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Number theory / *Théorie des nombres*

On sections of arithmetic fundamental groups of open p-adic annuli

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Abstract. We show the non-existence of sections of arithmetic fundamental groups of open p-adic annuli of small radii. This implies the non-existence of sections of arithmetic fundamental groups of formal boundaries of formal germs of p-adic curves.

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1. Introduction/Statement of the Main Result

Let $p \geq 2$ be a prime integer, k a *p-adic local field* (i.e., $k/\mathbb{Q}_p$ is a finite extension), with ring of integers $\mathcal{O}_k$, uniformiser π, and residue field F. Thus, F is a finite field of characteristic p.

Let $X \to \operatorname{Spec} \mathcal{O}_k$ be a flat, proper, relative $\mathcal{O}_k$-curve, with X normal, and $X_k \stackrel{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} \operatorname{Spec} k$ geometrically connected. Assume $X(F) \neq \emptyset$. Let $x \in X^{\text{cl}}(F)$ be a closed point, $\mathcal{O}_{X,x}$ the local ring at x, $\widehat{\mathcal{O}}_{X,x}$ its completion, and $E \stackrel{\text{def}}{=} \widehat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_k} k = \widehat{\mathcal{O}}_{X,x}\left[\frac{1}{\pi}\right]$. Write $\mathscr{X} \stackrel{\text{def}}{=} \operatorname{Spec} E$, which we assume to be geometrically connected. We shall refer to $\mathscr{X}$ as the *formal germ of X at x*.

Let η be a geometric point of $\mathscr{X}$ with values in its generic point. Thus, η determines an algebraic closure $\bar{k}$ of k, and a geometric point $\bar{\eta}$ of $\mathscr{X}_{\bar{k}} \stackrel{\text{def}}{=} \mathscr{X} \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$. There exists a canonical exact sequence of profinite groups (cf. [3, Exposé IX, Théorème 6.1])

$$1 \longrightarrow \pi_1(\mathscr{X}_{\bar{k}}, \bar{\eta}) \longrightarrow \pi_1(\mathscr{X}, \eta) \longrightarrow G_k \longrightarrow 1. \tag{1}$$

Here, $\pi_1(\mathscr{X}, \eta)$ denotes the arithmetic étale fundamental group of $\mathscr{X}$ with base point η, $\pi_1(\mathscr{X}_{\bar{k}}, \bar{\eta})$ the étale fundamental group of $\mathscr{X}_{\bar{k}}$ with base point $\bar{\eta}$, and $G_k \stackrel{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$ the absolute Galois group of k.

The sequence (1) splits if $\mathscr{X}(k) \neq \emptyset$. This is for example the case if the morphism $X \to \operatorname{Spec} \mathcal{O}_k$ is smooth at x. If $\mathscr{X}(k) = \emptyset$; for instance if X is stable and regular, and x is an ordinary F-rational double point of $X_F \stackrel{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} F$, the existence of sections $s : G_k \to \pi_1(\mathscr{X}, \eta)$ of the projection $\pi_1(\mathscr{X}, \eta) \twoheadrightarrow G_k$ would provide examples of sections of the projection $\pi_1(X_k, \eta) \twoheadrightarrow G_k$ which are *non-geometric* (η induces a geometric point of X_k, denoted also η, via the morphism $\mathscr{X}_k \to X_k$), i.e., which do not arise from rational points. These in turn will provide *counter-examples to* *the p-adic version of the Grothendieck anabelian section conjecture.* This prompts the following question.

Question 1. *With the above notations, assume $\mathscr{X}(k)=\emptyset$. Does the exact sequence* (1) *split?*

In this note we investigate the case where $\mathscr{X}$ is a *p-adic open annulus*. Let

$$A \stackrel{\text{def}}{=} \mathscr{O}_k[\![S]\!], \quad B \stackrel{\text{def}}{=} A \otimes_{\mathscr{O}_k} k = A\left[\frac{1}{\pi}\right],$$

$D \stackrel{\text{def}}{=} \operatorname{Spf} A$ is the formal standard open disc, and $\mathscr{D} \stackrel{\text{def}}{=} D_k = \operatorname{Spec} B$ its "generic fibre" which is the standard open disc centred at the point "$S=0$". Let $\mathbb{A}^1_k = \operatorname{Spec} k[S]$, $Y_k = \mathbb{P}^1_k$ its smooth compactification with function field $k(S)$, and $Y = \mathbb{P}^1_{\mathscr{O}_k}$ the smooth compactification of $\mathbb{A}^1_{\mathscr{O}_k} = \operatorname{Spec} \mathscr{O}_k[S]$. We shall identify A with the completion of the local ring of Y at the closed point "$S=0$". We have a natural morphism $\mathscr{D} \to \mathbb{P}^1_k$, which induces an identification between the set of closed points of $\mathscr{D}$ and the set

$$\left\{x \in \mathbb{P}^1_k : |S(x)| < 1\right\}.$$

For an integer $n \geq 1$, let

$$A_n \stackrel{\text{def}}{=} \frac{\mathscr{O}_k[\![S,T]\!]}{(S^n T - \pi)}, \quad B_n \stackrel{\text{def}}{=} A_n \otimes_{\mathscr{O}_k} k, \quad \text{and} \quad \mathscr{C}_n \stackrel{\text{def}}{=} \operatorname{Spec} B_n.$$

The natural embedding $\mathscr{C}_n \hookrightarrow \mathscr{D}$ induces an identification between the set of closed points of $\mathscr{C}_n$ and the open annulus

$$\left\{x \in \mathscr{D} : |\pi|^{\frac{1}{n}} < |S(x)| < 1\right\}.$$

Further, let $P \stackrel{\text{def}}{=} A_{(\pi)}$ be the localisation of A at the ideal (π), and $\widehat{P}$ the completion of P, which is a complete discrete valuation ring isomorphic to

$$\mathscr{O}_k[\![S]\!]\{S^{-1}\} \stackrel{\text{def}}{=} \left\{\sum_{i=-\infty}^{\infty} a_i S^i : a_i \in \mathscr{O}_k, \lim_{\substack{\longrightarrow \\ i \to -\infty}} |a_i| = 0\right\},$$

where $|\cdot|$ is a normalised absolute value of $\mathscr{O}_k$ (cf. [2, §2, 5]). Let $L \stackrel{\text{def}}{=} \operatorname{Fr}(\widehat{P})$ be the fraction field of $\widehat{P}$, and $\mathscr{C}_\infty \stackrel{\text{def}}{=} \operatorname{Spec} L$. We shall refer to $\mathscr{C}_\infty$ as a *formal boundary* of the formal germs $\mathscr{D}$, and $\mathscr{C}_i$ for $i \geq 1$. We have natural scheme morphisms

$$\mathscr{C}_\infty \longrightarrow \cdots \longrightarrow \mathscr{C}_{n+1} \longrightarrow \mathscr{C}_n \longrightarrow \cdots \longrightarrow \mathscr{C}_1 \longrightarrow \mathscr{D} \longrightarrow \mathbb{P}^1_k.$$

Let η be a geometric point of $\mathscr{C}_\infty$, which induces a geometric point (denoted also η) of $\mathscr{C}_n$ for $n \geq 1$. For $i \in \mathbb{N} \cup \{\infty\}$, we have an exact sequence of arithmetic fundamental groups

$$1 \longrightarrow \pi_1(\mathscr{C}_{i,\bar{k}}, \bar{\eta}) \longrightarrow \pi_1(\mathscr{C}_i, \eta) \longrightarrow G_k \longrightarrow 1, \tag{2}$$

where $\pi_1(\mathscr{C}_i,\eta)$ denotes the arithmetic étale fundamental group of $\mathscr{C}_i$ with base point η, $\pi_1(\mathscr{C}_{i,\bar{k}},\bar{\eta})$ the étale fundamental group of $\mathscr{C}_{i,\bar{k}} \stackrel{\text{def}}{=} C_i \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ with base point $\bar{\eta}$; which is induced by η, and $G_k \stackrel{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$ the absolute Galois group of k. Here $\bar{k}$ is the algebraic closure of k determined by η.

Our main result in this note is the following.

Theorem 2. *We use notations as above. There exists an integer $N \geq 1$, such that for every integer $n \geq N$, the projection $\pi_1(\mathscr{C}_n,\eta) \twoheadrightarrow G_k$* doesn't split.

The author ignores, for the time being, if the projection $\pi_1(\mathscr{C}_1,\eta) \twoheadrightarrow G_k$ splits or not.

As a corollary of Theorem 2, we obtain the following.

Theorem 3. *The projection $\pi_1(\mathscr{C}_\infty,\eta) \twoheadrightarrow G_k$* doesn't split.

One of the consequences of Theorems 2 and 3 is that one can not produce examples of sections of hyperbolic curves over p-adic local fields, which arise from sections of arithmetic fundamental groups of boundaries of formal fibres, or open annuli with small radii. Those sections would be non-geometric, hence would provide counter-examples to the p-adic version of the Grothendieck anabelian section conjecture.

Finally we observe the following. For $i \in \mathbb{N} \cup \{\infty\}$, let $\pi_1(\mathscr{C}_{i,\bar{k}}, \bar{\eta})^{\mathrm{ab}}$ be the maximal abelian quotient of $\pi_1(\mathscr{C}_{i,\bar{k}}, \bar{\eta})$, and consider the push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathscr{C}_{i,\bar{k}}, \bar{\eta}) & \longrightarrow & \pi_1(\mathscr{C}_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(\mathscr{C}_{i,\bar{k}}, \bar{\eta})^{\mathrm{ab}} & \longrightarrow & \pi_1(\mathscr{C}_i, \eta)^{(\mathrm{ab})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array} \tag{3}$$

Thus, $\pi_1(\mathscr{C}_i, \eta)^{(\mathrm{ab})}$ is the geometrically abelian quotient of $\pi_1(\mathscr{C}_i, \eta)$.

Proposition 4. *The projection* $\pi_1(\mathscr{C}_i, \eta)^{(\mathrm{ab})} \twoheadrightarrow G_k$ splits, $\forall\, i \in \mathbb{N}$.

The author ignores, for the time being, if the projection $\pi_1(\mathscr{C}_\infty, \eta)^{(\mathrm{ab})} \twoheadrightarrow G_k$ splits or not.

2. Proof of Theorem 2

In this section we shall prove Theorem 2. We use the notations used in Section 1. We argue by contradiction, and *assume* that the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ *splits*, $\forall\, n \geq 1$.

Proposition 5. *There exists a relative curve* $X \to \operatorname{Spec} \mathscr{O}_k$ *with the following properties.*

- (i) *The morphism* $X \to \operatorname{Spec} \mathscr{O}_k$ *is flat, proper,* stable, *and* $X_k \stackrel{\mathrm{def}}{=} X \times_{\operatorname{Spec} \mathscr{O}_k} \operatorname{Spec} k$ *is geometrically connected.*
- (ii) X *is* regular.
- (iii) *The set of singular points* X_F^{sing} *of the special fibre* $X_F \stackrel{\mathrm{def}}{=} X \times_{\operatorname{Spec} \mathscr{O}_k} \operatorname{Spec} F$ *of* X *consists of* F-rational ordinary double points, $U \stackrel{\mathrm{def}}{=} X_F \setminus X_F^{\mathrm{sing}}$ *is* F*-smooth, and* $U(F) = \emptyset$ *holds.*
- (iv) $X(k) = \emptyset$ *holds.*

Proof. First, assume $p \neq 2$. Let $\widetilde{C} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_F$ with function field $k(\widetilde{C})$. Thus, $\operatorname{Card}(\widetilde{C}(F)) = \operatorname{Card} F + 1$ is even. Arrange the set $\widetilde{C}(F)$ in pairs of F-rational points: $\widetilde{C}(F) = \{(x_i, y_i)\}_{1 \leq i \leq \frac{\operatorname{Card} F + 1}{2}}$. One can identify in $\widetilde{C}$ the points x_i and y_i; $1 \leq i \leq \frac{\operatorname{Card} F+1}{2}$, to construct a stable proper F-curve C which is geometrically connected and geometrically reduced, with normalisation $\widetilde{C} \to C$. Moreover, the set of singular points $C^{\mathrm{sing}} = \{c_i\}_{1 \leq i \leq \frac{\operatorname{Card} F+1}{2}}$ consists of F-rational ordinary double points, and the pre-image of c_i in $\widetilde{C}$ consists of the two F-rational points $\{x_i, y_i\}$. In particular, $C(F) = C^{\mathrm{sing}} = \{c_i\}_{1 \leq i \leq \frac{\operatorname{Card} F+1}{2}}$. More precisely, for $1 \leq i \leq \frac{\operatorname{Card} F+1}{2}$, let $\widetilde{\mathscr{O}}_i \stackrel{\mathrm{def}}{=} \mathscr{O}_{\widetilde{C}, x_i} \cap \mathscr{O}_{\widetilde{C}, y_i} \subset k(\widetilde{C})$, $\mathscr{N}_{x_i} \stackrel{\mathrm{def}}{=} \mathfrak{m}_{x_i} \cap \widetilde{\mathscr{O}}_i$, and $\mathscr{N}_{y_i} \stackrel{\mathrm{def}}{=} \mathfrak{m}_{y_i} \cap \widetilde{\mathscr{O}}_i$, where $\mathfrak{m}_{x_i}$ (resp. $\mathfrak{m}_{y_i}$) is the maximal ideal of $\mathscr{O}_{\widetilde{C}, x_i}$ (resp. $\mathscr{O}_{\widetilde{C}, y_i}$). Define $\mathscr{O}_{c_i} \stackrel{\mathrm{def}}{=} F + \mathscr{N}_{x_i} \mathscr{N}_{y_i} \subset \widetilde{\mathscr{O}}_i$. Then $\mathscr{O}_{c_i}$ is a local ring (with maximal ideal $\mathscr{N}_{x_i} \mathscr{N}_{y_i}$, and residue field F) whose integral closure is $\mathscr{O}_{c_i}$ (cf. [1, Proposition 3.1, Theorem 3.4, and the references therein] for the properties of $\mathscr{O}_{c_i}$, as well as the existence of C with the required properties).

In case $p = 2$. Consider the affine F-curve $\operatorname{Spec}\left(\frac{F[s,t]}{(st)}\right)$, and $\widetilde{C}$ its smooth compactification. Thus, $\widetilde{C}$ consists of two F-smooth irreducible components $\widetilde{C}_1 = \mathbb{P}^1_F$, and $\widetilde{C}_2 = \mathbb{P}^1_F$, which intersect at the F-rational ordinary double point $c = (s, t) \in \operatorname{Spec}\left(\frac{F[s,t]}{(st)}\right)$. On each irreducible component $\widetilde{C}_i$ of $\widetilde{C}$; $1 \leq i \leq 2$, the set of F-rational points of $\widetilde{C}_i \setminus \{c\}$ is non-empty and comes into pairs of rational points $\{(x_{i,j}, y_{i,j})\}_{1 \leq j \leq \frac{\operatorname{Card} F}{2}}$. As above we can identify each of those pairs of F-rational points $(x_{i,j}, y_{i,j})$ into an F-rational ordinary double point $c_{i,j}$ to construct a reducible and geometrically connected stable curve F-curve C such that the set of singular points C^{sing} consists of F-rational ordinary double points, a double point $c_{i,j}$ lies on a unique irreducible component of C, and $C^{\mathrm{sing}} = C(F)$ (the local ring at c_i is defined as above; the case $p \neq 2$. See [7, §4], for a discussion of this procedure and the existence of such a curve C in the case of reducible curves).

Now the stable F-curve C can be deformed to a semi-stable $\mathcal{O}_k$-curve $X \to \operatorname{Spec}\mathcal{O}_k$ with special fibre $X_F = C$ satisfying (i) and (ii) (cf. [9, Proposition 7.10, Corollary 7.11 and its proof]). By our construction (iii) holds also. If $x \in X(k)$, then X specialises in a point $\bar{x} \in C(F)$ which is a regular point of C and lies on a unique irreducible component of C (cf. [5, Corollary 9.1.32]). Thus, (iv) follows from (iii). □

Let $X \to \operatorname{Spec}\mathcal{O}_K$ be a regular, proper, flat, and stable $\mathcal{O}_k$-curve as in Proposition 5. Let $y \in X_F(F)$ be an F-rational point, which is an ordinary double point and a regular point of X (cf. Proposition 5 (ii) and (iii)). We fix an isomorphism $\rho : \widehat{\mathcal{O}}_{X,y} \xrightarrow{\sim} R[\![S,T]\!]/(ST-\pi)$, and identify $\mathscr{X} \overset{\text{def}}{=} \operatorname{Spec}(\widehat{\mathcal{O}}_{X,y} \otimes_{\mathcal{O}_k} k)$ with $\mathscr{C}_1$ via the isomorphism $\rho_k : \widehat{\mathcal{O}}_{X,y} \otimes_{\mathcal{O}_k} k \xrightarrow{\sim} \frac{R[\![S,T]\!]}{(ST-\pi)} \otimes_{\mathcal{O}_k} k$ induced by ρ. Thus, we have scheme morphisms

$$\mathscr{C}_\infty \longrightarrow \cdots \longrightarrow \mathscr{C}_{n+1} \longrightarrow \mathscr{C}_n \longrightarrow \cdots \longrightarrow \mathscr{C}_1 \longrightarrow X_k.$$

For $n \geq 1$, write

$$\pi_1(X_k \setminus \mathscr{S}_n, \eta) \overset{\text{def}}{=} \varprojlim_{S_n \subset X_k \setminus \mathscr{C}_n} \pi_1(X_k \setminus S_n, \eta),$$

where the projective limit is over all finite sets of closed points $S_n \subset X_k \setminus \mathscr{C}_n$, and $\pi_1(X_k \setminus S_n, \eta)$ is the arithmetic fundamental group of the affine curve $X_k \setminus S_n$ with base point η. (Here, we identify the set of closed points of $\mathscr{C}_n$; $n \geq 1$, with a subset of the set of closed points of X_k which specialise in y.) There is a natural projection $\pi_1(X_k \setminus \mathscr{S}_n, \eta) \twoheadrightarrow G_k$, and we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathscr{C}_n, \eta) & \longrightarrow & G_k \\ \downarrow & & \| \\ \pi_1(X_k \setminus \mathscr{S}_n, \eta) & \longrightarrow & G_k \end{array} \tag{4}$$

where the left vertical map is induced by the morphism $\mathscr{C}_n \to X_k$.

Further, we have a natural map

$$\varprojlim_{n \geq 1} \pi_1(\mathscr{C}_n, \eta) \longrightarrow \varprojlim_{n \geq 1} \pi_1(X_k \setminus \mathscr{S}_n, \eta),$$

and $\varprojlim_{n \geq 1} \pi_1(X_k \setminus \mathscr{S}_n, \eta)$ is naturally identified with the absolute Galois group $G_{k(X)} \overset{\text{def}}{=} \operatorname{Gal}\big(k(X)^{\mathrm{sep}}/k(X)\big)$, where $k(X)^{\mathrm{sep}}$ is the separable closure of the function field $k(X)$ of X determined by the geometric point η.

Lemma 6. *The projection $G_{k(X)} \twoheadrightarrow G_k$* splits.

Proof. First, our assumption that the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ *splits*, implies that the projection $\pi_1(X_k \setminus \mathscr{S}_n, \eta) \to G_k$ splits, $\forall n \geq 1$ (cf. diagram (4)).

Let $(H_i)_{i \in I}$ be a projective system of quotients $G_{k(X)} \twoheadrightarrow H_i$, where H_i sits in an exact sequence $1 \to F_i \to H_i \to G_k \to 1$ with F_i finite, and $G_{k(X)} = \varprojlim_{i \in I} H_i$. [More precisely, write $G_{k(X)}$ as a projective limit of finite groups $\{\widetilde{H}_i\}_{i \in I}$. Then $\widetilde{H}_i$ fits in an exact sequence $1 \to F_i \to \widetilde{H}_i \to G_i \to 1$, where G_i is a quotient of G_k, and F_i a quotient of $\operatorname{Gal}(k(X)^{\mathrm{sep}}/k(X)\bar{k})$. Let $1 \to F_i \to H_i \to G_k \to 1$ be the pull-back of the group extension $1 \to F_i \to \widetilde{H}_i \twoheadrightarrow G_i \to 1$ by $G_k \twoheadrightarrow G_i$. Then $G_{k(X)} = \varprojlim_{i \in I} H_i$]. The set $\operatorname{Sect}(G_k, G_{k(X)})$ of group-theoretic sections of the projection $G_{k(X)} \twoheadrightarrow G_k$ is naturally identified with the projective limit $\varprojlim_{i \in I} \operatorname{Sect}(G_k, H_i)$ of the sets $\operatorname{Sect}(G_k, H_i)$ of group-theoretic sections of the projection $H_i \twoheadrightarrow G_k$. For each $i \in I$, the set $\operatorname{Sect}(G_k, H_i)$ is non-empty. Indeed, H_i (being a quotient of $G_{k(X)}$) is a quotient of $\pi_1(X_k \setminus \mathscr{S}_n, \eta)$ for some $n \geq 1$, this quotient $\pi_1(X_k \setminus \mathscr{S}_n, \eta) \twoheadrightarrow H_i$ commutes with the projections onto G_k, and we know the projection

$\pi_1(X_k \setminus \mathscr{S}_n, \eta) \to G_k$ splits. Hence the projection $H_i \twoheadrightarrow G_k$ splits. Moreover, the set $\mathrm{Sect}(G_k, H_i)$ is, up to conjugation by the elements of F_i, a torsor under the group $H^1(G_k, F_i)$ which is finite since k is a p-adic local field (cf. [6, (7.1.8) Theorem (iii)]). Thus, $\mathrm{Sect}(G_k, H_i)$ is a nonempty finite set. Hence the set $\mathrm{Sect}(G_k, G_{k(X)})$ is nonempty being the projective limit of nonempty finite sets. This finishes the proof of Lemma 6. (See also [8, the proof of Proposition 1.5] for similar arguments in a slightly different context.) □

Let $s : G_k \to G_{k(X)}$ be a section of the projection $G_{k(X)} \twoheadrightarrow G_k$ (cf. Lemma 6).

Lemma 7. *The section s is* geometric*, i.e., $s(G_k) \subset D_x$, where $D_x \subset G_{k(X)}$ is a decomposition group associated to a (unique) rational point $x \in X(k)$. In particular, $X(k) \neq \emptyset$.*

Proof. This follows from [4, Proposition 2.4 (2)]. □

Now the conclusion of Lemma 7 that $X(k) \neq \emptyset$ contradicts the assertion (iv) in Proposition 5 that $X(k) = \emptyset$. This is a contradiction. Thus, our assumption that the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ *splits*, $\forall n \geq 1$, can not hold. Let $N \geq 1$ be such that the projection $\pi_1(\mathscr{C}_N, \eta) \twoheadrightarrow G_k$ doesn't splits. Then the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ doesn't splits, $\forall n \geq N$ as required. Indeed, this follows from the fact that for $n \geq N$ we have a natural homomorphism $\pi_1(\mathscr{C}_n, \eta) \to \pi_1(\mathscr{C}_N, \eta)$ which commutes with the projections onto G_k. Hence if the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ splits then the projection $\pi_1(\mathscr{C}_N, \eta) \twoheadrightarrow G_k$.

This finishes the proof of Theorem 2. □

3. Proof of Theorem 3

Next, we explain how Theorem 3 can be derived from Theorem 2. We have, $\forall n \geq 1$, a commutative diagram

$$\begin{tikzcd} \pi_1(\mathscr{C}_\infty, \eta) \arrow[r] \arrow[d] & G_k \arrow[d, equal] \\ \pi_1(\mathscr{C}_n, \eta) \arrow[r] & G_k \end{tikzcd}$$

where the horizontal maps are the natural projections, and the left vertical map is induced by the morphism $\mathscr{C}_\infty \to \mathscr{C}_n$.

Now assume that the projection $\pi_1(\mathscr{C}_\infty, \eta) \twoheadrightarrow G_k$ splits. Then the projection $\pi_1(\mathscr{C}_n, \eta) \twoheadrightarrow G_k$ splits, $\forall n \geq 1$, by the above diagram. But this contradicts Theorem 2.

This finishes the proof of Theorem 3. □

4. Proof of Proposition 4

Let $n \geq 1$ be an integer, and ℓ_1, ℓ_2, distinct prime integers such that $\ell_1 \geq 2n$, and $\ell_2 \geq 2n$. Let $\mathscr{O}_1$, and $\mathscr{O}_2$, be totally ramified extensions of $\mathscr{O}_k$ of degree ℓ_1, and ℓ_2, with fraction fields $L_1 = \mathrm{Fr}(\mathscr{O}_1)$, and $L_2 = \mathrm{Fr}(\mathscr{O}_2)$; respectively. Thus, the extensions L_1/k and L_2/k, are disjoint and $\mathscr{C}_n(L_i) \neq \emptyset$, for $i \in \{1,2\}$. A restriction-corestriction argument shows that the class $[\pi_1(\mathscr{C}_n, \eta)^{(\mathrm{ab})}]$ of the group extension $\pi_1(\mathscr{C}_n, \eta)^{(\mathrm{ab})}$ in $H^2(G_k, \pi_1(\mathscr{C}_{n,\bar{k}}, \bar{\eta})^{\mathrm{ab}})$ is trivial. Thus the group extension $\pi_1(\mathscr{C}_n, \eta)^{(\mathrm{ab})}$ splits.

This finishes the proof of Proposition 4. □

References

[1] Y. Aubry, A. Iezzi, "On the Maximum Number of Rational Points on Singular Curves over Finite Fields", *Mosc. Math. J.* **15** (2015), no. 4, p. 615-627.

[2] N. Bourbaki, *Eléments de mathématique. Algèbre commutative. Chapitre 9: Anneaux locaux noethériens complets*, Masson, 1983.

[3] A. Grothendieck (ed.), *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, vol. 224, Springer, 1971.

[4] J. Koenigsmann, "On the 'Section Conjecture' in anabelian geometry", *J. Reine Angew. Math.* **588** (2005), p. 221-235.

[5] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2002.

[6] J. Neukirch, A. Schmidt, K. Wingberg, "Cohomology of number fields", 2000.

[7] M. Rosenlicht, "Equivalence relations on algebraic curves", *Ann. Math.* **56** (1952), no. 1, p. 169-191.

[8] M. Saïdi, "On the existence of non-geometric sections of arithmetic fundamental groups", *Math. Z.* **277** (2014), no. 1-2, p. 361-372.

[9] M. Talpo, A. Vistoli, "Deformation theory from the point of view of fibered categories", in *Handbook of moduli. Volume III*, Advanced Lectures in Mathematics (ALM), vol. 26, International Press, 2013, p. 281-397.