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Comptes Rendus
Mathématique

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Volume 360 (2022), p. 53-57

<https://doi.org/10.5802/crmath.282>

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www.centre-mersenne.org

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Number theory / *Théorie des nombres*

On the denominators of harmonic numbers. IV

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Abstract. Let $\mathscr{L}$ be the set of all positive integers n such that the denominator of $1+1/2+\cdots+1/n$ is less than the least common multiple of $1,2,\ldots,n$. In this paper, under a certain assumption on linear independence, we prove that the set $\mathscr{L}$ has the upper asymptotic density 1. The assumption follows from Schanuel's conjecture.

Keywords. harmonic numbers, least common multiples, upper asymptotic density.

Mathematical subject classification (2010). 11B05, 11B75.

Funding. This work is supported by the National Natural Science Foundation of China (Grant Nos. 12101332 and 12101009), the Natural Science Foundation in Jiangsu Province (Grant No. BK20200748), the Natural Science Foundation for Colleges and Universities in Jiangsu Province (Grant No. 20KJB110003), the Anhui Provincial Natural Science Foundation (Grant No. 2108085QA02) and the Talent Foundation of Anhui Normal University (Grant No.752038).

Manuscript received 11th August 2021, revised 25th August 2021 and 18th September 2021, accepted 12th October 2021.

1. Introduction

For any positive integer n, let

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = \frac{u_n}{v_n}, \quad (u_n, v_n) = 1,\ v_n > 0.$$

The number H_n is called *n-th harmonic number*. Shiu [6] proved that $v_n = v_{n+1}$ for infinitely many positive integers n. Recently, Wu and Chen [9] showed that the set of positive integers n with $v_n = v_{n+1}$ has asymptotic density one. For related research, one may refer to [1, 2, 5, 7, 8]. Especially, Eswarathasan and Levine [2] conjectured that the set J_p of positive integers n such that $p \mid u_n$ is finite for any prime number p. Sanna [5] proved that $J_p(x) \le 129 p^{2/3} x^{0.765}$, where

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$J_p(x)$ denotes the number of integers which are in J_p and do not exceed x. Later, Wu and Chen [7] improved this result to $J_p(x) \le 3x^{2/3+1/(25\log p)}$.

Let $\mathscr{L}$ be the set of all positive integers n such that v_n is less than the least common multiple of $1,2,\dots,n$. Note that v_n divides the least common multiple of $1,2,\dots,n$. Let

$$\bar{d}(\mathscr{L}) = \limsup_{x\to\infty} \frac{\mathscr{L}(x)}{x}.$$

In this paper, under a certain assumption on linear independence, we prove that the set $\mathscr{L}$ has the upper asymptotic density 1. Firstly, we introduce a special case of Schanuel's conjecture [4, p. 30–31].

Weak Schanuel's Conjecture. *If $\beta_1,\dots,\beta_m$ are non-zero, multiplicatively independent algebraic numbers, then $\log\beta_1,\dots,\log\beta_m$ are algebraically independent.*

It is clear that the set of prime numbers is multiplicatively independent. For any distinct primes $q_1,q_2,\dots,q_l$, it follows from weak Schanuel's conjecture that $\log q_1,\dots,\log q_l$ are algebraically independent and so are $1/\log q_1,\dots,1/\log q_l$. So we mention the following conjecture.

Conjecture 1. *For any distinct primes $q_1,q_2,\dots,q_l$, the l real numbers $1/\log q_1,\dots,1/\log q_l$ are linear independent over $\mathbb{Q}$.*

In this paper, we prove the following result.

Theorem 2. *Assuming Conjecture 1, we have $\bar{d}(\mathscr{L}) = 1$.*

2. Preliminaries

Lemma 3 ([3, Theorem 429], Mertens' theorem).

$$\prod_{\substack{p\le x\\ p \text{ is a prime}}} \left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x} \text{ as } x\to+\infty.$$

Lemma 4 ([3, Theorem 442], Kronecker's theorem). *If $\vartheta_1,\vartheta_2,\dots,\vartheta_k,1$ are real numbers and linearly independent over $\mathbb{Q}$, $\alpha_1,\alpha_2,\dots,\alpha_k$ are arbitrary real numbers, and N and ϵ are positive real numbers, then there are integers $n>N$, $s_1,s_2,\dots,s_k$ such that $|n\vartheta_m - s_m - \alpha_m| < \epsilon$ $(m=1,2,\dots,k)$.*

Lemma 5. *Let a,b,c,d be positive real numbers. Then*

$$|(a,b)\setminus(c,d)| \le |c-a|+|b-d|+2,$$

where (x,y) denotes the set of all integers n with $x<n<y$.

Proof. If $b\le c$ or $d\le a$, then the claim is obvious, since $(a,b)\cap(c,d)=\emptyset$. So assuming $b>c$ and $d>a$ one gets

$$\begin{aligned}|(a,b)\setminus(c,d)| &= \sum_{\substack{a<n<b\\ n\le c \text{ or } n\ge d}} 1 \le \sum_{a<n\le\min\{b,c\}} 1 + \sum_{\max\{a,d\}\le n<b} 1 = \sum_{a<n\le c} 1 + \sum_{d\le n<b} 1\\ &\le |c-a|+|b-d|+2.\end{aligned}$$

This completes the proof of Lemma 5. □

3. Proof of Theorem 2

Let L_n be the least common multiple of $1,2,\dots,n$. Let p_i be i-th prime, $a_1 = 1$ and

$$a_i = \prod_{j=2}^{i}\left(1-\frac{1}{p_j}\right), \quad i = 2,3,\dots.$$

By Lemma 3, $a_i \to 0$ as $i \to +\infty$. For any $0<\varepsilon<1$, we can choose k (fixed) such that $a_k < \frac{1}{2}\varepsilon$. In view of the assumption,

$$\frac{\log p_2}{\log p_i}, \quad i = 2,3,\dots,k$$

are linear independent over $\mathbb{Q}$. By Lemma 4, for any

$$0<\delta<\frac{\varepsilon}{16k\log p_k}, \tag{1}$$

there are infinitely many k tuples $q>0$, $s_2,\dots,s_k$ of integers such that

$$\left|q\frac{\log p_2}{\log p_i} - s_i + \frac{\log a_{i-1}}{\log p_i}\right| \le \delta, \quad i = 2,3,\dots,k.$$

That is,

$$p_i^{s_i-\delta} \le a_{i-1}p_2^q \le p_i^{s_i+\delta}, \quad i = 2,3,\dots,k. \tag{2}$$

It is clear that $s_i \to +\infty$ as $q \to +\infty$ $(i = 2,3,\dots,k)$. Now we prove that

$$((p_i-1)p_i^{s_i-1}, p_i^{s_i}) \subseteq \mathscr{L}. \tag{3}$$

Let $n \in ((p_i-1)p_i^{s_i-1}, p_i^{s_i})$. Then $p_i^{s_i-1} | L_n$. Since

$$1+\frac{1}{2}+\cdots+\frac{1}{p_i-1} = \sum_{1\le j\le (p_i-1)/2}\left(\frac{1}{j}+\frac{1}{p_i-j}\right) = \sum_{1\le j\le (p_i-1)/2}\frac{p_i}{j(p_i-j)},$$

it follows that

$$\begin{aligned} H_n &= \sum_{j=1, p_i^{s_i-1}\nmid j}^{n}\frac{1}{j} + \frac{1}{p_i^{s_i-1}}\left(1+\frac{1}{2}+\cdots+\frac{1}{p_i-1}\right) \\ &= \sum_{j=1, p_i^{s_i-1}\nmid j}^{n}\frac{1}{j} + \frac{1}{p_i^{s_i-2}}\sum_{1\le j\le (p_i-1)/2}\frac{1}{j(p_i-j)} \\ &= \frac{a}{b} + \frac{1}{p_i^{s_i-2}}\frac{c}{d}, \end{aligned}$$

where $p_i^{s_i-1}\nmid b$ and $p_i\nmid d$. Then $\nu_{p_i}(H_n)\ge -(s_i-2)$. Hence, v_n is not divisible by $p_i^{s_i-1}$, that is, $p_i^{s_i-1}\nmid v_n$. Noting that $v_n|L_n$ and $p_i^{s_i-1}|L_n$, we have $v_n<L_n$. Hence $n\in\mathscr{L}$. So (3) holds. It follows from (3) that

$$\begin{aligned}\mathscr{L}(p_2^q) &\ge \sum_{i=2}^{k}|\mathscr{L}\cap(a_ip_2^q,a_{i-1}p_2^q)|\\ &\ge \sum_{i=2}^{k}|((p_i-1)p_i^{s_i-1},p_i^{s_i})\cap(a_ip_2^q,a_{i-1}p_2^q)|\\ &= \sum_{i=2}^{k}\left(|(a_ip_2^q,a_{i-1}p_2^q)|-|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})|\right)\\ &\ge \sum_{i=2}^{k}\left(a_{i-1}p_2^q-a_ip_2^q-1-|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})|\right)\\ &= a_1p_2^q-a_kp_2^q-k+1-\sum_{i=2}^{k}|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})|\\ &\ge p_2^q-\frac{1}{2}\varepsilon p_2^q-k-\sum_{i=2}^{k}|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})|.\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})| &\le |a_ip_2^q-(p_i-1)p_i^{s_i-1}|+|a_{i-1}p_2^q-p_i^{s_i}|+2\\ &=\left(1-\frac{1}{p_i}\right)|a_{i-1}p_2^q-p_i^{s_i}|+|a_{i-1}p_2^q-p_i^{s_i}|+2\\ &\le 2|a_{i-1}p_2^q-p_i^{s_i}|+2.\end{aligned}$$

If $a_{i-1}p_2^q\ge p_i^{s_i}$, then by (2) we have

$$0\le a_{i-1}p_2^q-p_i^{s_i}\le p_i^{s_i+\delta}-p_i^{s_i}=\left(p_i^{\delta}-1\right)p_i^{s_i}.$$

If $a_{i-1}p_2^q< p_i^{s_i}$, then by (2) we have

$$0\le p_i^{s_i}-a_{i-1}p_2^q\le p_i^{s_i}-p_i^{s_i-\delta}=p_i^{-\delta}\left(p_i^{\delta}-1\right)p_i^{s_i}\le\left(p_i^{\delta}-1\right)p_i^{s_i}.$$

In all cases, we have

$$|a_{i-1}p_2^q-p_i^{s_i}|\le\left(p_i^{\delta}-1\right)p_i^{s_i}.$$

By (2),

$$p_i^{s_i}\le a_{i-1}p_2^qp_i^{\delta}\le p_2^qp_i^{\delta}.$$

Hence,

$$|a_{i-1}p_2^q-p_i^{s_i}|\le\left(p_i^{\delta}-1\right)p_i^{\delta}p_2^q.$$

In view of (1),

$$0<\delta\log p_i\le\delta\log p_k<\frac{\varepsilon}{16k}<\frac{1}{16}.$$

It follows from $e^x-1\le 2x$ $(0\le x\le\frac{1}{2})$ that

$$\left(p_i^{\delta}-1\right)p_i^{\delta}\le 2\delta(\log p_i)e^{\delta\log p_i}<\frac{\varepsilon}{8k}e^{1/16}<\frac{\varepsilon}{4k}.$$

Thus,

$$|a_{i-1}p_2^q-p_i^{s_i}|<\frac{\varepsilon}{4k}p_2^q.$$

It follows that

$$\sum_{i=2}^{k}|(a_ip_2^q,a_{i-1}p_2^q)\setminus((p_i-1)p_i^{s_i-1},p_i^{s_i})|<k\left(\frac{\varepsilon}{2k}p_2^q+2\right)=\frac{1}{2}\varepsilon p_2^q+2k.$$

Hence,

$$\mathscr{L}(p_2^q) \geq p_2^q - \frac{1}{2}\varepsilon p_2^q - k - \frac{1}{2}\varepsilon p_2^q - 2k = p_2^q - \varepsilon p_2^q - 3k.$$

Thus,

$$\bar{d}(\mathscr{L}) \geq \lim_{q\to\infty} \frac{\mathscr{L}(p_2^q)}{p_2^q} \geq 1-\varepsilon,$$

where $q \to \infty$ and q satisfying (2). Therefore, $\bar{d}(\mathscr{L}) = 1$. This completes the proof.

Acknowledgments

We would like to thank the referee for his/her helpful comments.

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