



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mécanique

Julie Diani

Free vibrations of linear viscoelastic polymer cantilever beams


Volume 348, issue 10-11 (2020), p. 797-806.

<https://doi.org/10.5802/crmeca.15>

Part of the Thematic Issue: Contributions in mechanics of materials

Guest editors: Julie Diani, Olivier Castelnau and Francisco Chinesta

© Académie des sciences, Paris and the authors, 2020.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mécanique sont membres du
Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org



Contributions in mechanics of materials

Free vibrations of linear viscoelastic polymer cantilever beams

Julie Diani^a

^a LMS, CNRS UMR 7649, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau, France

E-mail: julie.diani@polytechnique.edu

Abstract. The free vibrations of cantilever slender beams of polymers, which are viscoelastic materials, are theoretically described using the simple Euler–Bernoulli assumption. The comparison between the theory and the experimental data collected for a thermoplastic elastomer, polyether block amide, shows very satisfactory results. Consequently, the theory is used for a thoughtful analysis of the impact of the material parameters and the beam geometry on its free vibration. Finally, the comparison of the dynamic behaviors of two polymers, using the free vibration test and a simple uniaxial tension/relaxation test, is discussed.

Résumé. Les vibrations libres d'une poutre encastree de polymère, mesurées expérimentalement sont reproduites théoriquement à l'aide de l'hypothèse des poutres d'Euler–Bernoulli, une fois le comportement viscoélastique du matériau identifié classiquement. La théorie permet alors de simplement faire varier les paramètres matériaux et géométriques de la poutre afin de tester leurs impacts sur le test de vibration libre. En utilisant l'analyse théorique et en observant la réponse de deux matériaux lors d'un essai simple de traction/relaxation, il est possible de prédire leur comportement relatif en vibration libre.

Keywords. Polymer, Viscoelasticity, Vibration, Damping, Cantilever beam.

2020 Mathematics Subject Classification. 00X99.

Funding. This work benefited from the support of the chair “Modelling advanced polymers for innovative material solutions” led by the Ecole polytechnique (l’X) and the Fondation de l’Ecole polytechnique and sponsored by Arkema.

Manuscript received 17th April 2020, revised 12th June 2020, accepted 15th June 2020.

1. Introduction

In shoe sole applications, polymers are preferred for various reasons such as comfort, lightness, resistance to wear, cushioning effect, and so on. Good shock absorption is often desired, but in some sport applications, like running, elastic energy recovery is also crucial to providing good bouncing. For this reason, sport brands together with polymer companies are looking for materials offering a perfect compromise. To compare materials, the free vibration of a slender cantilever beam was introduced by Arkema as a characterization test. To obtain a quantitative analysis of this test, the vibration of a linear viscoelastic cantilever beam was calculated using the Euler–Bernoulli beam theory. The theory was first validated on actual experimental data and

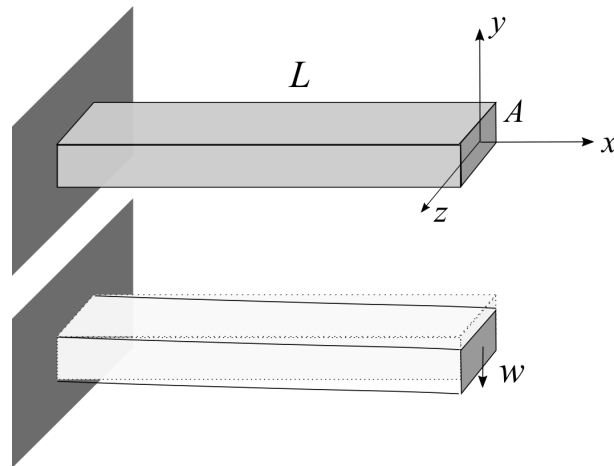


Figure 1. Free vibration test: the cantilever beam is subjected to an initial vertical displacement at $x = L$ and then let go.

then used to understand better the impact of the material viscoelastic parameters and of the beam geometry on its free vibration. Note that most contributions focusing on the vibrations of viscoelastic cantilever beams deal with theory (see [1–4]) without any experimental validation or application perspective. However, [5] provides a comparison between experiments and the Euler–Bernoulli theory for an excited laminated beam whose behavior is defined by a simple Kelvin–Voigt model.

In Section 2, the basic equations are briefly recalled for elastic and linear viscoelastic materials. In Section 3, the theory is validated on experimental data recorded on metal and polymer beams. Then, the impacts of the material parameters and of the beam geometry are analyzed. Finally, a simple uniaxial tension/relaxation characterization is used to predict the performance of different materials during the free vibration test.

2. Theory

2.1. Elastic problem

Let us consider a homogeneous elastic beam of length L , uniform cross-section A , mass density ρ , and Young modulus E (Figure 1). Using the Euler–Bernoulli beam theory [6], the equation of motion of the beam is given by

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0, \quad 0 \leq x \leq L, \quad (1)$$

where I is the second moment of area of the beam cross-section. Seeking for a solution of the form $w(x, t) = \varphi(x)\psi(t)$, Equation (1) transforms into

$$-\frac{EI}{\rho A \varphi(x)} \frac{\partial^4 \varphi(x)}{\partial x^4} = \frac{1}{\psi(t)} \frac{\partial^2 \psi(t)}{\partial t^2} = -\omega^2 \quad (2)$$

with ω being constant. Therefore, the problem consists in solving a system of two differential equations

$$\begin{cases} \frac{\partial^4 \varphi(x)}{\partial x^4} - \delta^4 \varphi(x) = 0, & \delta^4 = \frac{\rho A \omega^2}{EI} \\ \frac{\partial^2 \psi(t)}{\partial t^2} + \omega^2 \psi(t) = 0 \end{cases} \quad (3)$$

of general solution

$$\begin{cases} \varphi(x) = \alpha_1 \sinh \delta x + \alpha_2 \cosh \delta x + \alpha_3 \sin \delta x + \alpha_4 \cos \delta x \\ \psi(t) = \alpha_5 \sin \omega t + \alpha_6 \cos \omega t \end{cases} \quad (4)$$

with parameters $\alpha_1, \alpha_2, \alpha_3,$ and α_4 being determined by the boundary conditions and parameters α_5 and α_6 being defined by the initial conditions. For a cantilever beam clamped at one end ($x = 0$) and free at the other end ($x = L$), the boundary conditions can be written as

$$w(0, t) = 0, \quad \frac{\partial w}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial^2 w}{\partial x^2} \Big|_{x=L} = 0, \quad \frac{\partial^3 w}{\partial x^3} \Big|_{x=L} = 0. \quad (5)$$

Introducing these boundary conditions into (4), parameters α_i satisfy

$$\alpha_1 + \alpha_3 = 0, \quad \alpha_2 + \alpha_4 = 0 \quad (6)$$

$$\begin{pmatrix} \sinh \delta L + \sin \delta L & \cosh \delta L + \cos \delta L \\ \cosh \delta L + \cos \delta L & \sinh \delta L - \sin \delta L \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (7)$$

and the non-trivial solution is written as

$$\cos \delta_n L \cosh \delta_n L = -1. \quad (8)$$

The first few roots have already been calculated: $\delta_1 L = 1.87510, \delta_2 L = 4.69409,$ and $\delta_3 L = 7.85340$ (see for instance [4] for a report of solutions for different boundary conditions).

Let us focus on the first mode of vibration. For the initial conditions $w(L, 0) = \Delta_i$ and $\partial w / \partial t|_{t=0} = 0,$ the solution is written as

$$\begin{aligned} w(x, t) = & \frac{\Delta_i}{\cosh \delta_1 L - \cos \delta_1 L + (\sin \delta_1 L - \sinh \delta_1 L) \frac{\cos \delta_1 L + \cosh \delta_1 L}{\sinh \delta_1 L + \sin \delta_1 L}} \\ & \times \cos \left(\sqrt{\frac{EI}{\rho A}} \delta_1^2 t \right) \left(\cosh \delta_1 x - \cos \delta_1 x + (\sin \delta_1 x - \sinh \delta_1 x) \frac{\cos \delta_1 x + \cosh \delta_1 x}{\sinh \delta_1 x + \sin \delta_1 x} \right). \end{aligned} \quad (9)$$

In order to compare the solution to an actual experimental result obtained on an elastic beam, one needs to calculate the solution, taking into account air friction and grip system friction. In such a case, Equation (1) transforms into

$$EI \frac{\partial^4 w}{\partial x^4} + c_f \frac{\partial w}{\partial t} + \rho A \frac{\partial^2 w}{\partial t^2} = 0, \quad 0 \leq x \leq L. \quad (10)$$

Variable separation still applies and by writing the solution as $w(x, t) = e^{j\omega t} \varphi(x)$ (with $j^2 = -1$), one obtains

$$EI \frac{\partial^4 \varphi(x)}{\partial x^4} + (c_f j \omega - \rho A \omega^2) \varphi(x) = 0 \quad (11)$$

$$\Leftrightarrow \frac{\partial^4 \varphi(x)}{\partial x^4} - \delta^4 \varphi(x) = 0 \quad \text{with } \delta^4 = \frac{\rho A \omega^2 - c_f j \omega}{EI}. \quad (12)$$

The solution of the latter equation has been defined in (3), with values of δ_n still being the roots of (8). Therefore, for each vibration mode $\delta_n,$ the value of ω_n is obtained by solving

$$\rho A \omega_n^2 - c_f j \omega_n - EI \delta_n^4 = 0. \quad (13)$$

Note that the values of ω_n are now complex. Focusing on the first mode δ_1 only and considering the same boundary and initial conditions, the free vibration of the cantilever beam is written as

$$\begin{aligned} w(x, t) = & e^{-\text{Im}(\omega_1)t} \cos(\text{Re}(\omega_1)t) \\ & \times \frac{\Delta_i}{\cosh \delta_1 L - \cos \delta_1 L + (\sin \delta_1 L - \sinh \delta_1 L) \frac{\cos \delta_1 L + \cosh \delta_1 L}{\sinh \delta_1 L + \sin \delta_1 L}} \\ & \times \left(\cosh \delta_1 x - \cos \delta_1 x + (\sin \delta_1 x - \sinh \delta_1 x) \frac{\cos \delta_1 x + \cosh \delta_1 x}{\sinh \delta_1 x + \sin \delta_1 x} \right). \end{aligned} \quad (14)$$

2.2. Linear viscoelastic materials

The viscoelastic behavior of a polymer at infinitesimal strain is well described by a generalized Maxwell model. The latter consists of a parallel scheme of an elastic branch characterized by Young modulus E_∞ and viscoelastic branches defined by relaxation times and associated Young moduli (τ_i, E_i) . The vibration problem (10) is now written as [7, 8]

$$I \int_0^t \left(E_\infty + \sum_i E_i e^{-\frac{t-\tau}{\tau_i}} \right) \frac{\partial}{\partial \tau} \left[\frac{\partial^4 w(x, \tau)}{\partial x^4} \right] d\tau + c_f \frac{\partial w}{\partial t} + \rho A \frac{\partial^2 w}{\partial t^2} = 0, \quad 0 \leq x \leq L. \quad (15)$$

Then, introducing the solution of the form $w(x, t) = e^{j\omega t} \varphi(x)$ leads to the differential equation

$$I \left(E_\infty + \sum_i E_i \frac{j\omega\tau_i}{1+j\omega\tau_i} \right) \frac{\partial^4 \varphi(x)}{\partial x^4} + (c_f j\omega - \rho A \omega^2) \varphi(x) = 0 \quad (16)$$

$$\Leftrightarrow \frac{\partial^4 \varphi(x)}{\partial x^4} - \delta^4 \varphi(x) = 0, \quad \text{with } \delta^4 = \frac{\rho A \omega^2 - c_f j\omega}{I \left(E_\infty + \sum_i E_i \frac{j\omega\tau_i}{1+j\omega\tau_i} \right)}. \quad (17)$$

Therefore, the problem to solve is similar to the previous case, and solution (14) still holds with ω_1 satisfying

$$\rho A \omega_1^2 - c_f j\omega_1 - I \left(E_\infty + \sum_i E_i \frac{j\omega_1\tau_i}{1+j\omega_1\tau_i} \right) \delta_1^4 = 0. \quad (18)$$

The model is now tested against experimental data recorded on metal and polymer beams.

3. Model validation

3.1. Free vibration of a metal elastic beam

A metal reglet of thickness $T = 0.5$ mm, width $W = 13$ mm, and length $L = 65$ mm was subjected to a free vibration test. After applying an initial displacement $\Delta_i = -1.5$ mm at its free end ($x = L$), the reglet was set free from vibrations. The vertical motion of the tip of the reglet was measured with respect to time by a laser beam. Although the Young modulus of the reglet was unknown, it was easily estimated thanks to the vibration frequency; a realistic value of $E = 121$ GPa was obtained. The air friction and the clamp system damping were taken into account with the parameter c_f , which is fitted on the experimental data. A very satisfactory comparison between the experiment and the theory equation (14) is displayed in Figure 2, showing the relevance of the theory including the Euler–Bernoulli assumption.

3.2. Free vibration of a polymer beam

The thermoplastic elastomer commercialized by Arkema under the reference PEBAX®4033 was used for experimental testing. The linear viscoelasticity of the polymer was characterized using dynamic mechanical analysis in torsion. Rectangular specimens were subjected to torsion frequency sweeps, from 10^{-2} to 10 Hz, at given temperatures, from -80 °C to 80 °C with 5 °C temperature increments, using an Anton Paar MCR 502 rheometer. The time–temperature superposition assumption [9] was successfully applied. The experimental master curves for the material built at 20 °C reference temperature, as well as the linear viscoelastic fit by a generalized Maxwell model using 40 viscoelastic branches, are displayed in Figure 3. The values of the relaxation spectrum are given in Appendix A. Moreover, the elastic shear modulus at high temperature, G_∞ , was recorded at 21 MPa. Note that the values of the shear moduli (G_∞, G_i) were then multiplied by a correction parameter $x = 0.95$ depending on the specimen geometry, rationally defined in [10] to account for the grip clamping when applying the Saint-Venant assumption [11] for rectangular

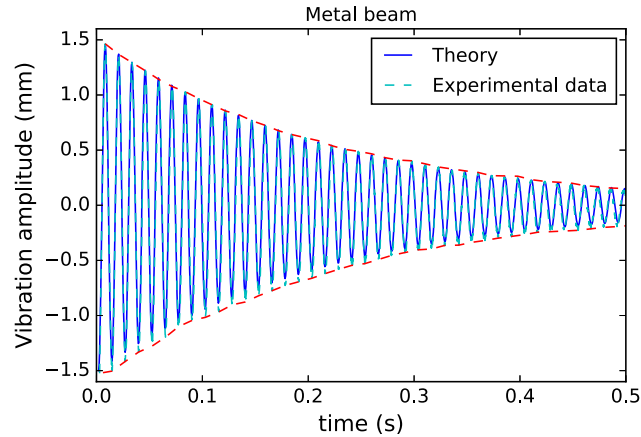


Figure 2. Comparison between the Euler–Bernoulli beam theory and the experimental data for the free vibration of a metal reglet (elastic material), taking into account air friction and clamp damping through the fit coefficient c_f .

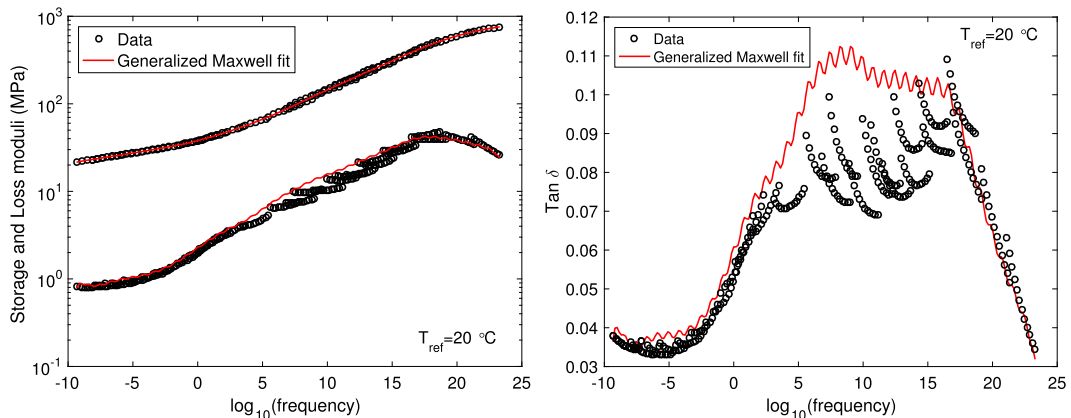


Figure 3. Linear viscoelasticity master curves for PEBAX®4033 at 20 °C and generalized Maxwell model fit.

specimens in torsion. Finally, the Young moduli E_∞ and E_i are simply assumed as thrice the values of G_∞ and G_i , respectively. Although this assumption is exact for E_∞ and some values of E_i , it probably overestimates some other E_i values. Nonetheless, the comparison between the free vibration experimental results and the theory will prove that this assumption is reasonable.

A rectangular beam of thickness $T = 2.1$ mm, width $W = 10$ mm, and length $L = 65$ mm of PEBAX®4033 was subjected to the free vibration test. Theoretical solutions were calculated using (14) with the initial displacement and the characterized linear viscoelastic behavior of the polymer as inputs. A very satisfactory comparison between the model and the experimental data is shown in Figure 4. Note that the theoretical solution was calculated with damping parameter c_f set to zero, assessing the first order of the material viscoelasticity in the vibration damping of the polymer beam.

The simple Euler–Bernoulli theory is relevant to reproducing the experimental data obtained with the free vibration experiment for the cantilever beam. Therefore, the theory will be used to discuss the impact of the material behavior and of the beam geometry on its free vibration.

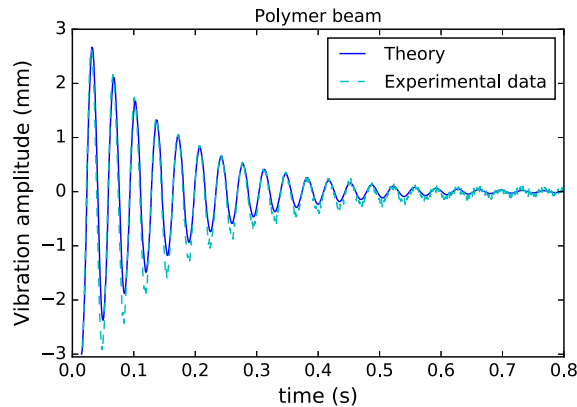


Figure 4. Comparison between the Euler–Bernoulli beam theory and the experimental data for the free vibration of a PEBAX@4033 polymer beam.

4. Analysis and discussion

4.1. Impact of material parameters

For the purpose of simplicity, the polymer beam geometry is considered as reference. To better understand the impact of material parameters on the free vibration test, a simple Zener model is considered for the viscoelastic behavior, consisting of an elastic branch characterized by stiffness E_∞ in parallel with a viscous branch characterized by stiffness E_v and relaxation time τ_v . For practical purposes, the reference parameters are set to $E_\infty = 80$ MPa, $E_v = 20$ MPa, and $\tau_v = 0.002$ s. This relaxation time is chosen to obtain a realistic duration for total damping in comparison to the experimental data displayed in Figure 4. Finally, since the air friction and clamp damping are observed to be of second order for the PEBAX® elastomer, the friction parameter is set to zero ($c_f = 0$). The beam is theoretically subjected to an initial vertical displacement approximately -3 mm as in the experiment. The free vibration is plotted with respect to time according to the material parameters.

First, several values of E_∞ and E_v are considered while keeping the relaxation time τ_v constant. Figure 5 shows a comparison of the theoretical free vibrations when keeping E_∞ constant and increasing E_v and when decreasing E_∞ while keeping E_v constant. As one can expect, the stiffer the viscous branch, the faster the damping. Moreover, the damping seems to be controlled by the absolute value of E_v . For instance, the relative viscosity, defined by $E_v/(E_\infty + E_v)$, has an insignificant impact. Finally, the stiffness of the elastic branch E_∞ affects the vibration frequency, which decreases when E_∞ decreases and all other parameters are kept constant.

Second, the stiffnesses E_∞ and E_v are kept constant and τ_v is varied. Figure 6 shows the comparisons of the free vibrations of the cantilever beam when multiplying or dividing the relaxation time τ_v by a factor of five for reference relaxation times $\tau_{\text{ref}} = 0.001$ s and $\tau_{\text{ref}} = 0.05$ s. One notes that the damping depends on the absolute value of the relaxation time and not on a relative increase or decrease in a given reference value. In fact, considering the material and beam dimensions, the best damping was obtained for $\tau_v \approx 0.006$ s. This can be observed from Figure 7 displaying the values of the imaginary part of ω as a function of τ_v .

Note that materials of different relaxation times τ_v and the same viscosity $\eta_v = E_v \times \tau_v$ may show similar free vibration but not necessarily. Two limit cases are worth mentioning. When τ becomes very large, the beam behaves like an elastic beam of Young modulus $E_\infty + E_v$. When τ is very small, the beam behaves like an elastic beam of Young modulus E_∞ . Finally, note that

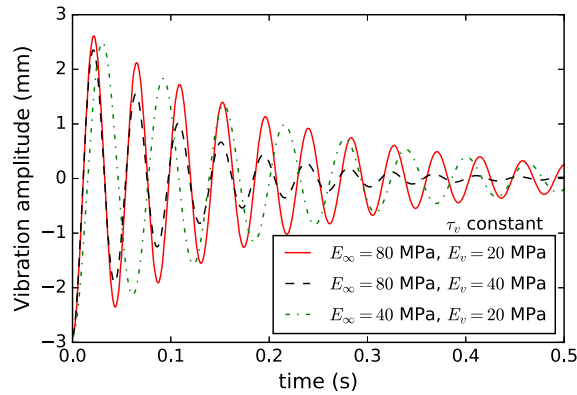


Figure 5. Impact of the Zener material stiffnesses E_∞ and E_v on the free vibration of the cantilever beam.

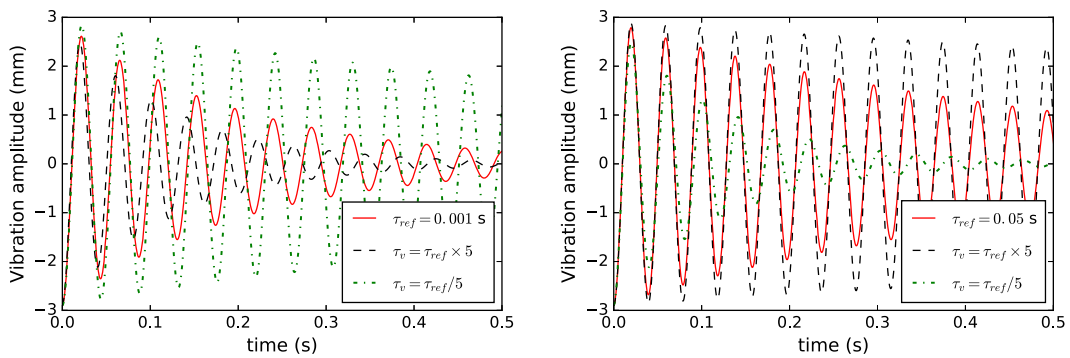


Figure 6. Impact of the relaxation time of the Zener material on the beam vibration.

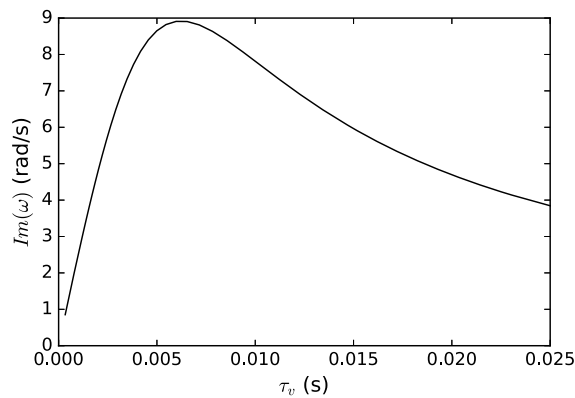


Figure 7. Estimates of $Im(\omega)$, characterizing the beam vibration damping, with respect to the relaxation time τ_v .

it could be difficult to extend the later analysis to real polymers that present a spectrum of relaxation times.

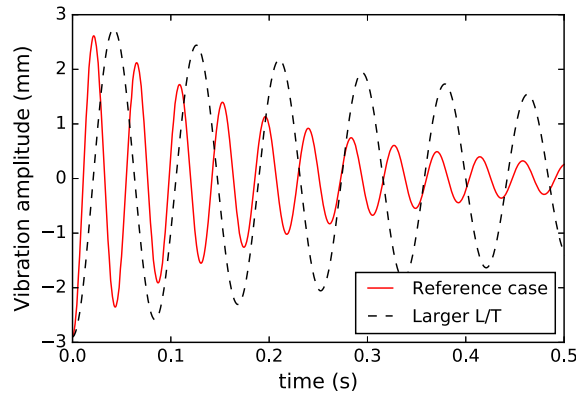


Figure 8. Impact of the cantilever beam geometry on its free vibration.

4.2. Impact of beam geometry

Since the behavior of different materials may be compared using this test, let us focus on the impact of the beam geometry on its free vibration. First, let us note that in (17) and (18), the solution is independent of W . Moreover, two beams presenting the same ratios L/T show the same vibration behavior. When increasing L/T , the frequency decreases and the damping is delayed. It is therefore important when comparing two materials experimentally to consider similar geometry in terms of dimension ratio L/T .

4.3. Material comparison

In Section 3.2, it was shown that it is possible to predict the free vibration of a cantilever polymer beam when the linear viscoelasticity of the material is known. Nonetheless, the characterization of the linear viscoelasticity of a polymer, as presented in Figure 3, is rather time-consuming. Therefore, this section aims at proposing a test, simpler and faster to run, to predict how different materials will perform in terms of the free vibration test. Since most labs studying materials are equipped with standard uniaxial tensile machines and since relaxation tests are relevant to characterizing the viscoelasticity of materials, only a uniaxial tension/relaxation test is performed. For this purpose, PEBAX®4033 is compared with another PEBAX®, which is labeled 70R53. Both materials are subjected to uniaxial tension up to a small deformation ϵ of approximately 1% at a constant crosshead speed of 100 mm/min. Then the stress relaxation is recorded for 30 s. The comparison of the material stiffness with respect to time, defined as $(F/S_0)/\epsilon = E(t)$ and recorded during the relaxation step, is shown in Figure 9. PEBAX®70R53 appears significantly stiffer and undergoes more stress relaxation at a faster rate. This result is in agreement with the ratios of polyamide hard segments to polyether soft segments in both materials. PEBAX®4033 contains significantly fewer polyamide hard segments than PEBAX®70R53 [12]. From this behavior characterization combined with the previous vibration analyses, one expects the vibration frequency of PEBAX®70R53 to be significantly higher than that of PEBAX®4033 due to its higher stiffness and its vibration damping to be faster due to its higher viscosity ($E_v = E(0) - E(\infty)$).

PEBAX®70R53 slender beams of the same dimension as that of PEBAX®4033 ($T = 2.1$ mm, $W = 10$ mm, and $L = 65$ mm) were subjected to the same vibration test protocol. The comparison of the vibrating behaviors of both PEBAX® elastomers is displayed in Figure 10. As expected, PEBAX®70R53 vibrates at a higher frequency but for a significantly shorter duration.

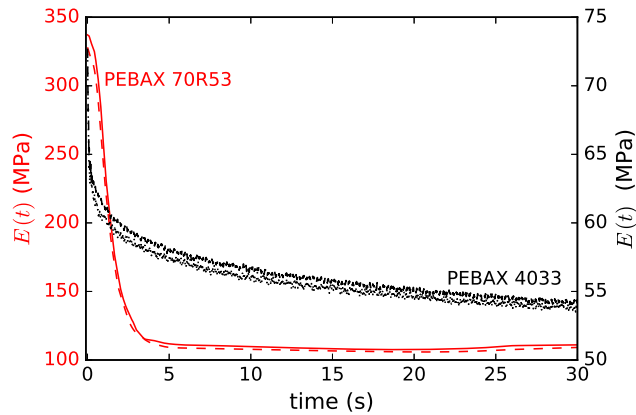


Figure 9. Relaxation moduli of PEBAx® elastomers 4033 and 70R53 obtained during relaxation tests at approximately 1% strain, attained with a uniaxial tensile test at a constant crosshead speed of 100 mm/min.

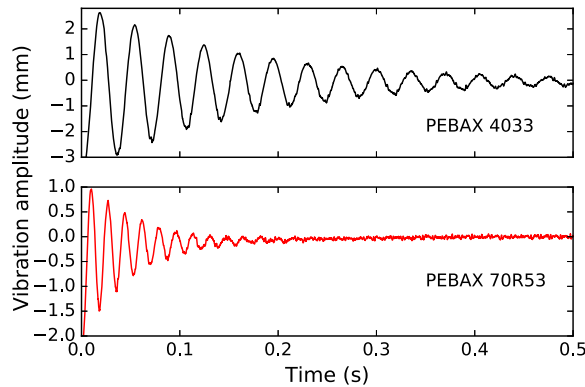


Figure 10. Comparison of the free vibration of PEBAx® elastomers 4033 and 70R53 for beams having the same geometries.

As a consequence, sport equipment suppliers looking to increase the damping will favor materials with high absolute viscosity.

5. Conclusion

This study aimed at providing quantitative insight into the test of free vibration of a viscoelastic cantilever beam to compare polymers for dynamic applications. Applying the simple Euler–Bernoulli beam theory for linear viscoelastic materials, a very satisfactory quantitative comparison between the theory and the experimental data was obtained for a homogeneous slender beam of PEBAx® showing a rather extended spectrum of relaxation times.

The validated theory was then used to analyze the impact of the material parameters and the beam geometry on the free vibration behavior of a homogeneous rectangular beam described by Zener viscoelastic behavior. On the one hand, the analysis showed that the vibration frequency is related to the instantaneous stiffness of the material, and the frequency increases with the material stiffness. On the other hand, the vibration damping increases with increase in the difference between the instantaneous and long-term material stiffnesses. Although significant,

the impact of the relaxation time is more difficult to analyze; no general trend has been discerned. This is therefore especially true when considering an actual polymer beam presenting a spectrum of relaxation times. However, it was shown that the free vibration of a polymer cantilever beam is directly connected to quantities that may be measured by just a tension/relaxation test. It was actually possible to predict how two materials would compare in terms of their free vibration responses by simply comparing their mechanical responses to a tension/relaxation test.

Appendix A. Viscoelastic spectrum of PEBAX®4033 for reference temperature 20 °C

τ_i (s)	G_i (MPa)	τ_i (s)	G_i (MPa)
8.083E-25	3.522E+01	3.776E-08	1.096E+01
5.507E-24	3.812E+01	2.573E-07	9.180E+00
3.754E-23	4.119E+01	1.754E-06	7.412E+00
2.559E-22	4.413E+01	1.195E-05	6.096E+00
1.744E-21	4.678E+01	8.144E-05	5.239E+00
1.188E-20	4.918E+01	5.553E-04	4.576E+00
8.098E-20	5.132E+01	3.783E-03	3.952E+00
5.517E-19	5.196E+01	2.578E-02	3.312E+00
3.761E-18	4.898E+01	1.757E-01	2.698E+00
2.563E-17	4.331E+01	1.198E+00	2.208E+00
1.747E-16	3.795E+01	8.160E+00	1.848E+00
1.190E-15	3.375E+01	5.563E+01	1.596E+00
8.114E-15	3.010E+01	3.791E+02	1.434E+00
5.532E-14	2.670E+01	2.583E+03	1.335E+00
3.769E-13	2.340E+01	1.761E+04	1.270E+00
2.568E-12	2.075E+01	1.200E+05	1.215E+00
1.750E-11	1.885E+01	8.175E+05	1.126E+00
1.193E-10	1.700E+01	5.573E+06	1.011E+00
8.129E-10	1.475E+01	3.798E+07	9.898E-01
5.542E-09	1.271E+01	2.588E+08	1.380E+00

References

- [1] L. Struik, "Free damped vibrations of linear viscoelastic materials", *Rheol. Acta* **6** (1967), p. 119-129.
- [2] S. Mahmoodi, S. Khadem, M. Kokabi, "Non-linear free vibrations of Kelvin-Voigt visco-elastic beams", *Int. J. Mech. Sci.* **49** (2007), p. 722-732.
- [3] M. Ilyasov, "Vibrations of linear viscoelastic materials for any hereditary property", *Mech. Time-Depend Mater.* **11** (2007), p. 249-263.
- [4] M. Avcar, "Free vibration analysis of beams considering different geometric characteristics and boundary conditions", *Int. J. Mech. Appl.* **4** (2014), p. 94-100.
- [5] S. Mahmoodi, N. Jalili, S. Khadem, "An experimental investigation of nonlinear vibration and frequency response analysis of cantilever viscoelastic beams", *J. Sound Vib.* **311** (2008), p. 1409-1419.
- [6] S. Timoshenko, *History of Strength of Materials*, McGraw-Hill, New York, 1953.
- [7] R. M. Christiansen, *Theory of Viscoelasticity*, 2 ed., Dover, New York, 2003.
- [8] Y. Lei, S. Adhikari, M. Friswell, "Dynamic characteristics of damped viscoelastic nonlocal Euler-Bernoulli beams", *Eur. J. Mech. A/Solids* **42** (2013), p. 125-136.
- [9] J. D. Ferry, *Viscoelastic Properties of Polymers*, 3 ed., John Wiley & Sons, New York, 1980.
- [10] J. Diani, P. Gilormini, "On necessary precautions when measuring solid polymer linear viscoelasticity with dynamic analysis in torsion", *Polym. Test.* **63** (2017), p. 275-280.
- [11] A. B. de Saint-Venant, *De la torsion des prismes avec des considérations sur leur flexion ainsi que sur l'équilibre des solides élastiques en général et des formules pratiques*, Mémoires des Savants Etrangers, Paris, 1855.
- [12] J. P. Sheth, J. Xu, G. L. Wilkes, "Solid state structure-property behavior of semicrystalline poly(ether-block-amide) PEBAX® thermoplastic elastomers", *Polymer* **44** (2003), p. 743-756.