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Partial differential equations / *Equations aux dérivées partielles*

A sharp relative isoperimetric inequality for the square

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Abstract. We compute the exact value of the least “relative perimeter” of a shape S, with a given area, contained in a unit square; the relative perimeter of S being the length of the boundary of S that does not touch the border of the square.

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1. Introduction

Let Q denote the unit cube in $\mathbb{R}^N$, $N \geq 2$. Given a measurable set $S \subset Q$ (S stands for shape) we denote by $\mathbb{1}_S$ the characteristic function of S, by $|S| = \|\mathbb{1}_S\|_{L^1}$ the volume of S (i.e., the area of S when $N = 2$), and by $P(S,Q)$, or simply $P(S)$, the relative perimeter of S, i.e., taking into account only the part of the boundary of S inside Q; in other words $P(S)$ is the total mass of the measure $\nabla \mathbb{1}_S$ (possibly infinite if S is not rectifiable).

Our goal is to give an *explicit* formula when $N = 2$, for the function $f_N(t)$ defined for $0 \leq t \leq 1$ by

$$f_N(t) = \inf\left\{P(S);\ S \text{ is a measurable subset of } Q \text{ such that } |S| = t\right\}. \tag{1}$$

Clearly

$$f_N(t) = f_N(1-t) \quad \forall\ t \in [0,1]\,; \tag{2}$$

just replace S by $Q \setminus S$, and thus we will often assume that $0 \leq t \leq \frac{1}{2}$.

The main result of this note is the following:

[*] Corresponding author.

Theorem 1. *Assume $N = 2$, then*

$$f_2(t) = \begin{cases} (\pi t)^{\frac{1}{2}} & \text{if } 0 \le t \le \frac{1}{\pi} \\ 1 & \text{if } \frac{1}{\pi} \le t \le \frac{1}{2} \end{cases}. \tag{3}$$

Moreover the infimum in (1) *is achieved by explicit shapes whose boundary consists or arcs of circles or line segments.*

Remark 2. We do not know any similar result for $f_N(t)$ when $N \ge 3$. In particular, it is not known whether $f_N(t) \equiv 1$ in a neighborhood of $t = \frac{1}{2}$. There is however a simple lower bound for $f_N(t)$ valid for all $N \ge 2$. More precisely

$$f_N(t) \ge 4t(1-t) \quad \forall\, N \ge 2,\ \forall\, t \in \left[0, \frac{1}{2}\right], \tag{4}$$

and the constant 4 in (4) is sharp. This inequality is originally due to H. Hadwiger [7] when the infimum in (1) is restricted to polyhedral subsets S of the cube Q. Far-reaching variants appeared subsequently in the literature (see e.g. S. G. Bobkov [5, 6], D. Bakry and M. Ledoux [3], F. Barthe and B. Maurey [4], and their references). The version stated as (4) (i.e., for measurable sets S) was proved in its full generality by L. Ambrosio, J. Bourgain, H. Brezis and A. Figalli in [2, Appendix], where it plays an essential role. Note that when $N = 2$ inequality (4) is consistent with the explicit formula (3) since

$$(\pi t)^{\frac{1}{2}} \ge 4t(1-t) \ \ \forall\, t \in \left(0, \frac{1}{\pi}\right), \tag{5}$$

or equivalently

$$\frac{\pi^{\frac{1}{2}}}{4} \ge s\left(1-s^2\right) \ \ \forall\, s \in \left(0, \frac{1}{\pi^{\frac{1}{2}}}\right). \tag{6}$$

Indeed the function $s(1-s^2)$ is increasing on the interval $(0, \frac{1}{3^{\frac{1}{2}}})$ and thus (5) reduces to

$$\frac{\pi^{\frac{1}{2}}}{4} \ge \frac{1}{\pi^{\frac{1}{2}}}\left(1 - \frac{1}{\pi}\right),$$

which is obvious.

Remark 3. Y. Altshuler and A. Bruckstein [1] established earlier a version of Theorem 1 where the infimum in (1) is restricted to "nice" connected sets S. Their strategy of proof enters as an ingredient in this note.

Remark 4. The conclusion of Theorem 1 is probably known to the experts even though we could not find a reference in the literature. E. Milman suggested an alternative approach by considering the result of H. Howards cited in [8, Section 7], and concerning the isoperimetric problem on a flat $2D$ torus.

Acknowledgments

We thank Emanuel Milman for useful discussions concerning the proof of Theorem 1. We also thank Jean Mawhin and Petru Mironescu for enlightening exchanges related to Lemma 6.

2. Some simple facts

The proof of Theorem 1 relies on three simple facts.

Fact 1 (The classical planar isoperimetric inequality). *Given any shape S in the plane we have*

$$P(S) = P\left(S, \mathbb{R}^2\right) \ge 2\sqrt{\pi}\sqrt{|S|}.$$

Fact 2 (The half-plane isoperimetric inequality). *Given any shape S in the half-plane denoted* $\frac{1}{2}\mathbb{R}^2$ *we have*

$$P(S) = P\left(S, \frac{1}{2}\mathbb{R}^2\right) \geq \sqrt{2\pi}\sqrt{|S|}.$$

Proof. If S touches the boundary of the half-plane, we reflect it across the boundary, thereby generating a (symmetric) shape S' of area $2|S|$ and such that

$$P\left(S', \mathbb{R}^2\right) = 2P\left(S, \frac{1}{2}\mathbb{R}^2\right).$$

Applying 1 to S' we obtain

$$P\left(S', \mathbb{R}^2\right) \geq 2\sqrt{\pi}\sqrt{|S'|},$$

which yields

$$P\left(S, \frac{1}{2}\mathbb{R}^2\right) \geq \sqrt{\pi}\sqrt{2|S|}.$$

□

Fact 3 (The quarter-plane isoperimetric inequality). *Given any shape S in a quater-plane denoted* $\frac{1}{4}\mathbb{R}^2$ *we have*

$$P\left(S, \frac{1}{4}\mathbb{R}^2\right) \geq \sqrt{\pi}\sqrt{|S|}.$$

Proof. If S touches the two orthogonal boundaries of the quarter-plane we reflect it symmetrically into the three quarters plane, generating a shape S' of area $4|S|$ and such that

$$P\left(S', \mathbb{R}^2\right) = 4P\left(S, \frac{1}{4}\mathbb{R}^2\right).$$

Applying 1 to S' we obtain

$$P\left(S', \mathbb{R}^2\right) \geq 2\sqrt{\pi}\sqrt{|S'|},$$

which yields

$$P\left(S, \frac{1}{4}\mathbb{R}^2\right) \geq \sqrt{\pi}\sqrt{|S|}.$$

□

3. Proof of Theorem 1

Since we consider only the case $N = 2$, we will write simply $f(t)$ instead of $f_2(t)$. Set

$$g(t) = \begin{cases} (\pi t)^{\frac{1}{2}} & \text{if } 0 \leq t \leq \frac{1}{\pi} \\ 1 & \text{if } \frac{1}{\pi} \leq t \leq \frac{1}{2} \end{cases}. \tag{7}$$

The goal is to prove that $f(t) = g(t)\ \forall\ t \in [0, \frac{1}{2}]$. The proof is divided into 7 steps.

Step 1. We have

$$f(t) \leq g(t) \quad \forall\ t \in \left[0, \frac{1}{2}\right]. \tag{8}$$

Assume first that $t \leq \frac{1}{\pi}$ and consider the set S as in Figure 1, where $R = 2\sqrt{\frac{t}{\pi}} \leq 1$, so that $|S| = \frac{\pi R^2}{4} = t$ and $P(S) = \frac{2\pi R}{4} = (\pi t)^{\frac{1}{2}}$. Therefore (by definition of $f(t)$), $f(t) \leq (\pi t)^{\frac{1}{2}} = g(t)$.

Assume now that $\frac{1}{\pi} \leq t \leq \frac{1}{2}$ and consider the set S as in Figure 2, so that $|S| = t$. On the other hand $P(S) = 1$. Therefore, (by definition of $f(t)$), $f(t) \leq 1 = g(t)$. □

In what follows we concentrate on the lower bound

$$f(t) \geq g(t) \quad \forall\ t \in \left[0, \frac{1}{2}\right]. \tag{9}$$

Let S be a minimizer in (1). We know from abstract theory (see A. Ros [12, Theorem 1] and the references therein) that ∂S is smooth and consists of arcs of circle - possibly straight lines;

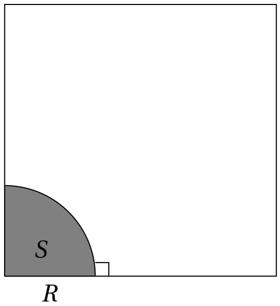

Figure 1.

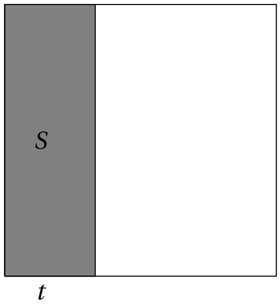

Figure 2.

moreover if $\partial Q \cap \partial S \neq \emptyset$, then ∂Q meets ∂S orthogonally. In view of this fundamental result it suffices to establish the lower bound

$$P(S) \geq g(|S|) \tag{10}$$

when S is restricted to the above class i.e., ∂S is smooth, ∂S consists of arcs of circle (or straight lines) and ∂S meets ∂Q orthogonally; but S need not be connected. Our next step allows to assume that S is also connected.

Step 2. *Reduction to the case where S is also connected*

Assume we have established (10) under the additional assumption that the shape is connected. Consider now some S which is *not* connected, and write $S = \bigcup_i S_i$ where here (S_i) are the connected components of S. Then

$$|S| = \sum_i |S_i| \tag{11}$$

and

$$P(S) = \sum_i P(S_i). \tag{12}$$

Assume first that $|S| \leq \frac{1}{\pi}$; then $|S_i| \leq \frac{1}{\pi}$ $\forall\ i$ and by (10) applied to S_i we have

$$P(S_i) \geq \sqrt{\pi |S_i|} \quad \forall\ i$$

Thus

$$P(S) \geq \sum_i \sqrt{\pi |S_i|} \geq \sqrt{\pi \sum_i |S_i|} = \sqrt{\pi |S|}$$

i.e., (10) holds for S.

Assume next that $\frac{1}{\pi} \leq |S| \leq \frac{1}{2}$. We distinguish two cases:

Case 1. $|S_i| \le \frac{1}{\pi}\ \forall\ i$.

Then, as above,

$$P(S) \ge \sqrt{\pi|S|} \ge 1 = g(|S|),$$

i.e., (10) holds for S.

Case 2. $|S_i| > \frac{1}{\pi}$ *for some* $i = i_0$.

By (10) applied to S_{i_0} we have $P(S_{i_0}) \ge 1$, and thus $P(S) \ge P(S_{i_0}) \ge 1$, i.e., (10) also holds for S.

We are therefore reduced to the situation investigated by Altshuler and Bruckstein [1], even under the additional assumption that ∂S consists of arcs of circle (or straight lines) meeting ∂Q orthogonally. We follow the strategy of their argument. □

Step 3. *S touches 0 side of Q.*

In this case the classical isoperimetric inequality (1 in Section 2) yields

$$P(S) \ge 2\sqrt{\pi|S|} \ge g(|S|) \quad \forall\ S, \text{ with } |S| \le \frac{1}{2}.$$ □

A typical example is as in Figure 3:

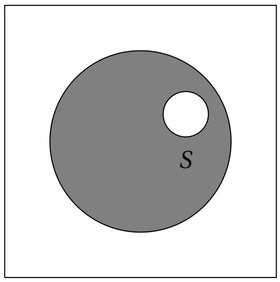

Figure 3.

Step 4. *S touches 1 side of Q*

A typical example is as in Figure 4:

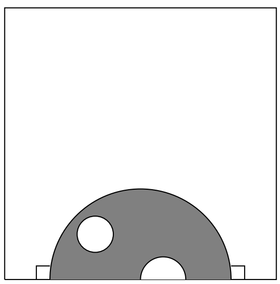

Figure 4.

In this case, the half-plane isoperimetric inequality (see 2 in Section 2) yields

$$P(S) \ge \sqrt{2\pi|S|} \ge g(|S|) \quad \forall\ S, \text{with } |S| \le \frac{1}{2}.$$ □

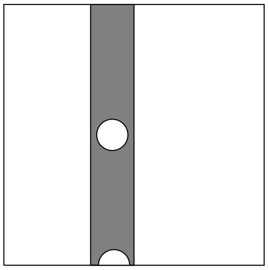

Figure 5.

Step 5. *S touches 2 sides of Q.*

In this case we have either S touches two opposite sides of Q or two adjacent sides of Q. The two opposite sides correspond to Figure 5.

(Here we use the assumption that S is connected.) In this configuration

$$P(S) \geq 2 \geq g(|S|) \quad \text{for all such } S.$$

The two adjacent sides correspond to Figure 6.

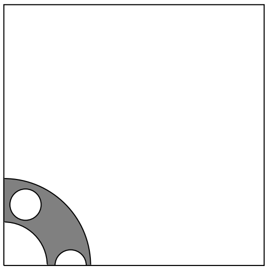

Figure 6.

In this configuration the quarter plane isoperimetric inequality (see 3 in Section 2) yields

$$P(S) \geq \sqrt{\pi |S|} \geq g(|S|) \quad \forall\, S, \quad \text{with} \quad |S| \leq \frac{1}{2}. \qquad \square$$

Step 6. *S touches 3 sides of Q.*

A typical example is as in Figure 7, where a portion of the boundary of S has to join two opposite sides of Q.

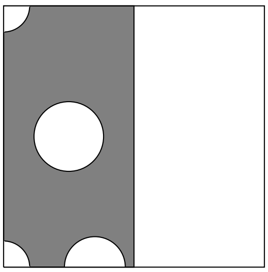

Figure 7.

In this case

$$P(S) \geq 1 \geq g(|S|) \quad \forall\, S, \quad \text{with} \quad |S| \leq \frac{1}{2} \qquad \square$$

Step 7. *S touches the 4 sides of Q.*

A typical example is as in Figure 8.

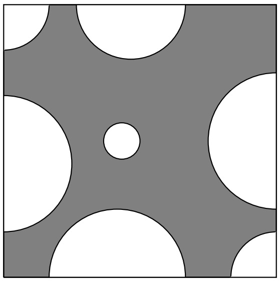

Figure 8.

Let $T = Q \setminus S$ and denote by (T_i) the connected components of T.

Lemma 5. *Each T_i touches 0,1, or 2 adjacent sides of Q.*

The proof of Lemma 5 relies on the following assertion which appears without proof in a paper by H. Poincaré [11, p. 67].

Lemma 6. *Let A_1, B_1 be points of ∂Q belonging to opposite sides of Q, and let $\mathscr{C}_1$ be a curve in Q connecting A_1 to B_1. Let A_2, B_2 be points of ∂Q belonging to a distinct pair of opposite sides of Q, and let $\mathscr{C}_2$ be a curve in Q connecting A_2 to B_2. Then*

$$\mathscr{C}_1 \cap \mathscr{C}_2 \neq \emptyset. \tag{13}$$

(see Figure 9)

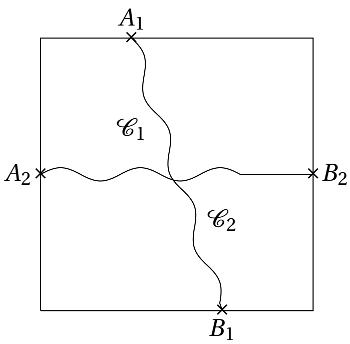

Figure 9.

Proof of Lemma 6. Let $(p_1(t), q_1(t))$ (resp. $(p_2(t), q_2(t))$), $0 \le t \le 1$, be a parametrization of $\mathscr{C}_1$ (resp. $\mathscr{C}_2$) such that $(p_1(0), q_1(0)) = A_1$, $(p_1(1), q_1(1)) = B_1$, (resp. $(p_2(0), q_2(0)) = A_2$, $(p_2(1), q_2(1)) = B_2$). Consider the map $F : \overline{Q} \to \mathbb{R}^2$ defined by

$$F(t,s) = (F_1(t,s), F_2(t,s)) = \big(p_1(t) - p_2(s), q_1(t) - q_2(s)\big).$$

We have to show that there exists some $(t,s)\in Q$ such that $F(t,s)=0$. Note that

$$F_1(t,0)=p_1(t)-p_2(0)\;=p_1(t)>0,\ \forall\ t\in[0,1], \tag{14}$$

$$F_1(t,1)=p_1(t)-p_2(1)\;=p_1(t)-1<0,\ \forall\ t\in[0,1], \tag{15}$$

$$F_2(0,s)=q_1(0)-q_2(s)\;=1-q_2(s)>0,\ \forall\ s\in[0,1], \tag{16}$$

and

$$F_2(1,s)=q_1(1)-q_2(s)\;=-q_2(s)<0,\ \forall\ s\in[0,1]. \tag{17}$$

We deduce from the Poincaré–Miranda theorem (see W. Kulpa [9], J. Mawhin [10], and the references therein) that there exists $(t,s)\in\overline{Q}$ (and in fact $(t,s)\in Q$) such that $F(t,s)=0$. □

Proof of Lemma 5. Assume by contradiction that T_i touches (at least) 2 opposite sides of Q. Fix a path $\mathscr{C}_1$ connecting these 2 opposite sides within T_i (this is possible because T_i is connected). Consider the remaining 2 opposite sides of Q and fix a path $\mathscr{C}_2$ connecting them within S; this is possible because S touches (by assumption) the 4 sides of Q and S is connected. From Lemma 6 we know that $\mathscr{C}_1\cap\mathscr{C}_2\neq\emptyset$. But this is impossible since $\mathscr{C}_1\subset T_i$, $\mathscr{C}_2\subset S$ and $T_i\cap S=\emptyset$. □

Proof of Step 7. We now complete the proof of Step 7. By 1, 2 and 3 in Section 2, we have for every i

$$P(T_i)\geq\min\left\{\sqrt{\pi},\sqrt{2\pi},2\sqrt{\pi}\right\}\sqrt{|T_i|}=\sqrt{\pi}\sqrt{|T_i|}. \tag{18}$$

Thus

$$P(S)=P(T)=\sum_i P(T_i)\geq\sqrt{\pi}\sum_i\sqrt{|T_i|}. \tag{19}$$

From the obvious inequality

$$\sum_i\sqrt{|T_i|}\geq\sqrt{\sum_i|T_i|}, \tag{20}$$

we deduce that

$$P(S)\geq\sqrt{\pi}\sqrt{\sum_i|T_i|}=\sqrt{\pi}\sqrt{|T|}=\sqrt{\pi}\sqrt{1-|S|}. \tag{21}$$

On the other hand $\sqrt{1-|S|}\geq\sqrt{|S|}$ since $|S|\leq\frac{1}{2}$, and therefore

$$P(S)\geq\sqrt{\pi}\sqrt{|S|}\geq g(|S|)\quad\forall\ S,\quad\text{with}\quad|S|\leq\frac{1}{2} \tag{22}$$

□

Remark 7. The same argument as above applies to the case where Q is replaced by a rectangle $D(X,Y)$ of dimensions X and Y such that $X\leq Y$. By analogy with the above we define for $0\leq t\leq XY$,

$$f(t)=\inf\{P(S);\ S\text{ is a measurable subset of }D(X,Y)\text{ such that}|S|=t\}.$$

Clearly

$$f(t)=f(XY-t)\quad\forall\ t\in[0,XY].$$

The analogue of Theorem 1 is:

Theorem 8. *We have*

$$f(t)=\begin{cases}(\pi t)^{\frac{1}{2}} & \textit{if } 0\leq t\leq\frac{X^2}{\pi}\\ X & \textit{if } \frac{X^2}{\pi}\leq t\leq\frac{1}{2}XY\end{cases}.$$

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