INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Jiu Liu, Jia-Feng Liao, Hui-Lan Pan and Chun-Lei Tang

A bifurcation-type result for Kirchhoff equations

Volume 360 (2022), p. 247-254

<https://doi.org/10.5802/crmath.294>

Comptes Rendus Mathématique
2022, 360, p. 247-254
https://doi.org/10.5802/crmath.294

Partial differential equations / *Équations aux dérivées partielles*

A bifurcation-type result for Kirchhoff equations

Jiu Liu [a], Jia-Feng Liao [b], Hui-Lan Pan [c] and Chun-Lei Tang [*, d]

[a] School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun, Guizhou 558000, People's Republic of China

[b] School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637002, People's Republic of China

[c] School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, People's Republic of China

[d] School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

E-mail: tangcl@swu.edu.cn

Abstract. This paper deals with a class of Kirchhoff type elliptic Dirichlet boundary value problems where the combined effects of Kirchhoff term and nonlinear term allow us to establish a bifurcation-type result as the positive parameter varies.

2020 Mathematics Subject Classification. 35J60, 35J20.

Funding. Supported by National Natural Science Foundation of China (11861052,11971393), Science and Technology Foundation of Guizhou Province ([2019]1300, [2019]5653), Innovation group project of Education Department of Guizhou ([2019]067) and Science and technology research project of Chongqing Education Commission (KJQN201800636).

Manuscript received 31st July 2021, revised 18th September 2021, accepted 27th October 2021.

1. Introduction and main result

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $2^* = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ +\infty, & N = 1, 2. \end{cases}$ We consider the following Kirchhoff type boundary value problems

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $a > 0, b > 0$ and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a suitable continuous function.

[*] Corresponding author.

When $N=1, \Omega=(0,L)$, $a=1$ and $g(x,u)=g(x)$, solutions of Eq. (1) are related to the stationary states of the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}-\left(1+b\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=g,\ u(0,t)=u(L,t)=0,$$

which was proposed by Kirchhoff in 1883 to describe the small transversal oscillations of an elastic clamped string. For this reason, Eq. (1) is often called Kirchhoff equation.

Over the past 20 years Eq. (1) has been extensively studied by using variational methods, see for example, [1, 2, 6–10, 12] and the references therein. See also [11] and references therein for a broad survey.

In particular, Ambrosetti and Arcoya investigated the following special case of Eq. (1)

$$\begin{cases}-\left(a+b\int_\Omega |\nabla u|^2\mathrm{d}x\right)\Delta u=|u|^{p-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{2}$$

where $2<p<\min\{4,2^*\}$, and got that

(I) there exists $\gamma_0>0$ such that for any $b\in(0,\gamma_0)$, I_b has a mountain pass critical point u_1 with $I_b(u_1)>0$, as well as a global minimum u_2 with $I_b(u_2)<0$, see [2, Theorem 4.4], where I_b is the Euler functional associated with Eq. (2);

(II) if b is large, then Eq. (2) has no nontrivial solution, see [2, Remark 4.5 (ii)].

Conclusions (I) and (II) motivate us to study bifurcation-type results on Eq. (2) as the parameter b varies. This is the object of this paper to which we give a positive answer.

Define

$$S_p=\inf_{u\in H\setminus\{0\}}\frac{\|u\|^2}{|u|_p^2}.$$

where $\|u\|^2=\int_\Omega|\nabla u|^2\mathrm{d}x$ and $|u|_p^p=\int_\Omega|u|^p\mathrm{d}x$ denote the usual norms of u in $H:=H_0^1(\Omega)$ and $L^p(\Omega)$ respectively. Then

$$\int_\Omega |u|^p\mathrm{d}x\le A_p\|u\|^p \text{ for all } u\in H,\tag{3}$$

where $A_p=S_p^{-\frac{p}{2}}$. Define

$$\Lambda=\frac{p-2}{2}A_p^{\frac{2}{p-2}}\left(\frac{4-p}{2a}\right)^{\frac{4-p}{p-2}}.$$

By using the variational methods, we obtain the following bifurcation-type theorem.

Theorem 1. *Suppose that $a>0, b>0, 2<p<\min\{4,2^*\}$. Then*

(i) *Eq.* (2) *has no nontrivial solutions for any $b>\Lambda$;*

(ii) *Eq.* (2) *has at least two positive solutions for any $b<\Lambda$;*

(iii) *Eq.* (2) *has at least a positive solution for $b=\Lambda$.*

According to Remark 7 below, we can give the rough graphs of the mountain pass value c_b and the local minimum value m_b.

From [2] we know that for any $b\in(0,\gamma_0)$, I_b possesses a global minimum. But if $b\in(0,\Lambda)$ is close to Λ, then 0 is a global minimum. So we need to select an appropriate constraint in order to obtain a local minimum.

The Euler functional associated with Eq. (2) is

$$I_b(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{p}\int_\Omega|u|^p\mathrm{d}x.$$

Obviously, I_b is of class C^1 and

$$\langle I_b'(u),v\rangle=\left(a+b\|u\|^2\right)\int_\Omega\nabla u\cdot\nabla v\mathrm{d}x-\int_\Omega|u|^{p-2}uv\mathrm{d}x$$

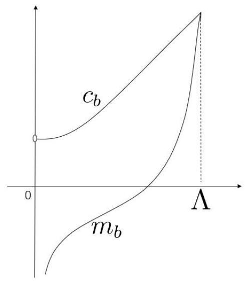

Figure 1.

for all $u, v \in H$.

2. Proof of Theorem 1

To prove Theorem 1, five lemmas are in order.

Lemma 2. *Suppose that* $b > 0$ *and* $2 < p < \min\{4, 2^*\}$. *Then* I_b *is coercive on* H.

Proof. From (3) we have

$$\begin{aligned} I_b(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{A_p}{p}\|u\|^p. \end{aligned}$$

Since $2 < p < \min\{4, 2^*\}$, I_b is coercive on H. □

Lemma 3. *Suppose that* $a > 0, 0 < b < \Lambda, 2 < p < \min\{4, 2^*\}$ *and*

$$T_b = \left[\frac{(p-2)A_p}{2b}\right]^{\frac{1}{4-p}}.$$

Then there exist $t_b \in (0, T_b)$ *and* $t'_b \in (T_b, +\infty)$ *such that all nontrivial solution of Eq.* (2) *belongs to the set* $\{u \in H : t_b \leq \|u\| \leq t'_b\}$. *Moreover, if* $0 < b_1 < b_2 < \Lambda$, *then* $t_{b_1} < t_{b_2}$ *and* $M_{b_2} \subset M_{b_1}$, *where* $M_b := \{u \in H : \|u\| \geq t_b\}$.

Proof. Suppose that $u \in H$ is a nontrivial solution of Eq. (2). Then combining with (3) we have

$$a\|u\|^2 + b\|u\|^4 = \int_\Omega |u|^p \mathrm{d}x \leq A_p\|u\|^p.$$

Thus

$$b\|u\|^2 - A_p\|u\|^{p-2} + a \leq 0.$$

For $t > 0$, define $h_b(t) = bt^2 - A_p t^{p-2} + a$. Then $h'_b(T_b) = 2bT_b - (p-2)A_p T_b^{p-3} = 0$, $h'_b(t) < 0$ for all $t \in (0, T_b)$ and $h'_b(t) > 0$ for all $t \in (T_b, +\infty)$.

When $b < \Lambda$, $h_b(T_b) = T_b^{p-2}(bT_b^{4-p} - A_p) + a < 0$. Combining with $h_b(0) = a > 0$ and $\lim_{t\to+\infty} h_b(t) = +\infty$ implies that there exist $t_b \in (0, T_b)$ and $t'_b \in (T_b, +\infty)$ such that $h_b(t_b) = h_b(t'_b) = 0$, $h_b(t) > 0$ for all $t \in (0, t_b) \cup (t'_b, +\infty)$ and $h_b(t) < 0$ for all $t \in (t_b, t'_b)$. So $t_b \leq \|u\| \leq t'_b$.

If $0 < b_1 < b_2 < \Lambda$, then $h_{b_1}(t_{b_2}) < h_{b_2}(t_{b_2}) = 0$. So there exists $t_{b_1} \in (0, t_{b_2})$ such that $h_{b_1}(t_{b_1}) = 0$ and $h_{b_1}(t) > 0$ for all $t \in (0, t_{b_1})$. Hence $M_{b_2} \subset M_{b_1}$. □

From Lemma 2, $m_b := \inf_{u\in M_b} I_b(u)$ is well defined. We consider the constraint problem $m_b = \inf_{u\in M_b} I_b(u)$ in order to obtain a local minimum solution of Eq. (2).

Lemma 4. *Suppose that $a > 0, 0 < b < \Lambda$ and $2 < p < \min\{4, 2^*\}$. Then I_b satisfies mountain pass geometry, i.e., there exist $\alpha_b > 0$ and $v_b \in H$ such that $\|v_b\| > t_b$ and $\inf_{\|u\|=t_b} I_b(u) \geq \alpha_b > I_b(v_b)$. Moreover, $I_b(v_b) \geq m_b$.*

Proof. From (3) we have

$$\begin{aligned}\inf_{\|u\|=t_b} I_b(u) &= \inf_{\|u\|=t_b}\left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x\right)\\ &\geq \inf_{\|u\|=t_b}\left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{A_p}{p}\|u\|^p\right)\\ &= \frac{at_b^2}{2} + \frac{bt_b^4}{4} - \frac{A_p t_b^p}{p}.\end{aligned}$$

For $t > 0$, define $f_b(t) = \frac{at^2}{2} + \frac{bt^4}{4} - \frac{A_p t^p}{p}$. Then $f_b'(t) = at + bt^3 - A_p t^{p-1} = th_b(t) > 0$ for all $t \in (0, t_b) \cup (t_b', +\infty)$ and $f_b'(t) < 0$ for all $t \in (t_b, t_b')$. Thus $f_b(t_b) > f_b(0) = 0$ and $f_b(t_b) > f_b(T_b) > f_b(t)$ for all $t \in (T_b, t_b']$.

The infimum S_p can be achieved by a positive function $e \in H$ and $\int_\Omega |e|^p \mathrm{d}x = A_p\|e\|^p$. Let $v_b = \frac{T_b}{\|e\|} e$. Then $\|v_b\| = T_b > t_b$ and

$$I_b(v_b) = \frac{aT_b^2}{2} + \frac{bT_b^4}{4} - \frac{A_p T_b^p}{p} = f_b(T_b).$$

Hence $\inf_{\|u\|=t_b} I_b(u) \geq \alpha_b := f_b(t_b) > f_b(T_b) = I_b(v_b) \geq m_b$. □

Lemma 5. *Suppose that $a > 0, 0 < b < \Lambda$ and $2 < p < \min\{4, 2^*\}$. Then there exists $\{u_n\} \subset M_b$ such that $I_b(u_n) \to m_b$ and $I_b'(u_n) \to 0$.*

Proof. By Ekeland's variational principle [5], there exists $u_n \in M_b$ such that

$$m_b \leq I_b(u_n) \leq m_b + \frac{1}{n}$$

and for any $v \in M_b$,

$$I_b(v) - I_b(u_n) \geq -\frac{1}{n}\|u_n - v\|.$$

Thus $I_b(u_n) \to m_b$. From Lemma 4 we know that for n large enough, $\|u_n\| > t_b$. For any $h \in H$ and $\|h\| = 1$, let $v = u_n + th$ for t small enough, we have

$$\langle I_b'(u_n), h\rangle = \lim_{t\to 0^+} \frac{I_b(u_n + th) - I_b(u_n)}{t} \geq -\frac{1}{n}.$$

Similarly,

$$\langle I_b'(u_n), -h\rangle = \lim_{t\to 0^+} \frac{I_b(u_n - th) - I_b(u_n)}{t} \geq -\frac{1}{n}.$$

Then

$$|\langle I_b'(u_n), h\rangle| \leq \frac{1}{n} = o(1).$$

According to the arbitrariness of h we have $I_b'(u_n) \to 0$. □

Lemma 6. *Suppose that $a > 0, b > 0$ and $2 < p < \min\{4, 2^*\}$. If $\{u_n\}$ satisfies $\sup_n I_b(u_n) < +\infty$ and $I_b'(u_n) \to 0$, then $\{u_n\}$ contains a convergent subsequence.*

Proof. Since $\{u_n\}$ satisfies $\sup_n I_b(u_n) < +\infty$, from Lemma 2 we have $\{u_n\}$ is bounded in H. Then there exists $u \in H$ such that up to a subsequence, $u_n \rightharpoonup u$ in H, $u_n \to u$ in $L^p(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. Combining with the Hölder inequality implies

$$\begin{aligned}\left|\int_\Omega |u_n|^{p-2}u_n(u_n-u)\mathrm{d}x\right| &\le \int_\Omega |u_n|^{p-1}|u_n-u|\mathrm{d}x\\ &\le |u_n|_p^{p-1}|u_n-u|_p\\ &= o(1).\end{aligned}$$

Note that

$$\int_\Omega \nabla u_n\cdot\nabla u\mathrm{d}x = \int_\Omega |\nabla u|^2\mathrm{d}x + o(1)$$

and

$$o(1) = \langle I_b'(u_n), u_n-u\rangle = (a+b\|u_n\|^2)\int_\Omega \nabla u_n\cdot\nabla(u_n-u)\mathrm{d}x - \int_\Omega |u_n|^{p-2}u_n(u_n-u)\mathrm{d}x.$$

Thus $\|u_n\| \to \|u\|$. Therefore, $u_n \to u$ in H. □

Proof of Theorem 1.

Conclusion (i). Eq. (2) has no nontrivial solutions for any $b > \Lambda$.

Suppose that $u \in H$ is a nontrivial solution of Eq. (2). Then using (3) and the Young inequality we have

$$\begin{aligned}a\|u\|^2 + b\|u\|^4 &= \int_\Omega |u|^p\mathrm{d}x\\ &\le A_p\|u\|^p\\ &= \left(\frac{2a}{4-p}\right)^{\frac{4-p}{2}}\|u\|^{4-p}A_p\left(\frac{4-p}{2a}\right)^{\frac{4-p}{2}}\|u\|^{2p-4}\\ &\le \frac{4-p}{2}\frac{2a}{4-p}\|u\|^2 + \frac{p-2}{2}A_p^{\frac{2}{p-2}}\left(\frac{4-p}{2a}\right)^{\frac{4-p}{p-2}}\|u\|^4\\ &= a\|u\|^2 + \frac{p-2}{2}A_p^{\frac{2}{p-2}}\left(\frac{4-p}{2a}\right)^{\frac{4-p}{p-2}}\|u\|^4.\end{aligned}\tag{4}$$

So $b \le \Lambda$. Conclusion (i) holds.

Conclusion (ii). Eq. (2) has at least two positive solutions for any $b < \Lambda$.

From Lemma 5 and Lemma 6 we have that there exists $u \in M_b$ such that $I_b(u) = m_b$ and $I_b'(u) = 0$. Note that $I_b(|u|) = I_b(u) = m_b$ and $\||u|\| = \|u\| > t_b$. For any $h \in H$,

$$\langle I_b'(|u|), h\rangle = \lim_{t\to 0^+}\frac{I_b(|u|+th) - I_b(|u|)}{t} \ge 0$$

and

$$\langle I_b'(|u|), -h\rangle = \lim_{t\to 0^+}\frac{I_b(|u|-th) - I_b(|u|)}{t} \ge 0.$$

Then $I_b'(|u|) = 0$. That $|u| > 0$ follows from the strong maximum principle. In fact, $|u|$ is a ground state solution. Indeed, from Lemma 3 one has $v \in M_b$ for any $v \in \{v \in H\backslash\{0\} : I_b'(v) = 0\}$ and then $I_b(v) \ge m_b = I_b(|u|)$.

According to Lemma 4, we define

$$c_b = \inf_{\gamma\in\Gamma_b}\max_{t\in[0,1]} I_b(\gamma(t)),$$

where

$$\Gamma_b = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = v_b\}.$$

Then $c_b \ge \alpha_b > m_b$ and combining with Lemma 6, we know that c_b is a critical value by the mountain pass lemma [3]. From Theorem 10 in [4] we want to state positivity property of the mountain pass solution. We take $p(u) = |u|$ in [4, Theorem 10] and obtain a critical point $u \geq 0$. The strong maximum principle implies $u > 0$. Conclusion (ii) holds.

Conclusion (iii). Eq. (2) has at least a positive solution for $b = \Lambda$.

Because S_p can be achieved by some positive normalized function $v \in H$, i.e., $S_p = \|v\|^2$ and $|v|_p = 1$, $S_p^{\frac{1}{p-2}} v$ is a positive solution of the following semi-linear elliptic equation

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

By scaling the function $w := \left(\frac{4-p}{2a}\right)^{\frac{1}{2-p}} S_p^{\frac{1}{p-2}} v$ solves the following equation

$$\begin{cases} -\dfrac{2a}{4-p}\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by calculating one has $a + b\|w\|^2 = \frac{2a}{4-p}$. It shows that w is a positive solution of Eq. (2). Conversely, if w is a positive solution of Eq. (2), then $(a + b\|w\|^2)^{\frac{1}{2-p}} w$ is the one of Eq. (5). Then when the positive solution of Eq. (5) is unique, the one of Eq. (2) is also unique, for example, when Ω is ball or p is close to 2. Conclusion (iii) holds. □

Remark 7. We prove some properties of critical values m_b and c_b.

(i) m_b and c_b are monotone increasing functions on $(0, \Lambda)$;
(ii) $m_\Lambda := \lim_{b\to\Lambda^-} m_b$ and $c_\Lambda := \lim_{b\to\Lambda^-} c_b$ are critical values of Eq. (2) with $b = \Lambda$;
(iii) $m_\Lambda = c_\Lambda = \frac{p-2}{4p}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} (aS_p)^{\frac{p}{p-2}} = I_\Lambda(u)$ for all $u \in \{u \in H\backslash\{0\} : I'_\Lambda(u) = 0\}$;
(iv) $\lim_{b\to 0^+} m_b = -\infty$ and $c_0 := \lim_{b\to 0^+} c_b > 0$.

Proof of (i). For any $b_1, b_2 \in (0, \Lambda)$ and $b_1 < b_2$, from Lemma 3 we have $M_{b_2} \subset M_{b_1}$. Then

$$m_{b_1} = \inf_{u\in M_{b_1}} I_{b_1}(u) \leq \inf_{u\in M_{b_2}} I_{b_1}(u) \leq \inf_{u\in M_{b_2}} I_{b_2}(u) = m_{b_2}.$$

Fix $b_2 \in (0, \Lambda)$. For any $\gamma \in \Gamma_{b_2}$, $\max_{0\leq t\leq 1} I_{b_2}(\gamma(t)) \geq f_{b_2}(t_{b_2}) > f_{b_2}(T_{b_2}) > f_{b_2}(t)$ for all $t \in (T_{b_2}, t'_{b_2}]$, where the symbols are from Lemma 3 and 4. According to the definition of T_b, there exists $b_1 \in (0, b_2)$ such that $T_{b_2} < T_{b_1} < t'_{b_2}$. For any $b \in (b_1, b_2)$, we define

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \dfrac{1}{2}, \\ \dfrac{[T_{b_2} + (2t-1)(T_b - T_{b_2})]e}{\|e\|}, & \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\widetilde{\gamma} \in \Gamma_b$, $T_{b_2} \leq T_{b_2} + (2t-1)(T_b - T_{b_2}) \leq T_b < T_{b_1} < t'_{b_2}$ for all $t \in \left[\frac{1}{2}, 1\right]$. Thereby, one has

$$\max_{\frac{1}{2}\leq t\leq 1} I_{b_2}(\widetilde{\gamma}(t)) = \max_{\frac{1}{2}\leq t\leq 1} f_{b_2}(T_{b_2} + (2t-1)(T_b - T_{b_2})) \leq f_{b_2}(T_{b_2}) < \max_{0\leq t\leq 1} I_{b_2}(\gamma(t))$$

and then

$$\max_{0\leq t\leq 1} I_{b_2}(\gamma(t)) = \max_{0\leq t\leq \frac{1}{2}} I_{b_2}(\widetilde{\gamma}(t)) = \max_{0\leq t\leq 1} I_{b_2}(\widetilde{\gamma}(t)) > \max_{0\leq t\leq 1} I_b(\widetilde{\gamma}(t)) \geq c_b.$$

In view of the arbitrariness of γ, we have $c_{b_2} \geq c_b$.

Proof of (ii). For any $b \in \left(\frac{\Lambda}{2}, \Lambda\right)$, there exists a positive function $u_b \in H$ such that $I_b(u_b) = m_b$ and $I'_b(u_b) = 0$. Combining with (3) implies

$$\frac{\Lambda}{2}\|u_b\|^4 \leq a\|u_b\|^2 + b\|u_b\|^4 = \int_\Omega |u_b|^p \mathrm{d}x \leq A_p\|u_b\|^p.$$

Thus $\|u_b\| \le \left(\frac{2A_p}{\Lambda}\right)^{\frac{1}{4-p}}$. From

$$m_b = I_b(u_b) \le \frac{a}{2}\|u_b\|^2 + \frac{\Lambda}{4}\|u_b\|^4$$

it follows that m_b has a upper bound on $\left(\frac{\Lambda}{2},\Lambda\right)$. We set $m_\Lambda = \sup_{\frac{\Lambda}{2}<b<\Lambda} m_b$. For any $\varepsilon > 0$, there exists $b_\varepsilon \in \left(\frac{\Lambda}{2},\Lambda\right)$ such that $m_\Lambda - \varepsilon < m_{b_\varepsilon} \le m_\Lambda$. Take $\delta = \Lambda - b_\varepsilon$. When $\Lambda - \delta < b < \Lambda$, by using the monotonicity we have $m_\Lambda - \varepsilon < m_{b_\varepsilon} \le m_b \le m_\Lambda < m_\Lambda + \varepsilon$, i.e., m_b is left continuous at Λ ($m_\Lambda = \lim_{b\to\Lambda^-} m_b$). Let $\{b_n\} \subset \left(\frac{\Lambda}{2},\Lambda\right)$ be an increasing sequence and $b_n \to \Lambda$. Then $m_{b_n} \to m_\Lambda$ and there exists a positive sequence $\{u_n\} \subset H$ such that $I_{b_n}(u_n) = m_{b_n}$ and $I'_{b_n}(u_n) = 0$. So $\|u_n\| \le \left(\frac{2A_p}{\Lambda}\right)^{\frac{1}{4-p}}$ and then

$$I_\Lambda(u_n) = I_{b_n}(u_n) + \frac{\Lambda - b_n}{4}\|u_n\|^4 = m_{b_n} + \frac{\Lambda - b_n}{4}\|u_n\|^4 \to m_\Lambda,$$

$$\|I'_\Lambda(u_n)\|_* = \sup_{\|v\|=1}\langle I'_\Lambda(u_n), v\rangle = \sup_{\|v\|=1}(\Lambda - b_n)\|u_n\|^2\int_\Omega \nabla u_n\cdot\nabla v\,\mathrm{d}x \to 0.$$

From Lemma 6, we know that there exists a nonnegative $u \in H$ such that up to a subsequence, $u_n \to u$ in H. Hence $I_\Lambda(u) = m_\Lambda$ and $I'_\Lambda(u) = 0$. The strong maximum principle implies that u is positive.

The case of c_b is proved to be completely similar.

Proof of (iii). Let $I'_\Lambda(u) = 0$, then from (4) we have

$$a\|u\|^2 + \Lambda\|u\|^4 = \int_\Omega |u|^p\,\mathrm{d}x = A_p\|u\|^p.$$

Let $h(t) = \Lambda t^2 - A_p t^{p-2} + a$ for $t > 0$. Then $h'(T) = 0$, $h'(t) < 0$ for all $t \in (0,T)$ and $h'(t) > 0$ for all $t \in (T,+\infty)$, where

$$T = \left[\frac{(p-2)A_p}{2\Lambda}\right]^{\frac{1}{4-p}}.$$

Since $h(T) = T^{p-2}(\Lambda T^{4-p} - A_p) + a = 0$, T is a unique positive solution of equation $\Lambda t^2 - A_p t^{p-2} + a = 0$. Thereby, for any $u \in \{u \in H\backslash\{0\} : I'_\Lambda(u) = 0\}$ we have $\|u\| = T$ and

$$\begin{aligned} I_\Lambda(u) &= I_\Lambda(u) - \frac{1}{p}\langle I'_\Lambda(u), u\rangle \\ &= \frac{(p-2)a}{2p}T^2 - \frac{(4-p)\Lambda}{4p}T^4 \\ &= \frac{p-2}{4p}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}(aS_p)^{\frac{p}{p-2}}. \end{aligned}$$

Then

$$m_\Lambda = c_\Lambda = \frac{p-2}{4p}\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}(aS_p)^{\frac{p}{p-2}}.$$

Proof of (iv). From Lemma 3 we know that t_b is a monotone increasing function on $(0,\Lambda)$. Fix $w \in H$, $\|w\| = 1$. For any $L > 0$ there exists $B > t_{\frac{\Lambda}{2}}$ such that

$$I_0(Bw) = \frac{aB^2}{2}\|w\|^2 - \frac{B^p}{p}\int_\Omega |w|^p\,\mathrm{d}x < -L - 1.$$

Then there exists $b_0 \in \left(0,\frac{\Lambda}{2}\right)$ such that

$$I_{b_0}(Bw) = I_0(Bw) + \frac{b_0B^4}{4}\|w\|^4 < -L.$$

So for any $b \in (0, b_0)$, we have $\|Bw\| = B > t_{\frac{\Lambda}{2}} > t_{b_0} > t_b$ and then

$$m_b = \inf_{u \in M_b} I_b(u) \le I_b(Bw) \le I_{b_0}(Bw) < -L,$$

i.e., $\lim_{b\to 0^+} m_b = -\infty$.

Since c_b is a monotone increasing function on $(0,\Lambda)$ and $c_b > 0$ for all $b \in (0,\Lambda)$, we obtain that $\lim_{b\to 0^+} c_b$ exists. In the following, we prove $c_0 := \lim_{b\to 0^+} c_b > 0$. We adopt the symbols of Lemma 3 and Lemma 4. Define $t_0 = \left(\frac{a}{A_p}\right)^{\frac{1}{p-2}}$. Then $t_0 < T_b$ and $h_b(t_0) = bt_0^2 > 0$. Since $t_b < T_b$, $h_b(t_b) = 0$ and $h_b'(t) < 0$ for all $t \in (0, T_b)$, we have $t_0 < t_b$. Combining with $f_b'(t) > 0$ for all $t \in (0, t_b)$ implies $f_b(t_b) > f_b(t_0)$. Then we get

$$\inf_{\|u\|=t_b} I_b(u) \ge \alpha_b = f_b(t_b) > f_b(t_0) \ge t_0^2\left(\frac{a}{2} - \frac{A_p}{p} t_0^{p-2}\right) = \frac{p-2}{2p}\frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}} > 0.$$

Thereby $c_b \ge \alpha_b \ge \frac{p-2}{2p}\frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}}$. Hence $c_0 \ge \frac{p-2}{2p}\frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}} > 0$.

Acknowledgments

The authors express their gratitude to the reviewer who made many helpful suggestions and pointed out kindly to us a simple proof of Theorem 1 (iii) which led to an improvement of the original manuscript.

References

[1] A. Ambrosetti, D. Arcoya, "Remarks on non homogeneous elliptic Kirchhoff equations", *NoDEA, Nonlinear Differ. Equ. Appl.* **23** (2016), no. 6, article no. 57 (11 pages).
[2] ———, "Positive solutions of elliptic Kirchhoff equations", *Adv. Nonlinear Stud.* **17** (2017), no. 1, p. 3-15.
[3] A. Ambrosetti, P. H. Rabinowitz, "Dual variational methods in critical point theory and applications", *J. Funct. Anal.* **14** (1973), p. 349-381.
[4] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, "Variational methods for indefinite superlinear homogeneous elliptic problems", *NoDEA, Nonlinear Differ. Equ. Appl.* **2** (1995), no. 4, p. 553-572.
[5] I. Ekeland, "On the variational principle", *J. Math. Anal. Appl.* **47** (1974), p. 324-353.
[6] G. M. Figueiredo, A. Suárez, "Some remarks on the comparison principle in Kirchhoff equations", *Rev. Mat. Iberoam.* **34** (2018), no. 2, p. 609-620.
[7] Z. Liang, F. Li, J. Shi, "Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **31** (2014), no. 1, p. 155-167.
[8] ———, "Positive solutions of Kirchhoff-type non-local elliptic equation: a bifurcation approach", *Proc. R. Soc. Edinb., Sect. A, Math.* **147** (2017), no. 4, p. 875-894.
[9] D. Naimen, "On the Brezis-Nirenberg problem with a Kirchhoff type perturbation", *Adv. Nonlinear Stud.* **15** (2015), no. 1, p. 135-156.
[10] K. Perera, Z. Zhang, "Nontrivial solutions of Kirchhoff-type problems via the Yang index", *J. Differ. Equations* **221** (2006), no. 1, p. 246-255.
[11] P. Pucci, V. D. Rădulescu, "Progress in nonlinear Kirchhoff problems [Editorial]", *Nonlinear Anal., Theory Methods Appl.* **186** (2019), p. 1-5.
[12] J.-J. Sun, C.-L. Tang, "Existence and multiplicity of solutions for Kirchhoff type equations", *Nonlinear Anal., Theory Methods Appl.* **74** (2011), no. 4, p. 1212-1222.