



INSTITUT DE FRANCE  
Académie des sciences

# *Comptes Rendus*

---

## *Mathématique*


Dario Corona and Alessandro Della Corte

### **The critical exponent functions**

Volume 360 (2022), p. 315-332

<<https://doi.org/10.5802/crmath.286>>

© Académie des sciences, Paris and the authors, 2022.  
*Some rights reserved.*

 This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mathématique* sont membres du  
Centre Mersenne pour l'édition scientifique ouverte  
[www.centre-mersenne.org](http://www.centre-mersenne.org)



---

Combinatorics, Dynamical systems / *Combinatoire, Systèmes dynamiques*

# The critical exponent functions

Dario Corona \*, <sup>a</sup> and Alessandro Della Corte <sup>a</sup>

<sup>a</sup> University of Camerino, School of Science and Technology Camerino (MC), Italy

E-mails: [dario.corona@unicam.it](mailto:dario.corona@unicam.it) (D. Corona), [alessandro.dellacorte@unicam.it](mailto:alessandro.dellacorte@unicam.it)  
(A. Della Corte)

**Abstract.** The *critical exponent* of a finite or infinite word  $w$  over a given alphabet is the supremum of the reals  $\alpha$  for which  $w$  contains an  $\alpha$ -power. We study the maps associating to every real in the unit interval the inverse of the critical exponent of its base- $n$  expansion. We strengthen a combinatorial result by J.D. Currie and N. Rampersad to show that these maps are left- or right-Darboux at every point, and use dynamical methods to show that they have infinitely many nontrivial fixed points and infinite topological entropy. Moreover, we show that our model-case map is topologically mixing.

**2020 Mathematics Subject Classification.** 37B40, 37B20, 68R15, 26A21, 26A18.

*Manuscript received 4th September 2021, revised 13th October 2021, accepted 19th October 2021.*

## 1. Introduction

The base- $n$  expansion of real numbers has often been used to define sets and functions having interesting properties. In many cases, it has been through this path that remarkable counterexamples to appealing but mistaken intuition in the theory of real functions has been identified. For instance the Cantor set, which is basic for the construction of counterexamples in analysis and topology, can be very simply defined by means of base-3 expansion. Early examples of transcendental numbers have been defined through simple recursive procedures based on the decimal expansion ([15]) and the rich theory of normal numbers, which has been very fruitful in measure theory and measurable dynamics, exploits the base- $n$  expansion at its very beginning (a very nice reference work is [17]).

In the last decades, attention has been devoted to the concept of *critical exponent* of a word  $w$  over a given alphabet, that is the supremum of the reals  $\alpha$  for which  $w$  contains an  $\alpha$ -power. The study of critical exponents is natural for understanding the repetition threshold of an alphabet (see [5, p. 126]). The long-standing problem of finding the repetition threshold as a function of the alphabet size was recently solved by M. Rao ([18]) (and independently by J. D. Currie and N. Rampersad ([8])), who covered the last cases of a general conjecture by F. Dejean ([9]), and the answer is that the repetition threshold for size  $n$  is  $\frac{n}{n-1}$  for  $n = 2$  and  $n \geq 5$ , while it is  $7/4$  for

---

\* Corresponding author.

$n = 3$  and  $7/5$  for  $n = 4$ . The critical exponent and related concepts (such as the Diophantine exponent of a sequence) proved significant in symbolic dynamics ([2]) and transcendental number theory ([1]).

In this paper we explore the relations between the combinatorial concept of critical exponent and the analytical and dynamical properties of objects defined through it. We do so by considering, as a prototype, the  $n$ -critical exponent function  $\kappa_n : [0, 1] \rightarrow [0, 1]$ , i.e. the map associating to every real in the unit interval the inverse of the critical exponent of its (infinite) base- $n$  expansion (we assume  $1/\infty = 0$ ). We mainly focus on the model case  $\kappa_2$ . In particular, we strengthen a combinatorial result by J. D. Currie and N. Rampersad ([7]) to prove, using dynamical methods, that the critical exponent functions have infinite topological entropy and infinitely many fixed points.

The very fine dependence shown by the critical exponent on the combinatorial properties of the word makes the analytical and dynamical properties of functions defined through it quite complicated in principle. For instance, while simple normality of reals is concerned with asymptotic densities, and thus is unaffected by changes limited to prefixes of any finite length, the critical exponent may depend sharply on a single digit of an infinite word. This is the main reason for which methods developed for the study of dynamics of highly irregular maps (typically having dense discontinuities) prove useful in our framework. The most relevant feature of our approach is indeed the interplay between methods and results from combinatorics on words and quite recent results on the study of the dynamics of Baire 1 maps ([21, 22]) and Darboux maps ([16]). In particular, we establish the existence of nontrivial fixed points for  $\kappa_n$ , which is in itself a genuinely combinatorial property, by exploiting dynamical properties of Darboux–Baire 1 maps.

The structure of the paper is as follows. In Section 2 we present the notation and the basic concepts that we use throughout the paper. In Section 3 we define the critical exponent and prove a combinatorial result which will be used throughout. In Section 4 we study the model case  $\kappa_2$ , in which the critical exponent function is defined using base-2 expansion, investigating its basic analytical and dynamical properties. In Section 5 we generalize most of our results to the critical exponent function in base  $n$ , indicated by  $\kappa_n$ . In Section 6 we introduce and study the map defined as the pointwise supremum of  $\kappa_n$ , which is independent of the choice of a base. Finally, in Section 7 some open questions are presented.

## 2. Notation and preliminaries

For an alphabet  $\mathcal{A} = \{a_0, a_1, \dots, a_n\}$ , we denote by  $\mathcal{A}^*$  and  $\mathcal{A}^\omega$  the set of the finite and infinite words over  $\mathcal{A}$  respectively. We use the symbol  $\epsilon$  for the empty word and denote by  $\mathcal{A}^+$  the set of finite words different from  $\epsilon$ . We set

$$\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega.$$

If  $w \in \mathcal{A}^\infty$ , we denote by  $w_i$  its  $i^{\text{th}}$  digit. An element of an indexed set of words will be denoted by  $w_{(i)}$ . We indicate by  $\ell : \mathcal{A}^* \rightarrow \mathbb{N}$  the map associating to the nonempty finite word  $a = a_1 \dots a_n$  the natural number  $n$ , while we set  $\ell(\epsilon) = 0$ .

Let  $y, w \in \mathcal{A}^\infty$ . We say that  $y$  is a *subword* (in the literature the term *factor* is also used with the same meaning) of  $w$  if there exist  $w_{(1)} \in \mathcal{A}^*$  and  $w_{(2)} \in \mathcal{A}^\infty$  such that  $w = w_{(1)}yw_{(2)}$ . We say that  $y \in \mathcal{A}^+$  is a *prefix* (*suffix*) of  $w$  if  $w = yw_{(2)}$  ( $w = w_{(1)}y$ ). For every  $w \in \mathcal{A}^\omega$ , we denote by  $\mathcal{L}(w)$  the language of  $w$ , namely the set of all finite subwords of  $w$ .

For every  $w \in \mathcal{A}^*$  and  $n \in \mathbb{N}$ , we denote by  $w^n$  the concatenation of  $n$ -copies of  $w$ , namely

$$w^n := \underbrace{ww \dots w}_{n\text{-times}}.$$

We denote by  $\delta: \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty$  the deletion operator that removes the first digit of a word. Hence, if  $w = w_0 w_1 w_2 w_3 \dots \in \mathcal{A}^\infty$ , we have

$$\delta(w) = \delta(w_0 w_1 w_2 w_3 \dots) = w_1 w_2 w_3 \dots,$$

while  $\delta(\epsilon) = \epsilon$ . Accordingly, for every  $n \in \mathbb{N}$ , the operator  $\delta^n$  removes the first  $n$  letters of a word, hence

$$\delta^n(w) = \delta^n(w_0 w_1 \dots w_n w_{n+1} \dots) = w_n w_{n+1} w_{n+2} \dots.$$

For every two real numbers  $a < b$ , we denote by  $]a, b[$  the open interval with endpoints  $a$  and  $b$ . For each  $x \in [0, 1]$ , we denote by  $w_x \in \{0, 1\}^\omega$  the binary expansion of  $x$ , that is

$$x = (0.w_x)_2 = \sum_{i=0}^{\infty} \frac{(w_x)_i}{2^i}.$$

If  $x \in (0, 1]$  is a dyadic rational, we consider its binary expansion  $w_x$  whose digits are ultimately 1, for instance  $w_{\frac{1}{2}} = 01111\dots$

**Definition 1.** A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be of Baire class 0 if it is continuous, and for every positive integer  $N$  it is said to be of Baire class  $N$  if there exists a sequence  $(f_n)$  of functions of Baire class  $N-1$  which converges pointwise to  $f$ . For every  $N \geq 1$ , we denote by  $\mathcal{B}_N$  the set of all functions of Baire class  $N$  defined over  $[0, 1]$ .

**Definition 2.** A function  $f: [0, 1] \rightarrow \mathbb{R}$  is a Darboux function if it satisfies the intermediate value property. More precisely,  $f$  is a Darboux function if for all  $x, y \in [0, 1]$  and  $\alpha$  between  $f(x)$  and  $f(y)$ , there exists a number  $z$  between  $x$  and  $y$  such that  $f(z) = \alpha$ .

If  $f: [0, 1] \rightarrow [0, 1]$ , for every  $n \in \mathbb{N}$  we denote by  $f^n: [0, 1] \rightarrow [0, 1]$  the  $n^{\text{th}}$  iteration of  $f$ , hence

$$f^n(x) = \underbrace{f(f(\dots(f(x))))}_{n \text{ times}}.$$

### 3. The critical exponent and a combinatorial result

We recall here the definition of critical exponent of a word (cf. [5]). Let us fix an alphabet  $\mathcal{A}$ . For positive integers  $p$  and  $q$ , a word  $w \in \mathcal{A}^+$  is a  $p/q$ -power if it is of the form  $w = x^n y$ , where  $x \in \mathcal{A}^+$ ,  $y \in \mathcal{A}^*$  is a prefix of  $x$ ,  $\ell(w) = p$  and  $\ell(x) = q$  (cf. [9]). Let  $\alpha \in \mathbb{Q}^+$ . A word  $w \in \mathcal{A}^\infty \setminus \epsilon$  avoids  $\alpha$ -powers if none of its subwords is an  $r$ -power for any rational  $r \geq \alpha$ . Otherwise, we say that  $w$  contains an  $\alpha$ -power.

**Definition 3.** The critical exponent of a word  $E: \mathcal{A}^\infty \rightarrow \mathbb{R}$  is defined as

$$E(w) = \begin{cases} \sup\{r \in \mathbb{Q} : w \text{ contains an } r\text{-power}\}, & \text{if } w \neq \epsilon, \\ 0, & \text{if } w = \epsilon. \end{cases}$$

The repetition threshold of an alphabet  $\mathcal{A}$  is the least possible critical exponent for infinite words over  $\mathcal{A}$ . The answer to the question about the repetition threshold of  $\{0, 1\}$  was given by the well-known Thue–Morse sequence, defined below, which has proven significant in as different fields as combinatorics of words, number theory, differential geometry and Morse theory ([3]).

**Definition 4.** The Thue–Morse sequence is the element  $w_\tau = w_0 w_1 w_2 \dots w_n \dots \in \{0, 1\}^\omega$  given by

$$\begin{cases} w_0 = 0, \\ w_{2n} = w_n, \\ w_{2n+1} = 1 - w_n. \end{cases}$$

The Thue–Morse sequence starts as follows

$$w_\tau = 0 \ 1 \ 10 \ 1001 \ 10010110 \ 1001011001101001 \ \dots$$

It has been proved (see the classical paper [23]) that  $E(w_\tau) = 2$ .

**Remark 5.** The Thue–Morse sequence can be also defined recursively using the bitwise negation as follows. The first digit is  $w_0 = 0$ . Then, if the first  $2^n$  digits are given, then the next  $2^n$  digits are their bitwise negation. More precisely, if we have

$$w_0, w_1, \dots, w_{2^n-1},$$

then

$$w_k = 1 - w_{k-2^n}, \quad \forall k = 2^n, 2^n + 1, \dots, 2^{2^n} - 1.$$

**Remark 6.** Let  $w \in \{0, 1\}^+$  and  $y \in \{0, 1\}^\omega$  such that  $E(y) = \alpha > E(w)$ . If  $E(wy) > \alpha$ , then there exists a finite subword  $v$  of  $wy$  such that  $E(v) > \alpha$ . Moreover,  $v$  is not a subword of  $w$  or a subword of  $y$ . As a consequence,  $v = sp$ , with  $s$  a suffix of  $w$  and  $p$  a prefix of  $y$ .

The next theorem guarantees the existence of binary words constrained to a given prefix  $w$  having any critical exponent compatible with  $w$ . This theorem is a strengthening of [7, Theorem 6], which corresponds to the particular case  $w = \epsilon$ .

**Theorem 7.** Let  $w \in \{0, 1\}^*$ . For every  $\alpha > \max\{2, E(w)\}$ , there exists  $y \in \{0, 1\}^\omega$  such that  $E(wy) = \alpha$ .

**Proof.** As usual in this context, the tricky part is to make sure that there is no suffix of  $w$  which, concatenated with a prefix of  $y$ , is a  $\beta$ -power, with  $\beta > \alpha$ .

By the construction given in the proof of [7, Theorem 6], there exists an infinite binary word  $z$  such that

$$E(z) = \alpha, \quad \text{and} \quad E(\delta^n(z)) = \alpha, \quad \forall n \in \mathbb{N}.$$

Moreover, for every  $m$  sufficiently large, we can choose  $z$  from a sequence  $(z_{(m)})_m$ , where

$$z_{(m)} = w_{\tau, (m)} w_{\tau, (m)} z_{(3, m)} \tag{1}$$

and  $w_{\tau, (m)}$  is the prefix of length  $2^m$  of the Thue–Morse sequence, with  $z_{(3, m)}$  an infinite binary word<sup>1</sup>.

The idea of the proof is to find  $n, m \in \mathbb{N}$  such that

$$E(w\delta^n(z_{(m)})) = \alpha.$$

Indeed, the word  $wz_{(m)}$  could have a critical exponent greater than  $\alpha$ . By Remark 6, this issue could occur only if  $w = \tilde{w}b$ , where  $\tilde{w}$  and  $b$  are a prefix and a suffix of  $w$  respectively, and  $z_{(m)} = pbp\tilde{b}\dots$ , where  $p$  is a prefix of  $z_{(m)}$  and  $\tilde{b}$  is a prefix of  $b$ , so that

$$wz_{(m)} = \tilde{w} \overbrace{b| \overbrace{p b \dots p b p}^{\lfloor \alpha \rfloor \text{ times}} \tilde{b}} = q \dots,$$

with  $E(q)$  greater than  $\alpha$ .

To avoid this problem, we proceed as follows. Let  $\tilde{b}$  be the longest suffix of  $w$  which is a subword of the Thue–Morse sequence. Notice that  $\tilde{b}$  is well defined, since the last digit of  $w$  is of course inside Thue–Morse and  $w$  is a finite word. We write  $w = \tilde{w}\tilde{b}$ . By (1), there exists  $m_w \in \mathbb{N}$  such that, for every  $m \geq m_w$ , the first  $2^m$  digits of  $z_{(m)}$  contain at least one occurrence of  $\tilde{b}$ . Then, for every  $m \geq m_w$ ,  $z_{(m)}$  can be written as in (1) with

$$w_{\tau, (m)} = z_{(1)} \tilde{b} z_{(2, m)},$$

<sup>1</sup>In the notation of [7], set  $s_0 = m$ ,  $r_0 = 3$ ,  $t_0 = 2^{m-1} + c$ , with  $c$  such that  $\beta_0$  is obtainable, and remove the first  $2^m - (t_0)$  digits so as to start with the beginning of the Thue–Morse sequence.

where  $z_{(1)}$  is the longest prefix of the Thue–Morse sequence such that  $z_{(1)}\bar{b}$  contains only one occurrence of  $\bar{b}$ . For the following steps, notice that this construction gives only a lower bound for the length of  $z_{(2,m)}$ , hence we are still free to extend it as we desire.

Removing the prefix  $z_{(1)}\bar{b}$  from  $z$ , we obtain  $y_{(m)} = z_{(2,m)}w_{\tau,(m)}z_{(3,m)}$ . By construction, there exists  $n \in \mathbb{N}$  such that  $y_{(m)} = \delta^n(z)$  for every  $m \geq m_w$ . As a consequence,  $E(y_{(m)}) = \alpha$ . We will show that, choosing properly the length of  $z_{(2,m)}$ , we obtain  $E(wy_{(m)}) = \alpha$ . In other words, we will prove the existence of an  $m_0 \geq m_w$  such that

$$E(wy_{(m_0)}) = \alpha. \tag{2}$$

Arguing by contradiction, assume that  $E(wy_{(m)}) > \alpha$  for every  $m \geq m_w$ . By Remark 6, for every  $m \geq m_w$  there exist a suffix  $s_{(m)}$  of  $w$  and a prefix  $p_{(m)}$  of  $y_{(m)}$  such that

$$E(s_{(m)}p_{(m)}) > \alpha.$$

Set  $u_{(m)} = s_{(m)}p_{(m)}$ . The suffix  $s_{(m)}$  of  $w$  cannot be a subword of the Thue–Morse sequence. Otherwise, by definition of  $y_{(m)}$ ,  $u_{(m)}$  should be a subword of  $z_{(m)}$ , hence  $E(z_{(m)}) \geq E(u_{(m)}) > \alpha$ , which is a contradiction. As a consequence,  $s_{(m)}$  is not a subword of  $z_{(2,m)}$ . Since  $E(u) > \alpha > 2$ ,  $s_{(m)}$  must appear at least twice in  $u_{(m)}$ , so  $p_{(m)}$  is of the form

$$p_{(m)} = x_{(m)}s_{(m)} \dots x_{(m)}\tilde{s}_{(m)},$$

with  $\tilde{s}_{(m)}$  a prefix of  $s_{(m)}$  and  $z_{(2,m)}$  a subword of  $x_{(m)}$ . As a consequence, we have  $\ell(x_{(m)}) > \ell(z_{(2,m)})$  and

$$E(u_{(m)}) = E(p_{(m)}) + \frac{\ell(s_{(m)})}{\ell(x_{(m)}s_{(m)})} \leq E(p_{(m)}) + \frac{\ell(w)}{\ell(x_{(m)}s_{(m)})} < E(p) + \frac{\ell(w)}{\ell(z_{(2,m)})}.$$

Thus,

$$\alpha \leq \limsup_{m \rightarrow \infty} E(u_{(m)}) \leq \limsup_{m \rightarrow \infty} E(p_{(m)}). \tag{3}$$

To obtain a contradiction and complete the proof, it remains to show that

$$\limsup_{m \rightarrow \infty} E(p_{(m)}) < \alpha. \tag{4}$$

Since  $p_{(m)}$  is a subword of  $y_{(m)}$ ,  $E(p_{(m)}) \leq \alpha$ . Hence, if  $\alpha \notin \mathbb{N}$ , then

$$E(p_{(m)}) = \lfloor \alpha \rfloor + \frac{\ell(\tilde{s}_{(m)})}{\ell(x_{(m)}s_{(m)})} < \lfloor \alpha \rfloor + \frac{\ell(w)}{\ell(z_{(2,m)})},$$

and therefore

$$\limsup_{m \rightarrow \infty} E(p_{(m)}) = \lfloor \alpha \rfloor < \alpha.$$

If  $\alpha \in \mathbb{N}$ ,  $\alpha \geq 3$ , then

$$E(p) = \alpha - 1 + \frac{\ell(\tilde{s}_{(m)})}{\ell(x_{(m)}s_{(m)})},$$

so

$$\limsup_{m \rightarrow \infty} E(p_{(m)}) = \alpha - 1 < \alpha.$$

By (3) and (4) we obtain a contradiction, so there exists an  $m_0 \geq m_w$  such that (2) holds, and we are done. □

**Remark 8.** By the construction of the proof of Theorem 7, the length of  $z_{(2,m)}$  can be chosen arbitrarily large. More formally, let  $w \in \{0, 1\}^*$  and  $\alpha > \max\{2, E(w)\}$ . Using the same notation of the proof of Theorem 7, for every  $m \geq m_w$ , the word  $y_{(m)}$  is well defined and  $E(y_{(m)}) = \alpha$ . The proof of 7 shows not only that there exists an  $m_0 \geq m_w$  such that  $E(wy_{(m_0)}) = \alpha$  holds, but also that there exists an increasing sequence  $(m_k)_k \subset \mathbb{N}$  such that  $m_k \geq m_w$  and  $E(wy_{(m_k)}) = \alpha$ . In the

following, such a sequence will be denoted by  $(y_{(\alpha,k)})_k \subset \{0, 1\}^\omega$ . We notice that every  $y_{(\alpha,k)}$  is of the form

$$y_{(\alpha,k)} = z_{(2,m_k)} w_{\tau,(m_k)} z_{(3,m_k)}.$$

Being  $(m_k)_k$  an increasing sequence, the length of  $z_{(2,m_k)}$  can be arbitrarily large.

#### 4. The model case $\kappa_2$

In this section, we define and analyze the 2-critical exponential function, which will be denoted by  $\kappa_2$ . Its definition is based on the critical exponent of a word, and many properties we shall present stem from Theorem 7. We remind that, given  $x \in [0, 1]$ ,  $w_x \in \{0, 1\}^\omega$  is its binary expansion and  $\mathcal{L}(w_x)$  is the set of all the finite subwords of  $w_x$ .

**Definition 9.** *The critical exponent function  $\kappa_2: [0, 1] \rightarrow [0, 1]$  is defined as*

$$\kappa_2(x) := \inf_{u \in \mathcal{L}(w_x)} \frac{1}{E(u)}.$$

**Remark 10.** Using the convention  $1/\infty = 0$ , we can write  $\kappa_2(x) = 1/E(w_x)$ .

##### 4.1. Analytical properties for $\kappa_2$

**Proposition 11.** *The function  $\kappa_2$  is upper semi-continuous and of Baire class 1.*

**Proof.** Let us fix  $x \in [0, 1]$  and  $\varepsilon > 0$ . By definition of  $\kappa_2$ , there exists a prefix of  $w_x$ , say  $w_\varepsilon \in \{0, 1\}^+$ , such that

$$\frac{1}{E(w_\varepsilon)} \leq \kappa_2(x) + \varepsilon.$$

If  $x = 0.a01^\infty = 0.a10^\infty$  ( $a \in \{0, 1\}^*$ ) is a nonzero dyadic rational, then, for any integer  $M > 0$ , the binary expansion of any real close enough to  $x$  will have a prefix of type  $a10^M$  or  $a01^M$ . Therefore,  $\lim_{n \rightarrow \infty} \kappa_2(x_n) = 0 = \kappa_2(x)$ , so  $\kappa_2$  is continuous at  $x$ . A similar argument of course applies if  $x = 0$ . Otherwise, there exist  $N_\varepsilon > 0$  and a sequence  $(y_n)_{n \geq N_\varepsilon} \subset \{0, 1\}^\omega$  such that

$$w_{x_n} = w_\varepsilon y_n, \quad \forall n \geq N_\varepsilon.$$

As a consequence,

$$\kappa_2(x_n) = \inf_{u \in \mathcal{L}(w_{x_n})} \frac{1}{E(u)} \leq \frac{1}{E(w_\varepsilon)} \leq \kappa_2(x) + \varepsilon, \quad \forall n \geq N_\varepsilon.$$

By the arbitrariness of  $\varepsilon$ , we obtain

$$\limsup_{n \rightarrow \infty} \kappa_2(x_n) \leq \kappa_2(x),$$

and by the arbitrariness of the sequence  $(x_n)_n$  we obtain that  $\kappa_2$  is upper semi-continuous. By a classical result (cf. [4]), every semi-continuous function is of Baire class 1, so  $\kappa_2 \in \mathcal{B}_1$ .  $\square$

Let us define the sets

$$C_2 := \{x \in I : \kappa_2(x) = 0\}$$

and

$$D_2 := \{x \in I : \kappa_2(x) \neq 0\}.$$

Since the binary expansion of every rational number in  $[0, 1]$  has a period, which is a finite word repeated infinitely many times, then

$$\mathbb{Q} \cap [0, 1] \subset C_2.$$

In the following Propositions 12-16, we will show that  $C_2$  coincides in fact with the continuity set of  $\kappa_2$  and prove some of its topological and measure-theoretic properties.

**Proposition 12.** *The function  $\kappa_2$  is continuous at  $x \in [0, 1]$  if and only if  $x \in C_2$ .*

**Proof.** Since  $\kappa_2(x) \geq 0$  for every  $x \in [0, 1]$ , Proposition 11 implies that if  $x \in C_2$ , then  $\kappa_2$  is continuous at  $x$ . On the other hand, for every  $x \notin C_2$  there exists a sequence  $(x_n)_n \subset \mathbb{Q} \cap [0, 1] \subset C_2$  such that  $x_n \rightarrow x$ . Therefore,

$$\lim_{n \rightarrow \infty} \kappa_2(x_n) = 0 \neq \kappa_2(x),$$

and  $\kappa_2$  is not continuous at  $x$ . □

**Remark 13.** By Proposition 12, the function  $\kappa_2$  is not quasi-continuous. Indeed, the graph of the restriction of every quasi-continuous function to its continuity set is dense in the graph of the function (see [19, p. 632]), whereas  $\kappa_2|_{C_2}$  is identically zero.

Before proving the next result, we introduce the following notation. We say that a number  $x \in [0, 1]$  is normal in base 2 if every word  $w \in \{0, 1\}^+$  has density  $2^{-\ell(w)}$  in  $w_x$  (cf. [17]). We set

$$\mathcal{N}_2 := \{x \in I : x \text{ is a normal number in base } 2\}.$$

**Proposition 14.** *The set  $C_2$  is a co-meagre set and it has full Lebesgue measure.*

**Proof.** Since the set of continuity of every real function is a  $G_\delta$  set, Proposition 12 implies that  $C_2$  is a  $G_\delta$ . Moreover,  $C_2$  is dense, since  $\mathbb{Q} \cap [0, 1] \subset C_2$ . Then,  $D_2$  is an  $F_\sigma$  subset of  $[0, 1]$ . As a consequence, it is either meagre or else has nonempty interior. Since for every interval  $]a, b[ \subset [0, 1]$  we have  $]a, b[ \cap C_2 \neq \emptyset$ , we obtain that  $D_2$  has empty interior, hence it is meagre. Thus,  $C_2$  is co-meagre.

Since  $\mathcal{N}_2$  has full Lebesgue measure (for an elementary proof see [12]) and  $\mathcal{N}_2 \subset C_2$ ,  $C_2$  has full Lebesgue measure. □

**Proposition 15.** *The sets  $C_2$  and  $D_2$  are dense in  $[0, 1]$ .*

**Proof.** Since  $\mathbb{Q} \cap [0, 1] \subset C_2$ , the set  $C_2$  is dense in  $[0, 1]$ . To prove that  $D_2$  is dense in  $[0, 1]$ , let us fix  $x \in [0, 1]$ . For every  $\varepsilon > 0$ , there exists a prefix of  $w_x$ , say  $w_\varepsilon$ , such that

$$|(0.w_\varepsilon z)_2 - x| \leq \varepsilon, \quad \forall z \in \{0, 1\}^\omega.$$

Hence, we obtain

$$x_\varepsilon = (0.w_\varepsilon w_\tau)_2 \in D_2 \cap [x - \varepsilon, x + \varepsilon].$$

By the arbitrariness of  $x$  and  $\varepsilon$ , we obtain the thesis. □

**Proposition 16.** *The sets  $C_2$  and  $D_2$  are neither closed nor open in  $[0, 1]$ .*

**Proof.** Since  $C_2$  and  $D_2$  are complementary sets, it suffices to show that  $C_2$  and  $D_2$  are not closed. To show that  $C_2$  is not closed, let us consider  $x_0 \in D_2$  and a sequence  $(x_k)_k$  of dyadic rationals which converges to  $x_0$ . Then  $x_k \in C_2$  for every  $k$ , and  $x_k \rightarrow x_0 \notin C_2$ . On the other hand, to show that  $D_2$  is not closed, let us consider the sequence  $(x_k)_k \subset [0, 1]$  defined as  $x_k = (0.0^k w_\tau)_2$ . Then,  $x_k \in D_2$  for every  $k$  and  $\lim_{k \rightarrow \infty} x_k = 0 \notin D_2$ . □

**Remark 17.** The number  $x_\tau = (0.w_\tau)_2 \in [0, 1]$  is such that  $\kappa_2(x_\tau) = 1/2$ , and  $x_\tau$  is a maximum of  $\kappa_2$ , since there is not a word  $w \in \{0, 1\}^\omega$  such that  $E(w) < 2$ .

**Proposition 18.** *The image of  $\kappa_2$  is  $[0, 1/2]$ . Moreover, the function  $\kappa_2$  has the following symmetry:*

$$\kappa_2(x) = \kappa_2(1 - x), \quad \forall x \in [0, 1].$$

**Proof.** By [7, Theorem 6], for each  $\alpha > 2$  there is an infinite binary word, say  $w_\alpha \in \{0, 1\}^\omega$ , with critical exponent  $\alpha$ . Therefore,  $x_\alpha = (0.w_\alpha)_2 \in [0, 1]$  is such that  $\kappa_2(x_\alpha) = 1/\alpha$ . By the arbitrariness of  $\alpha > 2$ , we obtain  $]0, 1/2[ \subset \kappa_2(I)$ . Since  $\kappa_2(x_\tau) = 1/2$  and  $C_2$  is non-empty, we have  $\kappa_2([0, 1]) = [0, 1/2]$ .

To show the desired symmetry, it suffices to notice that for every  $x \in [0, 1]$  the word  $w_{1-x}$  is the bitwise negation of  $w_x$ . □



For every function  $f: [0, 1] \rightarrow \mathbb{R}$ , we denote by  $R^-(f, x)$  and  $R^+(f, x)$  the left and right range of  $f$  at  $x$ , respectively. More precisely,

$$R^-(f, x) := \left\{ \alpha \in \mathbb{R} : f^{-1}(\alpha) \cap (x - \delta, x) \neq \emptyset, \forall \delta > 0 \right\},$$

$$R^+(f, x) := \left\{ \alpha \in \mathbb{R} : f^{-1}(\alpha) \cap (x, x + \delta) \neq \emptyset, \forall \delta > 0 \right\}.$$

Sufficiently well-behaved functions will have typically “small” left and right ranges at every point. For instance, if the left (right) limit of  $f$  at  $x$  exists, then its left (right) range at  $x$  consists at most of one point. On the contrary, the map  $\kappa_2$  has “large” left and right ranges at every point  $x \in D_2$ , as proved in the following proposition. We will exploit this to get other properties of  $\kappa_2$ , such as the existence of almost-fixed points and fixed points for  $\kappa_2$ .

**Proposition 19.** *If  $x \in C_2$ , then*

$$R^-(\kappa_2, x) \cup R^+(\kappa_2, x) = \{0\}. \tag{5}$$

*Otherwise, if  $x \in D_2$ , then at least one of the following holds:*

$$[0, \kappa_2(x)[ \subset R^-(\kappa_2, x) \quad \text{or} \quad [0, \kappa_2(x)[ \subset R^+(\kappa_2, x). \tag{6}$$

To prove the previous proposition, we need the following lemma.

**Lemma 20.** *Let  $x \in [0, 1]$  and let us denote by  $w_{(n)}$  the finite prefix of  $w_x$  of length  $n$ . Then for every  $n$  at least one of the following holds:*

- (a) *for every  $\alpha > \max\{2, E(w_{(n)})\}$  there exists a word  $y_{\alpha, n} \in \{0, 1\}^\omega$  such that  $E(w_{(n)}y_{\alpha, n}) = \alpha$  and  $(w_{(n)}y_{\alpha, n})_2 < x$ ;*
- (b) *for every  $\alpha > \max\{2, E(w_{(n)})\}$  there exists a word  $y_{\alpha, n} \in \{0, 1\}^\omega$  such that  $E(w_{(n)}y_{\alpha, n}) = \alpha$  and  $(w_{(n)}y_{\alpha, n})_2 > x$ .*

**Proof.** Let us fix  $n \geq 1$ . Using the same notation of the proof of Theorem 7, for every  $m \geq m_{w_{(n)}}$ , let  $w_{\tau, (m)}$  be the prefix of the Thue–Morse sequence of length  $2^m$  and  $z_{(2, m)}$  such that

$$w_{\tau, (m)} = z_{(1)} \bar{b} z_{(2, m)},$$

being  $z_{(1)}$  and  $\bar{b}$  uniquely determined by  $w_{(n)}$ . Moreover, let us denote by  $z_{(2)}$  the infinite word over  $\{0, 1\}$  such that

$$w_\tau = z_{(1)} \bar{b} z_{(2)}.$$

Notice that  $z_{(2, m)} \rightarrow z_{(2)}$  as  $m \rightarrow \infty$ . By Remark 8, for every  $\alpha > \{\max\{2, E(w_{(n)})\}$  there exists an increasing sequence  $(m_k)_k \subset \mathbb{N}$  such that, setting

$$y_{(\alpha, k)} = z_{(2, m_k)} w_{\tau, (m_k)} z_{(3, m_k)},$$

we have  $E(w_{(n)}y_{(\alpha, k)}) = \alpha$ , for every  $k \in \mathbb{N}$ . Notice that  $z_{2, m_k}$  does not depend on  $\alpha$ . We divide the remaining part of the proof in two cases:

- (i) if  $w_x \neq w_{(n)}z_{(2)}$ , then these two words differ on at least one digit. Let  $j > n$  be the index of the first different digit, hence  $(w_x)_j \neq (w_{(n)}z_{(2)})_j$ . Let  $z_{(2, j)}$  be the prefix of  $z_{(2)}$  with  $\ell(w_{(n)}z_{(2, j)}) = j$ . Then either  $(0.w_{(n)}z_{(2, j)}r)_2 < x$  for all  $r \in \{0, 1\}^\omega$  or  $(0.w_{(n)}z_{(2, j)}r)_2 > x$  for all  $r \in \{0, 1\}^\omega$ . As a consequence, since  $z_{(2, m_k)} \rightarrow z_{(2)}$  as  $k \rightarrow \infty$  and since for every  $\alpha$  we can choose  $y_{(\alpha, k)}$  such that  $\ell(w_{(n)}z_{(2, m_k)}) > j$ , we obtain the thesis.
- (ii) if  $w_x = w_{(n)}z_{(2)}$ , then  $w_x$  is definitely equal to the Thue–Morse sequence  $w_\tau$ . By the bitwise negation construction of the Thue–Morse sequence presented in Remark 5, for every  $m \geq m_{w_{(n)}}$  we have

$$w_x = w_{(n)}z_{(2, m)}1001\dots$$

At the same time, for every  $\alpha > \max\{2, E(w_{(n)})\}$  and  $m \geq m_{w_{(n)}}$  the word  $y_{(\alpha, m)}$  is given by  $w_{(n)}z_{(2, m)}w_{\tau, (m)}z_{(3, m)}$ , hence  $w_{(n)}y_{(\alpha, m)} = w_{(n)}z_{(2, m)}0110\dots$ . As a consequence, for every  $\alpha$  we obtain  $(w_{(n)}y_{(\alpha, m)})_2 < (w_x)_2 = x$ , so a) holds.

□

**Remark 21.** Let  $x_\tau = (0.w_\tau)_2$ . Then for all  $n \in \mathbb{N}$  the construction given at the point (ii) of the previous proof applies. As a consequence, denoting by  $w_{\tau, n}$  the prefix of  $w_\tau$  such that  $\ell(w_{\tau, n}) = n$ , for every  $n \in \mathbb{N}$  and every  $\alpha > 2$  there exists an infinite word  $y_{(\alpha, n)}$  such that  $E(w_{\tau, n}y_{(\alpha, n)}) = \alpha$  and  $(w_{\tau, n}y_{(\alpha, n)})_2 < x_\tau$ .

**Proof of Proposition 19.** If  $x \in C_2$ ,  $\kappa_2$  is continuous at  $x$  by Proposition 12. As a consequence, for every  $\alpha \neq 0$  there exists a neighbourhood  $I_\alpha$  of  $x$  such that  $\alpha \notin \kappa_2(I_\alpha)$ . This implies that  $\alpha \notin R^-(\kappa_2, x) \cup R^+(\kappa_2, x)$ . On the other hand, for every neighbourhood of  $x$  there exists a number  $y \neq x$  such that  $\kappa_2(y) = 0$  and (5) holds.

Let  $x \in D_2$ , hence  $\kappa_2(x) > 0$ . Let us fix  $n \in \mathbb{N}$  and denote by  $w_{(n)}$  the finite prefix of  $w_x$  of length  $n$ . Moreover, let us set  $A = [0, \kappa_2(x)[$ . Since  $E(w_x) \geq E(w_{(n)})$ , by Lemma 20, there exists a set of words  $(y_{\alpha, n})_{\alpha \in A} \subset \{0, 1\}^\omega$  such that  $\kappa_2((w_{(n)}y_{\alpha, n})_2) = \alpha$  and either

$$(0.w_{(n)}y_{\alpha, n})_2 < x, \quad \forall \alpha \in A, \tag{7}$$

or

$$(0.w_{(n)}y_{\alpha, n})_2 > x, \quad \forall \alpha \in A. \tag{8}$$

Therefore, we can define the sets

$$L = \{n \in \mathbb{N} : (7) \text{ holds}\} \quad \text{and} \quad R = \{n \in \mathbb{N} : (8) \text{ holds}\}.$$

Since  $L \cup R = \mathbb{N}$ , at least one between  $L$  and  $R$  has infinite elements. Without loss of generality, let us assume that it is  $L$ . Then, for every  $\varepsilon > 0$  there exists a number  $n_\varepsilon \in L$  such that the interval characterized by  $w_{n_\varepsilon}$ , namely

$$I_{w_{n_\varepsilon}} = \{(w_{n_\varepsilon}y)_2 : y \in \{0, 1\}^\omega\},$$

satisfies

$$I_{w_{n_\varepsilon}} \subset ]x - \varepsilon, x + \varepsilon[.$$

Since  $n_\varepsilon \in L$ , for every  $\alpha \in [0, \kappa_2(x)[$  there exists  $y_{\alpha, n_\varepsilon}$  such that

$$E(0.w_{(n)}y_{\alpha, n}) = \frac{1}{\alpha}, \quad \text{and} \quad (0.w_{(n)}y_{\alpha, n})_2 < x,$$

hence

$$\kappa_2((0.w_{(n)}y_{\alpha, n})_2) = \alpha, \quad \text{and} \quad (0.w_{(n)}y_{\alpha, n})_2 \in I_{w_{n_\varepsilon}} \cap [0, x[ \subset ]x - \varepsilon, x[.$$

By the arbitrariness of  $\alpha \in [0, \kappa_2(x)[$  and  $\varepsilon > 0$ , we have  $R^-(\kappa_2, x) = [0, \kappa_2(x)[$ . Otherwise, if  $R$  is an infinite set, the same argument shows that  $R^+(\kappa_2, x) = [0, \kappa_2(x)[$ . □

By Remark 21 and Proposition 19, we obtain the following result.

**Corollary 22.** Let  $x_\tau = (0.w_\tau)_2$ . Then  $R^-(\kappa_2, x_\tau) = [0, 1/2[$ .

### 4.2. Dynamical properties for $\kappa_2$

We give the following definition (cf. [16]).

**Definition 23.** Let  $f: [0, 1] \rightarrow \mathbb{R}$ . A point  $x \in [0, 1]$  is said an almost fixed point of  $f$ , and we write  $x \in \text{aFix}(f)$ , if and only if

$$x \in \text{Int}(R^-(f, x)) \cup \text{Int}(R^+(f, x)). \tag{9}$$

If  $x = 0$  or  $x = 1$ , then we only consider  $R^+(f, x)$  or  $R^-(f, x)$ , respectively.

**Corollary 24.** A point  $x \in [0, 1]$  is an almost fixed point for  $\kappa_2$  if and only if  $\kappa_2(x) > x$ .

**Proof.** If  $x$  is an almost fixed point, then there exists a sequence  $(x_n)_n \subset [0, 1]$  which converges to  $x$  and such that  $\kappa_2(x_n) = x$ , for every  $n \in \mathbb{N}$ . By the upper semi-continuity of  $\kappa_2$  we have

$$x = \limsup_{n \rightarrow \infty} \kappa_2(x_n) \leq \kappa_2(x).$$

It remains to show that  $x \neq \kappa_2(x)$ . By (9), there exists an  $\varepsilon > 0$  such that  $[x - \varepsilon, x + \varepsilon] \subset R^-(\kappa_2, x)$ , or  $[x - \varepsilon, x + \varepsilon] \subset R^+(\kappa_2, x)$ . Assuming by contradiction that  $x = \kappa_2(x)$ , this implies the existence of a sequence  $(x_n)_n \subset [0, 1]$  which converges to  $x$  and such that  $\kappa_2(x_n) = x + \varepsilon = \kappa_2(x) + \varepsilon$ . Using again the upper semi-continuity of  $\kappa_2$  we obtain

$$\kappa_2(x) + \varepsilon = x + \varepsilon = \limsup_{n \rightarrow \infty} \kappa_2(x_n) \leq \kappa_2(x),$$

which is absurd.

If  $\kappa_2(x) > x$ , then Proposition 19 implies that in every neighbourhood of  $x$  there exists a number  $y$  such that  $\kappa_2(y) = x$ . Thus  $x$  is an almost fixed point.  $\square$

**Corollary 25.** The set  $\text{aFix}(\kappa_2)$  is uncountable.

**Proof.** Since  $x_\tau = (0.w_\tau)_2 = (0.01101\dots)_2 < 1/2$ , we have that  $\kappa_2(x_\tau) = 1/2 > x_\tau$ . Fix a  $\delta > 0$  such that  $x_\tau + \delta < 1/2$ . By Corollary 22, for every  $\alpha \in [x_\tau + \delta, 1/2[$  there exists a number  $y_\alpha \in ]x_\tau - \delta, x_\tau[$  such that  $\kappa_2(y_\alpha) = \alpha$ . As a consequence,  $\kappa_2(y_\alpha) > y_\alpha$ . By Corollary 24,  $y_\alpha$  is an almost fixed point. Since  $\alpha$  can be chosen from an uncountable set, and  $\alpha \neq \beta$  implies that  $y_\alpha \neq y_\beta$ , we are done.  $\square$

[16, Theorem 3.2] ensures that, assuming the Darboux property, in every neighbourhood of an almost fixed point of a Baire 1 function there exists a fixed point. It is not known to us whether  $\kappa_2$  has the Darboux property. However, by Proposition 19, it is left- or right-Darboux at every point. This in fact suffices to prove an analogous existence result for fixed points.

**Corollary 26.** For every  $x_0 \in \text{aFix}(\kappa_2)$  and every  $\varepsilon > 0$ , there exists  $z \in ]x_0 - \varepsilon, x_0 + \varepsilon[$  such that

$$\kappa_2(z) = z.$$

**Proof.** Let us fix  $x_0 \in \text{aFix}(\kappa_2)$  and  $\varepsilon > 0$ . Using Proposition 19, we can assume either  $R^-(\kappa_2, x_0) = [0, \kappa_2(x_0)[$  or  $R^+(\kappa_2, x_0) = [0, \kappa_2(x_0)[$ . If  $R^-(\kappa_2, x_0) = [0, \kappa_2(x_0)[$ , the following procedure proves the existence of a fixed point  $c \in ]x_0 - \varepsilon, x_0[$ . Otherwise, an analogous procedure shows the existence of a fixed point  $c \in ]x_0, x_0 + \varepsilon[$ .

Let us assume  $R^-(\kappa_2, x) = [0, \kappa_2(x_0)[$ . By Corollary 24,  $\kappa_2(x_0) > x_0$ , so there exists two points  $x_1, x_2$  such that

$$0 < x_0 - \varepsilon < x_1 < x_2 < x_0 < \kappa_2(x_0)$$

Hence,  $[x_1, x_0] \subset R^-(\kappa_2, x_0)$ . We denote by  $K$  the set of points on the diagonal of  $[0, 1] \times [0, 1]$  between  $(x_2, x_2)$  and  $(x_0, x_0)$ , namely

$$K = \left\{ (x_2 + t(x_0 - x_2), x_2 + t(x_0 - x_2)) \in [0, 1] \times I : t \in [0, 1] \right\}.$$

We remark that  $K$  is a compact subset of  $[0, 1] \times [0, 1]$ . By Proposition 19, there exist  $y_1, y_0 \in ]x_2, x_0[$  such that  $\kappa_2(y_1) = x_1$  and  $\kappa_2(y_0) = x_0$ . Since  $\kappa_2$  is a function of Baire class one, there exists a

sequence of continuous functions, say  $(f_n)_n$ , that converges pointwise to  $\kappa_2$ . Since  $x_1 < x_2 < y_1$  and  $y_0 < x_0$ , for  $n$  sufficiently large we have  $f_n(y_1) < y_1$  and  $f_n(y_0) > y_0$ . Since  $(f_n)$  is a sequence of continuous functions, for every  $n$  sufficiently large there exists  $z_n$  such that  $(z_n, f_n(z_n)) \in K$ . Since  $K$  is a compact set, going if necessary to a subsequence, there exists  $z \in [x_2, x_0]$  such that  $z_n \rightarrow z$ . By the pointwise convergence of  $(f_n)_n$  to  $\kappa_2$ , we obtain

$$\kappa_2(z) = \lim_{n \rightarrow \infty} \kappa_2(z_n) = \lim_{n \rightarrow \infty} z_n = z,$$

hence  $z$  is a fixed point for  $\kappa_2$ , and  $c \in ]x_0 - \varepsilon, x_0[$ . □

Since  $\text{aFix}(\kappa_2)$  is uncountable and every neighbourhood of an almost fixed point contains a fixed point, we have the following result.

**Corollary 27.** *The set of the fixed points of  $\kappa_2$  is infinite.*

We address now the problem of estimating the complexity of the dynamics of  $\kappa_2$  from the point of view of its topological entropy. We recall that the topological entropy  $h$  of an interval map  $f$  can be written as ([6, 10]):

$$h(f) = \lim_{\varepsilon \rightarrow 0} \left( \limsup_{n \rightarrow \infty} \frac{1}{n} \log |(n, \varepsilon)| \right), \tag{10}$$

where  $|(n, \varepsilon)|$  indicates the maximum cardinality of an  $\varepsilon$ -separated set in the metric  $d_n$  defined below:

$$d_n(x_1, x_2) := \max \left\{ \left| f^i(x_1) - f^i(x_2) \right| : 0 \leq i \leq n \right\}. \tag{11}$$

The topological entropy can therefore be interpreted as an estimate of the exponential increase rate of the number of topologically distinguishable orbits when the resolution power diverges.

We first establish a technical result concerning the existence of horseshoes, which are defined as follows.

**Definition 28.** *Let  $f: [0, 1] \rightarrow [0, 1]$ . For each  $m \in \mathbb{N}$ ,  $m \geq 2$ , an  $m$ -horseshoe for  $f$  is an ordered pair  $(J, D)$ , where  $J \subset [0, 1]$  is an interval and  $D$  is a family of pairwise disjoint closed intervals  $I_1, \dots, I_m \subset J$  such that  $J \subset f(I_k)$ , for all  $k = 1, \dots, m$ .*

**Remark 29.** Some works give a slightly more general definition of  $m$ -horseshoe, allowing the closed intervals  $I_1, \dots, I_m$  to be adjacent.

**Proposition 30.** *The function  $\kappa_2$  has an  $m$ -horseshoe for each  $m \geq 2$ .*

**Proof.** Noting that  $x_\tau < 7/16$ , we set  $J = [0, 7/16]$ . By Proposition 19, there exists a  $\delta$  such that  $I = [x_\tau - \delta, x_\tau + \delta] \subset J$  and

$$J \subsetneq \kappa_2(I). \tag{12}$$

If  $I$  satisfies (12), there exist two numbers  $x_1, x_2 \in I^\circ$  (the interior of  $I$ ) such that  $\kappa_2(x_1) > 7/16$ ,  $\kappa_2(x_2) > 7/16$  and  $x_1 \neq x_2$ . By Proposition 19, there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $I_1 = [x_1 - \delta_1, x_1 + \delta_1]$  and  $I_2 = [x_2 - \delta_2, x_2 + \delta_2]$  are disjoint subsets of  $I$  and they both satisfy (12). Since we can repeat this construction as many times we need, the thesis follows by induction. □

It is well known in topological dynamics of interval maps that the existence of horseshoes of every order implies infinite topological entropy (cf. [20, Proposition 4.6]). We remark that, although usually continuity is assumed (as for instance in the nice reference work [20]), this implication does not depend on any regularity assumption on the function, as it is immediately clear considering the expression (10) for the entropy. Therefore, the following result holds.

**Proposition 31.** *The function  $\kappa_2$  has infinite topological entropy.*

We recall that a point is called *transitive* if its orbit is dense, and that a *Li-Yorke pair* is a pair of points  $(x, y)$  such that

$$\limsup_{n \rightarrow \infty} |\kappa_2^n(x) - \kappa_2^n(y)| > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} |\kappa_2^n(x) - \kappa_2^n(y)| = 0.$$

When continuity is assumed for interval maps, positive topological entropy implies Li-Yorke chaos and topological mixing implies transitivity ([20]). In this case, however, the implications do rely on the regularity assumption. In the following, we prove some results which imply topological mixing, but only establish a sort of “finite version” of transitivity and Li-Yorke chaos, as we will see.

**Lemma 32.** *For all  $n \in \mathbb{N}$ ,  $n \geq 1$ , and for all  $y \in [0, 1/2^n]$ , there exists  $x \in [0, 1/2^{2^n}]$  such that  $\kappa_2(x) = y$ .*

**Proof.** If  $y = 0$ , then  $x = 0$ . If  $y \in ]0, 1/2^n]$ , then  $1/y \geq 2^n$ . By Theorem 7, there exists  $\alpha_y \in \{0, 1\}^\omega$  such that  $E(0^{2^n} \alpha_y) = 1/y$ . As a consequence, the number  $x = (0.0^{2^n} \alpha_y)_2$  is such that  $\kappa_2(x) = y$  and  $x \in [0, 1/2^{2^n}]$ .  $\square$

**Proposition 33.** *For each non-empty open interval  $I \subset [0, 1]$ , there exists an  $n \in \mathbb{N}$  such that  $\kappa_2^n(I) = [0, 1/2]$ .*

**Proof.** Let  $I$  be a non-empty open interval. By Proposition 15, there exists a number  $x \in I$  such that  $\kappa_2(x) > 0$ . By Theorem 7, there exists a neighbourhood  $U \subset I$  of  $x$  such that  $[0, \kappa_2(x)] \subset \kappa_2(U)$ . Let us use the following notation for the tetration operation:

$${}^n 2 := \begin{cases} 1, & \text{if } n = 0, \\ 2^{n-1} 2, & \text{if } n > 0, \end{cases}$$

hence

$${}^n 2 = \underbrace{2^{2^{\dots^2}}}_{n\text{-times}}.$$

Let  $n \in \mathbb{N}$  be such that  $1/({}^n 2) < \kappa_2(x)$ . Hence,

$$[0, 1/({}^n 2)] \subset [0, \kappa_2(x)] \subset \kappa_2(U) \subset \kappa_2(I),$$

and by Lemma 32 we have  $[0, 1/({}^{n-1} 2)] \subset \kappa_2([0, 1/({}^n 2)]) \subset \kappa_2^2(I)$ . By an iterative procedure involving Lemma 32 we have

$$[0, 1/({}^{n-k+1} 2)] \subset \kappa_2^k(I), \quad \forall k = 1, \dots, n,$$

hence  $[0, 1/({}^1 2)] = [0, 1/2] \subset \kappa_2^n(I)$ . Since also  $\kappa_2^n(x) \in [0, 1/2]$ , we are done.  $\square$

The following result directly follows from Proposition 33.

**Corollary 34.** *The function*

$$\bar{\kappa}_2 := \kappa_2|_{[0, 1/2]} : [0, 1/2] \rightarrow [0, 1/2]$$

*is topologically mixing, namely for every two non-empty open sets  $U, V \subset [0, 1/2]$ , there exists  $N \in \mathbb{N}$  such that  $\bar{\kappa}_2^n(U) \cap V \neq \emptyset$ , for all  $n \geq N$ .*

Before stating the next result, we need the following definition.

**Definition 35.** *For all  $w \in \{0, 1\}^+$ , we define the cylinder characterized by  $w$  as the open interval*

$$I_w := \left] (0.w)_2, (0.w)_2 + 2^{-\ell(w)} \right[.$$

We remark that if  $x \in I_w$ , then there exists  $z_x \in \{0, 1\}^\omega$  such that  $x = (0.wz_x)_2$ .

**Lemma 36.** *For every  $w_{(1)}, w_{(2)} \in \{0, 1\}^+$  such that  $(0.w_{(1)})_2, (0.w_{(2)})_2 < 1/2$ , there exist  $x_1, x_2 \in I_{w_{(1)}}$  and a number  $n \in \mathbb{N}$  such that:*

- (i)  $\kappa_2^{n+2}(x_1), \kappa_2^{n+2}(x_2) \in I_{w_{(2)}}$ ;
- (ii)  $\kappa_2^n(x_2) - \kappa_2^n(x_1) > 1/8$ .

**Proof.** Since  $(0.w_{(2)})_2 < 1/2$ , by Theorem 7, there exist  $\alpha_1, \alpha_2 \in \{0, 1\}^\omega$  such that

$$1/E(00\alpha_1), 1/E(01\alpha_2) \in I_{w_{(2)}}.$$

Let  $z_{(1)} = (0.00\alpha_1)_2$  and  $z_{(2)} = (0.01\alpha_2)_2$ . Since  $z_{(1)} < 1/4$ , by Lemma 32 there exists  $y_1 \in [0, 1/16]$  such that  $\kappa_2(y_1) = z_{(1)}$ . Moreover, using again Theorem 7, there exists  $\beta_1 \in \{0, 1\}^\omega$  such that  $1/E(010\beta_2) = z_{(2)}$ . As a consequence, setting  $y_2 = (0.010\beta_2)_2$  we have  $\kappa_2^2(y_1), \kappa_2^2(y_2) \in I_{w_{(2)}}$  and  $y_2 - y_1 > 1/4 - 1/16 > 1/8$ . Therefore, it remains to prove that there exist  $x_1, x_2 \in I_{w_{(1)}}$  and a number  $n \in \mathbb{N}$  such that  $\kappa_2^n(x_1) = y_1$  and  $\kappa_2^n(x_2) = y_2$ . Let  $\mu = \max\{2, E(w_{(1)})\}$ . By Theorem 7, for any  $\alpha > \mu$  there exists a word  $w \in \{0, 1\}^\omega$  such that  $E(w_{(1)}w) = \alpha$ . As a consequence, we have  $[0, 1/\mu] \subset \kappa_2(I_{w_{(1)}})$ . Since  $w_{(1)}$  is a finite word,  $1/\mu > 0$ . By Proposition 33, there exists an  $n \in \mathbb{N}$  such that  $\kappa_2^{n-1}([0, 1/\mu]) = [0, 1/2]$ , hence  $\kappa_2^n(I_{w_{(1)}}) = [0, 1/2]$ . As a consequence, there exist  $x_1, x_2 \in I_{w_{(1)}}$  such that  $\kappa_2(x_1) = y_1$  and  $\kappa_2(x_2) = y_2$ , and we are done.  $\square$

The result proven in Lemma 36 can be interpreted as the existence of “finite-type transitivity” and “finite-type Li–Yorke pairs”, in the sense that, by a backward iterative procedure, it ensures that we can find:

- (1) points whose orbit visits an arbitrarily large (but only finite) set of open intervals of arbitrarily small size;
- (2) pairs of points which, for as many iterates as we want (but only finitely many), are, alternatively, arbitrarily close and at some finite distance.

Whether it can be established the existence of proper Li–Yorke chaos, and/or of a point with a dense orbit, is an open question for the authors. In fact, even the existence of an infinite orbit, although extremely plausible on general ground, is not directly achievable through the technique developed in the previous lemmas. Notice that the topologically generic (in the  $C^0$  topology) bounded Baire 1 map is not Li–Yorke chaotic (as proven in [21]).

We conclude this section with a remark concerning dimensional aspects. Let us recall that the box-counting dimension of a subset  $A \subset I$  can be defined as:

$$\dim_B(A) = \lim_{n \rightarrow \infty} \frac{\log N(n)}{n \log 2}, \tag{13}$$

where  $N(n)$  is the number of intervals of length  $2^{-n}$ , having dyadic rationals as endpoints, needed to cover the set  $A$ . This is equivalent to the standard definition of box-counting dimension, in which no constraint is imposed on the endpoints of the covering intervals, (see [11, p. 41]).

For rational numbers  $\alpha < 7/3$ , the number of words of length  $n$  which are  $\alpha$ -power free grows polynomially with  $n$  (cf. [13])<sup>2</sup>. This, using (13), immediately implies that

$$\dim_H \left( \left\{ x \in [0, 1] : \kappa_2(x) > \frac{3}{7} \right\} \right) \leq \dim_B \left( \left\{ x \in [0, 1] : \kappa_2(x) > \frac{3}{7} \right\} \right) = 0,$$

where  $\dim_H$  indicates the Hausdorff dimension. It is unknown to the authors which is  $\dim_H(D_2)$ .

<sup>2</sup>The authors thank Boris Adamczewski for having indicated this result.

### 5. The functions $\kappa_n$

In this section we extend our investigation to maps generated using bases other than 2. We will derive most of the properties from the ones established for  $\kappa_2$ , although the deduction is not trivial as, similarly to what happens for properties linked to the expansion of reals in different bases (such as, for instance, normality), no easy procedure allows a direct generalization.

Let  $n \in \mathbb{N}$ ,  $n \geq 2$ , and consider  $\mathcal{A}_n = \{a_0, a_1, \dots, a_{n-1}\}$ . For every  $x \in [0, 1]$ , we denote by  $w_{x,n} \in \mathcal{A}_n^\omega$  the expansion of  $x$  in base  $n$ , namely

$$x = (0.w_{x,n})_n.$$

Since the definition of critical exponent does not depend on the chosen alphabet, we give the following definition.

**Definition 37.** For every integer  $n \geq 2$ , the function  $\kappa_n : [0, 1] \rightarrow [0, 1]$  is defined by:

$$\kappa_n(x) = \inf_{u \in \mathcal{L}(w_{x,n})} \frac{1}{E(u)}.$$

We denote by  $C_n$  the set of the zeros of  $\kappa_n$ , hence  $C_n = \{x \in I : \kappa_n(x) = 0\}$ , and we set  $D_n = [0, 1] \setminus C_n$ .

**Remark 38.** As for the  $\kappa_2$  function, the following properties hold:  $\kappa_n$  is upper semi-continuous;  $\mathcal{N}_n \subset C_n$ , where  $\mathcal{N}_n$  is the set of points in  $[0, 1]$  that are normal real numbers in base  $n$ ;  $C_n$  is a co-meagre set of full Lebesgue measure;  $\kappa_n$  is continuous at  $x$  if and only if  $x \in C_n$ .

For all  $n \in \mathbb{N}$ ,  $n \geq 2$ , we denote by  $r_n \in [0, 1]$  the supremum over the set of  $y \in [0, 1]$  such that there exists  $x \in [0, 1]$  with  $\kappa_n(x) \geq y$ . More formally,

$$r_n = \left\{ y \in [0, 1] : \{x \in I : \kappa_n(x) \geq y\} \neq \emptyset \right\}, \quad \forall n \in \mathbb{N}, n \geq 2. \tag{14}$$

Using the Dejean’s theorem [9], proved by M. Rao [18], we have the following proposition.

**Proposition 39.** Let  $r_n$  be defined by (14). Then

$$r_n = \begin{cases} \frac{4}{7}, & \text{if } n = 3, \\ \frac{5}{7}, & \text{if } n = 4, \\ \frac{n-1}{n}, & \text{otherwise.} \end{cases}$$

**Remark 40.** By the results given in [7] and [24], for  $n = 2, 3$  we have  $\kappa_n([0, 1]) = [0, r_n]$ , namely

$$\kappa_2([0, 1]) = [0, 1/2], \quad \kappa_3([0, 1]) = [0, 4/7].$$

However, at the best of our knowledge, there are not similar results for  $n \geq 4$ .

Since for every  $\alpha \in ]1, \infty[$  it is possible to find a finite alphabet and an infinite word  $w$  on it such that  $\alpha$  is its critical exponent (c.f. [14]), we have the following result.

**Proposition 41.** For every  $\alpha \in [0, 1)$ , there exist  $n \geq 2$  and  $x \in [0, 1]$  such that  $\kappa_n(x) = \alpha$ .

**Proof.** If  $\alpha = 0$ , then  $\kappa_n(0) = \alpha$ , for all  $n \geq 2$ . Hence, using the result of [14], there exists an  $n \geq 2$  and a word  $w \in \{a_0, a_1, \dots, a_{n-1}\}^\omega$  such that  $E(w) = 1/\alpha \in ]1, \infty[$ . Hence, setting  $x = (0.w)_n$ , we have  $\kappa_n(x) = 1/E(w) = \alpha$ . □

Some analytical and dynamical properties of the general function  $\kappa_n$  can be obtained by considering its restriction on the set of points whose  $n$ -base expansion is an infinite binary word, more precisely on the set

$$A_2^n := \{x \in [0, 1] : w_{x,n} \in \{0, 1\}^\omega\}.$$

In particular, Proposition 19 can be used to prove the following.

**Corollary 42.** *Let  $n \geq 2$ . If  $x \in C_n$ , then*

$$R^-(\kappa_n, x) \cup R^+(\kappa_n, x) = \{0\}. \tag{15}$$

*If  $x \in A_2^n \cap D_n$ , then at least one of the following holds:*

$$[0, \kappa_n(x)[ \subset R^-(\kappa_n, x) \quad \text{or} \quad [0, \kappa_n(x)[ \subset R^+(\kappa_n, x). \tag{16}$$

**Proof.** If  $n = 2$ , then the thesis is equivalent to Proposition 19. So let us assume  $n > 2$ .

As in Proposition 19, if  $x \in C_n$ , then the upper semi-continuity and the density of  $\mathbb{Q} \cap [0, 1]$  in  $[0, 1]$  implies (15).

If  $x \in A_2^n \cap D_n$ , then  $w_{x,n} \in \{0, 1\}^\omega$  and  $\kappa_n(x) \in ]0, 1/2[$ . Let  $y = (0.w_{x,n})_2$ , hence  $\kappa_2(y) = \kappa_n(x)$ . By Proposition 19, either  $[0, \kappa_2(y)[ \subset R^-(\kappa_2, y)$  or  $[0, \kappa_2(y)[ \subset R^+(\kappa_2, y)$ . Without loss of generality, let us assume that  $[0, \kappa_2(y)[ \subset R^-(\kappa_2, y)$  (the other case is similar). In this case, we shall prove that  $[0, \kappa_n(x)[ \subset R^-(\kappa_n, x)$ . This is equivalent to prove that for every  $\alpha \in [0, \kappa_n(x)[$ , there exists a sequence  $(x_{(\alpha,k)})_k \subset [0, 1]$  such that  $x_{(\alpha,k)} < x$  and  $\kappa_n(x_{(\alpha,k)}) = \alpha$  for all  $k$ , and  $\lim_{k \rightarrow \infty} x_{(\alpha,k)} = x$ . Let us fix  $\alpha \in [0, \kappa_n(x)[ = [0, \kappa_2(y)[$ . Since  $[0, \kappa_2(y)[ \subset R^-(\kappa_2, y)$ , there exists a sequence  $(y_{(\alpha,k)})_k \subset [0, 1]$  such that  $y_{(\alpha,k)} < y$  and  $\kappa_2(y_{(\alpha,k)}) = \alpha$  for all  $k$ , and  $\lim_{k \rightarrow \infty} y_{(\alpha,k)} = y$ . Now, set  $x_{(\alpha,k)} = (0.w_{y_{(\alpha,k)}})_n$ . Therefore,  $\kappa_n(x_{(\alpha,k)}) = \alpha$  and  $x_{(\alpha,k)} < x$  for all  $k$ .

From  $x \in A_2^n \cap D_n$  it follows  $\kappa_n(x) = \kappa_2(y) > 0$ , so  $y$  is not rational, and in particular it is not a dyadic rational. This implies that  $w_{y_{(\alpha,k)}}$  converges to  $w_{x,n}$  in standard product topology on the Cantor space  $\{0, 1\}^\omega$ . This implies that  $\lim_{k \rightarrow \infty} x_{(\alpha,k)} = x$ . Then, (16) holds.  $\square$

Corollary 42 plays for  $\kappa_n$  the same role of Proposition 19 for  $\kappa_2$ . Indeed, it implies the following results for the function  $\kappa_n$ .

- The existence of an uncountable set of almost-fixed points: as for  $\kappa_2$ , if  $\kappa_n(x) > x$  then  $x$  is an almost-fixed point (see Corollary 24). Let  $x_{\tau,n} = (0.w_\tau)_n$ . Then  $\kappa_n(x_{\tau,n}) = 1/2 > x_{\tau,n}$ , so  $x_{\tau,n}$  is an almost fixed point. With a similar procedure exploited in the proof of Corollary 25, Corollary 42 implies the existence of an uncountable set of almost-fixed points for  $\kappa_n$ .
- The existence of an infinite set of fixed-points: Corollary 26 holds since the function  $\kappa_2$  is left- or right-Darboux at every point, is of Baire class one and it has almost-fixed points. Since each  $\kappa_n$  verifies these properties, results analogous to Corollary 26 and Corollary 27 hold.
- The existence of horseshoes of every order: Proposition 30 is based on Proposition 19 and on the fact that  $x_\tau < 7/16$ . Since  $x_{\tau,n} = (0.w_\tau)_n < x_\tau < 7/16$ , a similar result holds for  $\kappa_n$ . So also the next result holds.
- For every  $n \geq 2$ ,  $\kappa_n$  has infinite topological entropy.

As for what concerns the generalizations of results 32-36, they are more difficult, since they are concerned with multiple iterations of  $\kappa_n$ . Indeed, for  $p > 1$ ,  $\kappa_n^p((0.w)_n)$  depends on  $n$  in a complicated way.

### 6. The function $\kappa$

We finally want to study an object again defined through the critical exponent, but independent of the choice of a particular base. The simplest way to achieve that is the following.

**Definition 43.** *We define  $\kappa : [0, 1] \rightarrow [0, 1]$  as*

$$\kappa(x) = \sup \{ \kappa_n(x) : n \in \mathbb{N}, n \geq 2 \}.$$

**Remark 44.** The function  $\kappa$  clearly vanishes on absolutely normal real numbers. As a consequence,  $\kappa$  is Lebesgue-measurable and  $\int_0^1 \kappa(x) dx = 0$ .



**Proposition 45.** *Set  $C_\kappa = \{x \in I : \kappa(x) = 0\}$ . Then  $I \setminus C_\kappa$  is an uncountable set.*

**Proof.** It is based on the fact that every real number greater than 1 is a critical exponent (see [14]). More formally, for every  $\alpha > 1$ , there exists  $n > 1$  and  $w_\alpha \in \{a_0, a_1, \dots, a_{n-1}\}^\infty$  such that  $\alpha = E_n(0.w_\alpha)$ . As a consequence, being  $\kappa(x) = \sup_{n>1} \kappa_n(x)$ , we have  $\bigcup_{\alpha>1} \{0.w_\alpha\} \subset I \setminus C_\kappa$ , so  $I \setminus C_\kappa$  is uncountable.  $\square$

**Proposition 46.** *The set  $C_\kappa$  is co-meagre.*

**Proof.** By definition of  $\kappa$ ,  $x \in C_\kappa$  if and only if  $\kappa_n(x) = 0$  for every  $n > 1$ . Hence

$$C_\kappa = \bigcap_{n \in \mathbb{N}, n \geq 2} C_n,$$

and

$$I \setminus C_\kappa = \bigcup_{n \in \mathbb{N}, n \geq 2} ([0, 1] \setminus C_n).$$

Since the countable union of meagre sets is meagre,  $I \setminus C_\kappa$  is meagre, thus  $C_\kappa$  is a co-meagre set.  $\square$

**Proposition 47.** *For all  $x \in [0, 1]$ ,  $\kappa$  is not continuous in  $x$ .*

**Proof.** If  $\kappa(x) \neq 0$ , then for every  $\varepsilon > 0$ , there exists  $x_\varepsilon \in ]x - \varepsilon, x + \varepsilon[ \cap \mathbb{Q}$ . As a consequence,

$$\kappa(x) - \kappa(x_\varepsilon) = \kappa(x) > 0, \quad \forall \varepsilon > 0,$$

hence  $\kappa$  is not continuous in  $x$ . Now, we consider  $x \in [0, 1]$  such that  $\kappa(x) = 0$  and we are going to prove that for every  $\varepsilon > 0$  there exists an  $x_\varepsilon \in ]x - \varepsilon, x + \varepsilon[$  such that  $\kappa(x_\varepsilon) \geq 1/2$ . For every  $\varepsilon > 0$ , there exists an  $\bar{n} \geq 2$  and an  $a_i \in \mathcal{A}_{\bar{n}}$  such that, for every  $w \in \mathcal{A}_{\bar{n}}^\infty$ , we have  $(0.a_i w)_{\bar{n}} \in ]x - \varepsilon, x + \varepsilon[$ . Since the word  $w_\tau$  defined using the Thue–Morse sequence

$$w_\tau = a_0 a_1 a_1 a_0 a_1 a_0 a_0 a_1 a_0 a_1 a_1 a_0 a_1 a_0 a_0 a_1 \dots,$$

belongs to  $\mathcal{A}_n^\infty$  for all  $n \geq 2$ , we obtain

$$x_\varepsilon = (0.a_i w_\tau)_{\bar{n}} \in ]x - \varepsilon, x + \varepsilon[, \quad \text{and} \quad \kappa(x_\varepsilon) \geq \kappa_{\bar{n}}(x_\varepsilon) = \frac{1}{2}.$$

As a consequence,  $\kappa$  is not continuous in  $x$ , and we are done.  $\square$

**Lemma 48.** *For every  $n \in \mathbb{N}$ , the Baire class of order  $n$  is closed with respect to the maximum between a finite number of functions. In other words, if  $f_1, \dots, f_k \in \mathcal{B}_n$ , with  $k \in \mathbb{N}, k \geq 1$ , then  $h = \max\{f_1, \dots, f_k\} \in \mathcal{B}_n$ .*

**Proof.** Set  $n = 0$ . Since the maximum of two continuous functions is a continuous function,  $\mathcal{B}_0$  is closed with respect to the maximum between two elements. Now suppose that  $\mathcal{B}_{n-1}$  is closed with respect to the maximum between two elements. Let  $f, g \in \mathcal{B}_n$  and  $h = \max\{f, g\}$ . By definition, there exist two sequences  $(f_k), (g_k) \subset \mathcal{B}_{n-1}$  such that

$$f(x) = \lim_{k \rightarrow \infty} f_k(x), \quad g(x) = \lim_{k \rightarrow \infty} g_k(x), \quad \forall x.$$

We define  $h_k = \max\{f_k, g_k\}$ , which is in  $\mathcal{B}_{n-1}$  by induction hypothesis. Since we have

$$h(x) = \max\{f(x), g(x)\} = \lim_{k \rightarrow \infty} \max\{f_k(x), g_k(x)\} = \lim_{k \rightarrow \infty} h_k(x),$$

then  $h \in \mathcal{B}_n$ . This proves that  $\mathcal{B}_n$  is closed with respect to the maximum between two functions,  $\forall n \in \mathbb{N}$ .

To extend this result to the maximum between a finite number of functions, it suffices to notice that

$$\max\{f_1, f_2, \dots, f_k\} = \max\left\{\max\left\{\dots\left\{\max\{\max\{f_1, f_2\}, f_3\}\dots\right\}, f_k\right\}\right\}.$$

$\square$

**Proposition 49.** *The function  $\kappa$  belongs to  $\mathcal{B}_2 \setminus \mathcal{B}_1$ .*

**Proof.** By Proposition 47,  $k \notin \mathcal{B}_1$ . For every  $n \in \mathbb{N}$ ,  $n \geq 2$ , let us define  $h_n: [0, 1] \rightarrow [0, 1]$  as

$$h_n(x) = \max\{\kappa_2(x), \dots, \kappa_n(x)\}.$$

By Lemma 48,  $h_n \in \mathcal{B}_1$ . Since

$$\kappa(x) = \sup_{n \geq 2} \kappa_n(x) = \lim_{n \rightarrow \infty} h_n(x),$$

then  $\kappa \in \mathcal{B}_2$ . □

## 7. Some open questions

This work opens some questions that link combinatorial, analytical and dynamical properties. Some of them are listed below.

- What is the Hausdorff dimension of  $D_2$  (and in general of  $D_n$ )?
- Is  $\kappa_2$  (and in general  $\kappa_n$ ) a Darboux function?
- Are the fixed points of  $\kappa_2$  (and in general of  $\kappa_n$ ) uncountable?
- Does there exist a transitive point for  $\kappa_2$  (and, in general, for  $\kappa_n$ )?
- Does there exist a Li–Yorke pair for  $\kappa_2$  (and, in general, for  $\kappa_n$ )?
- Does  $\kappa$  attain the value 1?

## References

- [1] B. Adamczewski, “On the expansion of some exponential periods in an integer base”, *Math. Ann.* **346** (2010), p. 107–116.
- [2] B. Adamczewski, Y. Bugeaud, “Dynamics for  $\beta$ -shifts and Diophantine approximation”, *Ergodic Theory Dyn. Syst.* **27** (2007), no. 6, p. 1695–1711.
- [3] J.-P. Allouche, J. Shallit, “The ubiquitous Prouhet–Thue–Morse sequence”, in *Sequences and their applications. Proceedings of the international conference, SETA '98, Singapore, December 14–17, 1998* (C. Ding et al., eds.), Springer Series in Discrete Mathematics and Theoretical Computer Science, Springer, 1999, p. 1–16.
- [4] R. Baire, *Leçons sur les fonctions discontinues*, Gauthier-Villars, 1905, VIII u. 127 S.8°.
- [5] J. Berstel, A. Lauve, C. Reutenauer, F. V. Saliola, *Combinatorics on Words: Christoffel Words and Repetitions in Words*, CRM Monograph Series, vol. 27, American Mathematical Society, 2008.
- [6] R. Bowen, “Entropy for group endomorphisms and homogeneous spaces”, *Trans. Am. Math. Soc.* **153** (1971), p. 401–414.
- [7] J. D. Currie, N. Rampersad, “For Each  $\alpha > 2$  There Is an Infinite Binary Word with Critical Exponent  $\alpha$ ”, *Electron. J. Comb.* (2008), article no. N34.
- [8] ———, “A proof of Dejean’s conjecture”, *Math. Comput.* **80** (2011), no. 274, p. 1063–1070.
- [9] F. Dejean, “Sur Un Théorème de Thue”, *J. Comb. Theory, Ser. A* **13** (1972), no. 1, p. 90–99.
- [10] E. I. Dinaburg, “A correlation between topological entropy and metric entropy”, *Dokl. Akad. Nauk SSSR* **190** (1970), no. 1, p. 19–22.
- [11] K. Falconer, *Fractal geometry: mathematical foundations and applications*, second ed., John Wiley & Sons, 2003.
- [12] F. Filip, J. Šustek, “An elementary proof that almost all real numbers are normal”, *Acta Univ. Sapientiae, Math.* **2** (2010), p. 99–110.
- [13] J. Karhumäki, J. Shallit, “Polynomial versus exponential growth in repetition-free binary words”, *J. Comb. Theory, Ser. A* **105** (2004), no. 2, p. 335–347.
- [14] D. Krieger, J. Shallit, “Every Real Number Greater than 1 Is a Critical Exponent”, *Theor. Comput. Sci.* **381** (2007), no. 1, p. 177–182.
- [15] J. Liouville, “Sur des classes très-étendues de quantités dont la valeur n’est ni algébrique, ni même réductible à des irrationnelles algébriques.”, *J. Math. Pures Appl.* (1851), p. 133–142.
- [16] R. J. Pawlak, “On the Entropy of Darboux Functions”, *Colloq. Math.* **116** (2009), no. 2, p. 227–241.
- [17] M. Queffelec, “Old and new results on normality”, in *Dynamics and stochastics. Festschrift in honor of M. S. Keane* (D. Denteneer et al., eds.), Institute of Mathematical Statistics Lecture Notes - Monograph Series, vol. 48, Institute of Mathematical Statistics, 2006, p. 225–236.

- [18] M. Rao, “Last Cases of Dejean’s Conjecture”, *Theor. Comput. Sci.* **412** (2011), no. 27, p. 3010-3018.
- [19] H. Rosen, “Darboux quasicontinuous functions”, *Real Anal. Exch.* **23(1997-98)** (1998), no. 2, p. 631-639.
- [20] S. Ruette, *Chaos on the Interval*, University Lecture Series, vol. 67, American Mathematical Society, 2017.
- [21] T. H. Steele, “Dynamics of Typical Baire-1 Functions on the Interval”, *J. Appl. Anal.* **23** (2017), no. 2, p. 59-64.
- [22] ———, “Dynamics of Baire-1 Functions on the Interval”, *Eur. J. Math.* **5** (2019), no. 1, p. 138-149.
- [23] A. Thue, *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen*, J. Dybwad, 1912, 67 S. Lex. OCLC: 458299532.
- [24] E. Vaslet, “Critical Exponents of Words over 3 Letters”, *Electron. J. Comb.* (2011), article no. P125.