *For any set $\mathscr{B} \subset \mathscr{K}_\varepsilon^+$ bounded in $\mathscr{F}_+^b$, we have*

$$\operatorname{dist}_{L_2(0,M;H^{1-\delta})}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathscr{K}_\varepsilon) \to 0 \ (t \to \infty),$$
$$\operatorname{dist}_{C([0,M];H^{-\delta})}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathscr{K}_\varepsilon) \to 0 \ (t \to \infty),$$

where M is an arbitrary positive number.

3. Homogenization of attractors to a problem for reaction–diffusion equations in perforated domain

In this section, we study the limit behavior of trajectory attractors $\mathfrak{A}_\varepsilon$ of reaction–diffusion equations (1) as $\varepsilon \to 0+$ and their relation to the trajectory attractor of the corresponding homogenized equation.

To define the "strange term" (the potential in the limit equation), we consider the following problem:

$$\begin{cases} -\Delta_y v = 0, & y \in \mathbb{R}^n \setminus G_0, \\ \dfrac{\partial v}{\partial \nu_y} + b(x,y)v = b(x,y), & y \in \partial G_0, \\ v \to 0, & |y| \to \infty. \end{cases}$$

In this problem, the variable x plays the role of slow parameter. The limit potential $V(x)$ can be determined by the formula

$$V(x) = \int_{\partial G_0} \frac{\partial}{\partial \nu_y} v(x,y)\, \mathrm{d}\sigma_y. \tag{11}$$

The homogenized (limit) problem reads as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - f(u) - V(x)u + g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ u = U(x), & t = 0. \end{cases} \tag{12}$$

Clearly, problem (12) also has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to problem (12), and

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}},$$

where $\overline{\mathcal{K}}$ is the kernel of problem (12) in $\mathcal{F}^b$.

Let us formulate the main theorem regarding the initial boundary value problem for a reaction–diffusion system.

Theorem 3.1. *The following limit holds in the topological space* Θ_+^{loc}*:*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \quad \text{as } \varepsilon \to 0+. \tag{13}$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \quad \text{as } \varepsilon \to 0+ \text{ in } \Theta^{\mathrm{loc}}. \tag{14}$$

Remark 2. Recall that the spaces in the theorem depend on ε. All the functions can be continued inside the holes keeping the respective norms (see details in [12]).

The proof is based on the following. It is clear that (14) implies (13). Therefore, it is sufficient to prove (14); that is, for every neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc}, there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \quad \text{for } \varepsilon < \varepsilon_1. \tag{15}$$

Suppose that (15) is not true. Then there exist a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ $(k \to \infty)$, and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(s) \in \mathcal{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all } k \in \mathbb{N}. \tag{16}$$

The function $u_{\varepsilon_k}(s)$, $s\in\mathbb{R}$ is the solution to the problem

$$\begin{cases} \dfrac{\partial u_{\varepsilon_k}}{\partial t} = \Delta u_{\varepsilon_k} - f(u_{\varepsilon_k}) + g(x), & x\in\Omega_{\varepsilon_k}, \\ \dfrac{\partial u_{\varepsilon_k}}{\partial \nu} + \varepsilon_k^{n/(2-n)} b^j_{\varepsilon_k}(x) u_{\varepsilon_k} = 0, & x\in\partial G^j_{\varepsilon_k}, j\in\Upsilon_{\varepsilon_k}, \\ u_{\varepsilon_k} = 0, & x\in\partial\Omega, \end{cases} \tag{17}$$

on the whole time axis $t\in\mathbb{R}$. Now we prove the uniform estimate of the family of solutions (see [13] for such estimates). The ε-uniform estimate of the solution follows from the results in [14, Ch. III, §5] and [6]. More precisely, the sequence $\{u_{\varepsilon_k}(s)\}$ is bounded in $\mathscr{F}^b$, that is,

$$\begin{aligned} \|u_{\varepsilon_k}\|_{\mathscr{F}^b} &= \sup_{t\in\mathbb{R}} \|u_{\varepsilon_k}(t)\| \\ &+ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(s)\|_1^2 \,\mathrm{d}s\right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(s)\|_{\mathbf{L}_p}^p \,\mathrm{d}s\right)^{1/p} \\ &+ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\|\frac{\partial u_{\varepsilon_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^q \mathrm{d}s\right)^{1/q} \le C \quad \text{for all } k\in\mathbb{N}. \end{aligned} \tag{18}$$

Hence, there exists a subsequence $\{u_{\varepsilon'_k}(s)\}\subset\{u_{\varepsilon_k}(s)\}$, which we label the same, such that

$$u_{\varepsilon_k}(s)\to\bar{u}(s) \quad \text{as } n\to\infty \quad \text{in } \Theta^{\mathrm{loc}}, \tag{19}$$

where $\bar{u}(s)\in\mathscr{F}^b$ and $\bar{u}(s)$ satisfies (18) with the same constant C. Due to (18), we have $u_{\varepsilon_k}(s)\rightharpoonup\bar{u}(s)$ $(n\to\infty)$ weakly in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{V}_\varepsilon)$, weakly in $L_p^{\mathrm{loc}}(\mathbb{R};\mathbf{L}_{p,\varepsilon})$, and $*$-weakly in $L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$ and $\partial u_{\varepsilon_k}(s)/\partial t\rightharpoonup\partial\bar{u}(s)/\partial t$ $(k\to\infty)$ weakly in $L_{q,w}^{\mathrm{loc}}(\mathbb{R};\mathbf{H}_\varepsilon^{-r})$. We claim that $\bar{u}(s)\in\overline{\mathscr{K}}$. We have already proved that $\|\bar{u}\|_{\mathscr{F}^b}\le C$. Therefore, we have to establish that $\bar{u}(s)$ is a weak solution to (12). Using (18), we obtain that

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \Delta u_{\varepsilon_k} - g(x) \longrightarrow \frac{\partial\bar{u}}{\partial t} - \Delta\bar{u} - g(x) \quad \text{as } k\to\infty \tag{20}$$

in the space $D'(\mathbb{R};\mathbf{H}_\varepsilon^{-r})$ because the derivative operators are continuous in the space of distributions.

Since the function $f(v)$ is continuous with respect to $v\in\mathbb{R}$, we conclude that

$$f(u_{\varepsilon_k}(x,s))\to f(\bar{u}(x,s)) \quad \text{as } k\to\infty \quad \text{a.e. in } (x,s)\in\Omega\times(-M,M). \tag{21}$$

Following [5, 15], we can prove the following statement.

Lemma 3.2. *We have*

$$\left|\varepsilon^{\frac{n}{n-2}} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} b^j_\varepsilon(x)\varphi\,\mathrm{d}s - \int_\Omega V(x)\bar{\varphi}\,\mathrm{d}x\right| \le M\varepsilon\|\varphi\|_{\mathbf{H}_\varepsilon} \tag{22}$$

for $\varphi\in\mathbf{H}_\varepsilon$*, and for all* t*,*

$$\varepsilon^{\frac{n}{n-2}} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} b^j_\varepsilon(x) u_\varepsilon\psi\,\mathrm{d}s \longrightarrow \int_\Omega V(x)\bar{u}\psi\,\mathrm{d}x \tag{23}$$

as $\varepsilon\to0$ *for any* $\psi\in\mathscr{F}^b$*, where* $V(x)$ *is defined in* (11) *and the constant* M *is independent of* ε*.*

Using (20), (21), and (23) and passing to the limit in the equation of problem (17) as $k\to\infty$ in the space $D'(\mathbb{R}_+;\mathbf{H}^{-r})$, we obtain that the function $\bar{u}(x,s)$ satisfies the problem

$$\begin{cases} \dfrac{\partial\bar{u}}{\partial t} = \Delta\bar{u} - f(\bar{u}) - V(x)\,\bar{u} + g(x), & x\in\Omega, \\ \bar{u} = 0, & x\in\partial\Omega. \end{cases} \tag{24}$$

Consequently, $\bar{u} \in \overline{\mathscr{K}}$. We have proved above that $u_{\varepsilon_k}(s) \to \bar{u}(s)$ as $k \to \infty$ in Θ^{loc}. The hypothesis $u_{\varepsilon_k}(s) \notin \mathscr{O}'(\overline{\mathscr{K}})$ implies that $\bar{u} \notin \mathscr{O}'(\overline{\mathscr{K}})$; moreover, $\bar{u} \notin \overline{\mathscr{K}}$. We arrive at a contradiction. The theorem is proved.

Using compact inclusions (10), we can strengthen convergence (13).

Corollary 3.3. *For every* $0 < \delta \le 1$ *and for any* $M > 0$,

$$\text{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0, \tag{25}$$

$$\text{dist}_{C([0,M];\mathbf{H}^{-\delta})}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \quad (\varepsilon \to 0+). \tag{26}$$

To prove (25) and (26), we just repeat the proof of Theorem 3.1, replacing the topology Θ^{loc} with $L_2^{\text{loc}}(\mathbb{R}_+;\mathbf{H}^{1-\delta})$ or $C^{\text{loc}}(\mathbb{R}_+;\mathbf{H}^{-\delta})$.

Finally, we consider the reaction–diffusion equations for which the uniqueness theorem of the Cauchy problem is formulated. It is sufficient to assume that the nonlinear term $f(u)$ in (1) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \ge -C|v_1 - v_2|^2 \quad \text{for } v_1, v_2 \in \mathbb{R} \tag{27}$$

(see [7, 8]). In [7], it was proved that if (27) holds, then (1) and (12) generate the dynamical semigroups in $\mathbf{H}$, which have the global attractors $\mathscr{A}_\varepsilon$ and $\overline{\mathscr{A}}$ bounded in the space $\mathbf{V} = H_0^1(\Omega)$ (see also [9, 16]). We have

$$\mathscr{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \quad \overline{\mathscr{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Convergence (26) implies the following corollary.

Corollary 3.4. *Under the assumptions of Theorem* 3.1*, the following limit holds:*

$$\text{dist}_{\mathbf{H}^{-\delta}}(\mathscr{A}_\varepsilon, \overline{\mathscr{A}}) \to 0 \ (\varepsilon \to 0+).$$

Acknowledgments

The work of GAC and VVC was partially supported by the Russian Foundation for Basic Research (projects 18-01-00046 and 17-01-00515) and the Russian Science Foundation (project 20-11-20272). The work of KAB was supported in part by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (project AP05132071).

References

[1] V. A. Marchenko, E. Y. Khruslov, *Boundary Value Problems in Domains with Fine-Grain Boundary*, Naukova Dumka, Kiev, 1974, Russian.

[2] V. A. Marchenko, E. Y. Khruslov, *Homogenization of Partial Differential Equations*, Birkhäuser, Boston (MA), 2006.

[3] D. Cioranescu, F. Murat, "Un terme étrange venu d'ailleurs I & II", in *Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar, Volume II & III* (H. Berzis, J. L. Lions, eds.), Research Notes in Mathematics, vol. 60 & 70, Pitman, London, 1982, 98-138 & 154-178.

[4] A. G. Belyaev, A. L. Piatnitski, G. A. Chechkin, "Asymptotic behavior of a solution to a boundary value problem in a perforated domain with oscillating boundary", *Sib. Math. J.* **39** (1998), no. 4, p. 621-644, Translated from *Sib. Mat. Z.* **39** (1998) no. 4, p. 730-754).

[5] G. A. Chechkin, A. L. Piatnitski, "Homogenization of boundary–value problem in a locally periodic perforated domain", *Appl. Anal.* **71** (1999), no. 1-4, p. 215-235.

[6] A. G. Belyaev, A. L. Piatnitski, G. A. Chechkin, "Averaging in a perforated domain with an oscillating third boundary condition", *Russ. Acad. Sci. Sb. Math.* **192** (2001), no. 7, p. 933-949, Translated from *Mat. Sb.* **192** (2001) no. 7, p. 3-20).

[7] V. V. Chepyzhov, M. I. Vishik, "Trajectory attractors for reaction–diffusion systems", *Top. Meth. Nonlin. Anal. J. Julius Schauder Center* **7** (1996), no. 1, p. 49-76.

[8] V. V. Chepyzhov, M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence (RI), 2002.

[9] A. V. Babin, M. I. Vishik, *Attractors of Evolution Equations*, North–Holland, Amsterdam, 1992 (Moscow: Nauka; 1989).

[10] K. A. Bekmaganbetov, G. A. Chechkin, V. V. Chepyzhov, "Weak convergence of attractors of reaction–diffusion systems with randomly oscillating coefficients", *Appl. Anal.* **98** (2019), no. 1-2, p. 256-271.
[11] F. Boyer, P. Fabrie, "Mathematical tools for the study of the incompressible Navier–Stokes equations and related models", in *Applied Mathematical Sciences*, vol. 183, Springer, New York (NY), 2013.
[12] V. V. Jikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Verlag, Berlin–New York, 1994.
[13] É. Sanchez-Palencia, *Homogenization Techniques for Composite Media*, Springer–Verlag, Berlin, 1987.
[14] V. P. Mikhailov, *Partial Differential Equations*, Mir, Moscow, 1978.
[15] J. I. Diaz, D. Gomez-Castro, T. A. Shaposhnikova, M. N. Zubova, "Classification of homogenized limits of diffusion problems with spatially dependent reaction over critical-size particles", *Appl. Anal.* **98** (2018), no. 1-2, p. 232-255.
[16] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematics Series, vol. 68, Springer-Verlag, New York (NY), 1988.