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Partial differential equations / *Équations aux dérivées partielles*

On the limit spectrum of a degenerate operator in the framework of periodic homogenization or singular perturbation problems

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Abstract. In this paper we perform the analysis of the spectrum of a degenerate operator A_ε corresponding to the stationary heat equation in a ε-periodic composite medium having two components with high contrast diffusivity. We prove that although A_ε is a self-adjoint operator with compact resolvent, its limit A_0 when the size ε of the medium tends to zero is a non self-adjoint operator whose spectrum is bounded by positive constants depending on the first eigenvalue of the one-dimensional Laplacian in $H_0^1(0,L)$ and the first eigenvalue of the bi-dimensional Laplacian with mixed boundary conditions on the representative cell C. Furthermore, we show that the homogenized problem and the one-dimensional limit problem obtained by the reduction of dimension $3d-1d$ occurring locally are identical except for one boundary condition which is a homogeneous Neumann condition on the boundary of C in the $3d-1d$ problem and a periodicity condition in the case of homogenization.

Résumé. Dans ce travail, nous analysons le spectre d'un opérateur dégénéré A_ε correspondant à l'équation de la chaleur stationnaire dans un milieu composite ε-périodique ayant deux composantes avec des coefficients de conductivité à fort contraste. Nous montrons que bien que $\mathscr{A}_\varepsilon$ soit un opérateur auto-adjoint à résolvante compacte, sa limite A_0 lorsque la période ε tend vers 0 est un opérateur non auto-adjoint dont le spectre est borné par des constantes positives ne dépendant que de la première valeur propre du Laplacien uni-dimensionnel dans $H_0^1(0,L)$ et de la première valeur propre du Laplacien bi-dimensionnel avec conditions au bord mixtes sur la cellule de référence C. Nous montrons en outre que le problème homogénéisé et le problème limite obtenu après réduction de dimension $3d-1d$ intervenant localement sont identiques, à une condition aux limites près, la condition de Neumann homogène sur le bord de C dans le problème $3d-1d$ devant être remplacée dans le problème homogénéisé par une condition de périodicité.

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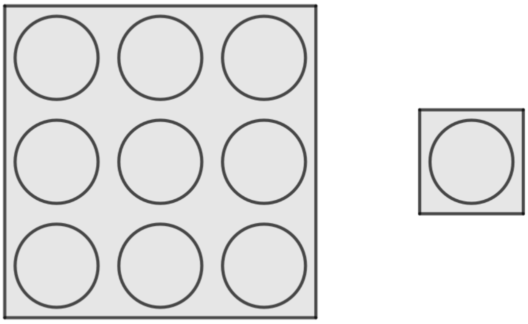

Figure 1. The projection of the domain $\Omega = \omega \times (0, L)$ on the x'-plane and the representative cell $C = M' \cup \overline{D}$.

1. Introduction, setting of the problem and statement of the results

The purpose of this work is the study of two problems: the first one is concerned by the asymptotic analysis of the eigenelements of a spectral problem in the framework of periodic homogenization; the problem writes as

$$\begin{cases} A_\varepsilon u_\varepsilon = \lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega, \\ \text{where } A_\varepsilon u = -\varepsilon^2 \Delta u \chi_{M_\varepsilon} - \Delta u \chi_{F_\varepsilon} \quad \forall\, u \in D(A_\varepsilon), \\ \text{with } D(A_\varepsilon) = \left\{ u \in H^2(\Omega) \cap V_h,\ \frac{\partial u}{\partial n} \chi_{\partial M_\varepsilon} = -\frac{1}{\varepsilon^2} \frac{\partial u}{\partial n} \chi_{\partial F_\varepsilon} \right\}. \end{cases} \tag{1}$$

The notations used in (1) are the following.

Ω denotes a bounded rectangular open set of $\mathbb{R}^3$ of the form $\Omega := \omega \times (0, L)$, ω being a domain of $\mathbb{R}^2$ and L is a positive number. The space V_h (h stands for homogenization) is defined by

$$V_h := \left\{ u \in H^1(\Omega),\ u(x', 0) = u(x', L) = 0 \text{ a.e. } x' = (x_1, x_2) \in \omega \right\}. \tag{2}$$

In the sequel, the variable x_3 will play a particular role and it must be distinguished from the horizontal variable $x' := (x_1, x_2)$ or $y = (y_1, y_2)$. The gradient and the Laplacian with respect to these variables will be denoted by ∇' and Δ'.

We assume that Ω is the reference configuration of a composite medium whose two components are a set F_ε of vertical cylindrical fibers and its complement, the matrix M_ε. Hence, the projection on thex'-plane of the set F_ε is made up of a ε-periodic set of disks while the complement of such set represents the projection of M_ε, see Figure 1. The characteristic functions of F_ε (resp. M_ε) are denoted by χ_{F_ε} (resp. χ_{M_ε}). The fibers are distributed in Ω with a period of size ε and the ratio between the conductivity coefficients of the two components is $\frac{1}{\varepsilon^2}$. Throughout the paper, for a measurable set B we denote by $|B|$ its Lebesgue measure and by χ_B its characteristic function. A generic positive constant the value of which may change from a line to another will be denoted by K.

Let C be a square of $\mathbb{R}^2$ and let D be a disk strictly contained in C. The complement of $\overline{D}$ in C will be denoted by M' in such a way that $C = M' \cup \overline{D}$. The geometry of the domain is described as follows.

$$\begin{cases} Y_\varepsilon^i = (\varepsilon C + \varepsilon i) \times (0, L);\ \omega = \bigcup_{i \in I_\varepsilon} (\varepsilon C + \varepsilon i);\ \Omega = \bigcup_{i \in I_\varepsilon} Y_\varepsilon^i = \omega \times (0, L), \\ F_\varepsilon = \bigcup_{i \in I_\varepsilon} F_\varepsilon^i,\ F_\varepsilon^i = (\varepsilon \overline{D} + \varepsilon i) \times (0, L),\ M_\varepsilon = \bigcup_{i \in I_\varepsilon} M_\varepsilon^i,\ M_\varepsilon^i = Y_\varepsilon^i \setminus \overline{F}_\varepsilon^i = \varepsilon (C \setminus \overline{D}) \times (0, L), \\ I_\varepsilon = \left\{ i \in \mathbb{Z}^2,\ Y_\varepsilon^i \subset \Omega \right\}, \qquad \Omega = F_\varepsilon \cup M_\varepsilon. \end{cases} \tag{3}$$

In Figure 1 we have represented the projection of the domain Ω over the horizontal plane x' and the representative cell $C = \overline{D} \cup M'$.

The second problem we consider in this work is a special case of the first one. Indeed, we now assume that the composite medium consists of a single fiber $F_\varepsilon := (\varepsilon\overline{D}) \times (0,L)$ surrounded by the matrix $M_\varepsilon = \varepsilon M' \times (0,L) = \big(\varepsilon(C \setminus \overline{D})\big) \times (0,L)$. In other words, the global domain depends now on the small parameter ε and it is defined by $\Omega_\varepsilon := (\varepsilon C) \times (0,L) = F_\varepsilon \cup M_\varepsilon$.

In this configuration, the spectral problem (1) takes the form

$$\begin{cases} A_\varepsilon v_\varepsilon = \lambda_\varepsilon v_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \text{where } A_\varepsilon v = -\varepsilon^2 \Delta v \chi_{M_\varepsilon} - \Delta v \chi_{F_\varepsilon} \quad \forall\, v \in D(A_\varepsilon), \\ \text{with } D(A_\varepsilon) = \left\{ v \in H^2(\Omega_\varepsilon) \cap V_s^\varepsilon,\ \frac{\partial v}{\partial n}\chi_{\partial(\varepsilon M')} = -\frac{1}{\varepsilon^2}\frac{\partial v}{\partial n}\chi_{\partial(\varepsilon D)} \right\}, \end{cases} \tag{4}$$

where the space V_s^ε (the subscript "s" stands for singular perturbation) is now defined by

$$V_s^\varepsilon := \left\{ v \in H^1(\Omega_\varepsilon),\ v(x',0) = v(x',L) = 0 \ \text{ a.e. } x' = (x_1,x_2) \in \varepsilon C \right\}. \tag{5}$$

We now introduce the classical scaling $u_\varepsilon(y',x_3) = v_\varepsilon(\varepsilon y', x_3)$, $y' \in C$, (this approach is of course not applicable in the homogenization setting) allowing to transform the problem (4) into the following singular perturbation problem posed on the fixed domain $\Omega := C \times (0,L)$,

$$\begin{cases} A_\varepsilon u = \lambda_\varepsilon u \text{ in } \Omega, \\ \text{where } A_\varepsilon u = \left(-\Delta' u - \varepsilon^2 \frac{\partial^2 u}{\partial x_3^2}\right)\chi_M + \left(-\frac{1}{\varepsilon^2}\Delta' u - \frac{\partial^2 u}{\partial x_3^2}\right)\chi_F, \quad \forall\, u \in D(A_\varepsilon), \\ \text{with } D(A_\varepsilon) = \left\{ u \in H^2(\Omega) \cap V_s,\ \frac{\partial u}{\partial n_{CD}} = -\frac{1}{\varepsilon^2}\frac{\partial u}{\partial n_{DC}} \text{ on } \partial D \times (0,L),\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial C \times (0,L) \right\}, \end{cases} \tag{6}$$

V_s being the space V_s^ε corresponding to $\varepsilon = 1$ and defined in (5) and where n_{CD} denotes the outward normal from $C \setminus D$ towards D and n_{DC} the outward normal from D towards $C \setminus D$, see the representative cell in Figure 1.

Note that the study of the asymptotic behavior of (4) is the so-called reduction of dimension problem $3d - 1d$ since when ε goes to zero the three dimensional domain $\Omega_\varepsilon = (\varepsilon C) \times (0,L)$ looks like the segment $(0,L)$.

Clearly, the homogenization process induces a local reduction of dimension and one of the main feature of this work is to prove that there is no significant difference between the homogenized problem and the one dimensional problem obtained in the reduction of dimension $3d - 1d$. The only difference here is that the homogeneous Neumann condition on the boundary of C in the one dimensional model must be replaced by a periodicity condition in the homogenized problem. In other words, studying the behavior of the solutions of (1) is the same as the study of the behavior of the solutions of (4) which in turn is equivalent to the study of (6). This similarity between the two problems has already been explained in [20] in the framework of linear elasticity and in [17] in the framework of the conductivity equation.

Briefly this similarity may be explained by the fact that finding a two-scale limit (see [1,7,18]) taking into account the oscillations induced by the microscopic variable $y = \frac{x'}{\varepsilon}$ of any sequence $\frac{1}{\varepsilon^2}\overline{\phi}_\varepsilon(x',x_3)$ defined in Ω_ε is simply a matter of finding the weak limit in $L^2(C \times (0,L))$ of the rescaled sequence $\phi_\varepsilon(y,x_3) := \overline{\phi}_\varepsilon(\varepsilon y, x_3)$ defined on the fixed domain $C \times (0,L)$.

Note that the change of variable $x' = \varepsilon y$ which implies

$$\nabla'_y \phi_\varepsilon(y,x_3) = \varepsilon \nabla' \overline{\phi}_\varepsilon(\varepsilon y, x_3) = \varepsilon \nabla' \overline{\phi}_\varepsilon(x',x_3),\ \forall\, (x',x_3) \in (\varepsilon C) \times (0,L), \tag{7}$$

is the reason for which the two problems (4) and (6) are equivalent. The advantage of dealing with the reduction of dimension instead of the homogenization problem lies on the absence of the oscillations due to the fast variable which needs the use of specific tools from the homogenization theory.

This kind of problems was intensively studied during the last years through an abundant literature see [3,4,6,8,10,17,20–22,24] and it is well known that the high contrast between the conductivity coefficients of the components leads in general to a limit problem which may be very different from the equation modeling the composite structure at the scale ε. In particular a nonlocal phenomenon and/or strange term or memory effects may appear highlighting the gap between the properties of the fibers with those of the matrix.

We give here another example by studying the problem of the behavior of the spectrum when the operator is not uniformly bounded below with the respect to the small parameter ε. We choose to perform the study of a simple operator (the Laplacian) in order to focus on the main difficulty and for the sake of brevity and clarity of the presentation.

Unlike the study addressed in [24] where the geometry considered and the choice of the degenerate operator lead to an unbounded spectrum with gaps, we show here that the limit operator is still of a different nature from the original one despite the absence of gaps at the limit (see also [15] and the references therein); the limit spectrum we get in our case is bounded and described by two coupled equations related respectively to the vibrations in the fibers and in the matrix. Other settings have been studied in [2, 11, 12, 16].

Roughly speaking, we show here that a phenomenon similar to that observed in the static case for degenerate operators, see [17, 20, 21] occurs at the limit: the limiting spectrum is given by solving two equations: the main equation is a one-dimensional equation in $(0, L)$ describing the vibrations in the fibers but this equation alone is not sufficient to describe the phenomenon since it is strongly coupled to another one posed in the matrix. In other words, although the vibrations in the matrix are negligible at the scale ε, compared to those of the fibers, we still have a trace of the matrix vibrations at the limit; in other words, a memory effect appears at the limit.

For uniformly (with respect to the small parameter ε) bounded operators, the structure of the limit spectrum looks like the one of the homogenized problem in the static case, as seen for instance in [11, 12, 23].

Let us point out that the asymptotic analysis and the techniques leading to the limit problem in the local 3d-1d dimensional reduction, i.e., the singular perturbation problem, are exactly those giving the homogenized problem, except for a few minor differences due to the oscillations of the fast variable. Moreover as in the static case, see [17, 20], the homogenized problem is nothing but the exact copy of the one-dimensional model obtained in the local reduction of dimension.

Let us also point out that our main result (see Theorem 3 below) states that the limit spectrum is essentially that of the Laplacian in $(0, L)$ with Dirichlet boundary condition since we prove that all the limit eigenvalues are simple eigenvalues. The latter is true for the $3d-1d$ reduction dimension problem as well as for the homogenization problem (see Theorem 5 below); hence one can say that the local reduction of dimension takes precedence over the homogenization process itself.

It is easily seen that for a fixed ε, A_ε defined either by (1) or by (6) is a selfadjoint operator with compact resolvent and hence the following classical result takes place.

Proposition 1. *Problem* (1) *(or problem* (6)*) admits a sequence of eigenvalues* $(\lambda_\varepsilon^k)_k$, $0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \cdots \leq \lambda_\varepsilon^n \leq \ldots$, *with* $\lim_{k\to\infty} \lambda_\varepsilon^k = +\infty$ *while the associate eigenvectors* $(u_\varepsilon^k)_k$ *may be chosen as an orthonormal basis of* $L^2(\Omega)$.

Taking into account this result, the variational formulation of (1) and of (6) are respectively the following ones

$$\begin{cases} u_\varepsilon^k \in V_h, \\ \displaystyle\int_\Omega (\varepsilon^2 \nabla u_\varepsilon^k \nabla \phi \chi_{M_\varepsilon} + \nabla u_\varepsilon^k \nabla \phi \chi_{F_\varepsilon}) \mathrm{d}x = \lambda_\varepsilon^k \int_\Omega u_\varepsilon^k \phi \, \mathrm{d}x, \\ \forall\, \phi \in V_h, \end{cases} \tag{8}$$

$$\begin{cases} u_\varepsilon^k \in V_s, \\ \displaystyle\int_\Omega \left(\left(\nabla' u_\varepsilon^k \nabla' \phi + \varepsilon^2 \frac{\partial u_\varepsilon^k}{\partial x_3} \frac{\partial \phi}{\partial x_3} \right) \chi_M + \left(\frac{1}{\varepsilon^2} \nabla' u_\varepsilon^k \nabla' \phi + \frac{\partial u_\varepsilon^k}{\partial x_3} \frac{\partial \phi}{\partial x_3} \right) \chi_F \right) \mathrm{d}y \mathrm{d}x_3 = \lambda_\varepsilon^k \int_\Omega u_\varepsilon^k \phi \, \mathrm{d}y \mathrm{d}x_3, \\ \forall \, \phi \in V_s, \end{cases} \tag{9}$$

where $F := D \times (0,L)$ and $M := (C \setminus \overline{D}) \times (0,L)$.

We prove that the spectrum σ_0 of the homogenized problem is bounded: any eigenvalue $\lambda_k \in \sigma_0$ is such that $\mu_0 \le \lambda_k < \mu_1$ where the lower bound μ_0 is a positive constant depending both on the first eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ in $H_0^1(0,L)$ and the first eigenvalue μ_1 of the two-dimensional Laplacian operator (denoted in the sequel by Δ'_y) with mixed boundary conditions on the boundary of $C \setminus \overline{D}$ (periodicity on the sides of the square C and homogeneous Dirichlet condition on the boundary of the disk D) and on the other hand the calculation of these eigenvalues takes into account the vibrations outside the fibers through the mean value $\int_{C \setminus D} u_0^k \, \mathrm{d}y$ of u_0^k obtained by solving a second order equation posed in $C \setminus \overline{D}$, in such a way that the limit spectral problem is given by the following coupled system of two equations

$$\begin{cases} u_0^k \in H^1_\#(C), \quad -\Delta'_y u_0^k = \lambda_k u_0^k + 1 \ \text{ in } C \setminus \overline{D}, \\ u_0^k = 0 \text{ on } \partial D, \qquad u_0^k \text{ is } C-\text{periodic}, \\ v_k \in L^2(\omega; H^1_0(0,L)), \quad -\dfrac{\partial^2 v_k}{\partial x_3^2} = \lambda_k \left(1 + \dfrac{|C \setminus D|}{|D|} + \dfrac{\lambda_k}{|D|} \displaystyle\int_{C \setminus D} u_0^k \, \mathrm{d}y \right) v_k \quad \text{in } \Omega. \end{cases} \tag{10}$$

In the case of the singular perturbation problem (6), the limit spectral problem is the following one

$$\begin{cases} u_0^k(y) \in H^1(C)), \quad -\Delta'_y u_0^k = \lambda_k u_0^k + 1 \ \text{ in } C \setminus \overline{D}, \\ u_0^k = 0 \quad \text{on } \partial D, \\ \dfrac{\partial u_0^k}{\partial n} = 0 \quad \text{on } \partial C, \\ v_k \in H^1_0(0,L)), \quad -\dfrac{\mathrm{d}^2 v_k}{\mathrm{d}x_3^2} = \lambda_k \left(1 + \dfrac{|C \setminus D|}{|D|} + \dfrac{\lambda_k}{|D|} \displaystyle\int_{C \setminus D} u_0^k \, \mathrm{d}y \right) v_k \quad \text{in } (0,L). \end{cases} \tag{11}$$

Note the very close analogy between the two limit problems. The only difference between (10) and (11) lies in the Neumann boundary condition on ∂C of (11) replaced by a periodicity condition in (10) and in the fact that in the homogenized problem (10), the function v depends also on the variable x'; this simply means that the homogenized problem is a duplication of the phenomenon occurring in each cell.

In other words, the homogenization process does not affect the vertical variable and does not bring any new features to the limit model.

Note also that the existence and the uniqueness of u_0^k both in (10) and in (11) are ensured by the fact that λ_k belongs to the resolvent $\rho(-\Delta'_y)$ of $-\Delta'_y$ since $\lambda_k < \mu_1$. We will prove in Section 2 (see (40)) that $\int_{C \setminus D} u_0^k \mathrm{d}y \neq 0$ by the use of the property that $\frac{1}{\mu_1}$ is the best constant in the Poincaré inequality.

Furthermore, the last equation of (10) or (11) shows that $\lambda_k \left(1 + \frac{|C \setminus D|}{|D|} + \frac{\lambda}{|D|} \int_{C \setminus D} u_0^k \mathrm{d}y\right) \ge \lambda_0^1$ where $\lambda_0^1 := \frac{\pi^2}{L^2}$ is the first eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ in $H_0^1(0,L)$.

From now on and based on the previous comments comparing the homogenization problem with the problem of reduction of the dimension arising locally, we will focus on the asymptotic analysis of the singular perturbation problem (9) (the study of the reduction of dimension occurring in each cell). The reduction of dimension problem is usually encountered in the study of thin structures, see for instance [14] and [19].

We will prove in the next sections that in the case of homogenization, there exists a subsequence of ε still denoted by ε such that $(\lambda_\varepsilon^k)_\varepsilon$ tends to λ_k when ε goes to zero and the associate sequence of eigenvectors strongly two-scale converges to $u_k(x,y) = (\lambda_k u_0^k + 1)v_k$ where $\big((u_0^k(y), v_k(x)), \lambda_k\big)$ solves the system (10). For the reduction of dimension problem the associate sequence of eigenvectors u_ε^k converges strongly in $L^2(\Omega)$ to $u_k(y,x_3) = (\lambda_k u_0^k + 1)v_k$ where $\big((u_0^k(y), v_k(x_3)), \lambda_k\big)$ solves the system (11).

Remark 2. One can reasonably and legitimately ask what is the link between the problem (11) (or the problem (10)) and the classical formulation of eigenvalue problems. In fact, (11) is derived from the system (30) which in turn is derived from the equation (29) satisfied by the pair (u_k, v_k), see the details of the proof in Section 2 below. If one integrates the first equation of (30) over $C\setminus\overline{D}$, we get an equivalent formulation of (30) as follows

$$\begin{cases} u_k(y,x_3) \in L^2((0,L);H^1(C)), \quad -\Delta'_y u_k(y,x_3) = \lambda_k u_k \ \text{ in } (C\setminus\overline{D})\times(0,L), \\ u_k = v_k \quad \text{on } \partial D\times(0,L), \\ \dfrac{\partial u_k}{\partial n} = 0 \quad \text{on } \partial C\times(0,L), \\ v_k \in H_0^1(0,L)), \quad -\dfrac{\mathrm{d}^2 v_k}{\mathrm{d}x_3^2} + \dfrac{1}{|D|}\displaystyle\int_{\partial D}\dfrac{\partial u_k}{\partial n}\,\mathrm{d}\sigma = \lambda_k v_k \quad \text{in } (0,L). \end{cases} \tag{12}$$

Another equivalent formulation of (12) is the following

$$A_0\begin{pmatrix} u_k \\ v_k\end{pmatrix} = \lambda_k \begin{pmatrix} u_k \\ v_k\end{pmatrix} \tag{13}$$

where the operator A_0 is defined by $A_0 : D(A_0) \to L^2(\Omega)\times L^2(0,L)$ with

$$\begin{cases} D(A_0) = \left\{\begin{pmatrix} u \\ v\end{pmatrix} \in L^2(0,L;H^2(C\setminus\overline{D})\times H^2\cap H_0^1(0,L);\ u = v \text{ on } \partial D,\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial C\right\}, \\ A_0\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} -\Delta'_y u \\ -\frac{\mathrm{d}^2 v}{\mathrm{d}x_3^2} + \frac{1}{|D|}\int_{\partial D}\frac{\partial u}{\partial n}\,\mathrm{d}\sigma \end{pmatrix}, \qquad \forall \begin{pmatrix} u \\ v\end{pmatrix} \in D(A_0). \end{cases} \tag{14}$$

Although one can see from the structure of the limit spectrum which is bounded that the limit operator is very different from the starting one, once again we can see from (13) and (14) the sharp difference between the operator A_ε and the limit operator A_0 which is not even a self-adjoint operator.

Of course a similar remark applies for the homogenized problem given by (10).

From the technical point of view the main difficulty in the asymptotic analysis comes from the lack of compactness since the sequences of eigenvectors we have to consider are not bounded in $H^1(\Omega)$ so that the strong convergence in $L^2(\Omega)$ (or strong two-scale convergence in the case of homogenization) which allows to conclude that the limit of an eigenvector u_ε^k is still an eigenvector (i.e. $\neq 0$) is not straightforward. To overcome this difficulty, we will use an extension technique (see [9,25]) combined with another slightly more intricate argument. Our main results may be stated as follows.

Theorem 3. *For each $k = 1,2,\ldots$, the sequence of eigenvalues $(\lambda_\varepsilon^k)_\varepsilon$ of* (9) *is bounded and the associated sequence of eigenvectors $(u_\varepsilon^k)_\varepsilon$ is bounded in $L^2(0,L;H^1(C))$; there exists a solution $(\lambda_k, u_0^k, v_k) \in (\mu_0,\mu_1[\times L^2(0,L;H^1(C))\times H_0^1(0,L)$ of* (11) *with $v_k \neq 0$ such that for the whole sequence ε, one has*

$$\lambda_\varepsilon^k \longrightarrow \lambda_k, \tag{15}$$

$$u_\varepsilon^k \longrightarrow u_k(y,x_3) := (\lambda_k u_0^k + 1)v_k \text{ strongly in } L^2(0,L;H^1(C)), \tag{16}$$

$$u_\varepsilon^k \chi_F \longrightarrow v_k\chi_F \text{ strongly in } L^2(C;H_0^1(0,L)). \tag{17}$$

For all k, λ_k is a simple eigenvalue of the limit operator and $\lambda_k \in (\mu_0, \mu_1[$ where μ_0 depends only on μ_1 and on the first eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ in $H_0^1(0,L)$.

Conversely, any $\lambda \in (\mu_0, \mu_1[$ which is an eigenvalue of (11) is a limit of a sequence $(\lambda_\varepsilon^k)_\varepsilon$ of eigenvalues of (9).

The unique accumulation point of the sequence $(\lambda_k)_k$ is the first eigenvalue μ_1 of $-\Delta'_y$; hence $\lim_{k\to+\infty} \lambda_k = \mu_1$.

Remark 4. The property $v_k \neq 0$ may be deduced from the strong convergence (16) of the eigenvectors but we prefer to write it explicitly to highlight the fact that v_k is always an eigenvector of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ with Dirichlet condition.

Regarding the case of homogenization, exactly the same results obtained in the singular perturbation problem remain true but we have to formulate it taking into account the oscillations induced by the homogenization process. We use the notation $\overset{2-sc}{\rightharpoonup}$ (resp. $\overset{2-sc}{\longrightarrow}$) for the two-scale convergence (resp. the strong two-scale convergence), see [1, 18, 24].

Theorem 5. *For each $k = 1, 2, \ldots$, the sequence of eigenvalues $(\lambda_\varepsilon^k)_\varepsilon$ of (8) is bounded and the associated sequence of eigenvectors $(u_\varepsilon^k)_\varepsilon$ is bounded in $L^2(0,L;H^1(\omega))$; there exists a solution $(\lambda_k, u_0^k, v_k) \in (\mu_0, \mu_1[\times L^2(0,L;H^1_\#(C)) \times L^2(\omega;H_0^1(0,L)$ of (10) with $v_k \neq 0$ such that for the whole sequence ε, one has*

$$\lambda_\varepsilon^k \longrightarrow \lambda_k, \tag{18}$$

$$u_\varepsilon^k \overset{2-sc}{\longrightarrow} u_k(x,y) := (\lambda_k u_0^k + 1) v_k, \tag{19}$$

with the following corrector result

$$\int_\Omega \left(\left(\left| \varepsilon \nabla' u_\varepsilon^k - \nabla'_y u_k\left(x, \frac{x'}{\varepsilon}\right) \right|^2 + \varepsilon^2 \left| \frac{\partial u_\varepsilon^k}{\partial x_3} \right|^2 \right) \chi_{M_\varepsilon}(x') + \left(|\nabla' u_\varepsilon^k|^2 + \left| \frac{\partial u_\varepsilon^k}{\partial x_3} - \frac{\partial v_k}{\partial x_3} \right|^2 \right) \chi_{F_\varepsilon}(x') \right) \mathrm{d}x \to 0. \tag{20}$$

For all k, λ_k is a simple eigenvalue of the limit operator and $\lambda_k \in (\mu_0, \mu_1[$ where the constant μ_0 depends only on the first eigenvalue λ_0 of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ in $H_0^1(0,L)$ and the first eigenvalue μ_1 of $-\Delta'_y$ in $C \setminus \overline{D}$ with Dirichlet boundary condition on ∂D and periodicity on the sides of C.

Conversely, any eigenvalue $\lambda \in (\mu_0, \mu_1[$ of problem (10) is a limit of a sequence $(\lambda_\varepsilon^k)_\varepsilon$ of eigenvalues of (8).

The sequence $(\lambda_k)_k$ converges to μ_1.

Remark 6. Note that the structure of the limit spectrum is quite complicated because not only the mean value $\int_{C\setminus D} u_0 \,\mathrm{d}y$ arising in the second equation of the limit system must be calculated by the use of the first equation of the system but the function u_0 itself depends on the corresponding eigenvalue as shown by the first equation.

Remark also that in the homogenization setting, the analogous result of the convergence (17) is the convergence $\int_\Omega \left| \frac{\partial u_\varepsilon^k}{\partial x_3} - \frac{\partial v_k}{\partial x_3} \right|^2 \chi_{F_\varepsilon}(x')\,\mathrm{d}x \to 0$ obtained from the corrector result (20). However, the latter does not mean that the sequence $u_\varepsilon^k \chi_{F_\varepsilon}$ converges strongly in $L^2(\omega;H_0^1(0,L))$ to $\frac{|D|}{|C|} v_k = |D| v_k$ (we have assumed $|C| = 1$) in which case this convergence would be the exact analogue of (17). Unfortunately, because of the oscillations induced by the homogenization process, such exact analogue of (17) is false. This is one of the few differences between the $3d-1d$ problem and the homogenization problem.

Remark 7. Before proceeding to prove the results in the next sections, it should be pointed out that the study can be easily extended to the case of operators in divergence form. In that case, we have to take into account at the limit the contribution of the anisotropy of the fiber (or the fibers in the case of homogenization) as shown in [21]. On the other hand, one can consider other scalings of the form $\varepsilon^\gamma \chi_{F_\varepsilon} + \varepsilon^\delta \chi_{M_\varepsilon}$ as in [10,22]. For instance, if one considers coefficients of order ε^δ in the matrix M_ε instead of ε^2 and 1 in F_ε, then loosely speaking the structure of the limit problem depends on the limit of the ratio $\varepsilon^{\delta-2}$ as shown in [10,22], the critical case giving rise to a coupled system at the limit is the one corresponding to $\lim \varepsilon^{\delta-2} = l \in]0,+\infty[$. Here we address the critical case in the framework of the Laplacian operator for the sake of simplicity and brevity.

Remark 8. In order to highlight the close analogy between the $3d-1d$ limit problem and the homogenized problem, the macroscopic variable x will be denoted by $x = (y, x_3)$, $y \in C$ in the study of the $3d-1d$ problem for which $\Omega := C \times (0,L)$ while in the homogenization problem x will be denoted by $x = (x', x_3)$, $x' \in \omega := \bigcup_{i \in I_\varepsilon} (\varepsilon C + \varepsilon i)$ since $\Omega := \bigcup_{i \in I_\varepsilon} (\varepsilon C + \varepsilon i) \times (0,L)$ so that $x' = \varepsilon y + \varepsilon i$, $i \in I_\varepsilon$. Let us remember that in the reduction of dimension problem, Ω is obtained from the original thin domain $\Omega_\varepsilon = (\varepsilon C) \times (0,L)$ by the scaling $x' = \varepsilon y$, $y \in C$, which makes our notations homogeneous.

As mentioned above we study in detail the case of a single fiber; this is the purpose of the next section. In the last section, we will indicate the appropriate minor changes we have to do in the case of homogenization.

2. Proof of the results in the case of a single fiber: the reduction of dimension $3d-1d$

2.1. *Apriori estimate on the sequence of eigenvalues and eigenvectors*

Proposition 9. *For each $k = 1,2,\ldots,$ the sequence $(\lambda_\varepsilon^k, u_\varepsilon^k)$ of eigenpairs of* (9) *is bounded in $\mathbb{R} \times L^2(0,L;H^1(C))$. There exist $(\lambda_k, u_k, v_k) \in (0,\mu_1) \times L^2(0,L;H^1(C)) \times H_0^1(0,L)$ and a subsequence of ε still denoted by ε such that*

$$u_\varepsilon^k \rightharpoonup u_k \text{ weakly in } L^2(0,L;H^1(C)) \text{ and } u_k(y,x_3) = v_k(x_3) \text{ in } F = D \times (0,L), \tag{21}$$

$$\frac{\partial u_\varepsilon^k}{\partial x_3}\chi_F \rightharpoonup \frac{\mathrm{d} v_k}{\mathrm{d} x_3}\chi_F \quad \text{weakly in } L^2(\Omega), \tag{22}$$

$$\lambda_\varepsilon^k \to \lambda_k. \tag{23}$$

Proof. We first prove an apriori estimate on the sequence of eigenvalues which will play a key role in the sequel. Throughout the paper we refer to λ_k^0 as the k-th eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ in $(0,L)$ with Dirichlet boundary conditions and to μ_1 as the first eigenvalue of $-\Delta'_y$ in $C \setminus \overline{D}$ with Dirichlet boundary condition on the boundary of D and Neumann (respectively periodic) boundary condition on the sides of C in the case of a single fiber (respectively homogenization).

We claim that

$$\forall\, \varepsilon, \quad \forall\, k = 1,2,\ldots, \quad \lambda_\varepsilon^k \le \mu_1 + \varepsilon^2 \lambda_k^0. \tag{24}$$

Indeed, it is well known that the k-th eigenvalue λ_ε^k of (9) is given by the Min-Max formula

$$\lambda_\varepsilon^k = \min_{V^k \subset V_s} \max_{u \in V^k} \frac{\int_\Omega \left((|\nabla'_y u|^2 + \varepsilon^2 |\frac{\partial u}{\partial x_3}|^2)\chi_M + (\frac{1}{\varepsilon^2}|\nabla'_y u|^2 + |\frac{\partial u}{\partial x_3}|^2)\chi_F \right) \mathrm{d}y \mathrm{d}x_3}{\int_\Omega |u|^2 \,\mathrm{d}y \mathrm{d}x_3}, \tag{25}$$

where the space V_s is defined by (5) (with $\varepsilon = 1$) and the min runs over all subspaces V^k of V_s with finite dimension k.

Let $\phi(y)$ be an eigenvector associated to μ_1 extended by zero in D. Then $\phi(y)\psi(x_3)$ belongs to V_s for any $\psi \in H_0^1(0,L)$ and $\phi\psi = 0$ in F.

Let V^k be the subspace of V_s spanned by $\{\phi v^1, \phi v^2, \ldots, \phi v^k\}$ where $v^1, v^2, \ldots, v^k$ denote the associated eigenvectors to the first k eigenvalues $\lambda_1^0, \lambda_2^0, \ldots, \lambda_k^0$ of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$.

For any $u=\alpha_1\phi v^1+\cdots+\alpha_k\phi v^k\in V^k$, we have $u=0$ in F and since $v^1,v^2,\ldots,v^k,\ldots,$ is an orthonormal basis in $H^1_0(0,L)$ we also have

$$\begin{aligned}
&\int_\Omega u^2\,\mathrm{d}y\,\mathrm{d}x_3=\int_{C\setminus D}\phi^2\,\mathrm{d}y\int_0^L\big(\alpha_1^2(v^1)^2+\cdots+\alpha_k^2(v^k)^2\big)\,\mathrm{d}x_3=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\int_{C\setminus D}\phi^2\,\mathrm{d}y,\\
&\qquad\int_\Omega|\nabla'_y u|^2\mathrm{d}y\mathrm{d}x_3=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\int_{C\setminus D}|\nabla'_y\phi|^2\,\mathrm{d}y=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\mu_1\int_{C\setminus D}|\phi|^2\,\mathrm{d}y,\\
&\int_\Omega\varepsilon^2\Big|\frac{\partial u}{\partial x_3}\Big|^2\mathrm{d}y\mathrm{d}x_3=\varepsilon^2\int_0^L\Big(\alpha_1^2\Big(\frac{\mathrm{d}v^1}{\mathrm{d}x_3}\Big)^2+\cdots+\alpha_k^2\Big(\frac{\mathrm{d}v^k}{\mathrm{d}x_3}\Big)^2\Big)\,\mathrm{d}x_3\int_{C\setminus D}|\phi|^2\,\mathrm{d}y,\\
&\qquad\qquad=\varepsilon^2\big(\alpha_1^2\lambda_1^0+\cdots+\alpha_k^2\lambda_k^0\big)\int_{C\setminus D}|\phi|^2\,\mathrm{d}y\le\varepsilon^2\lambda_k^0\big(\alpha_1^2+\cdots+\alpha_k^2\big)\int_{C\setminus D}|\phi|^2\,\mathrm{d}y.
\end{aligned}\tag{26}$$

Note that the last equality in the second line of (26) is a consequence of the equation $-\Delta'_y\phi=\mu_1\phi\quad\text{in } C\setminus\overline{D}$. Hence, by this choice of V^k we obtain estimate (24) by the use of the Min-Max formula combined with (26).

We obtain that $\lambda_k\in(0,\mu_1)$ by passing to the limit (for a subsequence of ε) in (24). We will prove later that the value μ_1 cannot be attained by λ_k for all k and we will prove also that the sequence $(\lambda_k)_k$ admits a lower bound $\mu_0>0$ depending on μ_1 and on the first eigenvalue λ_0 of the onedimensional laplacian operator with Dirichlet boundary conditions in $(0,L)$.

Turning back to (9) and taking u^k_ε (with $\|u^k_\varepsilon\|_{L^2(\Omega)}=1$) as a test function, we get

$$\int_\Omega\Big(\Big(|\nabla' u^k_\varepsilon|^2+\varepsilon^2\Big|\frac{\partial u^k_\varepsilon}{\partial x_3}\Big|^2\Big)\chi_M+\Big(\frac{1}{\varepsilon^2}|\nabla' u^k_\varepsilon|^2+\Big|\frac{\partial u^k_\varepsilon}{\partial x_3}\Big|^2\Big)\chi_F\Big)\mathrm{d}y\mathrm{d}x_3=\lambda^k_\varepsilon\le K.\tag{27}$$

The last estimate implies that $\nabla' u^k_\varepsilon$ is bounded in $L^2(\Omega)$ and thus u^k_ε is bounded in $L^2(0,L;H^1(C))$. We then conclude that there exist a sequence of ε and $u_k\in L^2(C;H^1_0(0,L))$ such that the convergence (21) holds true.

The sequence $\nabla' u^k_\varepsilon\chi_F(y)\rightharpoonup\nabla' u_k\chi_F$ weakly in $L^2(\Omega)$. But $\nabla' u^k_\varepsilon\chi_F$ which is bounded in $L^2(\Omega)$ by $C\varepsilon$ strongly converges to zero in $L^2(\Omega)$. Hence, $\nabla' u_k\chi_F=0$ which means that $u_k=v_k(x_3)$ a.e. in F. The sequence $u^k_\varepsilon\chi_F(y)$ (note that the characteristic functions χ_F and χ_M depend only on the horizontal variable y) is bounded in $L^2(C;H^1_0(0,L))$ since $\frac{\partial u^k_\varepsilon}{\partial x_3}\chi_F$ is bounded in $L^2(\Omega)$ so that for a subsequence $\frac{\partial u^k_\varepsilon}{\partial x_3}\chi_F\rightharpoonup\frac{\partial u_k}{\partial x_3}\chi_F=\frac{\mathrm{d}v_k}{\mathrm{d}x_3}\chi_F$ weakly in $L^2\Omega)$. Hence $v_k\in H^1_0(0,L)$ and the convergence (22) holds true.This completes the proof. □

2.2. *The limit problem associated to (9)*

We choose a test function in (9) in the form $\phi=\bar u$ with $\bar u=\bar v(x_3)$ in F and $(\bar u,\bar v)\in V_s\times H^1_0(0,L)$. We get from (9)

$$\int_\Omega\Big(\Big(\nabla' u^k_\varepsilon\nabla'\bar u+\varepsilon^2\frac{\partial u^k_\varepsilon}{\partial x_3}\frac{\partial\bar u}{\partial x_3}\Big)\chi_M+\Big(\frac{\partial u^k_\varepsilon}{\partial x_3}\frac{\mathrm{d}\bar v}{\mathrm{d}x_3}\Big)\chi_F\Big)\mathrm{d}y\mathrm{d}x_3=\lambda^k_\varepsilon\int_\Omega u^k_\varepsilon\bar u\,\mathrm{d}y\mathrm{d}x_3.\tag{28}$$

Passing to the limit in this equation, we get

$$\begin{cases}(u_k,v_k)\in L^2(0,L;H^1(C))\times H^1_0(0,L),\ u_k=v_k\text{ in }F,\\ \displaystyle\int_\Omega\Big(\nabla' u_k\nabla'\bar u\chi_M+\frac{\mathrm{d}v_k}{\mathrm{d}x_3}\frac{\mathrm{d}\bar v}{\mathrm{d}x_3}\chi_F\Big)\mathrm{d}y\mathrm{d}x_3=\lambda_k\int_\Omega u_k\bar u\,\mathrm{d}y\mathrm{d}x_3,\\ \forall\ (\bar u,\bar v)\in V_s\times H^1_0(0,L),\ \bar u=\bar v\text{ in }F.\end{cases}\tag{29}$$

Finally a density argument allows to extend (29) to all test functions $\bar u\in L^2(0,L;H^1(C))$ such that $\bar u=\bar v$ in F and $\bar v\in H^1_0(0,L)$.

Choosing successively in (29) $\bar{u} \in L^2(0,L;H^1(C))$ such that $\bar{u} = 0$ in F and then $\bar{u} \in L^2(0,L;H^1(C))$ such that $\bar{u} = \bar{v} \in H^1_0(0,L)$ almost everywhere in Ω and bearing in mind the geometry of $\Omega := C \times (0,L) = \big((C\setminus D)\cup D\big)\times(0,L)$, we get that the limit problem (29) may be split into two equations leading to the following equivalent system

$$\begin{cases} u_k(y,x_3) \in L^2((0,L);H^1(C)), \quad -\Delta'_y u_k(y,x_3) = \lambda_k u_k \ \text{ in } (C\setminus\overline{D})\times(0,L),\\ u_k = v_k \quad \text{on } \partial D\times(0,L),\\ \dfrac{\partial u_k}{\partial n} = 0 \quad \text{on } \partial C\times(0,L),\\ v_k \in H^1_0(0,L)), \quad -\dfrac{\mathrm{d}^2 v_k}{\mathrm{d}x_3^2} = \lambda_k v_k + \dfrac{\lambda_k}{|D|}\displaystyle\int_{C\setminus D} u_k\,\mathrm{d}y \quad \text{in } (0,L). \end{cases} \tag{30}$$

We are now in a position to show that the values 0 and μ_1 cannot be achieved by the sequence of eigenvalues $(\lambda_k)_k$.

Proposition 10. *If λ_k is an eigenvalue of* (30)*, then* $0 < \lambda_k < \mu_1$.

Proof. If λ_k is an eigenvalue corresponding to the eigenvector u_k then $\lambda_k > 0$ as seen through (29) by taking $\bar{u} = u_k$, $\bar{v} = v_k$. On the other hand, by construction for each integer k, we have $\lambda_k \le \mu_1$.

Let us prove that μ_1 cannot be achieved. Let ϕ be an eigenvector associated to the first eigenvalue of $-\Delta'_y$, i.e., ϕ is a solution of

$$\begin{cases} -\Delta'_y\phi = \mu_1\phi \ \text{ in } C\setminus\overline{D}\\ \phi = 0 \quad \text{on } \partial D,\\ \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial C. \end{cases} \tag{31}$$

It is well known that μ_1 is a simple eigenvalue and that ϕ is regular and keeps a constant sign as well as its normal derivative on ∂D, see [13]. Multiplying (31) by $u_k(\cdot,x_3)$ and the first equation of (30) (which holds true for almost all $x_3 \in (0,L)$) by ϕ and integrating we infer

$$\begin{cases} \displaystyle\int_{C\setminus D} \nabla'_y\phi\nabla'_y u_k\,\mathrm{d}y - \int_{\partial D}\frac{\partial\phi}{\partial n}\,\mathrm{d}\sigma\, v_k(x_3) = \mu_1\int_{C\setminus D}\phi u_k\,\mathrm{d}y,\\ \displaystyle\int_{C\setminus D}\nabla'_y u_k\nabla'_y\phi\,\mathrm{d}y = \lambda_k\int_{C\setminus D} u_k\phi\,\mathrm{d}y, \end{cases} \tag{32}$$

so that for a.e. $x_3 \in (0,L)$,

$$(\lambda_k - \mu_1)\int_{C\setminus D} u_k\phi\,\mathrm{d}y = \int_{\partial D}\frac{\partial\phi}{\partial n}\,\mathrm{d}\sigma\ v_k(x_3). \tag{33}$$

If $\lambda_k = \mu_1$ then we get $v_k = 0$ a.e. in $(0,L)$ since the integral $\int_{\partial D}\frac{\partial\phi}{\partial n}\,\mathrm{d}\sigma$ does not vanish.

Turning back to the last equation of (30), we get $\int_{C\setminus D} u_k\,\mathrm{d}y = 0$ since $\lambda_k = \mu_1 \neq 0$. On the other hand u_k is an eigenvector for $-\Delta'_y$ associated to $\mu_1 = \lambda_k$ since $v_k = 0$. Therefore there exists $\eta \neq 0$ such that $u_k = \eta\phi$ which implies that $\int_{C\setminus D}\phi\,\mathrm{d}y = 0$. This is a contradiction since $\int_{C\setminus D}\phi\,\mathrm{d}y \neq 0$. □

Remark 11. Eigenvectors of (30) are pairs (u_k, v_k) made up of two inseparable elements. In particular, if $v_k = 0$ then $u_k = 0$ as shown by (30). Indeed, otherwise u_k should be an eigenvector of $-\Delta'_y$ associated to the eigenvalue $\lambda_k < \mu_1$ which is a contradiction. Conversely if $u_k = 0$ then $v_k = 0$ since almost everywhere in $(0,L)$, we have $v_k = u_k$ on the boundary of D.

We now prove that (11) and (30) are equivalent and then we will improve the lower bound of the limit eigenvalues using (11).

Proposition 12. *If (λ_k, u_k, v_k) solves the system* (30) *then $v_k \neq 0$ and u_k writes as*

$$u_k(y,x_3) = (\lambda_k u_0^k(y) + 1)\,v_k(x_3) \tag{34}$$

where (λ_k, u_0^k, v_k) *solves* (11)*. Furthermore, there exists a positive constant* μ_0 *depending on the first eigenvalues of* $-\Delta'_y$ *and* $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ *such that* $\lambda_k \geq \mu_0$ *for all k.*

Proof. Assume that (u_k, v_k) is a solution of (30). If $v_k = 0$ then $u_k = 0$ otherwise λ_k should be an eigenvector of $-\Delta'_y$ by virtue of (30) but this is a contradiction since we have proved that $\lambda_k < \mu_1$. Hence each solution of (30) is such that $u_k \neq 0$ and $v_k \neq 0$ unless $u_k = v_k = 0$. We will prove below that the later cannot happen, i.e., we have indeed an eigenvector at the limit, see Proposition 13.

Dividing by v_k in the first system of (30), one can check easily that $w_k := \frac{u_k}{v_k} - 1$ is the unique solution of

$$\begin{cases} -\Delta'_y w_k = \lambda_k w_k + \lambda_k \ \text{ in } C \setminus \overline{D} \\ w_k = 0 \quad \text{on } \partial D, \\ \frac{\partial w_k}{\partial n} = 0 \quad \text{on } \partial C. \end{cases} \tag{35}$$

Note that the uniqueness of w_k is ensured since $\lambda_k < \mu_1$ belongs to the resolvent of $-\Delta'_y$. On the other hand, the function $\lambda_k u_0^k$ where u_0^k is defined in (11) is also a solution of (35). Hence, we deduce that $w_k := \frac{u_k}{v_k} - 1 = \lambda_k u_0^k$ and therefore (34) follows. Using (34) in (30) we get (11).

We now make more precise the lower bound of the sequence of eigenvalues and we prove at the meanwhile that $\int_{C\setminus D} u_0^k(y)\,\mathrm{d}y > 0$. Multiplying the first equation of (11) by u_0 and bearing in mind that the best constant in the Poincaré's Inequality is equal to $\frac{1}{\mu_1}$, we get

$$\int_{C\setminus D} u_0^k(y)\,\mathrm{d}y = \int_{C\setminus D} |\nabla'_y u_0^k(y)|^2\,\mathrm{d}y - \lambda_k \int_{C\setminus D} |u_0^k(y)|^2\,\mathrm{d}y \geq \left(1 - \frac{\lambda_k}{\mu_1}\right) \int_{C\setminus D} |\nabla'_y u_0^k(y)|^2\,\mathrm{d}y. \tag{36}$$

On the other hand, the first eigenvalue μ_1 is characterized by

$$\mu_1 = \inf_{u \in \{\phi \in H^1(C\setminus \bar{D};\ \phi = 0 \text{ on } \partial D\}} \frac{\|\nabla'_y u\|^2_{L^2(C\setminus \bar{D})}}{\|u\|^2_{L^2(C\setminus \bar{D})}}. \tag{37}$$

Hence the following estimate holds true

$$\int_{C\setminus D} |\nabla'_y u_0^k(y)|^2\,\mathrm{d}y \geq \mu_1 \int_{C\setminus D} |u_0^k(y)|^2\,\mathrm{d}y. \tag{38}$$

From (36), we derive with the help of (38)

$$(\mu_1 - \lambda_k) \int_{C\setminus D} |u_0^k(y)|^2\,\mathrm{d}y \leq \int_{C\setminus D} u_0^k(y)\,\mathrm{d}y \leq \sqrt{|C\setminus D|} \left(\int_{C\setminus D} |u_0^k(y)|^2\,\mathrm{d}y \right)^{\frac{1}{2}}, \tag{39}$$

and then from (39) we deduce

$$0 < \int_{C\setminus D} u_0^k(y)\,\mathrm{d}y \leq \frac{|C\setminus D|}{\mu_1 - \lambda_k}. \tag{40}$$

By virtue of the last equation in (11), $\widehat{\lambda}_k := \lambda_k \big(1 + \frac{|C\setminus D|}{|D|} + \frac{\lambda_k}{|D|} \int_{C\setminus D} u_0^k\,\mathrm{d}y\big)$ is an eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ so that $\widehat{\lambda}_k \geq \lambda_0$ where λ_0 denotes the first eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$. Using the second inequality of (40) we get

$$\lambda_k \left(1 + \frac{|C\setminus D|}{|D|} + \lambda_k \frac{|C\setminus D|}{|D|(\mu_1 - \lambda_k)} \right) \geq \widehat{\lambda}_k \geq \lambda_0. \tag{41}$$

Hence, $\lambda_k \geq \mu_0 := \phi^{-1}(\lambda_0)$ where ϕ is the continuous increasing function defined on $(0, \mu_1)$ by $\phi(t) = t\big(1 + \frac{|C\setminus D|}{|D|} + t\frac{|C\setminus D|}{|D|(\mu_1 - t)}\big)$. □

So far, we have not yet proved that (u_k, v_k) is indeed an eigenvector; this is the purpose of the next subsection.

2.3. *The strong convergence of the eigenvectors*

We prove the following compactness result

Proposition 13. *For each k, there exists a subsequence of ε such that the sequence of solutions u_ε^k of* (9) *converges strongly in $L^2(\Omega)$ to the eigenvector u_k of* (30).

Proof. One can extend u_ε^k from F to the whole Ω in such a way that the extension U_ε^k fulfills $U_\varepsilon^k \in V_s$, $U_\varepsilon^k = u_\varepsilon^k$ in F and

$$\|\nabla' U_\varepsilon^k\|_{L^2(\Omega)} \leq K\|\nabla' u_\varepsilon^k\|_{L^2(F)}, \ \|\frac{\partial U_\varepsilon^k}{\partial x_3}\|_{L^2(\Omega)} \leq K\|\frac{\partial u_\varepsilon^k}{\partial x_3}\|_{L^2(F)}. \tag{42}$$

Note that the extension only affects the horizontal variable y and the Dirichlet boundary condition on the upper and lower faces of Ω ($x_3 = 0$ or $x_3 = L$) is preserved, see for instance [5, 9, 25].

In addition, one can assume that such extension satisfies in addition the two equations

$$\begin{cases} -\Delta'_y U_\varepsilon^k - \varepsilon^2 \frac{\partial^2 U_\varepsilon^k}{\partial x_3^2} = 0 & \text{in } M, \\ \frac{\partial U_\varepsilon^k}{\partial n} = 0 & \text{on } \partial C \times (0, L). \end{cases} \tag{43}$$

Indeed, if (43) is not true for U_ε^k, then one can introduce the function W_ε^k as the unique solution of

$$\begin{cases} W_\varepsilon^k \in V, \\ \displaystyle\int_M \left(\frac{1}{\varepsilon^2}\nabla'_y W_\varepsilon^k \nabla'_y \phi + \frac{\partial W_\varepsilon^k}{\partial x_3}\frac{\partial \phi}{\partial x_3}\right)\mathrm{d}y\mathrm{d}x_3 = \int_M \left(\frac{1}{\varepsilon^2}\nabla'_y U_\varepsilon^k \nabla'_y \phi + \frac{\partial U_\varepsilon^k}{\partial x_3}\frac{\partial \phi}{\partial x_3}\right)\mathrm{d}y\mathrm{d}x_3 \quad \forall\, \phi \in V, \end{cases} \tag{44}$$

where $V := \{u \in V_s, u = 0 \text{ on } \partial D \times (0, L)\}$. Hence, V is the subspace of V_s of functions vanishing in F. By the Lax–Milgram Theorem we get the existence and uniqueness for W_ε^k. Choosing $\phi \in C_0^\infty(M)$, the last equation leads to

$$-\frac{1}{\varepsilon^2}\Delta'_y W_\varepsilon^k - \frac{\partial^2 W_\varepsilon^k}{\partial x_3^2} = -\frac{1}{\varepsilon^2}\Delta'_y U_\varepsilon^k - \frac{\partial^2 U_\varepsilon^k}{\partial x_3^2} \quad \text{in } M. \tag{45}$$

Turning back to (44) and taking $\phi \in C^1(\overline{M})$, we get with the help of (45)

$$\frac{\partial W_\varepsilon^k}{\partial n} = \frac{\partial U_\varepsilon^k}{\partial n} \quad \text{on } \partial C \times (0, L). \tag{46}$$

On the other hand, using equation (44) with $\phi = W_\varepsilon^k$, we get the following estimate with the help of (42) and (27)

$$\begin{aligned} \left\|\frac{1}{\varepsilon}\nabla' W_\varepsilon^k\right\|_{L^2(M)} + \left\|\frac{\partial W_\varepsilon^k}{\partial x_3}\right\|_{L^2(M)} &\leq K\left(\left\|\frac{1}{\varepsilon}\nabla' U_\varepsilon^k\right\|_{L^2(M)} + \left\|\frac{\partial U_\varepsilon^k}{\partial x_3}\right\|_{L^2(M)}\right) \\ &\leq K\left(\left\|\frac{1}{\varepsilon}\nabla' u_\varepsilon^k\right\|_{L^2(F)} + \left\|\frac{\partial u_\varepsilon^k}{\partial x_3}\right\|_{L^2(F)}\right) \leq K. \end{aligned} \tag{47}$$

We now set $\tilde{u}_\varepsilon^k = U_\varepsilon^k - W_\varepsilon^k$. Multiplying equation (45) by ε^2, we see that $\tilde{u}_\varepsilon^k$ fulfills all the required properties including the two equations (43). In the sequel, we denote by U_ε^k the extension of u_ε^k satisfying (42) and (43).

Consider now the sequence defined in Ω by $z_\varepsilon^k = u_\varepsilon^k - U_\varepsilon^k$. If we prove that z_ε^k admits a strongly converging subsequence in $L^2(\Omega)$ then we can deduce the existence of such subsequence for u_ε^k since U_ε^k is bounded in $H^1(\Omega)$ by virtue of (42) and (27) and therefore it admits a strongly converging subsequence in $L^2(\Omega)$ according to the Rellich imbedding Theorem.

We first derive the following equation on z_ε^k by the use of (6) together with (43)

$$\begin{cases} z_\varepsilon^k \in V_s, \quad -\Delta'_y z_\varepsilon^k - \varepsilon^2 \frac{\partial^2 z_\varepsilon^k}{\partial x_3^2} = \lambda_\varepsilon^k z_\varepsilon^k + \lambda_\varepsilon^k U_\varepsilon^k \quad \text{in } M, \\ z_\varepsilon^k = 0 \quad \text{on } \partial D \times (0,L), \\ \frac{\partial z_\varepsilon^k}{\partial n} = 0 \quad \text{on } \partial C \times (0,L). \end{cases} \tag{48}$$

Since u_ε^k and U_ε^k are bounded respectively in $L^2(0,L;H^1(C))$ and $H^1(\Omega)$, the sequence z_ε^k is bounded in $L^2(0,L;H^1(C))$. Hence, there exist a subsequence and $z_k \in L^2(0,L;H^1(C))$ such that $z_\varepsilon^k \rightharpoonup z_k$ weakly in $L^2(0,L;H^1(C))$. Therefore, denoting by U_k the weak limit in $H^1(\Omega)$ of the corresponding subsequence U_ε^k, one can pass easily to the limit in (48) to get the equation

$$\begin{cases} z_k \in L^2(0,L;H^1(C)), \quad -\Delta'_y z_k = \lambda_k z_k + \lambda_k U_k \quad \text{in } M, \\ z_k = 0 \quad \text{on } \partial D \times (0,L), \\ \frac{\partial z_k}{\partial n} = 0 \quad \text{on } \partial C \times (0,L). \end{cases} \tag{49}$$

Note that by construction, $z_\varepsilon^k = 0$ in $F = D \times (0,L)$ so that the convergence $z_\varepsilon^k \chi_F(y) \rightharpoonup z_k \chi_D(y)$ weakly in $L^2(\Omega)$ shows that $z_k = 0$ in F which is equivalently written in the first boundary condition of (49). The Neumann boundary condition may be obtained in a classical way multiplying (48) first by a test function $\phi \in C_0^\infty(M)$ which allows to get equation (49) and then by a test function ϕ such that $\phi \in V_s$, $\phi = 0$ in F and $\phi \in C^1((0,L) \times \overline{C})$.

More generally, given a bounded sequence (f_ε) in $L^2(\Omega)$ and $f \in L^2(\Omega)$, we now consider equations of the form

$$\begin{cases} w_\varepsilon \in V_s, \quad -\Delta'_y w_\varepsilon - \varepsilon^2 \frac{\partial^2 w_\varepsilon}{\partial x_3^2} = \lambda_\varepsilon^k w_\varepsilon + f_\varepsilon \quad \text{in } M, \\ w_\varepsilon = 0 \quad \text{on } \partial D \times (0,L), \\ \frac{\partial w_\varepsilon}{\partial n} = 0 \quad \text{on } \partial C \times (0,L), \end{cases} \tag{50}$$

and

$$\begin{cases} w \in L^2(0,L;H^1(C)), \quad -\Delta'_y w = \lambda_k w + f \quad \text{in } M, \\ w = 0 \quad \text{on } \partial D \times (0,L), \\ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial C \times (0,L). \end{cases} \tag{51}$$

Regarding the sequence of solutions of (50), the following lemma holds true.

Lemma 14. *Assume that $\lambda_\varepsilon^k \to \lambda_k$ and that $f_\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega)$. Then the sequence w_ε is bounded in $L^2(0,L;H^1(C))$ and there exists a subsequence of ε such that $w_\varepsilon \rightharpoonup w$ weakly in $L^2(0,L;H^1(C))$ where w is the unique solution of* (51).

Proof. We only have to prove that w_ε is bounded in $L^2(0,L;H^1(C))$, the limit problem (51) satisfied by w can be established exactly by the same process already used in the proof of (49).

The main ingredient to get that apriori estimate relays on the Poincaré inequality

$$\int_{C\setminus D} |u|^2 \,\mathrm{d}y \le \frac{1}{\mu_1} \int_{C\setminus D} |\nabla'_y u|^2 \,\mathrm{d}y \quad \forall\, u \in H^1(C \setminus \overline{D}),\ u = 0 \text{ on } \partial D, \tag{52}$$

combined with the bound $\lambda_k < \mu_1$ proved in Proposition 10.

Note that we have used $\frac{1}{\mu_1}$ in (52) which is known as the best Poincaré's constant.

Multiplying equation (50) by w_ε and integrating, we get

$$\int_0^L \int_{C\setminus D} |\nabla' w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3 \le \lambda_\varepsilon^k \int_0^L \int_{C\setminus D} |w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3 + \|f_\varepsilon\|_{L^2(\Omega)} \|w_\varepsilon\|_{L^2(M)}. \tag{53}$$

Choosing in (52) $u = w_\varepsilon(\cdot, x_3)$ with $x_3 \in (0,L)$ and integrating over $0,L)$, we infer

$$\int_0^L \int_{C\setminus D} |w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3 \le \frac{1}{\mu_1} \int_0^L \int_{C\setminus D} |\nabla'_y w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3. \tag{54}$$

Let $\delta > 0$ be such that $0 < \lambda_k < \delta < \mu_1$. Turning back to (53) and using (54), we get for ε sufficiently small,

$$\left(1 - \frac{\delta}{\mu_1}\right) \int_0^L \int_{C\setminus D} |\nabla' w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3 \leq \|f_\varepsilon\|_{L^2(\Omega)} \|w_\varepsilon\|_{L^2(M)}. \tag{55}$$

Since f_ε is bounded in $L^2(\Omega)$, applying once again inequality (54) together with the Young inequality, we derive from (55) the estimate

$$\int_0^L \int_{C\setminus D} |\nabla' w_\varepsilon|^2 \,\mathrm{d}y\mathrm{d}x_3 \leq K. \tag{56}$$

(54) and (56) allow to conclude that w_ε is bounded in $L^2(0, L; H^1(C))$. □

We continue the proof of the Proposition 13 in the following way.

Multiplying the equations (48) and (50) respectively by w_ε and by z_ε^k and integrating we get

$$\begin{aligned}\int_M \left(\nabla' z_\varepsilon^k \nabla' w_\varepsilon + \varepsilon^2 \frac{\partial z_\varepsilon^k}{\partial x_3} \frac{\partial w_\varepsilon}{\partial x_3}\right) \mathrm{d}y\mathrm{d}x_3 &= \lambda_\varepsilon^k \int_M z_\varepsilon^k w_\varepsilon \,\mathrm{d}y\mathrm{d}x_3 + \lambda_\varepsilon^k \int_M U_\varepsilon^k w_\varepsilon \,\mathrm{d}y\mathrm{d}x_3 \\ &= \lambda_\varepsilon^k \int_M w_\varepsilon z_\varepsilon^k \,\mathrm{d}y\mathrm{d}x_3 + \int_M f_\varepsilon z_\varepsilon^k \,\mathrm{d}y\mathrm{d}x_3. \end{aligned} \tag{57}$$

Since U_ε^k is bounded in $H^1(\Omega)$, there exist a subsequence of ε and $U_k \in H^1(\Omega)$ such that $U_\varepsilon^k \rightharpoonup U_k$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Therefore for that a subsequence, we get from (57) with the help of Lemma 14

$$\lim \int_M f_\varepsilon z_\varepsilon^k \,\mathrm{d}y\mathrm{d}x_3 = \lim \lambda_\varepsilon^k \int_M U_\varepsilon^k w_\varepsilon \,\mathrm{d}y\mathrm{d}x_3 = \lambda_k \int_M U_k w \,\mathrm{d}y\mathrm{d}x_3. \tag{58}$$

On the other hand, one can multiply (49) and (51) respectively by w and by z_k and integrate to obtain

$$\begin{aligned}\int_M \nabla' z_k \nabla' w \,\mathrm{d}y\mathrm{d}x_3 = \int_M \nabla' w \nabla' z_k \,\mathrm{d}y\mathrm{d}x_3 &= \lambda_k \int_M z_k w \,\mathrm{d}y\mathrm{d}x_3 + \lambda_k \int_M U_k w \,\mathrm{d}y\mathrm{d}x_3 \\ &= \lambda_k \int_M w z_k \,\mathrm{d}y\mathrm{d}x_3 + \int_M f z_k \,\mathrm{d}y\mathrm{d}x_3. \end{aligned} \tag{59}$$

Combining (58) and (59), we get

$$\lim \int_M f_\varepsilon z_\varepsilon^k \,\mathrm{d}y\mathrm{d}x_3 = \lambda_k \int_M U_k w \,\mathrm{d}y\mathrm{d}x_3 = \int_M f z_k \,\mathrm{d}y\mathrm{d}x_3. \tag{60}$$

Choosing in particular $f_\varepsilon = z_\varepsilon^k$ which converges weakly in $L^2(\Omega)$ to $f = z_k$, we obtain

$$\lim \int_M (z_\varepsilon^k)^2 \,\mathrm{d}y\mathrm{d}x_3 = \int_M (z_k)^2 \,\mathrm{d}y\mathrm{d}x_3, \tag{61}$$

which implies the strong convergence of the subsequence z_ε^k and therefore the strong convergence of the corresponding subsequence of u_ε^k. Hence Proposition 13 is proved. □

We now proceed to the proof of Theorem 3.

2.4. *Proof of Theorem 3*

The strong convergence in $L^2(\Omega)$ of the eigenvectors is proved in Proposition 13. We use that convergence to improve it by proving the convergence of the sequence of energies from which we derive immediately (16) and (19).

Consider the sequence

$$J_\varepsilon = \int_\Omega \left(\left(|\nabla' u_\varepsilon^k - \nabla' u_k|^2 + \varepsilon^2 \left|\frac{\partial u_\varepsilon^k}{\partial x_3}\right|^2\right)\chi_M + \left(\frac{1}{\varepsilon^2}|\nabla' u_\varepsilon^k|^2 + \left|\frac{\partial u_\varepsilon^k}{\partial x_3} - \frac{\partial v_k}{\partial x_3}\right|^2\right)\chi_F\right) \mathrm{d}y\mathrm{d}x_3. \tag{62}$$

Choosing u_ε^k and (u_k, v_k) as test functions respectively in (9) and in (29), we get with the help of the weak convergences proved in Proposition 9 and of the strong convergence proved in Proposition 13,

$$J_\varepsilon = \lambda_\varepsilon^k \int_\Omega |u_\varepsilon^k|^2 \mathrm{d}y\mathrm{d}x_3 + \lambda_k \int_\Omega |u_k|^2 \mathrm{d}y\mathrm{d}x_3 - 2\int_\Omega \left(\nabla' u_\varepsilon^k \nabla' u_k \chi_M + \frac{\partial v_k}{\partial x_3}\chi_F\right)\mathrm{d}y\mathrm{d}x_3 \longrightarrow 2\lambda_k \int_\Omega |u_k|^2 \mathrm{d}y\mathrm{d}x_3 - 2\lambda_k \int_\Omega |u_k|^2 \mathrm{d}y\mathrm{d}x_3 = 0 \quad (63)$$

Hence the weak convergences stated in Proposition 9 are in fact strong convergences; in particular, keeping in mind Proposition 13, we get the strong convergences stated in Theorem 3.

We now prove that any $\lambda \in (\mu_0, \mu_1)$ which is an eigenvalue of (11) may be attained as a limit of a sequence $(\lambda_\varepsilon^k)_\varepsilon$; By this we can conclude that (11) has no other eigenvalues than those obtained as the limits of the eigenvalues λ_ε^k and thus we can list all its eigenvalues in increasing order. It is then clear that for a fixed k, we cannot have two subsequences ε and ε' with two different limits for λ_ε^k and $\lambda_{\varepsilon'}^k$ since this would lead to add a new element to the set of eigenvalues of (11); hence (15) holds for the whole sequence ε.

We argue by contradiction to prove that any $\lambda \in (\mu_0, \mu_1[$ which is an eigenvalue of (11) may be attained as a limit of a sequence $(\lambda_\varepsilon^k)_\varepsilon$.

If for any sequence, λ_ε^k does not converge to λ, then there exists a neighborhood of λ which does not contain any λ_ε^k for all k. In other words, λ belongs to the resolvent of the operator A_ε defined by (6). Hence, for any $f \in L^2(0,L) \subset L^2(\Omega)$, there exists $u_\varepsilon \in D(A_\varepsilon)$ such that

$$A_\varepsilon u_\varepsilon = \lambda u_\varepsilon + f \quad \text{in } \Omega. \quad (64)$$

Multiplying (64) by $\phi \in V_s$ and integrating we get

$$\int_\Omega \left(\left(\nabla' u_\varepsilon \nabla' \phi + \varepsilon^2 \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \phi}{\partial x_3}\right)\chi_M + \left(\frac{1}{\varepsilon^2}\nabla' u_\varepsilon \nabla' \phi + \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \phi}{\partial x_3}\right)\chi_F\right)\mathrm{d}y\mathrm{d}x_3 = \lambda \int_\Omega u_\varepsilon \phi \,\mathrm{d}y\mathrm{d}x_3 + \int_\Omega f\phi \,\mathrm{d}y\mathrm{d}x_3, \quad \forall\, \phi \in V_s. \quad (65)$$

To get apriori estimates on the sequence u_ε, we will use the following Poincaré type inequality

Lemma 15. *There exists a positive constant K such that*

$$\|u\|_{L^2(\Omega)} \le K\left(\|\nabla' u\|_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial x_3}\chi_F\right\|_{L^2(\Omega)}\right), \quad \forall\, u \in L^2(0,L;H^1(C)) \cap L^2(D;H_0^1(0,L)). \quad (66)$$

Proof. We argue by contradiction. Assuming inequality (66) false, one can find a sequence $u_n \in L^2(0,L;H^1(C)) \cap L^2(D;H_0^1(0,L)$ such that

$$\|u_n\|_{L^2(\Omega)} = 1 \quad \forall\, n, \quad \text{and} \quad \left(\|\nabla' u_n\|_{L^2(\Omega)} + \left\|\frac{\partial u_n}{\partial x_3}\chi_F\right\|_{L^2(\Omega)}\right) \longrightarrow 0. \quad (67)$$

Thanks to the classical Poincaré inequality $\|u - \frac{1}{|D|}\int_D u\,\mathrm{d}y\|_{L^2(C)} \le K\|\nabla' u\|_{L^2(C)}$, $\forall\, u \in H^1(C)$ applied to $u = u_n(\cdot, x_3)$, $x_3 \in (0,L)$, we get after integrating with respect to x_3, (remember that $\Omega = C \times (0,L)$)

$$\left\|u_n - \frac{1}{|D|}\int_D u_n \,\mathrm{d}y\right\|_{L^2(\Omega)} \le K\|\nabla' u_n\|_{L^2(\Omega)}. \quad (68)$$

On the other hand, the Poincaré inequality for functions of $H_0^1(0,L)$ applied with $u(x_3) = \int_D u_n(y,x_3)\,\mathrm{d}y$ leads to the estimate

$$\left\|\int_D u_n \,\mathrm{d}y\right\|_{L^2(\Omega)} \le K\left\|\frac{\partial u_n}{\partial x_3}\right\|_{L^2(F)}. \quad (69)$$

Combining (68) and (69) with (67), we come to a contradiction. ☐

Taking $\phi = u_\varepsilon$ in (65) and applying (66) with $u = u_\varepsilon$ (note that $V_s \subset L^2(0,L;H^1(C)) \cap L^2(D;H_0^1(0,L))$, we get the same apriori estimates as those obtained for the sequence u_ε^k in (27). Indeed all the apriori estimates on the sequence u_ε^k are based on its $L^2(\Omega)$- apriori estimate which still holds true for the sequence u_ε. Hence we can use the same arguments that led to the system (30) to pass to the limit $\varepsilon \to 0$ in (65) and we obtain at the limit

$$\begin{cases} u(y,x_3) \in L^2((0,L);H^1(C)), \quad -\Delta'_y u(y,x_3) = \lambda u + f \text{ in } (C\setminus\overline{D}) \times (0,L), \\ u = v \quad \text{on } \partial D \times (0,L), \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial C \times (0,L), \\ v \in H_0^1(0,L), \quad -\dfrac{\mathrm{d}^2 v}{\mathrm{d}x_3^2} = \lambda_k v + \dfrac{\lambda_k}{|D|}\displaystyle\int_{C\setminus D} u\,\mathrm{d}y + \dfrac{1}{|D|}\displaystyle\int_C f\,\mathrm{d}y \quad \text{in } (0,L). \end{cases} \tag{70}$$

Choosing $f = \frac{|D|}{|C|} g(x_3)\chi_D(y)$ (which implies $f = 0$ in $C\setminus D$) with an arbitrary $g \in L^2(0,L)$, the second equation in (70) reduces to

$$v \in H_0^1(0,L), \quad -\frac{\mathrm{d}^2 v}{\mathrm{d}x_3^2} = \lambda v + \frac{\lambda}{|D|}\int_{C\setminus D} u\,\mathrm{d}y + g \quad \text{in } (0,L). \tag{71}$$

Note that $v \neq 0$ for $g \neq 0$ otherwise the first equation in (70) would implies $u = 0$ since $\lambda < \mu_1$ is not an eigenvalue of $-\Delta'_y$. Hence one can express u in terms of (λ, u_0, v) according to (34) as $u = (\lambda u_0 + 1)v$. Note that the function u_0^k arising in (11) depends on k only through the associated eigenvalue λ_k so that one can drop here the subscript k and denote the associated function depending on λ by u_0. Therefore (71) takes the form

$$v \in H_0^1(0,L), \quad -\frac{\mathrm{d}^2 v}{\mathrm{d}x_3^2} = \lambda\left(1 + \frac{|C\setminus D|}{|D|} + \frac{\lambda}{|D|}\int_{C\setminus D} u_0\,\mathrm{d}y\right)v + g \quad \text{in } (0,L). \tag{72}$$

On the other hand, by hypothesis, λ is an eigenvalue of (11) so that the last equation of (11) with the same u_0 which depends only on λ shows that $\lambda\big(1 + \frac{|C\setminus D|}{|D|} + \frac{\lambda}{|D|}\int_{C\setminus D} u_0\,\mathrm{d}y\big)$ is an eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$. This is a contradiction since equation (72) valid for all $g \in L^2(0,L)$ means that the number $\lambda\big(1 + \frac{|C\setminus D|}{|D|} + \frac{\lambda}{|D|}\int_{C\setminus D} u_0^k\,\mathrm{d}y\big)$ belongs to the resolvent of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$.

We now prove that $\lim\limits_{k\to+\infty} \lambda_k = \mu_1$.

Since $\lambda_k \in (\mu_0, \mu_1)$ for any k, the sequence $(\lambda_k)_k$ admits at least an accumulation point and each accumulation point λ is such that $\mu_0 \le \lambda \le \mu_1$. Assume that there exists an accumulation point λ such that $\lambda < \mu_1$. There exists a subsequence $(\lambda_{k_n}, u_0^{k_n}, v_{k_n})$ of solutions of (11) such that $\lim_{n\to+\infty} \lambda_{k_n} = \lambda$. Hence the following equation takes place for all n

$$-\Delta' u_0^{k_n} = \lambda_{k_n} u_0^{k_n} + 1 \quad \text{in } C\setminus\overline{D}. \tag{73}$$

Let δ be a positive number such that $\lambda < \delta < \mu_1$. For n large enough we have $\lambda_{k_n} \ge \delta$ so that applying the Poincaré inequality

$$\int_{C\setminus D} |u|^2\,\mathrm{d}y \le \frac{1}{\mu_1}\int_{C\setminus D} |\nabla'_y u|^2\,\mathrm{d}y \quad \forall\, u \in H^1(C\setminus\overline{D}),\ u = 0 \text{ on } \partial D, \tag{74}$$

after multiplying (73) by $u_0^{k_n}$, we get for n large enough

$$\int_{C\setminus D} |\nabla'_y u_0^{k_n}|^2\,\mathrm{d}y \le \frac{\delta}{\mu_1}\int_{C\setminus D} |\nabla'_y u_0^{k_n}|^2\,\mathrm{d}y + \int_{C\setminus D} |u_0^{k_n}|\,\mathrm{d}y. \tag{75}$$

Applying successively the Cauchy–Schwarz inequality and (74) in the last integral of (75), we infer

$$\left(1 - \frac{\delta}{\mu_1}\right)\int_{C\setminus D} |\nabla'_y u_0^{k_n}|^2\,\mathrm{d}y \le \sqrt{|C\setminus D|}\sqrt{\frac{1}{\mu_1}}\sqrt{\int_{C\setminus D} |\nabla'_y u_0^{k_n}|^2\,\mathrm{d}y}. \tag{76}$$

Therefore, $(u_0^{k_n})_n$ is bounded in $H^1(C \setminus \overline{D})$ and one can assume that (possibly for another subsequence) $(u_0^{k_n})_n$ converges weakly to u_0 in $H^1(C \setminus \overline{D})$. In particular we have that $\lim_{n\to+\infty} \int_{C\setminus D} u_0^{k_n}\,\mathrm{d}y = \int_{C\setminus D} u_0\,\mathrm{d}y$. On the other hand $(\lambda_{k_n}, u_0^{k_n}, v_{k_n})$ being a solution of (11), the following equation (recall that $v_{k_n} \neq 0$)

$$-\frac{\mathrm{d}^2 v_{k_n}}{\mathrm{d}x_3^2} = \lambda_{k_n}\left(1 + \frac{|C\setminus D|}{|D|} + \frac{\lambda_{k_n}}{|D|}\int_{C\setminus D} u_0^{k_n}\,\mathrm{d}y\right) v_{k_n} \quad \forall\, n, \tag{77}$$

shows that the number μ defined by $\mu := \lambda\big(1+\frac{|C\setminus D|}{|D|}+\frac{\lambda}{|D|}\int_{C\setminus D} u_0\,\mathrm{d}y\big)$ is a finite accumulation point of the spectrum of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ since $\mu = \lim_{n\to+\infty}\mu_n$ where $\mu_n := \lambda_{k_n}\big(1+\frac{|C\setminus D|}{|D|}+\frac{\lambda_{k_n}}{|D|}\int_{C\setminus D} u_0^{k_n}\,\mathrm{d}y\big)$. This is a contradiction since it is well known that such spectrum is in fact an increasing sequence which tends to $+\infty$.

The last point which remains to prove is that all the limiting eigenvalues are simple and that u_ε^k converges to u_k for the whole sequence ε. Assuming that λ_k is a simple eigenvalue, the proof of the convergence of the eigenvectors for the whole sequence ε is known since the work of [23] (see also [9]). We sketch it for the convenience of the reader.

Assume that u_k is an eigenvector associated to the simple eigenvalue λ_k. Using the fact that the eigenvalues converge for the whole sequence ε, it is easy to check that the multiplicity of λ_k is equal or greater than that of λ_ε^k; hence λ_ε^k is simple and there are only two eigenvectors satisfying $\int_\Omega |u_\varepsilon^k|^2\,\mathrm{d}x = 1$, namely u_ε^k and $-u_\varepsilon^k$. Among these two eigenvectors, we choose the one satisfying the inequality

$$\int_\Omega u_\varepsilon^k u_k\,\mathrm{d}x > 0. \tag{78}$$

Therefore if ε' is a subsequence such that $u_{\varepsilon'}^k$ strongly converges in $L^2(\Omega)$ to the eigenvector $\widehat{u}$ associated to λ_k, we get by passing to the limit in (78),

$$\int_\Omega \widehat{u}u_k\,\mathrm{d}x > 0. \tag{79}$$

On the other hand, $u_k = \widehat{u}$ or $u_k = -\widehat{u}$ since λ_k is a simple eigenvalue. The last equality is excluded thanks to (79) so that any subsequence converges to u_k.

Let us now prove that all the limit eigenvalues are simple eigenvalues.

Assume that for some k, (13) holds true for two orthogonal eigenvectors $\binom{u_k}{v_k}$ and $\binom{\bar{u}_k}{\bar{v}_k}$. By hypothesis, we have

$$\int_0^L\int_{C\setminus D} u_k\overline{u}_k\mathrm{d}y\mathrm{d}x_3 + \int_0^L v_k\overline{v}_k\mathrm{d}x_3 = 0. \tag{80}$$

We know that u_k and $\overline{u}_k$ are given respectively by $u_k(y,x_3) = (\lambda_k u_0^k(y)+1))\,v_k(x_3)$ and $\overline{u}_k(y,x_3) = (\lambda_k u_0^k(y)+1))\,\overline{v}_k(x_3)$ where $u_0^k(y)$ given by the first equation of (11) depends only on the eigenvalue λ_k.

Turning back to (80), we infer

$$\begin{aligned}\int_0^L\left(\int_{C\setminus D}(\lambda_k u_0^k(y)+1)^2\mathrm{d}y\right)v_k(x_3)\,\overline{v}_k(x_3)\mathrm{d}x_3 + \int_0^L v_k\overline{v}_k\mathrm{d}x_3\\ = \int_0^L\left(\int_{C\setminus D}(\lambda_k u_0^k(y)+1)^2\mathrm{d}y+1\right)v_k(x_3)\,\overline{v}_k(x_3)\mathrm{d}x_3 = 0.\end{aligned} \tag{81}$$

As remarked above v_k and $\overline{v}_k$ are always eigenvectors of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ with Dirichlet condition so that (81) and the second equation of (11) would mean that v_k and $\overline{v}_k$ eigenvectors associated to the eigenvalue $\lambda_k\big(1+\frac{|C\setminus D|}{|D|}+\frac{\lambda_k}{|D|}\int_{C\setminus D} u_0^k\,\mathrm{d}y\big)$ are othogonal in $L^2(0,L)$. This is a contradiction since all the eigenvalues of $-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}$ with Dirichlet condition are simple eigenvalues.

The proof of Theorem 3 is now complete.

3. The case of homogenization

In this section we only take up the key points of the previous proofs to highlight the few minor changes to be made, changes due to the oscillations induced by the homogenization process. The points of Theorem 5 whose proofs are identical to the corresponding proofs done in the $3d-1d$ case will not be mentioned here for the sake of brevity.

3.1. *Proof of the estimate (24)*

In the spirit of the above section, the natural idea is to choose a test function vanishing over the set F_ε of fibers. To that aim, we consider an eigenvector $\phi(y)$ corresponding to the first eigenvalue of $-\Delta'_y$ in $C\setminus\overline{D}$ with the corresponding boundary conditions on the boundary: $\phi=0$ on ∂D and ϕ is C-periodic. We extend ϕ by zero inside D and by periodicity to the whole $\mathbb{R}^2$. The k-th eigenvalue λ^k_ε of (8) is now given by the same Min-Max formula as above, namely

$$\lambda^k_\varepsilon = \min_{V^k\subset V_h}\max_{u\in V^k}\frac{\int_\Omega\big(\varepsilon^2|\nabla u|^2\chi_{M_\varepsilon}+|\nabla u|^2\chi_{F_\varepsilon}\big)\,\mathrm{d}x'\,\mathrm{d}x_3}{\int_\Omega|u|^2\,\mathrm{d}x'\,\mathrm{d}x_3}. \tag{82}$$

For each ε, consider the subspace $V^k_\varepsilon\subset V_h$ spanned by $\{\phi(\frac{x'}{\varepsilon})v^1,\phi(\frac{x'}{\varepsilon})v^2,\dots,\phi(\frac{x'}{\varepsilon})v^k\}$ with the same $v^1,v^2,\dots,v^k$ as those defined in the previous section.

The functions of V^k_ε vanish in F_ε and similar calculations to those of (26) hold true. Indeed, making in each cell the change of variable $x':=\varepsilon y+\varepsilon i$ with $y\in C\setminus D$, we get

$$\begin{aligned}
\int_\Omega u^2\mathrm{d}x'\,\mathrm{d}x_3 &= \sum_{i\in I_\varepsilon}\int_{\varepsilon(C\setminus D)+\varepsilon i}\phi^2\Big(\frac{x'}{\varepsilon}\Big)\mathrm{d}x'\int_0^L\big(\alpha_1^2(v^1)^2+\cdots+\alpha_k^2(v^k)^2\big)\,\mathrm{d}x_3\\
&=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\varepsilon^2\sum_{i\in I_\varepsilon}\int_{C\setminus D}\phi^2(y)\,\mathrm{d}y,\\
\int_\Omega\varepsilon^2|\nabla'_{x'}u|^2\mathrm{d}x'\,\mathrm{d}x_3 &= \big(\alpha_1^2+\cdots+\alpha_k^2\big)\sum_{i\in I_\varepsilon}\varepsilon^2\int_{\varepsilon(C\setminus D)+\varepsilon i}\Big|\nabla'_{x'}\phi\Big(\frac{x'}{\varepsilon}\Big)\Big|^2\mathrm{d}x'\\
&=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\varepsilon^2\sum_{i\in I_\varepsilon}\varepsilon^2\int_{C\setminus D}\frac{1}{\varepsilon^2}|\nabla'_y\phi(y)|^2\,\mathrm{d}y\\
&=\big(\alpha_1^2+\cdots+\alpha_k^2\big)\varepsilon^2\mu_1\sum_{i\in I_\varepsilon}\int_{C\setminus D}|\phi(y)|^2\,\mathrm{d}y,\\
\int_\Omega\varepsilon^2\Big|\frac{\partial u}{\partial x_3}\Big|^2\mathrm{d}x'\,\mathrm{d}x_3 &= \varepsilon^2\int_0^L\bigg(\alpha_1^2\Big(\frac{\mathrm{d}v^1}{\mathrm{d}x_3}\Big)^2+\cdots+\alpha_k^2\Big(\frac{\mathrm{d}v^k}{\mathrm{d}x_3}\Big)^2\bigg)\mathrm{d}x_3\,\varepsilon^2\sum_{i\in I_\varepsilon}\int_{C\setminus D}|\phi(y)|^2\,\mathrm{d}y,\\
&=\varepsilon^4\big(\alpha_1^2\lambda_1^0+\cdots+\alpha_k^2\lambda_k^0\big)\sum_{i\in I_\varepsilon}\int_{C\setminus D}|\phi|^2\,\mathrm{d}y\\
&\le\varepsilon^4\lambda_k^0\big(\alpha_1^2+\cdots+\alpha_k^2\big)\sum_{i\in I_\varepsilon}\int_{C\setminus D}|\phi|^2\,\mathrm{d}y,
\end{aligned} \tag{83}$$

in such a way we arrive to the following estimate

$$\lambda^k_\varepsilon\le\frac{\Big(\mu_1+\varepsilon^2\lambda^0_k\Big)\Big(\alpha_1^2+\cdots+\alpha_k^2\Big)\varepsilon^2\sum_{i\in I_\varepsilon}\int_{C\setminus D}|\phi|^2\,\mathrm{d}y}{\varepsilon^2\Big(\alpha_1^2+\cdots+\alpha_k^2\Big)\sum_{i\in I_\varepsilon}\int_{C\setminus D}\phi^2(y)\,\mathrm{d}y}=\mu_1+\varepsilon^2\lambda^0_k, \tag{84}$$

which is nothing but the estimate (24) in the homogenization setting.

3.2. *Proof of the analogous of Proposition 9*

To continue the proof in the homogenization case we use the two-scale convergence in order to describe the effect of the oscillations due to the fast variable.

Analogously to the previous section, the use of the estimate (24) in (8) leads to the estimates

$$\|\varepsilon\nabla u_\varepsilon^k\|_{L^2(\Omega)} \le K, \quad \|\nabla u_\varepsilon^k \chi_{F_\varepsilon}\|_{L^2(\Omega)} \le K. \tag{85}$$

Hence, by a classical result of the two-scale convergence method, see [1], there exists $u_k(x,y) \in L^2(\Omega; H^1_\#(C))$ such that for a subsequence of ε,

$$u_\varepsilon^k \overset{2-sc}{\rightharpoonup} u_k, \quad \varepsilon\nabla' u_\varepsilon^k \overset{2-sc}{\rightharpoonup} \nabla'_y u_k. \tag{86}$$

The sequence $\varepsilon\nabla' u_\varepsilon^k \chi_{F_\varepsilon}$ converges strongly (and thus two-scale converges) to zero in $L^2(\Omega)$. On the other hand, one can check that the two-scale limit of $\varepsilon\nabla' u_\varepsilon^k \chi_{F_\varepsilon}$ is also equal to $\nabla'_y u_k(x,y)\chi_D(y)$, see [21]. Hence we deduce that $\nabla'_y u_k(x,y)\chi_D(y) = 0$ in such a way that $u_k(x,y) = v_k(x)$ in $\Omega \times D$. Furthermore the sequence $u_\varepsilon^k \chi_{F_\varepsilon}$ is bounded in $L^2(\omega; H^1_0(0,L))$ since $\frac{\partial u_\varepsilon^k}{\partial x_3}\chi_{F_\varepsilon} = \frac{\partial}{\partial x_3}(u_\varepsilon^k \chi_{F_\varepsilon})$ is bounded in $L^2(\Omega)$ by virtue of (85) so that there exists $z \in L^2(\omega; H^1_0(0,L))$ such that $u_\varepsilon^k \chi_{F_\varepsilon}$ converges weakly to z in $L^2(\omega; H^1_0(0,L))$. Since the weak limit in $L^2(\Omega)$ is the average over the reference cell C of the two-scale limit, we get $z(x) = \frac{1}{|C|}\int_C u_k(x,y)\chi_D(y)\,\mathrm{d}y = \frac{1}{|C|}\int_C v_k(x)\chi_D(y)\,\mathrm{d}y = |D|v_k(x)$ which implies that v_k actually belongs to $L^2(\omega; H^1_0(0,L))$. Let us remark that since $\chi_{F_\varepsilon}(x') = \chi_D(\frac{x'}{\varepsilon}) \overset{2-sc}{\rightharpoonup} \chi_D(y)$ the natural statement of the convergence (22) is the following

$$\frac{\partial u_\varepsilon^k}{\partial x_3}\chi_{F_\varepsilon} \overset{2-sc}{\rightharpoonup} \frac{\partial v_k}{\partial x_3}\chi_D(y), \tag{87}$$

which implies the weak convergence

$$\frac{\partial u_\varepsilon^k}{\partial x_3}\chi_{F_\varepsilon} \rightharpoonup |D|\frac{\partial v_k}{\partial x_3} \quad \text{weakly in } L^2(\Omega). \tag{88}$$

The convergence (87) is finer and more advantageous. Hence replacing weak convergences by the corresponding two-scale convergences, Proposition 9 still holds true in the homogenization setting.

3.3. *The limit problem associated to (8)*

In contrast to the previous section where it was not necessary to specify the limit of the transversal gradient $\nabla' u_\varepsilon^k \chi_F$, here we have first to specify the two-scale limit of $\nabla' u_\varepsilon^k \chi_{F_\varepsilon}$ in order to pass to the limit in (8). We use the property proved in [21] that the two-scale limit is in fact given by $\nabla' w(x,y)\chi_D(y)$ where w belongs to $L^2(\Omega; H^1(D)/\mathbb{R})$ and that $w = 0$ when dealing with isotropic materials which is the case here.

Hence, using this remark and choosing a test function $\bar{u}(x, \frac{x'}{\varepsilon}) \in C_0^\infty(\Omega; H^1_\#(C))$, $\bar{u} = \bar{v}(x)$ in $\Omega \times D$ with $\bar{v} \in C_0^\infty(\Omega)$, we can pass to the limit in (8) and use a density argument to get

$$\begin{cases} (u_k, v_k) \in L^2(\Omega; H^1_\#(C)) \times L^2(\omega; H^1_0(0,L)), \ u_k = v_k \text{ in } \Omega \times D, \\ \displaystyle\int_{\Omega\times Y} \left(\nabla' u_k \nabla' \bar{u}\chi_{C\setminus D} + \frac{\partial v_k}{\partial x_3}\frac{\partial \bar{v}}{\partial x_3}\chi_D\right)\mathrm{d}y\mathrm{d}x = \lambda_k \int_{\Omega\times Y} u_k \bar{u}\,\mathrm{d}y\mathrm{d}x, \\ \forall\ (\bar{u}, \bar{v}) \in L^2(\Omega; H^1_\#(C)) \times L^2(\omega; H^1_0(0,L)), \ \bar{u} = \bar{v} \text{ in } \Omega \times D. \end{cases} \tag{89}$$

Analogously to the second section, (89) may be written as a system in the following strong form

$$\begin{cases} u_k(x,y) \in L^2(\Omega; H^1_\#(C)), \quad -\Delta'_y u_k(x,y) = \lambda_k u_k \ \text{ in } \Omega \times (C \setminus \overline{D}), \\ u_k(x,y) = v_k(x) \quad \text{on } \Omega \times \partial D, \\ \text{for a.e. } x \in \Omega, \ u_k(x,\cdot) \text{ is C-periodic}, \\ v_k \in L^2(\omega; H^1_0(0,L)), \quad -\dfrac{\partial^2 v_k}{\partial x_3^2} = \lambda_k v_k + \dfrac{\lambda_k}{|D|} \displaystyle\int_{C\setminus D} u_k \, \mathrm{d}y \quad \text{in } \Omega. \end{cases} \tag{90}$$

The main point now is on whether the new periodic boundary condition on ∂C replacing the homogenous Neumann one is likely to hamper the proof of the Proposition 10.

3.4. *Proof of the analogous of Proposition 10*

After multiplying the first equation in (90) by the eigenvector ϕ associated to the first eigenvalue μ_1 of $-\Delta'_y$ (with periodic boundary condition on ∂C) and the equation $-\Delta'_y \phi = \mu_1 \phi$ by u_k, we are led to the following equations by integration over $C \setminus D$

$$\begin{cases} \displaystyle\int_{C\setminus D} \nabla'_y \phi \nabla'_y u_k \, \mathrm{d}y - \int_{\partial C} \frac{\partial \phi}{\partial n} u_k \, \mathrm{d}\sigma - \int_{\partial D} \frac{\partial \phi}{\partial n} \mathrm{d}\sigma \, v_k(x) = \mu_1 \int_{C\setminus D} \phi u_k \, \mathrm{d}y, \\ \displaystyle\int_{C\setminus D} \nabla'_y u_k \nabla'_y \phi \, \mathrm{d}y - \int_{\partial C} \frac{\partial u_k}{\partial n} \phi \, \mathrm{d}\sigma = \lambda_k \int_{C\setminus D} u_k \phi \, \mathrm{d}y. \end{cases} \tag{91}$$

Due to the periodicity of u_k and ϕ, the two boundary integrals on ∂C arising in (91) vanish since the normal derivative has opposite values on opposites faces of C. Except this remark, there is no other change in the rest of the proof.

3.5. *Proof of the analogous of Proposition 13*

The proof of Proposition 12 is based on the use of $\frac{1}{\mu_1}$ as the best constant in the Poincaré inequality for functions of $H^1(C)$ vanishing on ∂D. We have applied that property to the sequence of solutions of (11). In the homogenization setting, we apply the same property to solutions of (10) which also vanish on the boundary ∂D.

The analogous of Proposition 13 will be established in the proof of Theorem 5 below.

3.6. *Proof of Theorem 5*

The approach is identical as that used in the proof of Theorem 3, in particular the extension results are still valid, see [5, 9, 25]. Despite this, we would like to indicate how to deal with the question of the compactness of the solutions, which is a key point of the proof. More precisely, we now consider the analogous system of (50) which takes the following form in the homogenization setting

$$\begin{cases} w_\varepsilon \in V_h, \quad -\varepsilon^2 \Delta w_\varepsilon = \lambda^k_\varepsilon w_\varepsilon + f_\varepsilon \quad \text{in } M_\varepsilon, \\ w_\varepsilon = 0 \quad \text{on } \partial F_\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial n} = 0 \quad \text{on } \partial M_\varepsilon, \end{cases} \tag{92}$$

Remark that the second equality is in fact the Dirichlet boundary condition on the common part of the lateral boundary of M_ε with that of F_ε. The last boundary condition concerns the rest of the lateral boundary of M_ε. The equivalent of system (51) now writes as

$$\begin{cases} w \in L^2(\Omega; H^1_\#(C)), \quad -\Delta'_y w = \lambda_k w + f \quad \text{in } \Omega \times (C \setminus \overline{D}), \\ w = 0 \quad \text{on } \Omega \times \partial D, \\ \text{for a.e. } x \in \Omega, \ w(x,\cdot) \text{ is C-periodic}. \end{cases} \tag{93}$$

Regarding the sequence of solutions of (92), the following lemma which is the equivalent of Lemma 14 holds true.

Lemma 16. *Assume that* $\lambda_\varepsilon^k \to \lambda_k$ *and that* $f_\varepsilon \overset{2-sc}{\rightharpoonup} f$. *Then the sequences* w_ε *and* $\varepsilon\nabla w_\varepsilon$ *are bounded in* $L^2(\Omega)$ *and they two-scale converge respectively to* w *and* $\binom{\nabla'_y w}{0}$ *where* w *is the unique solution of* (93).

Proof. We only focus on the apriori estimate on w_ε and $\varepsilon\nabla w_\varepsilon$. Once again, we use the Poincaré inequality

$$\int_{C\setminus D} |u(y)|^2 \mathrm{d}y \le \frac{1}{\mu_1}\int_{C\setminus D} |\nabla'_y u(y)|^2 \mathrm{d}y, \quad \forall\, u \in H^1(C) \text{ such that } u = 0 \text{ on } \partial D. \tag{94}$$

Applying (94) with $u(y) := w_\varepsilon(\varepsilon y + \varepsilon i, x_3)$, $i \in I_\varepsilon$, $x_3 \in (0, L)$ and making the change of variable $x' := \varepsilon y + \varepsilon i$, we get

$$\int_{C_\varepsilon^i \setminus D_\varepsilon^i} |w_\varepsilon(x', x_3)|^2 \mathrm{d}x' \le \frac{\varepsilon^2}{\mu_1}\int_{C_\varepsilon^i\setminus D_\varepsilon^i} |\nabla'_x w_\varepsilon(x)|^2 \,\mathrm{d}x'. \tag{95}$$

Therefore, summing over $i \in I_\varepsilon$ and integrating (95) over $(0, L)$, we get the inequality

$$\int_{M_\varepsilon} |w_\varepsilon(x', x_3)|^2 \mathrm{d}x \le \frac{\varepsilon^2}{\mu_1}\int_{M_\varepsilon} |\nabla'_x w_\varepsilon(x)|^2 \,\mathrm{d}x. \tag{96}$$

From now on, the proof may be continued as in the previous section. We multiply equation (92) by w_ε and integrate over M_ε so that bearing in mind that $\lambda_\varepsilon^k \to \lambda_k < \mu_1$, we get for ε small enough and for some δ such that $\lambda_k < \delta < \mu_1$,

$$\left(1 - \frac{\delta}{\mu_1}\right)\varepsilon^2 \int_{M_\varepsilon} |\nabla'_x w_\varepsilon(x)|^2 \,\mathrm{d}x \le K. \tag{97}$$

Estimates (96) and (97) lead to the estimates stated in Lemma 16. We can then obtain the limit problem (93) by passing to the two-scale limit in (92). □

As in the second section, the use of the Lemma 16 allows to get the strong two-scale convergence of the sequence $z_\varepsilon^k := u_\varepsilon^k - U_\varepsilon^k$, i.e., $\lim \int_\Omega |z_\varepsilon^k|^2 \,\mathrm{d}x = \int_\Omega \int_C |z_k(x,y)|^2 \,\mathrm{d}x\mathrm{d}y$ which in turn implies the strong two-scale convergence of the sequence u_ε^k since U_ε^k strongly converges in $L^2(\Omega)$. As a consequence of the strong two-scale convergence of the eigenvectors, we derive easily the corrector result stated in Theorem 5 with the help of the limit problem (89).

Finally, to prove that any eigenvalue of (10) may be attained as a limit of a sequence of λ_ε^k, following the same argument used in the corresponding step of Section 2, we need to prove an equivalent of the Poincaré inequality (66) in order to get the $L^2(\Omega)$-boundedness for the solutions of the equivalent equation of (65) which now writes as

$$\int_\Omega \left(\varepsilon^2 \nabla u_\varepsilon \nabla\phi \chi_{M_\varepsilon} + \nabla u_\varepsilon \nabla \phi \chi_{F_\varepsilon}\right) \mathrm{d}x' \,\mathrm{d}x_3 = \lambda_k \int_\Omega u_\varepsilon \phi \,\mathrm{d}x' \,\mathrm{d}x_3 + \int_\Omega f\phi \,\mathrm{d}x' \,\mathrm{d}x_3, \quad \forall\, \phi \in V_h. \tag{98}$$

Lemma 17. *The sequence* u_ε *of solutions of* (98) *satisfies the following inequality*

$$\| u_\varepsilon^k \|_{L^2(\Omega)} \le K\left(\|\varepsilon \nabla' u_\varepsilon^k\|_{L^2(\Omega)} + \left\| \frac{\partial u_\varepsilon^k}{\partial x_3} \chi_{F_\varepsilon} \right\|_{L^2(\Omega)} \right). \tag{99}$$

Proof. We apply the following Poincaré's inequality

$$\int_C \left| u(y) - \frac{1}{|D|}\int_D u(y)\mathrm{d}y \right|^2 \mathrm{d}y \le K \int_C |\nabla'_y u(y)|^2 \,\mathrm{d}y, \quad \forall\, u \in H^1(C), \tag{100}$$

with $u(y) := u_\varepsilon(\varepsilon y + \varepsilon i, x_3)$, $i \in I_\varepsilon$, $x_3 \in (0, L)$ so that making the change of variable $x' := \varepsilon y + \varepsilon i$, we get as in (95),

$$\int_{C_\varepsilon^i} \left| u_\varepsilon(x', x_3) - \frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} u_\varepsilon(x', x_3)\mathrm{d}x' \right|^2 \mathrm{d}x' \le K\varepsilon^2 \int_{C_\varepsilon^i} |\nabla'_x u_\varepsilon(x)|^2 \,\mathrm{d}x'. \tag{101}$$

On the other hand, applying the one-dimensional Poincaré inequality to the function $\int_{D_\varepsilon^i} |u_\varepsilon(x', x_3) \mathrm{d}x'$, we infer

$$\int_0^L \sum_{i\in I_\varepsilon} \int_{C_\varepsilon^i} \frac{1}{|D_\varepsilon^i|^2} \left| \int_{D_\varepsilon^i} u_\varepsilon \, \mathrm{d}x' \right|^2 \mathrm{d}x' \mathrm{d}x_3 \le K \sum_{i\in I_\varepsilon} \frac{|C_\varepsilon^i|}{|D_\varepsilon^i|} \int_0^L \int_{D_\varepsilon^i} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^2 \mathrm{d}x' \mathrm{d}x_3. \tag{102}$$

Taking the sum over $i \in I_\varepsilon$, we derive the inequality (99) as a consequence of (101) and (102). □

The rest of the proof of Theorem 5 is done exactly as in the previous section for the proof of Theorem 3.

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References

[1] G. Allaire, "Homogenization and Two-Scale Convergence", *SIAM J. Math. Anal.* **23** (1992), no. 6, p. 1482-1518.

[2] G. Allaire, Y. Capdeboscq, "Homogenization of a spectral problem in neutronic multigroup diffusion", *Comput. Methods Appl. Mech. Eng.* **187** (2000), no. 1-2, p. 91-117.

[3] T. Arbogast, J. Douglas, U. Hornung, "Derivation of the double porosity model of single phase flow via homogenization theory", *SIAM J. Math. Anal.* **21** (1990), no. 4, p. 823-836.

[4] A. Braides, V. Chiadò Piat, A. Piatnitski, "A variational approach to double-porosity problems", *Asymptotic Anal.* **39** (2004), no. 3-4, p. 281-300.

[5] H. Brézis, *Analyse Fonctionnelle, Théorie et applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, 1983.

[6] D. Caillerie, B. Dinari, "A perturbation problem with two small parameters in the framework of the heat conduction of a fiber reinforced body", in *Partial differential equations*, Banach Center Publications, vol. 19, Polish Scientific Publishers, 1984, p. 59-78.

[7] J. Casado-Díaz, "Two-scale convergence for nonlinear Dirichlet problems", *Proc. R. Soc. Edinb., Sect. A, Math.* **130** (2000), no. 2, p. 249-276.

[8] H. Charef, A. Sili, "The effective equilibrium law for a highly heterogeneous elastic periodic medium", *Proc. R. Soc. Edinb., Sect. A, Math.* **143** (2013), no. 3, p. 507-561.

[9] D. Cioranescu, J. Saint Jean Paulin, *Homogenization of reticulated structures*, Applied Mathematical Sciences, vol. 136, Springer, 1999.

[10] A. Gaudiello, A. Sili, "Homogenization of highly oscillating boundaries with strongly contrasting diffusivity", *SIAM J. Math. Anal.* **47** (2015), no. 3, p. 1671-1692.

[11] S. Kesavan, "Homogenization of elliptic eigenvalue problems. I, II", *Appl. Math. Optim.* **5** (1979), p. 153-167, 197-216.

[12] S. Kesavan, N. Sabu, "Two-dimensional approximation of eigenvalue problems in shell theory: Flexural shells", *Chin. Ann. Math., Ser. B* **21** (2000), no. 1, p. 1-16.

[13] M. Kreĭn, M. Rutman, "Linear operators leaving invariant a cone in a Banach space", *Amer. Math. Soc. Transl. Ser.* **10** (1962), p. 1-128.

[14] H. Le Dret, *Problèmes variationnels dans les multi-domaines: modélisation des jonctions et applications*, Recherches en Mathématiques Appliquées, vol. 19, Masson, 1991.

[15] G. Leugering, S. A. Nazarov, J. Taskinen, "The band-gap structures of the spectrum in a periodic medium of Masonry type", *Netw. Heterog. Media* **15** (2020), no. 4, p. 555-580.

[16] T. A. Mel'nik, S. A. Nazarov, "Asymptotics of the Neumann spectral problem solution in a domain of "thick comb" type", *J. Math. Sci., New York* **85** (1997), no. 6, p. 2326-2346.

[17] F. Murat, A. Sili, "A remark about the periodic homogenization of certain composite fibered media", *Netw. Heterog. Media* **15** (2020), no. 1, p. 125-142.

[18] G. Nguetseng, "A General Convergence Result for a Functional Related to the Theory of Homogenization", *SIAM J. Math. Anal.* **20** (1989), no. 3, p. 608-623.

[19] G. Panasenko, *Multi-scale modelling for structures and composites*, Springer, 2005.

[20] R. Paroni, A. Sili, "Nonlocal effects by homogenization or 3D-1D dimension reduction in elastic materials reinforced by stiff fibers", *J. Differ. Equations* **260** (2016), no. 3, p. 2026-2059.

[21] A. Sili, “Homogenization of a nonlinear monotone problem in an anisotropic medium”, *Math. Models Methods Appl. Sci.* **14** (2004), no. 3, p. 329-353.
[22] ———, “A diffusion equation through a highly heterogeneous medium”, *Appl. Anal.* **89** (2010), no. 6, p. 893-904.
[23] M. Vanninathan, “Homogenization of eigenvalue problems in perforated domains”, *Proc. Indian Acad. Sci., Math. Sci.* **90** (1981), no. 3, p. 239-271.
[24] V. V. Zhikov, “On an extension and application of the two-scale convergence method”, *Mat. Sb.* **191** (2000), no. 7, p. 973-1014.
[25] V. V. Zhikov, S. M. Kozlov, O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer, 1994, translated from the Russian by G.A. Yosifian.