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Lower bound estimates of blow-up time for a quasilinear hyperbolic equation with superlinear sources

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Abstract. This paper deals with the lower bound for blow-up solutions to a quasilinear hyperbolic equation with strong damping. An inverse Hölder inequality with a correction constant is employed to overcome the difficulty caused by the failure of the embedding inequality. Moreover, a lower bound for blow-up time is obtained by constructing a new control functional with a small dissipative term and by applying an inverse Hölder inequality as well as energy inequalities. This result gives a positive answer to the open problem presented in [1].

Keywords. Inverse Hölder inequality, Energy estimate method, Energy inequality, Lower bound estimate, Quasilinear hyperbolic equation.

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1. Introduction

In this paper, the following quasilinear hyperbolic equation with strong damping is studied:

$$\begin{cases} u_{tt} - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) - \Delta u_t = |u|^{q(x,t)-2}u, & (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $T > 0$. It will be assumed throughout this paper that the exponents $p(x,t)$ and $q(x,t)$ satisfy the following conditions:

$$2 \leqslant p^- \leqslant p(x,t) \leqslant p^+ < \infty, \quad 1 < q^- \leqslant q(x,t) \leqslant q^+ < \infty.$$

Problem (1.1) models many physical problems such as viscoelastic fluids, electrorheological fluids, processes of filtration through porous media, fluids with temperature-dependent viscosity, and so on. The interested reader may refer to [2–4] and the references therein. In the case where

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p, q are fixed constants, many authors discussed the existence of solutions, finite-time blow-up of solutions for low initial energy and arbitrarily high initial energy, and some estimate of a lower bound for blow-up times. The interested reader may refer to [5–12]. In the case where p, q are continuous functions, S. N. Antontsev [13, 14] studied the following problem:

$$\begin{cases} u_{tt} = \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u) + \alpha\Delta u_t + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t), & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Antontsev proved the existence and the blow-up of weak solutions for negative initial energy. Later, Guo–Gao [15] discussed the blow-up properties of solutions to the above problems for the case where the initial energy is positive. In addition, Messaoudi and Talahmeh [16, 17] discussed blow-up properties of solutions to Problem (1.2) in the absence of a strong damping term.

It is well known that the source term causes finite-time blow-up of the solution while the damping term may drive the equation toward stability. Therefore, it is of interest to explore the mechanism of how sources dominate the dissipation (the damping term Δu_t), which has attracted considerable attention. In fact, the upper bound ensures the occurrence of blow-up while the lower bound may provide us a safe time interval for operation when we use Problem (1.1) to model a physical process. Hence, it is more interesting to give a lower bound estimate for hyperbolic problems than to give a upper bound. In 2017, Guo [1] applied the modified version of the Gagliardo–Nirenberg inequality for non-constant cases and energy inequalities to obtain some estimates of lower bounds for blow-up time in the case where $2 < p^- < q^+ < p^-(1 + (2 + p^{-*})/2N)$ with $p^{-*} = (Np^-/(N - p^-))(2 < p^- < N)$. In particular, Remark 1.1 of [1] gives an unsolved problem, namely, as follows.

Remark 1.1. Since $p \in [p^-(1 + (2 + p^{-*})/2N), p^{-*}]$, it seems that we cannot obtain results similar to those of Lemma 1.5 [1] unless we may obtain more information about $\|u_t\|_2$. Therefore, we need to develop a new method or technique to discuss this problem.

In this paper, we first follow along the lines of the proof of Lemma 1.3 [1] to obtain an inverse Hölder inequality with correction constants in the case where p lies in $[p^-(1+(2+p^{-*})/2N), p^{-*}]$. Second, we construct a new control functional with a small dissipative term and then apply the inverse Hölder inequality as well as energy inequalities to establish a differential inequality. Finally, we obtain an estimate of lower bounds for blow-up time.

This paper is organized as follows. First, in Section 2, we present some preliminaries. Section 3 is devoted to giving an estimation of a lower bound.

2. Preliminaries

Define the energy functional as

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \int_\Omega \frac{1}{p(x,t)}|\nabla u|^{p(x,t)}\,\mathrm{d}x - \int_\Omega \frac{1}{q(x,t)}|u|^{q(x,t)}\,\mathrm{d}x.$$

For simplicity, we give some notation and the embedding inequality to be used later. By Corollary 3.34 in [3], we know that $W_0^{1,p(x,0)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^r(\Omega)(1 < r \leqslant (Np^-/(N - p^-)))$. Let B be the best constant of the embedding inequality

$$\|u\|_r \leqslant B\|\nabla u\|_{p(\cdot)}, \quad \forall\ u \in W_0^{1,p(x,0)}(\Omega). \tag{2.1}$$

Set $E_1 = (q^+ - p^+)/q^+p^+\alpha_1, \alpha_1 = B_1^{(p^+q^+)/(p^+-q^+)}$, where $B_1 = \max\{B, 1\}$. The following conclusions are presented to shorten the statement of our main results and their proofs.

Lemma 2.1 ([15]). *Suppose that $u \in L^{q(x,t)}(Q_T) \cap L^\infty(0,T;W_0^{1,p(x,t)}(\Omega))$, and $u_t \in L^2(0,T;H^1(\Omega))$ is a solution to Problem (1.1). Then $E(t)$ satisfies the identity*

$$E(t) + \int_0^t \int_\Omega |\nabla u_s|^2 \,\mathrm{d}x\,\mathrm{d}s = E(0) + \int_0^t \int_\Omega \frac{p_s(\cdot)}{p^2(\cdot)} |\nabla u|^{p(\cdot)} \left(\ln|\nabla u|^{p(\cdot)} - 1\right) \mathrm{d}x\,\mathrm{d}s$$
$$- \int_0^t \int_\Omega \frac{q_s(\cdot)}{q^2(\cdot)} |u|^q (\ln|u|^q - 1)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.2}$$

Theorem 2.1 ([15]). *Assume that the initial data (u_0,u_1) and the exponents $p(x,t)$ and $q(x,t)$ satisfy the following conditions:*

$$(H_1)\ u_0 \in W_0^{1,p(x,0)}(\Omega), \quad u_1 \in L^2(\Omega), \quad E(0) + \frac{|\Omega|}{p^-} + \frac{|\Omega|}{q^-} < E_1,$$
$$\min\left\{\|\nabla u_0\|_{p(x,0)}^{p^-}, \|\nabla u_0\|_{p(x,0)}^{p^+}\right\} > \alpha_1;$$
$$(H_2)\ \max\{2,p^+\} < q^- \leqslant q(x,t) \leqslant q^+ < \frac{Np^-}{N-p^-}, \quad \forall\, x \in \Omega, t \geqslant 0;$$
$$(H_3)\ p_t \leqslant 0, \quad q_t \geqslant 0, \quad \left|\frac{p_t}{p^2}\right| + \left|\frac{q_t}{q^2}\right| \in L^1_{loc}((0,\infty);L^1(\Omega)).$$

Then the solution to Problem (1.1) is not global.

Some ideas of this proof of Theorem 2.1 mainly come from the pioneering work of Levine [6,18] (see also the work of Ball [19]). For more details, the reader may refer to [15].

Lemma 2.2 ([15]). *If u is the solution to Problem (1.1) and (H_3) is satisfied, then the energy functional $E(t)$ satisfies*

$$E(t) + \int_0^t \int_\Omega |\nabla u_s|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant E(0) + \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|\Omega| := E_2, \quad t \geqslant 0. \tag{2.3}$$

Lemma 2.3 ([1]). *Assume that u is the solution to Problem (1.1) and condition (H_1) is fulfilled. Then there exists a positive constant C depending on $|\Omega|$, p^-, N, and B_1 such that for any $k > (N(q^+ - p^-))/p^-$,*

$$\int_\Omega \frac{1}{q(\cdot)} |u|^{q(\cdot)} \,\mathrm{d}x \leqslant \frac{1}{q^- - p^+} \max\{C^{\mu(k)}, C^{\nu(k)}\} \max\left\{\left(\int_\Omega |u|^k \,\mathrm{d}x\right)^{\alpha(k)}, \left(\int_\Omega |u|^k \,\mathrm{d}x\right)^{\beta(k)}\right\}$$
$$+ \frac{p^+}{q^- - p^+}\left(E_2 + \frac{|\Omega|}{q^-}\right). \tag{2.4}$$

Here, μ, ν, α, and β are defined as follows:

$$\mu(k) = \begin{cases} \dfrac{N(q^+ - k)}{kp^- - N(q^+ - p^-)}, & k < q^+; \\ 1 - \dfrac{q^+}{k}, & k \geqslant q^+. \end{cases}$$

$$\nu(k) = \begin{cases} \dfrac{Np^-(q^+ - k)}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)}, & k < q^+; \\ 1 - \dfrac{q^+}{k}, & k \geqslant q^+. \end{cases}$$

$$\alpha(k) = \begin{cases} \dfrac{Np^- - q^+(N - p^-)}{kp^- - N(q^+ - p^-)}, & k < q^+; \\ \dfrac{q^+}{k}, & k \geqslant q^+. \end{cases}$$

$$\beta(k) = \begin{cases} \dfrac{[Np^- - q^+(N-p^-)]p^+}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)}, & k < q^+; \\ \dfrac{q^+}{k}, & k \geqslant q^+. \end{cases}$$

In fact, when $k < q^+$, we follow along the lines of the proof of Lemma 1.3 [1] to obtain the above conclusions. When $k \geqslant q^+$, we apply condition (H_1) to prove that there exists a positive constant α_2 depending on $E(0), B_1$ such that the term $\|u\|_{q(\cdot)}$ is bigger than α_2. Then, we apply some inequalities $\min\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\} \leqslant \int_\Omega |u|^{q(\cdot)}\,\mathrm{d}x \leqslant \max\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\}$ and $\|u\|_{q(\cdot)} \leqslant (1+|\Omega|)\|u\|_{q^+}$ to finish our proof.

3. Lower bound estimates

In this section, we give our main results and their proof.

Theorem 3.1. *If $(N(p^-+2))/(2(N-p^-)) < q^+ < ((2N-p^-+2)p^-)/(2(N-p^-))$, then the blow-up time T^* satisfies the estimate*

$$\int_{F(0)}^{+\infty} \frac{1}{C_4 y^{2-\frac{2}{\theta}} + C_5 y^{2-\frac{2}{\theta}+\lambda} + C_6 y^{2-\frac{2}{\theta}+2\lambda}}\,\mathrm{d}y \leq T^*.$$

Here, the constants C_4 and C_5 and the initial data $F(0)$ are defined in (3.11) and

$$\lambda = \frac{N+2}{2(N-p^-)}, \quad \theta = \frac{2(Np^- - q^+N + q^+p^-)(N-p^-)}{(N+2)p^-p^- - 2N(q^+-p^-)(N-p^-)}.$$

Proof. This proof is divided into three steps.

Step 1. Equivalent of blow-up. Define

$$H(t) = \left(\int_\Omega |u|^k\,\mathrm{d}x\right)^\theta - \frac{1}{2M}\int_0^t\int_\Omega |\nabla u_\tau|^2\,\mathrm{d}x\,\mathrm{d}\tau,$$

where

$$\theta = \frac{2(Np^- - q^+N + q^+p^-)(N-p^-)}{(N+2)p^-p^- - 2N(q^+-p^-)(N-p^-)}, \quad M = \frac{1}{q^- - p^+}\max\{C^{\mu_1}, C^{\nu_1}\}2^{\theta-1},$$

$$\mu_1 = \frac{N(q^+-k)}{kp^- - N(q^+-p^-)}, \quad \nu_1 = \frac{Np^-(q^+-k)}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+-p^+)}, \quad k = \frac{N(p^-+2)}{2(N-p^-)}.$$

By Lemmas 2.2 and 2.3 and the definition of $E(t)$, we have

$$\int_0^t\int_\Omega |\nabla u_\tau|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq E_2 + \int_\Omega \frac{1}{q(\cdot)}|u|^{q(\cdot)}\,\mathrm{d}x \leq M\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^\theta + C_1, \tag{3.1}$$

where

$$C_1 = M + \frac{p^+}{q^- - p^+}\left(E_2 + \frac{|\Omega|}{q^-}\right) + E_2.$$

The definition of $H(t)$ and Inequality (3.1) yield

$$H(t) \geq \frac{1}{2}\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^\theta - \frac{C_1}{2M}. \tag{3.2}$$

Combining the conclusion of Theorem 1.7 [1] with Inequality (3.2), we have

$$\lim_{t\to T^*} H(t) = +\infty. \tag{3.3}$$

Step 2. A first-order differential inequality. A simple computation shows that

$$H'(t) = \theta k\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta-1}\int_\Omega |u|^{(k-2)}uu_t\,\mathrm{d}x - \frac{1}{2M}\int_\Omega |\nabla u_t|^2\,\mathrm{d}x. \tag{3.4}$$

By using the Hölder inequality, the Sobolev embedding theorem, and the Young inequality, it is not hard to verify that

$$
\begin{aligned}
H'(t) &\le \theta k\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta-1}\left(\int_\Omega |u|^{(k-1)\frac{2N}{N+2}}\,\mathrm{d}x\right)^{\frac{N+2}{2N}}\left(\int_\Omega |u_t|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} - \frac{1}{2M}\int_\Omega |\nabla u_t|^2\,\mathrm{d}x\\
&\le C\theta k\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta-1}\left(\int_\Omega |u|^{(k-1)\frac{2N}{N+2}}\,\mathrm{d}x\right)^{\frac{N+2}{2N}}\left(\int_\Omega |\nabla u_t|^{2}\,\mathrm{d}x\right)^{\frac{1}{2}} - \frac{1}{2M}\int_\Omega |\nabla u_t|^2\,\mathrm{d}x\\
&\le \frac{MC^2}{2}\left[\theta k\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta-1}\left(\int_\Omega |u|^{(k-1)\frac{2N}{N+2}}\,\mathrm{d}x\right)^{\frac{N+2}{2N}}\right]^2 + \frac{1}{2M}\int_\Omega |\nabla u_t|^2\,\mathrm{d}x - \frac{1}{2M}\int_\Omega |\nabla u_t|^2\,\mathrm{d}x\\
&\le \frac{MC^2}{2}\left[\theta k\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta-1}\left(\int_\Omega |u|^{(k-1)\frac{2N}{N+2}}\,\mathrm{d}x\right)^{\frac{N+2}{2N}}\right]^2,
\end{aligned}
\tag{3.5}
$$

where the constant C is the best embedding constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{(2N/N-2)}(\Omega)$. In addition, noting that $(2N(k-1))/(N+2) \le p^{-*}$ and applying embedding inequality (2.1), Lemmas 2.2 and 2.3, and the definition of $E(t)$, we have

$$
\begin{aligned}
\left(\int_\Omega |u|^{\frac{2N(k-1)}{N+2}}\,\mathrm{d}x\right)^{\frac{N+2}{2N}} &\le B\|\nabla u\|_{p(\cdot)}^{k-1} \le B\max\left\{\left(\int_\Omega |\nabla u|^{p(\cdot)}\,\mathrm{d}x\right)^{\frac{k-1}{p^-}}, \left(\int_\Omega |\nabla u|^{p}(\cdot)\,\mathrm{d}x\right)^{\frac{k-1}{p^+}}\right\}\\
&\le B\max\left\{\left(p^+E_2 + \int_\Omega \frac{p^+}{q(\cdot)}|u|^{q(\cdot)}\,\mathrm{d}x\right)^{\frac{k-1}{p^-}}, \left(p^+E_2 + \int_\Omega \frac{p^+}{q(\cdot)}|u|^{q(\cdot)}\,\mathrm{d}x\right)^{\frac{k-1}{p^+}}\right\}\\
&\le B\max\left\{1, M_1^{\frac{k-1}{p^+}-\frac{k-1}{p^-}}\right\}\left(M_1 + \int_\Omega \frac{p^+}{q(\cdot)}|u|^{q(\cdot)}\,\mathrm{d}x\right)^{\frac{k-1}{p^-}}\\
&\le B\max\left\{1, M_1^{\frac{k-1}{p^+}-\frac{k-1}{p^-}}\right\}\left(M_2 + p^+M\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\theta}\right)^{\frac{k-1}{p^-}}\\
&\le C_2 + C_3\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\frac{(k-1)\theta}{p^-}},
\end{aligned}
\tag{3.6}
$$

where the constants C_i $(i = 2, 3)$ are defined as follows:

$$
C_2 = 2^{\frac{k-1}{p^-}}\left(p^+E_2 + 2^\theta p^+M + \frac{p^+E_2}{q^- - p^+} + \frac{p^+|\Omega|}{(q^- - p^+)p^-}\right)^{\frac{k-1}{p^-}} B\max\left\{1, (p^+E_2)^{\frac{k-1}{p^+}-\frac{k-1}{p^-}}\right\},
$$
$$
C_3 = B\max\left\{1, (p^+E_2)^{\frac{k-1}{p^+}-\frac{k-1}{p^-}}\right\}(2^\theta p^+M)^{\frac{k-1}{p^-}}.
$$

Therefore, inserting (3.6) into (3.5), we get

$$
H'(t) \leqslant \frac{MC^2\theta^2k^2}{2}\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{2(\theta-1)}\left[C_2 + C_3\left(\int_\Omega |u|^k\,\mathrm{d}x\right)^{\frac{(k-1)\theta}{p^-}}\right]^2. \tag{3.7}
$$

Step 3. A lower bound for blow-up time.

By using Inequality (3.2), (3.7) is equivalent to the inequality

$$
H'(t) \leqslant \frac{MC^2\theta^2k^2}{2}\left(2H(t) + \frac{C_1}{M}\right)^{2(1-\frac{1}{\theta})}\left[C_2 + C_3\left(2H(t) + \frac{C_1}{M}\right)^{\frac{k-1}{p^-}}\right]^2. \tag{3.8}
$$

Furthermore, a simple computation indicates that Inequality (3.8) may be rewritten as

$$
\left(2H(t) + \frac{C_1}{M}\right)' \leqslant MC^2\theta^2k^2\left(2H(t) + \frac{C_1}{M}\right)^{2(1-\frac{1}{\theta})}\left[C_2 + C_3\left(2H(t) + \frac{C_1}{M}\right)^{\frac{k-1}{p^-}}\right]^2. \tag{3.9}
$$

Setting $F(t)=2H(t)+C_1/M$, we have

$$F'(t) \leqslant MC^2\theta^2k^2F^{2(1-\frac{1}{\theta})}(t)\left[C_2+C_3F^{\frac{k-1}{p^-}}(t)\right]^2$$
$$:= C_4F^{2(1-\frac{1}{\theta})}(t)+C_5F^{2(1-\frac{1}{\theta})+\frac{k-1}{p^-}}(t)+C_6F^{2(1-\frac{1}{\theta})+\frac{2(k-1)}{p^-}}(t), \tag{3.10}$$

where

$$C_4=MC^2k^2\theta^2C_2^2,\quad C_5=2MC^2\theta^2k^2C_2C_3,$$
$$C_6=MC^2\theta^2k^2C_3^2,\quad F(0)=2\left(\int_\Omega |u_0|^k\,\mathrm{d}x\right)^\theta+\frac{C_1}{M}. \tag{3.11}$$

Equation (3.10) implies

$$\int_{F(0)}^{+\infty}\frac{1}{C_4y^{2-\frac{2}{\theta}}+C_5y^{2-\frac{2}{\theta}+\frac{k-1}{p^-}}+C_6y^{2-\frac{2}{\theta}+\frac{2(k-1)}{p^-}}}\,\mathrm{d}y\le T^*.$$

This completes the proof of this theorem. □

Remark 3.1. The fact

$$\frac{(2N-p^-+2)p^-}{2(N-p^-)}-p^-\left(1+\frac{2+p^{-*}}{2N}\right)=\frac{p^-p^-}{N(N-p^-)}>0$$

shows that the result of this paper gives a positive answer to the unsolved problem in [1]. However, when q^+ lies in the interval $[((2N-p^-+2)p^-)/(2(N-p^-)),p^{-*}]$, due to technical reasons, at present, we cannot give any answer.

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