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## *Mathématique*


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Volume 360 (2022), p. 151-159

<https://doi.org/10.5802/crmath.274>

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Probability theory / *Probabilités*

# Density estimates for the exponential functionals of fractional Brownian motion

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**Abstract.** In this note, we investigate the density of the exponential functional of the fractional Brownian motion. Based on the techniques of Malliavin's calculus, we provide a log-normal upper bound for the density.

**Mathematical subject classification (2010).** 60G22, 60H07.

**Funding.** This research was funded by the Vietnam National University, Hanoi under grant number QG.20.21.

*Manuscript received 6th March 2021, revised 19th June 2021, accepted 21st September 2021.*

## 1. Introduction

Let  $B^H = (B_t^H)_{t \in [0, T]}$  be a fractional Brownian motion (fBm) with Hurst index  $H \in (0, 1)$ . We recall that  $B^H$  is a centered Gaussian process with covariance function

$$R_H(t, s) := E[B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad 0 \leq s, t \leq T.$$

We consider the exponential functional of the form

$$F = \int_0^T e^{as + \sigma B_s^H} ds, \quad (1)$$

where  $T > 0$ ,  $a \in \mathbb{R}$  and  $\sigma > 0$  are constants. It is known that this functional plays an important role in several domains. The special case, where  $H = \frac{1}{2}$ , has been well studied and a lot of fruitful properties of  $F$  can be founded in the literature, see e.g. [3, 4, 10]. However, to the best our knowledge, the deep properties of  $F$  for  $H \neq \frac{1}{2}$  are scarce. In a recent paper [8], we have proved the Lipschitz continuity of the cumulative distribution function of  $F$  with respect to the Hurst index

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*H*. The aim of the present paper is to investigate the density of  $F$ . Unlike the case  $H = \frac{1}{2}$ , it is not easy to find the density of  $F$  explicitly for  $H \neq \frac{1}{2}$  and hence, our work will focus on providing the estimates for the density function. It should be noted that, in the last years, the density estimates for random variables related to fBm has been extensively studied, see e.g. [1, 2, 6, 7] and references therein.

The rest of this article is organized as follows. In Section 2, we briefly recall some of the relevant elements of the Malliavin calculus and two general estimates for densities. Our main results are then stated and proved in Section 3. Our Theorems 6 and 8 point out that the density of  $F$  is bounded from above by log-normal densities.

## 2. Preliminaries

In the whole paper, we assume  $H > \frac{1}{2}$ . Under this assumption, fBm admits the Volterra representation

$$B_t^H = \int_0^t K(t, s) dB_s, \tag{2}$$

where  $(B_t)_{t \in [0, T]}$  is a standard Brownian motion and for some normalizing constant  $c_H$ , the kernel  $K$  is given by

$$K(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad 0 < s \leq t \leq T.$$

Let us recall some elements of Malliavin calculus with respect to Brownian motion  $B$ , where  $B$  is used to present  $B_t^H$  as in (2). We suppose that  $(B_t)_{t \in [0, T]}$  is defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , where  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  is a natural filtration generated by the Brownian motion  $B$ . For  $h \in L^2[0, T]$ , we denote by  $B(h)$  the Wiener integral

$$B(h) = \int_0^T h(t) dB_t.$$

Let  $\mathcal{S}$  denote the dense subset of  $L^2(\Omega, \mathcal{F}, P)$  consisting of smooth random variables of the form

$$F = f(B(h_1), \dots, B(h_n)), \tag{3}$$

where  $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), h_1, \dots, h_n \in L^2[0, T]$ . If  $F$  has the form (3), we define its Malliavin derivative as the process  $DF := \{D_t F, t \in [0, T]\}$  given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1), \dots, B(h_n)) h_k(t).$$

More generally, for each  $k \geq 1$ , we can define the iterated derivative operator by setting

$$D_{t_1, \dots, t_k}^k F = D_{t_1} \dots D_{t_k} F.$$

For any  $p, k \geq 1$ , we shall denote by  $\mathbb{D}^{k,p}$  the closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{k,p}^p := E|F|^p + E \left[ \int_0^T |D_{t_1} F|^p dt_1 \right] + \dots + E \left[ \int_0^T \dots \int_0^T |D_{t_1, \dots, t_k}^k F|^p dt_1 \dots dt_k \right].$$

A random variable  $F$  is said to be Malliavin differentiable if it belongs to  $\mathbb{D}^{1,2}$ . For any  $F \in \mathbb{D}^{1,2}$ , the Clark–Ocone formula says that

$$F - E[F] = \int_0^T E[D_s F | \mathcal{F}_s] dB_s.$$

Moreover, any  $F, G \in \mathbb{D}^{1,2}$ , we have the following covariance formula

$$\text{Cov}(F, G) = E \left[ \int_0^T D_s F E[D_s G | \mathcal{F}_s] ds \right]. \tag{4}$$

In order to obtain the density estimates for exponential functionals we need the following general results.

**Proposition 1.** Let  $q, \alpha, \beta$  be three positive real numbers such that  $\frac{1}{q} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$ . Let  $F$  be a random variable in the space  $\mathbb{D}^{2,\alpha}$ , such that  $E[\|DF\|_H^{-2\beta}] < \infty$ . Then the density  $\rho_F(x)$  of  $F$  can be estimated as follows

$$\rho_F(x) \leq c_{q,\alpha,\beta} (P(F \leq x))^{1/q} \times \left( E\left[\|DF\|_H^{-1}\right] + \|D^2F\|_{L^\alpha(\Omega; H \otimes H)} \left\| \|DF\|_H^{-2} \right\|_\beta \right), \quad x \in \mathbb{R}, \tag{5}$$

where  $c_{q,\alpha,\beta}$  is a positive constant and  $H = L^2[0, T]$ .

**Proof.** This proposition comes from the computations on [5, p. 87]. □

**Proposition 2.** Let  $F \in \mathbb{D}^{2,4}$  be such that  $E[F] = 0$ . Define the random variable

$$\Phi_F := \int_0^T D_s F E[D_s F | \mathcal{F}_s] ds.$$

Assume that  $\Phi_F \neq 0$  a.s. and the random variables  $\frac{F}{\Phi_F}$  and  $\frac{1}{\Phi_F^2} \int_0^T D_s \Phi_F E[D_s F | \mathcal{F}_s] ds$  belong to  $L^2(\Omega)$ . Then the law of  $F$  has a continuous density given by

$$\rho_F(x) = \rho_F(0) \exp\left(-\int_0^x h_F(z) dz\right) \exp\left(-\int_0^x w_F(z) dz\right), \quad x \in \text{supp } \rho_F, \tag{6}$$

where the functions  $w_F$  and  $h_F$  are defined by

$$w_F(z) := E\left[\frac{F}{\Phi_F} \middle| F = z\right], \quad h_F(z) := E\left[\frac{1}{\Phi_F^2} \int_0^T D_s \Phi_F E[D_s F | \mathcal{F}_s] ds \middle| F = z\right].$$

**Proof.** This proposition is Theorem 7 in our recent paper [9]. □

### 3. The main results

In this section, we provide explicit estimates for the density  $\rho_F(x)$  of the functional  $F$  defined by (1). Our idea is to consider the random variable  $X := \ln F - E[\ln F]$  and use the relation  $\rho_F(x) = \frac{1}{x} \rho_X(\ln x - E[\ln F])$ ,  $x > 0$ , where  $\rho_X$  denotes the density of  $X$ .

We need some technical results.

**Proposition 3.** Consider the random variable  $X := \ln F - E[\ln F]$ . It holds that

$$0 \leq D_\theta X \leq \sigma K(T, \theta) \text{ a.s.} \tag{7}$$

$$0 \leq D_r D_\theta X \leq 2\sigma^2 K(T, \theta) K(T, r) \text{ a.s.} \tag{8}$$

**Proof.** By the chain rule for Malliavin derivatives, we have, for  $0 \leq r, \theta \leq T$ ,

$$D_\theta X = \frac{\sigma \int_0^T K(s, \theta) e^{as + \sigma B_s^H} ds}{\int_0^T e^{as + \sigma B_s^H} ds} \tag{9}$$

and

$$D_r D_\theta X = \frac{\sigma^2 \int_{\theta \vee r}^T K(s, \theta) K(s, r) e^{as + \sigma B_s^H} ds}{\int_0^T e^{as + \sigma B_s^H} ds} - \frac{\sigma^2 \int_r^T K(s, r) e^{as + \sigma B_s^H} ds \int_\theta^T K(s, \theta) e^{as + \sigma B_s^H} ds}{\left(\int_0^T e^{as + \sigma B_s^H} ds\right)^2}.$$

Because the function  $s \mapsto K(s, \theta)$  is non-decreasing for each  $\theta$ , (7) follows directly from (9). We also have

$$D_r D_\theta X \leq 2\sigma^2 K(T, \theta) K(T, r) \text{ a.s.}$$

To prove the non-negativity of the second order Malliavin derivative, we let  $U$  be a random variable with the density function defined by

$$f(x) = \frac{e^{ax + \sigma B_x^H}}{\int_0^T e^{as + \sigma B_s^H} ds}, \quad 0 \leq x \leq T.$$

Denote by  $E_U$  the expectation with respect to  $U$ . We have

$$D_r D_\theta X = E_U [K(U, \theta)K(U, r)] - E_U [K(U, \theta)] E_U [K(U, r)].$$

Note that the functions  $s \mapsto K(s, \theta)$  and  $s \mapsto K(s, r)$  are non-decreasing. Hence, by Chebyshev's association inequality,  $D_r D_\theta X \geq 0$  a.s. The proof of Proposition 3 is complete.  $\square$

**Lemma 4.** *Define*

$$M_r := E[F|\mathcal{F}_r] = E \left[ \int_0^T e^{as+\sigma k f l 6 B_s^H} ds \middle| \mathcal{F}_r \right], \quad 0 \leq r \leq T.$$

Then, for every  $p \geq 2$ , we have

$$E \left[ \left( \max_{0 \leq r \leq T} M_r \right)^p \right] \leq C < \infty,$$

where  $C$  is a positive constant depending on  $p, T, a, \sigma$  and  $H$ .

**Proof.** The stochastic process  $M := (M_r)_{0 \leq r \leq T}$  is a martingale with  $M_0 = E[F]$  and  $M_T = F$ . Hence, by Burkholder–David–Gundy inequality, we have

$$E \left[ \left( \max_{0 \leq r \leq T} M_r \right)^p \right] \leq c_p \left( M_0^p + E \left[ \langle M \rangle_T^{p/2} \right] \right) = c_p \left( (E[F])^p + E \left[ \langle M \rangle_T^{p/2} \right] \right), \quad (10)$$

where  $c_p$  is a positive constant. Using the Clark–Ocone formula we have

$$\begin{aligned} M_T &= EM_T + \int_0^T E[D_r M_T | \mathcal{F}_r] dB_r \\ &= E[F] + \sigma \int_0^T E \left[ \int_r^T K(s, r) e^{as+\sigma B_s^H} ds \middle| \mathcal{F}_r \right] dB_r, \end{aligned}$$

which gives us

$$\begin{aligned} \langle M \rangle_T &= \int_0^T \sigma^2 \left( E \left[ \int_r^T K(s, r) e^{as+\sigma B_s^H} ds \middle| \mathcal{F}_r \right] \right)^2 dr \\ &\leq \int_0^T \sigma^2 K^2(T, r) M_r^2 dr \text{ a.s.} \end{aligned}$$

Then, by Hölder inequality, we have

$$\begin{aligned} E \left[ \langle M \rangle_T^{p/2} \right] &\leq \sigma^p E \left[ \left( \int_0^T K^{\frac{2p-4}{p}}(T, r) K^{\frac{4}{p}}(T, r) M_r^2 dr \right)^{p/2} \right] \\ &\leq \sigma^p \left( \int_0^T K^2(T, r) dr \right)^{\frac{p}{2}-1} \left( \int_0^T K^2(T, r) E[M_r^p] dr \right) \\ &\leq \sigma^p T^{(p-2)H} \left( \int_0^T K^2(T, r) E[F^p] dr \right) \\ &= \sigma^p T^{pH} E[F^p]. \end{aligned} \quad (11)$$

Here we used the fact that  $\int_0^T K^2(T, r) dr = E|B_T^H|^2 = T^{2H}$ . So we obtain the desired conclusion by inserting (11) into (10).  $\square$

**Proposition 5.** *Let  $X$  be as in Proposition 3. We define  $\Phi_X := \int_0^T D_s X E[D_s X | \mathcal{F}_s] ds$ . Then,*

$$|\Phi_X|^{-1} \in L^p(\Omega), \quad \forall p \geq 1.$$

We also have

$$\left( \int_0^T |D_\theta X|^2 d\theta \right)^{-1} \in L^p(\Omega), \quad \forall p \geq 1.$$

**Proof.** It follows from (9) that

$$D_\theta X \geq \frac{\sigma}{T} e^{-2|a|T+\sigma \min_{0 \leq s \leq T} B_s^H - \sigma \max_{0 \leq s \leq T} B_s^H} \int_\theta^T K(s, \theta) ds \text{ a.s.} \tag{12}$$

On the other hand, by using the Cauchy–Schwarz inequality, we have

$$E[D_\theta X | F_\theta] \geq \frac{\sigma \left( E \left[ \sqrt{\int_\theta^T K(s, \theta) e^{as+\sigma B_s^H} ds} \middle| F_\theta \right] \right)^2}{E \left[ \int_\theta^T e^{as+\sigma B_s^H} ds \middle| F_\theta \right]} \text{ a.s.}$$

and

$$\sqrt{\int_\theta^T K(s, \theta) e^{as+\sigma B_s^H} ds} \geq \frac{\int_\theta^T K(s, \theta) \sqrt{e^{as+\sigma B_s^H}} ds}{\sqrt{\int_\theta^T K(s, \theta) ds}} \text{ a.s.}$$

We therefore get

$$E[D_\theta X | F_\theta] \geq \frac{\sigma \left( \int_\theta^T K(s, \theta) E \left[ e^{as/2+\sigma B_s^H/2} \middle| F_\theta \right] ds \right)^2}{\int_\theta^T K(s, \theta) ds E \left[ \int_\theta^T e^{as+\sigma B_s^H} ds \middle| F_\theta \right]} \text{ a.s.}$$

Furthermore, by Lyapunov’s inequality,

$$E \left[ e^{as/2+\sigma B_s^H/2} \middle| F_\theta \right] \geq e^{as/2+\sigma E[B_s^H | F_\theta]/2} \text{ a.s.}$$

As a consequence,

$$E[D_\theta X | F_\theta] \geq \frac{\sigma e^{-|a|T+\sigma \min_{0 \leq \theta \leq s \leq T} N_{s,\theta}} \int_\theta^T K(s, \theta) ds}{\max_{0 \leq \theta \leq T} M_\theta} \text{ a.s.} \tag{13}$$

where

$$N_{s,\theta} := E \left[ B_s^H \middle| F_\theta \right] \quad \text{and} \quad M_\theta := E \left[ \int_0^T e^{as+\sigma B_s^H} ds \middle| F_\theta \right].$$

Combining (12) and (13) yields

$$D_\theta X E[D_\theta X | F_\theta] \geq \frac{\sigma^2 e^{-3|a|T+\sigma \min_{0 \leq s \leq T} B_s^H - \sigma \max_{0 \leq s \leq T} B_s^H + \sigma \min_{0 \leq \theta \leq s \leq T} N_{s,\theta}}}{\max_{0 \leq \theta \leq T} M_\theta} \left( \int_\theta^T K(s, \theta) ds \right)^2 \text{ a.s.}$$

and hence,

$$\Phi_X \geq \frac{\sigma^2}{T} e^{-3|a|T+\sigma \min_{0 \leq s \leq T} B_s^H - \sigma \max_{0 \leq s \leq T} B_s^H + \sigma \min_{0 \leq \theta \leq s \leq T} N_{s,\theta}} \frac{\int_0^T \left( \int_\theta^T K(s, \theta) ds \right)^2 d\theta}{\max_{0 \leq \theta \leq T} M_\theta} \text{ a.s.} \tag{14}$$

We observe that

$$\begin{aligned} \int_0^T \left( \int_\theta^T K(s, \theta) ds \right)^2 d\theta &= \int_0^T \int_\theta^T \int_\theta^T K(t, \theta) K(s, \theta) ds dt d\theta \\ &= \int_0^T \int_0^T \left( \int_0^{s \wedge t} K(t, \theta) K(s, \theta) d\theta \right) ds dt \\ &= \int_0^T \int_0^T E \left[ B_t^H B_s^H \right] ds dt = \frac{T^{2H+2}}{2H+2}. \end{aligned}$$

This, together with (14), yields

$$|\Phi_X|^{-1} \leq \frac{2H+2}{\sigma^2 T^{2H+1}} e^{3|a|T+2\sigma \max_{0 \leq s \leq T} B_s^H + \sigma \max_{0 \leq \theta \leq s \leq T} N_{s,\theta}} \max_{0 \leq \theta \leq T} M_\theta \text{ a.s.}$$

We have  $(N_{s,\theta})_{0 \leq \theta \leq s \leq T}$  is a Gaussian field with finite variances because  $N_{s,\theta} = \int_0^\theta K_H(s,r) dB_r$ . Hence, by Fernique's theorem, there exists  $\varepsilon > 0$  such that

$$E \left[ e^{\varepsilon \max_{0 \leq \theta \leq s \leq T} |N_{s,\theta}|^2} \right] < \infty.$$

Since

$$e^{p\sigma \max_{0 \leq \theta \leq s \leq T} N_{s,\theta}} \leq e^{\frac{p^2\sigma^2}{4\varepsilon} + \varepsilon \max_{0 \leq \theta \leq s \leq T} |N_{s,\theta}|^2},$$

this implies that

$$e^{\sigma \max_{0 \leq \theta \leq s \leq T} N_{s,\theta}} \in L^p(\Omega) \text{ for any } p \geq 1.$$

Similarly, we also have

$$e^{2\sigma \max_{0 \leq s \leq T} B_s^H} \in L^p(\Omega) \text{ for any } p \geq 1.$$

So, recalling Lemma 4, we conclude that  $|\Phi_X|^{-1} \in L^p(\Omega)$  for any  $p \geq 1$ .

We deduce from (12) that

$$\int_0^T |D_\theta X|^2 d\theta \geq \frac{\sigma^2}{(2H+2)T^{2H}} e^{-4|a|T+2\sigma \min_{0 \leq s \leq T} B_s^H - 2\sigma \max_{0 \leq s \leq T} B_s^H} \text{ a.s.} \tag{15}$$

Hence, we also have  $(\int_0^T |D_\theta X|^2 d\theta)^{-1} \in L^p(\Omega)$ ,  $\forall p \geq 1$ . The proof of Proposition 5 is complete.  $\square$

We now are in a position to bound the density  $\rho_F(x)$  of  $F$ . We first use Proposition 1 to estimate the left tail of the density.

**Theorem 6.** *We have*

$$\rho_F(x) \leq \frac{c}{x} \exp\left(-\frac{(\ln x - E[\ln F])^2}{8\sigma^2 T^{2H}}\right), \quad 0 < x \leq e^{E[\ln F]}, \tag{16}$$

where  $c$  is a positive constant.

**Proof.** It is known from Proposition 5 that

$$\|DX\|_H^{-2} = \left(\int_0^T |D_\theta X|^2 d\theta\right)^{-1} \in L^p(\Omega), \quad \forall p \geq 1.$$

In addition, from the estimate (8), we have

$$\|D^2 X\|_{L^2(\Omega; H \otimes H)}^2 = \int_0^T \int_0^T E|D_\theta D_r X|^2 d\theta dr \leq \sigma^4 \int_0^T \int_0^T K^2(T,\theta)K^2(T,r) d\theta dr = \sigma^4 T^{2H} < \infty.$$

The above estimates allow us to use Proposition 1 with  $q = \beta = 4, \alpha = 2$  and we obtain

$$\rho_X(x) \leq cP(X \leq x)^{\frac{1}{4}}, \quad x \in \mathbb{R}, \tag{17}$$

where  $c$  is a positive constant.

The remaining of the proof is to bound  $P(X \leq x)$  for  $x \leq 0$ . We consider the function  $\varphi(\lambda := E[e^{-\lambda X}], \lambda > 0$  (this function is well defined because  $F^{-1} \in L^p(\Omega)$ ,  $\forall p \geq 1$ ). By using repeatedly the covariance formula (4), we have

$$\sigma_X^2 := \text{Var}(X) = E[\Phi_X]$$

and

$$\begin{aligned} \varphi'(\lambda) &= -E\left[Xe^{-\lambda X}\right] \\ &= \lambda E\left[e^{-\lambda X}\Phi_X\right] \\ &= \lambda\sigma_X^2 E\left[e^{-\lambda X}\right] + \lambda E\left[e^{-\lambda X}\left(\Phi_X - \sigma_X^2\right)\right] \\ &= \lambda\sigma_X^2 E\left[e^{-\lambda X}\right] - \lambda^2 E\left[e^{-\lambda X} \int_0^T D_s X E[D_s \Phi_X | \mathcal{F}_s] ds\right] \end{aligned}$$

Since  $D_s X \geq 0$  and  $D_r D_s X \geq 0$ , those imply that  $\int_0^T D_s X E[D_s \Phi_X | \mathcal{F}_s] ds \geq 0$ , and hence,

$$\varphi'(\lambda) \leq \lambda\sigma_X^2 E\left[e^{-\lambda X}\right] = \lambda\sigma_X^2 \varphi(\lambda), \lambda > 0.$$

This, together the fact  $\varphi(0) = 1$ , gives us

$$\varphi(\lambda) \leq e^{\frac{\lambda^2 \sigma_X^2}{2}}, \lambda > 0.$$

By Markov's inequality we have, for all  $\lambda > 0$ ,

$$P(X \leq x) \leq e^{\lambda x} \varphi(\lambda) \leq e^{\lambda x + \frac{\lambda^2 \sigma_X^2}{2}}, x \leq 0.$$

When  $x \leq 0$ , we can choose  $\lambda = -\frac{x}{\sigma_X^2}$  to get

$$P(X \leq x) \leq e^{-\frac{x^2}{2\sigma_X^2}}, x \leq 0.$$

From the estimate (7), we have  $\sigma_X^2 = E[\Phi_X] \leq \int_0^T |D_s X|^2 ds \leq \sigma^2 T^{2H}$ . So we deduce

$$P(X \leq x) \leq e^{-\frac{x^2}{2\sigma^2 T^{2H}}}, x \leq 0. \tag{18}$$

Combining (17) and (18) yields

$$\rho_X(x) \leq c e^{-\frac{x^2}{8\sigma^2 T^{2H}}}, x \leq 0.$$

where  $c$  is a positive constant. Recalling  $X = \ln F - E[\ln F]$ , the density of  $F$  satisfies  $\rho_F(x) = \frac{1}{x} \rho_X(\ln x - E[\ln F])$ . When  $0 < x \leq e^{E[\ln F]}$ , we have  $y := \ln x - E[\ln F] \leq 0$ . We thus obtain

$$\rho_F(x) = \frac{1}{x} \rho_X(y) \leq \frac{c}{x} e^{-\frac{y^2}{8\sigma^2 T^{2H}}} = \frac{c}{x} e^{-\frac{(\ln x - E[\ln F])^2}{8\sigma^2 T^{2H}}}, 0 < x \leq e^{E[\ln F]}.$$

This completes the proof of Theorem 6. □

**Remark 7.** Replacing  $X$  by  $F - E[F]$  in the proof of Theorem 6, we obtain the following Gaussian bound for the left tail

$$\rho_F(x) \leq c e^{-\frac{(x - E[F])^2}{8\sigma_F^2}}, x \leq E[F],$$

where  $\sigma_F^2 := \text{Var}(F)$  and  $c$  is a positive constant.

We now use Proposition 2 to estimate the right tail of the density.

**Theorem 8.** *We have*

$$\rho_F(x) \leq \frac{c}{x} \exp\left(-\frac{(\ln x - E[\ln F])^2}{2\sigma^2 T^{2H}}\right), x > e^{E[\ln F]}, \tag{19}$$

where  $c$  is a positive constant.



**Proof.** Let  $X$  be as in Proposition 3. Obviously, we have  $\Phi_X \neq 0$  a.s. Moreover, from the estimates (7) and (8) we obtain

$$\begin{aligned} 0 \leq D_s \Phi_X &= \int_0^T D_s D_\theta X E[D_\theta X | \mathcal{F}_\theta] d\theta + \int_0^T D_\theta X E[D_s D_\theta X | \mathcal{F}_\theta] d\theta \\ &\leq 4\sigma^3 \int_0^T K^2(T, \theta) K(T, s) d\theta = 4\sigma^3 K(T, s) T^{2H} \end{aligned}$$

and

$$0 \leq \int_0^T D_s \Phi_X E[D_s X | \mathcal{F}_s] ds \leq 4\sigma^4 T^{2H} \int_0^T K^2(T, s) ds = 4\sigma^4 T^{4H}$$

Hence, it follows from Proposition 5 that the random variable  $\frac{1}{\Phi_X^2} \int_0^T D_s \Phi_X E[D_s X | \mathcal{F}_s] ds$  belong to  $L^2(\Omega)$ . We also have  $\frac{X}{\Phi_X} \in L^2(\Omega)$  because

$$-|a|T - \sigma \max_{0 \leq s \leq T} B_s^H + \ln T - E[\ln F] \leq X \leq |a|T + \sigma \max_{0 \leq s \leq T} B_s^H + \ln T - E[\ln F]$$

and hence,  $X \in L^p(\Omega)$  for all  $p \geq 2$ .

In view of Proposition 2, the density  $\rho_X(x)$  of  $X$  is given by

$$\rho_X(x) = \rho_X(0) \exp\left(-\int_0^x h_X(z) dz\right) \exp\left(-\int_0^x w_X(z) dz\right), \quad x \in \text{supp } \rho_X, \tag{20}$$

where

$$w_X(z) := E\left[\frac{X}{\Phi_X} \middle| X = z\right] \quad \text{and} \quad h_X(z) := E\left[\frac{1}{\Phi_X^2} \int_0^T D_s \Phi_X E[D_s X | \mathcal{F}_s] ds \middle| X = z\right].$$

Since  $h_X \geq 0$ , this implies that

$$\exp\left(-\int_0^x h_X(z) dz\right) \leq 1, \quad x \geq 0.$$

From the estimate (7) we have

$$0 \leq \Phi_X \leq \sigma^2 \int_0^T K^2(T, \theta) d\theta = \sigma^2 T^{2H} \quad \text{a.s.}$$

and we obtain

$$\exp\left(-\int_0^x w_F(z) dz\right) \leq e^{-\frac{x^2}{2\sigma^2 T^{2H}}}, \quad x \in \mathbb{R}.$$

So we can conclude that

$$\rho_X(x) \leq \rho_X(0) e^{-\frac{x^2}{2\sigma^2 T^{2H}}}, \quad x \geq 0,$$

and (19) follows because  $\rho_F(x) = \frac{1}{x} \rho_X(\ln x - E[\ln F])$ . The proof of Theorem 8 is complete. □

### Acknowledgments

The author would like to thank the anonymous referee for their valuable comments for improving the paper. A part of this paper was done while the first author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for financial support and hospitality.

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