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Comptes Rendus
Mathématique

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Volume 360 (2022), p. 333-341

<https://doi.org/10.5802/crmath.291>

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Algebra, Group theory / *Algèbre, Théorie des groupes*

Sandpile monomorphisms and limits

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Abstract. We introduce a tiling problem between bounded open convex polyforms $\widehat{P} \subset \mathbb{R}^2$ with colored directed edges. If there exists a tiling of the polyform $\widehat{P}_2$ by $\widehat{P}_1$, we construct a monomorphism from the sandpile group $G_{\Gamma_1} = \mathbb{Z}^{\Gamma_1}/\Delta(\mathbb{Z}^{\Gamma_1})$ on $\Gamma_1 = \widehat{P}_1 \cap \mathbb{Z}^2$ to the one on $\Gamma_2 = \widehat{P}_2 \cap \mathbb{Z}^2$. We provide several examples of infinite series of such tilings converging to $\mathbb{R}^2$, and thus define the limit of the sandpile group on the plane.

2020 Mathematics Subject Classification. 20K30, 60K35, 47D07.

Manuscript received 1st January 2021, revised 25th August 2021 and 24th October 2021, accepted 24th October 2021.

1. Introduction

1.1. *Background*

Let $\overline{\Gamma} = \Gamma \cup \{s\}$ be the vertices of a finite connected (multi-)graph with sink s. Denote by $\partial\Gamma$ the boundary of Γ – the set of all vertices adjacent to the sink. The standard discrete graph Laplacian $\overline{\Delta}_{\overline{\Gamma}}$ is then defined as the difference between the adjacency matrix of $\overline{\Gamma}$ and its degree/valency matrix. When we delete from $\overline{\Delta}_{\overline{\Gamma}}$ the row and column corresponding to the sink, we obtain the reduced graph Laplacian Δ_Γ. The sandpile group G_Γ is then defined as the cokernel of Δ_Γ acting on $\mathbb{Z}^\Gamma$ [8], i.e.

$$G_\Gamma = \mathbb{Z}^\Gamma/\Delta_\Gamma(\mathbb{Z}^\Gamma).$$

The sandpile group was rediscovered several times. As a consequence, it is sometimes referred to as the critical group, as the graph Jacobian, or (sometimes) as the Picard group [2, 4, 6].

Of specific interest are the sandpile groups defined on finite connected domains of the standard square lattice $\mathbb{Z}^2$, i.e. on graphs $\overline{\Gamma}$ obtained from $\mathbb{Z}^2$ by contracting all vertices $\mathbb{Z}^2 \setminus (\mathbb{Z}^2 \cap P)$ outside of some bounded open set $P \subset \mathbb{R}^2$ to the sink. This is also the original setup of the sandpile model, a cellular automaton introduced by Bak, Tang and Wiesenfeld in 1987 [3] as the first and archetypical example of a system showing self-organized criticality. Shortly after

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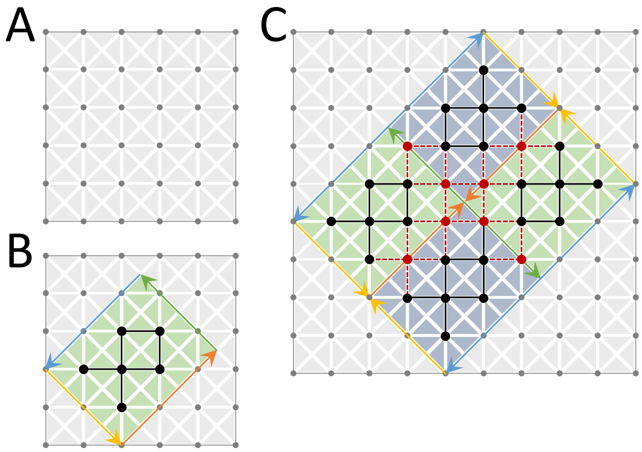

Figure 1. A) Dark-gray points represent the vertices of the standard square lattice $\mathbb{Z}^2$, while the gray isosceles triangles correspond to M. B) The isosceles triangles belonging to the M-polyform P_1 are highlighted by a green background. The black points and lines represent the vertices and edges of the graph $\Gamma(P_1) = \mathbb{Z}^2 \cap P_1$ defined by P_1. The sides of the M-polyform are directed and colored, exemplifying the definition of P_1^{DC}. C) DC-tiling of a M-polyform P_2 by four copies of the M-polyform P_1^{DC} from (B). Tiles have a green background if they can be obtained from P_1^{DC} by only translations and rotations, and blue otherwise. Note that the graph $\Gamma(P_2)$ also consists of the vertices lying on the common edges of pairs of tiles, and the corresponding edges (red points and lines).

the introduction of this cellular automaton, Dhar showed that its recurrent configurations form a group isomorphic to G_Γ, and laid the foundation for its analysis [8, 9]. On such domains, the sandpile group provides connections between various mathematical fields, including fractal geometry [7, 14], graph theory and algebraic geometry [4, 5], tropical geometry [12, 13], domino tilings [10], and others.

Only for few infinite families of graphs, the structure of the respective sandpile groups has been determined. In particular, the decomposition of the sandpile group on domains of $\mathbb{Z}^2$ is yet unknown. Further, results on the relationships between sandpile groups, in terms of homeomorphisms, on different domains of $\mathbb{Z}^2$ are lacking[1]. This is in stark contrast to the role of the sandpile model as the archetypical example for self-organized criticality, given that this concept itself is based on the notion of scaling.

1.2. *Sandpile monomorphisms (main result)*

In this article, we analyze the relationships between sandpile groups defined on different domains of the standard square lattice $\mathbb{Z}^2$. Specifically, given two domains $\Gamma_1, \Gamma_2 \subset \mathbb{Z}^2$, $\Gamma_1 \subseteq \Gamma_2$, our goal is to understand under which conditions group monomorphisms from G_{Γ_1} to G_{Γ_2} exist.

For this, let M be the unique tiling of $\mathbb{R}^2$ by isosceles triangles with base length 1 and height $\frac{1}{2}$ such that each vertex of $(\mathbb{Z}+0.5)^2$ coincides with the apecies of four triangles (Figure 1A). An

[1]As a consequence of the burning algorithm [8], the restriction of a recurrent configuration to a sub-domain is recurrent again. Similarly, a recurrent configuration can be extended to a super-domain by e.g. embedding it into the minimally stable configuration. However, these and similar transformations only define morphisms in the category of sets (of recurrent configurations), but not in the category of (sandpile) groups.

M-polyform $P \subset M$ then consists of a finite connected subset of triangles in M (Figure 1B). Note that M is *not* the usual triangular tiling of the plane, and that M-polyforms thus differ from the usual definition of polyiamonds. By a slight abuse of notation, we interpret each M-polyform P to directly correspond to the open subset of $\mathbb{R}^2$ enclosed by its isosceles triangles, i.e. to the interior of $\bigcup_{m\in P} m$. To each M-polyform P, we then associate the domain $\Gamma(P) = \mathbb{Z}^2 \cap P$. This domain is obtained from the standard square lattice $\mathbb{Z}^2$ as described in the Introduction, i.e. by contracting all vertices $\mathbb{Z}^2 \setminus (\mathbb{Z}^2 \cap P)$ to the sink (Figure 1B). We interchangeably denote by G_Γ and G_P the sandpile groups defined on the domain $\Gamma = \Gamma(P)$. Denote by P^{DC} the result of assigning directions and colors to the edges of an M-polyform P such that each edge has a different color (Figure 1B). Given two M-polyforms P_1 and P_2, we say that P_1 DC-tiles P_2 if there exists a tiling $T^{P_1\to P_2}$ of P_2 by copies of P_1^{DC} (allowing all transformations which correspond to automorphisms of M), such that every common edge of two adjacent tiles in $T^{P_1\to P_2}$ has the same color and direction (Figure 1C).

Theorem 1. *Let P_1 and P_2 be two convex M-polyforms, and assume that P_1 DC-tiles P_2. Then, there exists a group monomorphism $G_{P_1} \rightarrowtail G_{P_2}$ from G_{P_1} to G_{P_2}.*

In the proof of this theorem (Section 3), we construct an explicit mapping $\mu(T^{P_1\to P_2}) = (G_{P_1} \rightarrowtail G_{P_2})$ from DC-tilings to the corresponding group monomorphisms. Here, we only note two properties of this construction: (i) for two M-polyforms P_1 and P_2, there can exist more than one distinct DC-tiling of P_2 by P_1. For example, let the polyform P describe a square with width w and sides parallel to the standard axes of $\mathbb{R}^2$. Since the dihedral group D_4 of a square has order eight, there also exist eight different DC-tiling of P by itself. For $w > 2$, μ maps each of these tilings to a different automorphism of G_P, which directly correspond to the action of the respective element of D_4 on $\Gamma(P)$. For $w = 2$, the domain $\Gamma(P)$ however consists of only a single vertex, and all eight tilings are mapped to the trivial automorphism; (ii) denote by $\widehat{P}$ the result of extending a polyform P by one triangle in M adjacent to P such that $\Gamma(\widehat{P}) = \Gamma(P)$. Then, there exist no DC-tilings of $\widehat{P}$ by P, or vice versa. However, since $G_P = G_{\hat{P}}$, the set of automorphisms is non-empty. We thus conclude that the mapping μ is in general neither injective nor surjective.

1.3. *Limits of the sandpile group*

Let **Pol** denote the poset of bounded convex M-polyforms, with $P_1 \subseteq_{DC} P_2$ if there exists a DC-tiling $T^{P_1\to P_2}$ of the M-polyform P_2 by the M-polyform P_1 such that the position and orientation of one tile in $T^{P_1\to P_2}$ directly corresponds to P_1^{DC}, i.e. $P_1^{DC} \in T^{P_1\to P_2}$. We naturally identify **Pol** with its corresponding (small) category, with the (faithful) forgetful functor $U : \mathbf{Pol} \to \mathbf{Set}$ to the category of sets mapping each M-polyform to its corresponding open subset of $\mathbb{R}^2$ and $\subseteq_{DC}$ to set inclusions. Since the position and orientation of one tile uniquely identifies a DC-tiling (if it exists), the definition of **Pol** allows us to associate a DC-tiling $\nu(P_1 \subseteq_{DC} P_2) \in \{T^{P_1\to P_2}\}$ to each morphism $P_1 \subseteq_{DC} P_2$, i.e. the unique DC-tiling satisfying $P_1^{DC} \in T^{P_1\to P_2}$. We can then define the functor $F : \mathbf{Pol} \to \mathbf{Ab}$ from **Pol** to the category **Ab** of abelian groups, with $F(P) = G_P$ and $F(P_1 \subseteq_{DC} P_2) = \mu(\nu(P_1 \subseteq_{DC} P_2))$. To see that $F(\mathrm{id}_P) = \mathrm{id}_{G_P}$ and $F((P_2 \subseteq_{DC} P_3)\circ(P_1 \subseteq_{DC} P_2)) = (G_{P_2} \rightarrowtail G_{P_3}) \circ (G_{P_1} \rightarrowtail G_{P_2})$, we refer to the construction of the map μ in Section 3.

Of specific interest are infinite sequences $S = S_0 \subseteq_{DC} S_1 \subseteq_{DC} S_2 \ldots$ of M-polyforms in **Pol** (identity and composed morphisms omitted), i.e. functors $S \in \mathbf{Pol}^\omega$ from the usual linear order $\omega = \{0, 1, \ldots\}$ on the ordinal numbers to **Pol**. Trivially, each of these sequences, composed with the forgetful functor U, defines a direct limit $\varinjlim US = \bigcup_i U(S_i) \subseteq \mathbb{R}^2$ (in the category of sets; **Pol**does not admit all filtered colimits), which we denote by $\widehat{S}_\infty$. Furthermore, each sequence, composed with F, also defines a direct limit $\varinjlim FS$, denoted either as $G^S_{\hat{S}_\infty}$ or, equivalently, by $G^S_{\Gamma(\hat{S}_\infty)}$. We interpret $G^S_{\Gamma(\hat{S}_\infty)}$ as the limit of the sandpile group for $\Gamma(S_i) \to \Gamma(\widehat{S}_\infty)$ (with respect

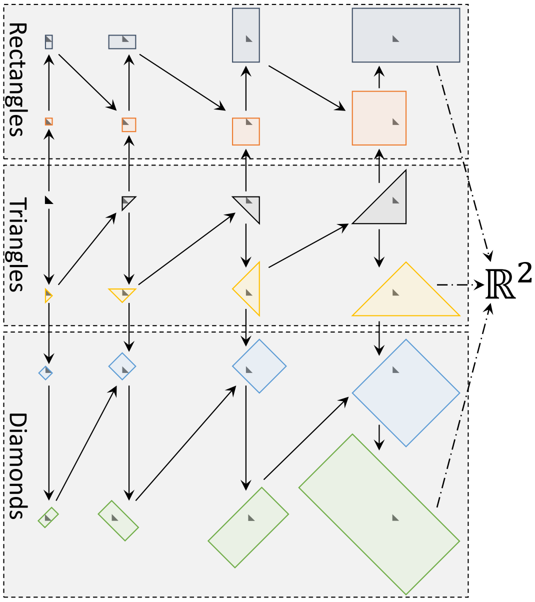

Figure 2. Depiction of a small part of the category **Pol**of M-polyforms. Each shape represents a M-polyform P, while arrows represent morphisms $P_1 \subseteq_{DC} P_2$ (identities and composed morphisms omitted). The position of the initial triangular M-polyform (black) is depicted by a gray background in each M-polyform. The composition of all non-bounded sequences S with the forgetful functor U in the depicted part of **Pol** has a direct limit of $\mathbb{R}^2$.

to the sequence S). In Figure 2, we depict the morphisms between four families of polyforms in **Pol**. The direct limit of each infinite sequence $S \in \mathbf{Pol}^{\omega}$ which only contains these polyforms and morphisms, with $S_{i+1} \neq S_i$ for all $i \in \omega$, is given by $\widehat{S}^{\infty} = \mathbb{R}^2$, and thus $\Gamma(\widehat{S}^{\infty}) = \mathbb{Z}^2$. To our knowledge, the respective limits of the sandpile group $G^S_{\mathbb{Z}^2}$ are the first[2] definitions of limits for the sandpile group on $\mathbb{Z}^2$.

If a given sequence S of M-polyforms is upper bounded, i.e. if there exists an $u \in \omega$ such that $US_j = US_u = \widehat{S}_{\infty}$ for all $j \geq u$, it directly follows that $G^S_{\hat{S}_{\infty}} \cong G_{S_u}$. Thus, for such upper bounded sequences, the limit of the sandpile group is completely determined (up to isomorphisms) by the upper bound, i.e. F preserves all finite direct limits. In such cases, we can drop the dependency of $G^S_{\hat{S}_{\infty}}$ on S and simply write $G_{\hat{S}_{\infty}}$. We may ask if the same also holds for unbounded sequences:

Question 2. *Let $S^A, S^B \in \mathbf{Pol}^{\omega}$ be two (possibly unbounded) sequences of M-polyforms with common limit $\widehat{S}_{\infty} = \varinjlim US^A = \varinjlim US^B$. Is $G^{S^A}_{\hat{S}_{\infty}}$ isomorphic to $G^{S^B}_{\hat{S}_{\infty}}$?*

Let $\widehat{\mathbf{Pol}}$ be the category with objects corresponding to all limits $\widehat{S}_{\infty} = \varinjlim US$ of sequences $S \in \mathbf{Pol}^{\omega}$ of polyforms, and morphisms $\widehat{S}^A_{\infty} \subseteq \widehat{S}^B_{\infty}$ if there exists a natural transformation $S^a \dot{\to} S^b$ between two sequences $S^a, S^b \in \mathbf{Pol}^{\omega}$ with $\widehat{S}^A_{\infty} = \varinjlim US^a$ and $\widehat{S}^B_{\infty} = \varinjlim US^b$, i.e. if $S^a_i \subseteq_{DC} S^b_i$ for all $i \in \omega$. We interpret **Pol** to represent a full subcategory of $\widehat{\mathbf{Pol}}$, with the object function of the (fully

[2]There exist several approaches to define sandpile models directly on $\mathbb{Z}^2$, which cope with the occurrence of infinite avalanches in various ways [1,11,16]. The (weak) limits of the sandpile measures for some of these models concentrate on certain abelian groups. The relationship between these "groups of recurrent configurations of sandpile models on infinite graphs" to the sandpile groups on finite graphs is, to our knowledge, however unclear. Specifically, to our knowledge, it was not shown if these groups correspond to some notion of direct or projective limit in the category of groups.

faithful) inclusion functor $I : \mathbf{Pol} \to \widehat{\mathbf{Pol}}$ given by $I(P) = \varinjlim U\delta P$, where $\delta : \mathbf{Pol} \to \mathbf{Pol}^\omega$ denotes the diagonal functor with $(\delta P)_i = P$ for all $i \in \omega$. Question 2 then asks if there exists a functor $\widehat{F} : \widehat{P} \to \mathbf{Ab}$ which preserves all direct limits, and for which F factors as $\widehat{F} \circ U$.

In case Question 2 can be answered in the affirmative, a unique limit $G_{\mathbb{Z}^2}$ (up to isomorphisms) of the sandpile group on $\mathbb{Z}^2$ would exist. In this case, the following would hold:

Corollary 3. *Assume that $G^{S^A}_{\hat{S}_\infty} \cong G^{S^B}_{\hat{S}_\infty}$ whenever $\widehat{S}_\infty = \varinjlim US^A = \varinjlim US^B$. Then, the limit of the sandpile group on $\mathbb{Z}^2$ is isomorphic to its limit on the upper-right quadrant of $\mathbb{Z}^2$, i.e. $G_{\mathbb{Z}^2} \cong G_{\mathbb{Z}^2_{\geq 0}}$.*

This corollary utilizes that, in morphisms $P_1 \subseteq_{DC} P_2$, the positions of the M-polyforms P_1 and P_2 are considered, and that thus two different morphisms can be mapped by ν to the same tiling $T^{P_1 \to P_2}$. This can be used to construct two sequences S^A and S^B such that there exists a natural isomorphism $FS^A \cong FS^B$, but for which $\widehat{S}^A_\infty \neq \widehat{S}^B_\infty$. Corollary 3 corresponds to choosing $S^A_0 = S^B_0$ to be square-shaped M-polyforms with side length w_0, S^A_{i+1} and S^B_{i+1} to have side lengths $w_{i+1} = 5w_i$, S^A_{i+1} to be positioned such that S^A_i is in its center, and S^B_{i+1} such that S^B_i is at its bottom-left.

1.4. *Relationship to harmonic functions and order of the sandpile group*

We say that a domain $\Gamma \subseteq \mathbb{Z}^2$ is convex if there exists a convex open set $P \subseteq \mathbb{R}^2$ such that $\Gamma = P \cap \mathbb{Z}^2$. Different to before, we do not require P to be an M-polyform anymore. We say that an R-valued function $H : \Gamma \to R$, $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, is harmonic (on Γ) if $\Delta_\Gamma H(v) = 0$ for all vertices $v \in \Gamma_0$ in the interior $\Gamma_0 = \Gamma \setminus \partial\Gamma$ of the domain. Note that, different to certain other literature, neither the sink nor the outer boundary $\partial(\mathbb{Z}^2 \setminus \Gamma)$ is in the domain of H. The R-valued harmonic functions on Γ form the R-module $\mathscr{H}^\Gamma_R$.

Lemma 4. *For every finite convex domain $\Gamma \subset \mathbb{Z}^2$, $-\Delta_\Gamma : \mathscr{H}^\Gamma_G \cong G^\Gamma$ is an isomorphism from $\mathscr{H}^\Gamma_G = \{H \in \mathscr{H}^\Gamma_{\mathbb{Q}} | \Delta_\Gamma H|_{\partial\Gamma} \in \mathbb{Z}^{\partial\Gamma}\} / \mathscr{H}^\Gamma_{\mathbb{Z}}$ to the sandpile group G_Γ, with $\mathscr{H}^\Gamma_G$ the subgroup of the rational-valued harmonic functions $\mathscr{H}^\Gamma_{\mathbb{Q}}$ with integer-valued Laplacians, modulo the integer-valued harmonic functions $\mathscr{H}^\Gamma_{\mathbb{Z}}$. This isomorphism corresponds to the exact sequence*

$$0 \longrightarrow G_\Gamma \longrightarrow \mathscr{H}^\Gamma_{\mathbb{Q}} / \mathscr{H}^\Gamma_{\mathbb{Z}} \longrightarrow (\mathbb{Q}/\mathbb{Z})^{\partial\Gamma} \longrightarrow 0.$$

This and the following lemma can be considered to be special cases of the general theory of toppling invariants introduced in [9]: By [7], every element of the sandpile group can be reached from the identity by only adding particles to the boundary of the domain. This allows to choose the (independent) toppling invariants generating the sandpile group such that their Laplacians only have support on the boundary $\partial\Gamma$ of the domain. This choice of toppling invariants (i.e. the exact sequence in Lemma 4) seems to be especially suited to further characterize the sandpile group as it e.g. significantly simplifies the proof of our main theorem. For this reason, an explicit construction of the isomorphism in Lemma 4 is provided in Section 2.

Denote by $\mathscr{B}^\Gamma_R = \{B_i\}_{i=1,\ldots,|\partial\Gamma|}$ a basis for the module $\mathscr{H}^\Gamma_R$ of R-valued harmonic functions on a finite convex domain $\Gamma \subset \mathbb{Z}^2$. By definition, the Laplacian $\Delta_\Gamma H$ of every harmonic function $H \in \mathscr{H}^\Gamma_R$, and thus also of every basis function in $\mathscr{B}^\Gamma_R$, only has support at the boundary $\partial\Gamma$ of the domain. The Laplacian of every basis function in $\mathscr{B}^\Gamma_R$ can thus be restricted to $\partial\Gamma$ without information loss, and we refer to $\Delta\mathscr{B}^\Gamma_R = (\Delta_\Gamma B_1|_{\partial\Gamma}, \ldots, \Delta_\Gamma B_{|\partial\Gamma|}|_{\partial\Gamma}) \in R^{|\partial\Gamma| \times |\partial\Gamma|}$ as the potential matrix of Γ (with respect to $\mathscr{B}^\Gamma_R$).

Lemma 5. *Let $\Gamma \subset \mathbb{Z}^2$ be a finite convex domain, and $\mathscr{B}^\Gamma_{\mathbb{Z}}$ be a basis for the module of integer-valued harmonic functions $\mathscr{H}^\Gamma_{\mathbb{Z}}$ on Γ. Then, the order of the sandpile group G_Γ is $|G_\Gamma| = |\det(\Delta\mathscr{B}^\Gamma_{\mathbb{Z}})|$.*

As for the previous lemma, this result can be considered to represent a special case of the general theory of toppling invariants [9].

2. Harmonic functions (proofs of Lemmata 4 and 5)

Recall that, in the sandpile model, every recurrent configuration can be reached from the empty configuration (or any other configuration) by only adding particles to the boundary of the domain and "relaxing" the sandpile [7]. Since the elements of the sandpile group correspond to the equivalence classes of the recurrent configurations of the sandpile model (see Introduction), this can be restated as follows: for every element $C \in G^\Gamma$ of the sandpile group, there exist (infinitely many) functions $X \in \mathbb{Z}^\Gamma$ which only have support at the boundary $\partial\Gamma$ of the domain, and which satisfy that $[X] = C$, with $[\cdot]: \mathbb{Z}^\Gamma \to G_\Gamma$ the canonical projection map to the sandpile group. Since Γ is assumed to be convex, there exist functions $B_X : \partial(\mathbb{Z}^2 \setminus \Gamma) \to \mathbb{Z}$ such that $X(v) = \sum_{w\sim v} B_X(w)$. When considering B_X as a boundary condition, it follows, by the existence and uniqueness of solutions to the discrete Dirichlet problem on convex domains [15], that, for every such $X \in \mathbb{Z}^\Gamma$, there exists a unique rational-valued harmonic function $H_X \in \mathcal{H}_{\mathbb{Q}}^\Gamma$ with $\Delta_\Gamma H_X = -X$.

The composition $[.] \circ -\Delta_\Gamma$ of the discrete Laplacian with the canonical projection map then maps two harmonic functions $H_{X,1}, H_{X,2} \in \mathcal{H}_{\mathbb{Q}}^\Gamma$, $\Delta_\Gamma H_{X,1}, \Delta_\Gamma H_{X,2} \in \mathbb{Z}^\Gamma$, to the same element of the sandpile group if and only if $-\Delta_\Gamma(H_{X,1} - H_{X,2}) \in \Delta_\Gamma(\mathbb{Z}^\Gamma)$. Since both $\Delta_\Gamma H_{X,1}$ and $\Delta_\Gamma H_{X,2}$ only have support at the boundary $\partial\Gamma$ of the domain, $H_{X,1} - H_{X,2}$ is thus an integer-valued harmonic function, which concludes our proof of Lemma 4.

We construct the inverse of $-\Delta_\Gamma : \mathcal{H}_G^\Gamma \cong G_\Gamma$ in two steps. For every configuration $C \in G_\Gamma$, we first define the coordinates $\sigma_\Gamma : G_\Gamma \to (\mathbb{Q}/\mathbb{Z})^{\partial\Gamma}$,

$$\sigma_\Gamma(C) \equiv -(\Delta\mathcal{B}_{\mathbb{Z}}^\Gamma)^{-1} X \pmod 1,$$

with respect to the basis $\mathcal{B}_{\mathbb{Z}}^\Gamma$. Note that, for two different choices $X^\alpha, X^\beta \in \mathbb{Z}^\Gamma$, $-(\Delta\mathcal{B}_{\mathbb{Z}}^\Gamma)^{-1}(X^\alpha - X^\beta) \in \mathbb{Z}^{\partial\Gamma}$, and that thus the coordinates σ_Γ don't depend on the specific choice for X. This also implies that σ_Γ correspond to toppling invariants as defined in [9].

In the second step, we then define the function $\phi_\Gamma : (\mathbb{R}/\mathbb{Z})^{\partial\Gamma} \to \mathcal{H}_{\mathbb{R}}^\Gamma/\mathcal{H}_{\mathbb{Z}}^\Gamma$, $\phi_\Gamma(s) = \sum_{i=1}^{|\partial\Gamma|} s_i B_i$. It is easy to check that the composition $\phi_\Gamma \circ \sigma_\Gamma : G_\Gamma \to \mathcal{H}_{\mathbb{R}}^\Gamma/\mathcal{H}_{\mathbb{Z}}^\Gamma$ is independent of the choice of the basis $\mathcal{B}_{\mathbb{Z}}^\Gamma$, and that $-\Delta_\Gamma\phi_\Gamma(\sigma_\Gamma([X])) = [X]$. The latter implies that $\phi_\Gamma \circ \sigma_\Gamma$ is the inverse of $-\Delta_\Gamma$.

The isomorphism between the sandpile group G_Γ and $\mathcal{H}_G^\Gamma$ proposes to consider the sandpile group as a discrete subgroup of a continuous Lie group isomorphic to $\mathcal{H}_{\mathbb{R}}^\Gamma/\mathcal{H}_{\mathbb{Z}}^\Gamma$, to which we refer to as the extended sandpile group $\widetilde{G}_\Gamma$ [14]. More precisely, the extended sandpile group is an extension of the torus $(\mathbb{R}\setminus\mathbb{Z})^{\partial\Gamma}$ by the usual sandpile group, and is defined by the exact sequence

$$0 \longrightarrow G_\Gamma \longrightarrow \widetilde{G}_\Gamma \longrightarrow (\mathbb{R}/\mathbb{Z})^{\partial\Gamma} \longrightarrow 0.$$

In terms of the sandpile model, this Lie group is obtained by allowing each vertex $b \in \partial\Gamma$ in the boundary $\partial\Gamma$ of the domain to carry a real value $\widetilde{C}(b) \in [0,4)$ of particles, while each vertex $v \in \Gamma_0$ in the interior $\Gamma_0 = \Gamma \setminus \partial\Gamma$ of the domain is still only allowed to carry an integer number of particles, i.e. $\widetilde{C}(v) \in \{0,1,2,3\}$ (the toppling rules are kept unchanged) [14]. This definition lifts $\phi_\Gamma : (\mathbb{R}/\mathbb{Z})^{\partial\Gamma} \cong \widetilde{G}_\Gamma$ to a group isomorphism, and a left-inverse of the inclusion map $G_\Gamma \to \widetilde{G}_\Gamma$ is given by the floor function $\lfloor\cdot\rfloor : \widetilde{G}_\Gamma \to G_\Gamma$. We thus naturally arrive at the function $f = \lfloor\cdot\rfloor \circ -\Delta_\Gamma \circ \phi_\Gamma : (\mathbb{R}/\mathbb{Z})^{\partial\Gamma} \to G_\Gamma$, $f(s) = -[\lfloor\sum_{i=1}^{|\partial\Gamma|} s_i \Delta_\Gamma B_i\rfloor]$, which justifies to interpret the usual sandpile group G_Γ as the discretization of an $|\partial\Gamma|$-dimensional torus [14].

Due to the properties of the floor function, the preimage $f^{-1}(C)$ of an element $C \in G_\Gamma$ of the sandpile group under f is connected. Denote by $\mathrm{vol}(f^{-1}(C))$ the volume of this preimage, with $\mathrm{vol}((\mathbb{R}/\mathbb{Z})^{\partial\Gamma}) = 1$. Since, for every $C \in G_\Gamma$, there exists a coordinate transformation $s \mapsto \tilde{s}$ such that C has coordinates $\tilde{s} = 0$, we get that $\mathrm{vol}(f^{-1}(C)) = \mathrm{vol}(f^{-1}(\mathbf{0})) = \frac{1}{|G_\Gamma|}$ for all $C \in G_\Gamma$, with $\mathbf{0}$ the identity of the sandpile group. The preimage $f^{-1}(\mathbf{0})$ of the identity under f forms a $|\partial\Gamma|$-parallelotope with edges g_i given by $(\Delta\mathcal{B}_{\mathbb{Z}}^\Gamma)g_i = e_i$, with $(e_i)_j = \delta_{ij}$ the i^{th} unit vector and δ_{ij} the Kronecker delta. The volume of this parallelotope is $\mathrm{vol}(f^{-1}(\mathbf{0})) = |\det(\Delta\mathcal{B}_{\mathbb{Z}}^\Gamma)^{-1}|$, and thus $|G_\Gamma| = |\det(\Delta\mathcal{B}_{\mathbb{Z}}^\Gamma)|$, which proves Lemma 5.

3. Construction of sandpile monomorphisms (proof of Theorem 1)

Let P_A and P_B be two convex M-polyforms, and assume that there exists a DC-tiling $T^{P_A \to P_B}$ of P_B by P_A. In this section, we then construct a monomorphism from the sandpile group on the domain $\Gamma_A = \Gamma(P_A) = \mathbb{Z}^2 \cap P_A$ to the sandpile group on $\Gamma_B = \Gamma(P_B) = \mathbb{Z}^2 \cap P_B$, and thus prove Theorem 1. Before starting this construction, we shortly state three properties of DC-tilings.

Corollary 6. *The domains of different tiles do not overlap, i.e.* $\Gamma_i \cap \Gamma_j = \{\}$ *for all* $i \neq j$.

The domains of the tiles in general don't cover Γ_B. Specifically, all vertices of Γ_B which lie directly on common edges (including their endpoints) of two tiles are not elements of any Γ_i (red vertices in Figure 1). We refer to the set $\partial^T \Gamma_B = \Gamma_B \setminus \bigcup_i \Gamma_i$ of these vertices as the internal boundaries of the tiling. They separate the domains Γ_i in the following sense:

Corollary 7. *The removal of all vertices in* $\partial^T \Gamma_B$ *splits* Γ_B *into the disconnected components* $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{|T^{P_A \to P_B}|}\}$.

By definition, each tile $P_i \in T^{P_A \to P_B}$ can be obtained from P_A^{DC} by a combination of translations, rotations and reflections. If this is possible by using only translations and rotations, we assign the sign $s(P_i) = +1$ to the tile, and otherwise the sign $s(P_i) = -1$. This definition also induces signs $s(v) = s(\Gamma_i) = s(P_i)$ for the domains Γ_i and vertices $v \in \Gamma_i$ belonging to the tiles. To the internal boundaries $\partial^T \Gamma_B$ and their vertices $b \in \partial^T \Gamma_B$, we assign the sign $s(\partial^T \Gamma_B) = s(b) = 0$. The relationship of each tile P_i with the polyform P_A (i.e. the translations, rotations and reflections mapping P_A on P_i) corresponds to a function $\psi_i : \Gamma_A \to \Gamma_B$ which maps vertices $v_A \in \Gamma_A$ of the polyform onto their corresponding vertices v_i of the tile. For two vertices $v, w \in \Gamma_B$, we then define the equivalence relation $\equiv_{DC}$ such that $v \equiv_{DC} w$ if there exists a $v_A \in \Gamma_A$ such that $v = \psi_i(v_A)$ and $w = \psi_j(v_A)$ for some tiles P_i and P_j, or if both vertices are part of the internal boundaries, i.e. $v, w \in \partial^T \Gamma_B$. We denote by $[v]_{DC}$ the equivalence class of v induced by $\equiv_{DC}$.

Corollary 8. *Let* $b \in \partial^T \Gamma_B$ *be a vertex of the internal boundaries. Then, the number of neighbors of* b *in every equivalence class* $[v_B]_{DC}$, $v_B \in \Gamma_B$ *carrying a positive sign is equal to the number of neighbors carrying a negative sign, i.e.* $\sum_{\substack{v \in [v_B]_{DC} \\ v \sim b}} s(v) = 0$.

Proof. Assume that b has at least one neighbor in $[v_B]_{DC}$; otherwise the corollary is trivially satisfied. Also, assume $v_B \notin \partial^T \Gamma_B$, since otherwise $s(v) = 0$ for all $v \in [v_B]_{DC}$, from which the corollary also trivially follows. Denote by $N(b, v_B) = \{v \in [v_B]_{DC} | v \sim b\}$ the set of neighbors of b in the equivalence class of v_B. Every vertex can have maximally four neighbors, thus $|N(b, v_B)| \leq 4$. Being part of the internal boundaries, b must lie on at least one common edge (including endpoints) of two tiles $P_i \neq P_j$. These two tiles can be mapped onto one another by reflection on the common edge, and thus must have opposite signs. If a vertex $v \in \Gamma_i$ of P_i is a neighbor of b, it follows that there must be a vertex $w \in \Gamma_j$ of P_j which is also a neighbor of b, and which has opposite sign, i.e. $s(w) = -s(v)$. This excludes the case $|N(b, v_B)| = 1$, and proves the corollary for $|N(b, v_B)| = 2$. For $|N(b, v_B)| \in \{3, 4\}$, the structure of M directly implies that b has to lie on a common corner of three, respectively four, tiles. The corresponding internal angles of the tiles have to be smaller or equal to $360°/3 = 120°$, respectively $360°/4 = 90°$. The definition of M only admits internal angles which are multiples of $45°$ (Figure 1A). Thus, in both cases, only angles of $45°$ or $90°$ are possible. An angle of $45°$ is only possible if all $v \in [v_B]$ lie on the internal boundaries, which implies $s(v) = 0$ (see above). If the angle is $90°$, $|N(b, v_B)| = 3$ would imply that P_B is not convex, which can thus be excluded. Finally, if the angle is $90°$ and $|N(b, v_B)| = 4$, each of the four tiles to which these vertices belong must have exactly two adjacent tiles with opposite signs. □

By Lemma 4, the sandpile group G_Γ is isomorphic to $\mathscr{H}_G^\Gamma = \{H \in \mathscr{H}_\mathbb{Q}^\Gamma | \Delta_\Gamma H|_{\partial\Gamma} \in \mathbb{Z}^{\partial\Gamma}\} / \mathscr{H}_\mathbb{Z}^\Gamma$. To prove Theorem 1, it thus suffices to construct a monomorphism from $\mathscr{H}_G^{P_A}$ to $\mathscr{H}_G^{P_B}$ given

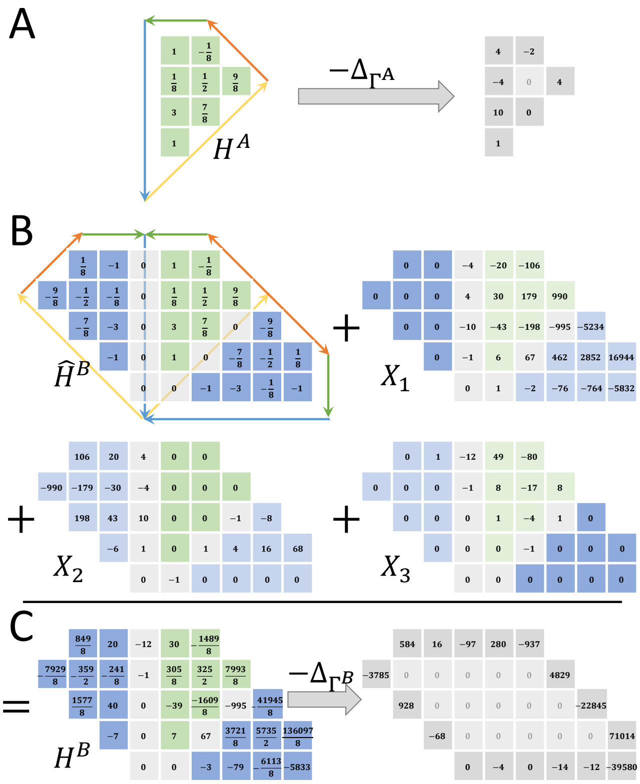

Figure 3. Construction of sandpile monomorphisms. A) A harmonic function $H_A \in \mathscr{H}_G^{\Gamma_A}$ (left) corresponding to the element $[-\Delta_{\Gamma_A} H_A] \in G_{P_A}$ of the sandpile group (right) on a given M-polyform P_A^{DC} (colored arrows indicate edges, green squares vertices of $\Gamma_A = \Gamma(P_A)$). B) The M-polyform P_A from (A) DC-tiles the depicted M-polyform P_B. Depicted is the rational-valued function $\widehat{H}_B$ (top-left), and the three integer-valued functions X_1, X_2 and X_3 (one for each tile) which cure the non-harmoniticity of $\widehat{H}_B$. C) The function $H_B = \widehat{H}_B + X_1 + X_2 + X_3$ is harmonic everywhere and can be reinterpreted as an element of $\mathscr{H}_G^{\Gamma_B}$ (left). It corresponds to the element $[-\Delta_{\Gamma_B} H_B] \in G_{P_B}$ of the sandpile group on P_B onto which $[-\Delta_{\Gamma_A} H_A] \in G_{P_B}$ H_A is mapped by the monomorphism (right).

a DC-tiling $T^{P_A \to P_B}$ of the M-polyform P_B by P_A. For a harmonic function $H_A \in \mathscr{H}_G^{\Gamma_A}$, define the rational-valued function $\widehat{H}_B \in \mathbb{Q}^{\Gamma_B}$ in the following way (Figure 3A&B): for each vertex $v \in \Gamma_i$ belonging to tile $P_i \in T^{P_A \to P_B}$, set $\widehat{H}_B(v) = s(P_i) H_A(v_A)$ with $v_A \in \Gamma_A$ the unique vertex satisfying $\psi_i(v_A) = v$. Otherwise, that is if v belongs to the internal boundaries, set $\widehat{H}_B(v) = 0$. Because $\widehat{H}_B(b) = 0$ for all vertices $b \in \partial^T \Gamma_B$ of the internal boundaries, Corollary 7 implies that $\Delta_{\Gamma_B} \widehat{H}_B(v) = s(\Gamma_i) \Delta_{\Gamma_A} H_A(v_A)$ for all vertices $v \in \Gamma_i$ belonging to the domain of a tile P_i, with $\psi_i(v_A) = v$. This implies that the Laplacian of $\widehat{H}_B$ is zero in the interior of the sub-domains Γ_i of Γ_B, and integer-valued at their boundaries. From Corollary 8, on the other hand, it follows that $\Delta_{\Gamma_B} \widehat{H}_B(b) = 0$ for every vertex $b \in \partial^T \Gamma_B$ of the internal boundaries. Thus, $\widehat{H}_B$ is harmonic nearly everywhere, except at the vertices directly adjacent to (but not including) the internal boundaries, for which $\Delta_{\Gamma_B} \widehat{H}_B$ is integer-valued.

The "harmonic deficit" of $\widehat{H}_B$ can be cured, one tile at a time: for a given tile P_i, we can define an integer-valued function $X_i \in \mathbb{Z}^{\Gamma_B}$ whose Laplacian is zero everywhere in the interior of Γ_B, except for those vertices $v \in \partial \Gamma_i \setminus \partial \Gamma_B$ at the boundary of Γ_i which are not also at the boundary of

Γ_B. Note that these vertices are all next to an internal boundary. For these vertices $v \in \partial\Gamma_i \setminus \partial\Gamma_B$, we require that $\Delta_{\Gamma_B} X_i(v) = -\Delta_{\Gamma_B} \widehat{H}_B(v)$. For example, X_i can be defined by the following iterative algorithm: first, set $X_i(v) = 0$ for all $v \in \Gamma_i$. Define $\Gamma^0 = \Gamma_i$ and $B^0 = \{v \in \Gamma_B \setminus \Gamma_i | \exists\, w \in \Gamma_i : v \sim w\}$. For every $v \in B^0$, choose an integer value $X_i(v)$ such that for every $w \in \partial\Gamma_i \setminus \partial\Gamma_B$, $\Delta_{\Gamma_B} \widehat{H}_B(w) = -\Delta_{\Gamma_B} X_i(w)$. Note that, since Γ_i is convex, the corresponding equation system always admits a solution (even though non-unique). Then, in every iteration $s = 1, 2, \ldots$, set $\Gamma^s = \Gamma^{s-1} \cup B^{s-1}$ and $B^s = \{v \in \Gamma_B \setminus \Gamma^s | \exists\, w \in \Gamma^s : v \sim w\}$. Note that Γ^s is convex again. Now, for every $v \in B^s$, choose an integer value $X_i(v)$ such that for every $w \in \partial\Gamma^s \setminus \partial\Gamma_B$, $\Delta_{\Gamma_B} X_i(w) = 0$. Again, this is possible due to convexity of Γ^s. Continue until all values of X_i are defined. In general, there is some degree of freedom to choose the values of X_i assigned in every step of this algorithm. However, note that the difference $X_i^\alpha - X_i^\beta$ of every two such possible choices X_i^α and X_i^β is integer-valued harmonic.

Given the functions X_i, we define $H_B \in \mathbb{Q}^\Gamma$ by $H_B = \widehat{H}_B + \sum_i X_i$. By construction, H_B is harmonic everywhere and has an integer-valued Laplacian. We can thus reinterpret H_B to be an element of $\mathscr{H}_{\mathbb{G}}^{\Gamma_B}$. It is then easy to see that the function $\xi : \mathscr{H}_G^{P_A} \to \mathscr{H}_G^{P_B}$, $\xi(H_A) = H_B$, is injective, and that it satisfies $\xi(H_B^1 + H_B^2) = \xi(H_B^1) + \xi(H_B^2)$. The function ξ is thus a group monomorphism, and with the isomorphism $-\Delta_\Gamma : \mathscr{H}_G^P \cong G_P$ from Lemma 4, we get that $\mu(T^{P_A \to P_B}) = \Delta_{\Gamma_B} \circ \xi \circ (\Delta_{\Gamma_A})^{-1} : G_{P_A} \rightarrowtail G_{P_B}$ is the group monomorphism stated to exist in Theorem 1.

Acknowledgements

The authors are grateful to Nikita Kalinin for carefully reading the manuscript.

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