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Approximate Ground States of Hypercube Spin Glasses are Near Corners

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Abstract. We show that with probability exponentially close to 1, all near-maximizers of any mean-field mixed p -spin glass Hamiltonian on the hypercube $[-1, 1]^N$ are near a corner. This confirms a recent conjecture of Gamarnik and Jagannath. The proof is elementary and extends to arbitrary polytopes with $e^{o(N^2)}$ faces.

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1. Introduction

The present paper concerns mixed p -spin glasses on the hypercube $[-1, 1]^N$. Such a model is specified by a sequence $\gamma_1, \gamma_2, \dots \geq 0$ of non-negative real numbers encapsulated in the mixture function

$$\xi(t) = \sum_{p=1}^{\infty} \gamma_p^2 t^p.$$

For each $p \in \mathbb{Z}^+$ we sample i.i.d. Gaussian variables $\{g_{i_1, i_2, \dots, i_p}\}_{i_1, i_2, \dots, i_p \in [N]}$ and study the resulting random Hamiltonian energy function

$$H_N(x) = \sum_{p=1}^{\infty} \frac{\gamma_p}{N^{(p+1)/2}} \sum_{i_1, \dots, i_p=1}^N g_{i_1, \dots, i_p} x_{i_1} \dots x_{i_p}.$$

Equivalently, $H_N(\cdot)$ is a Gaussian process with covariance

$$\mathbb{E}[H_N(x)H_N(x')] = \frac{1}{N} \xi(\langle x, x' \rangle).$$

We assume the γ_p decay exponentially, i.e. $\limsup_{p \rightarrow \infty} \frac{\log \gamma_p}{p} < 0$, so that there are no issues regarding convergence. Here and throughout we use a normalized inner product $\langle x, y \rangle = \frac{1}{N} \sum_i x_i y_i$ for $x, y \in \mathbb{R}^N$ and similarly define $\|x\|_2 = \sqrt{\frac{1}{N} \sum_i x_i^2}$. Hence the hypercube $[-1, 1]^N$ has diameter 2. This scaling is chosen for convenience as it makes all relevant quantities dimension-independent. We will further assume throughout that γ_p is strictly positive for some $p \geq 2$ so

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that the model is a genuine spin glass. The mixture function ξ is always taken to be fixed while sending $N \rightarrow \infty$.

We focus on the (random) set of near-maximizers of $H_N(x)$. This set is intimately related to the Gibbs measure $\mu(dx) \propto e^{\beta N H_N(x)} dx$ in the low temperature regime with β large. The Gibbs measure $\mu(dx)$ is typically studied not on the continuous cube $[-1, 1]^N$ but on the discrete cube $\{\pm 1\}^N$, where a great deal is known. A key quantity of interest is the free energy

$$F_N(\beta) = \frac{1}{N} \log \sum_{x \in \{-1, 1\}^N} e^{\beta N H_N(x)}.$$

The limiting value (in probability) of $F_N(\beta)$ is famously given by the Parisi formula proposed in [21] and proved in [20, 25]. The existence (but not the identification) of the limiting value for large N was established earlier in [15].

The Hamiltonian $H_N(\cdot)$ is non-convex and may have exponentially many near-maxima [8, 10, 13]. Moreover the structure of these near-maxima is highly nontrivial, as for each β the Gibbs measure on $\{-1, 1\}^N$ is known to concentrate on a random approximate ultrametric with high probability in so-called generic mixed p -spin models with $\sum_{p: \gamma_p > 0} \frac{1}{p} = \infty$ [9, 16]. Also of interest are the results [4, 6, 7, 22] which study the landscape of critical points for spherical spin glasses and the related spiked tensor models, computing the exponential growth rates for the number of local maxima and critical points with a given energy value.

Let us now turn from the discrete cube $\{-1, 1\}^N$ to the continuous cube $[-1, 1]^N$. The free energy in this case takes a similar form as in the Ising case by the work of [17, 19]. Regarding the ground states, it is not difficult to see that *some* near-maximum of H_N on $[-1, 1]^N$ must lie on a corner in $\{\pm 1\}^N$. Indeed, one may ignore the small contribution of terms of H_N which are *not* multi-linear and then observe that any multilinear function of the coordinates x_1, \dots, x_N is maximized at some corner of the cube. However this does not rule out the existence of other near-maxima of H_N which are far from a corner and therefore missed by considering the discrete cube.

It was conjectured in [14, Conjecture 3.6] that in fact *all* near-maxima of H_N on $[-1, 1]^N$ must occur near the corners with high probability as $N \rightarrow \infty$. In other words, to understand the set of near-maxima of H_N on $[-1, 1]^N$, it is in some sense sufficient to understand it on the discrete cube. Conditional on (an implication of) this result, [14] prove that approximate message passing algorithms fail to approximately optimize pure p -spin models with $\gamma_p \neq 0$ for exactly 1 value of p , over $[-1, 1]^N$ when $p \geq 4$ is even. Moreover their proof seems to apply to any ξ satisfying a suitable *overlap gap* property, perhaps with the requirement $\gamma_1 = 0$. By contrast for mixture functions ξ satisfying a strong *no overlap gap* condition, approximate message passing algorithms are able to efficiently locate near-maxima of H_N with high probability [1, 18].

Our main result is that all near-maxima of H_N on $[-1, 1]^N$ are close to a corner in $\{\pm 1\}^N$, confirming the conjecture of [14]. Moreover we obtain an explicit quantitative dependence, though we do not expect it to be tight. Below we use the notation $\Omega_{\varepsilon, \eta}(N)$ to represent a quantity bounded below by $C(\varepsilon, \eta)N$ for some constant $C(\varepsilon, \eta)$ independent of N when $N \geq N_0(\varepsilon, \eta)$ is sufficiently large.

Theorem 1. *Let ξ define a mixed p -spin model and fix $\varepsilon, \eta > 0$. Then for N sufficiently large, with probability $1 - e^{-\Omega_{\varepsilon, \eta}(N)}$ all $x \in [-1, 1]^N$ with*

$$H_N(x) \geq \max_{y \in [-1, 1]^N} H_N(y) - \int_{1-\varepsilon}^1 \sqrt{(1-t)\xi''(t)} dt + \eta$$

satisfy $|x|_2^2 \geq 1 - \varepsilon$.

The idea of the proof is based on that of [24], which uses uniform control of top eigenvalues of the Hessian $\nabla^2 H_N(x)$ to optimize mean field spin glasses on the sphere via small local steps.

Our main insight is that this idea continues to work when a constant fraction of coordinates are fixed at ± 1 , allowing us to substantially increase the energy H_N from any starting point far from a corner even after reaching the boundary of $[-1, 1]^N$. Our proof is elementary and avoids any reliance on complicated Parisi-type variational formulas which characterize much of the spin glass literature. In fact it does not even require the existence of a limiting value for $\max_{y \in [-1, 1]^N} H_N(y)$. Due to the simplicity of our approach, Theorem 1 extends to quite general polytopes as we explain in Section 3.

Finally let us mention two alternative approaches to our main result, at least on the cube. First, we believe that [12, Theorems 8, 9] should imply Theorem 1. In their language it suffices to check that $TAP^\infty(\mu)$ is bounded away from 0 for μ a probability measure on $[-1, 1]$ with L^2 norm bounded away from 1. See also [11, 23] for positive temperature and spherical analogs. Second, [17, Theorem 1.2] gives a Parisi-type formula for the ground state energy of mixed p -spin models on the subset of $[-1, 1]^N$ with any asymptotically fixed L^2 norm in $[0, 1]$, so showing that their formula is strictly increasing in this L^2 norm would imply Theorem 1. However a proof produced by either method would be far less elementary than the proof we present, and even with significant effort might not extend beyond highly structured classes of polytopes.

2. Proof of Theorem 1

We set $\zeta(t) = \sqrt{\xi''(t)}$. By our assumption that $\gamma_p > 0$ for some $p \geq 2$ it follows that $\zeta(t) > 0$ for any $t > 0$. Below and throughout, given $M \in \text{Mat}_{N \times N}(\mathbb{R})$ and a subspace $W \subseteq \mathbb{R}^N$ we set $M|_W = P_W^\top M P_W$ where $P_W : \mathbb{R}^N \rightarrow W$ is the orthogonal projection onto W . In other words $M|_W$ is the restriction of M to W as a bilinear form, and is a matrix of size $\dim(W) \times \dim(W)$.

Proposition 2. *For nonzero $x \in \mathbb{R}^N$ let x^\perp denote the orthogonal subspace to x . For any fixed subspace $W \subseteq x^\perp$, the restriction $\nabla^2 H_N(x)|_W$ of the Hessian of H_N to x^\perp has the distribution of a $GOE(\dim(W))$ matrix times*

$$\zeta(|x|_2^2) \sqrt{\frac{\dim(W)}{N}}.$$

By a $GOE(N)$ matrix we mean a symmetric $N \times N$ matrix of independent centered Gaussians in which diagonal entries have variance $\frac{2}{N}$ and off-diagonals have variance $\frac{1}{N}$. In the case $W = x^\perp$, Proposition 2 can be shown as in [24, Equation (3.10)] by setting $x = (x_1, 0, \dots, 0)$ using rotational invariance and performing a simple direct computation. See also the text following [24, Equation (1.8)]. Proposition 2 then follows for general subspaces $W \subseteq x^\perp$ because the upper $\dim(W) \times \dim(W)$ corner of a $GOE(N - 1)$ matrix is a $GOE(\dim(W))$ matrix up to scaling.

Denote the eigenvalues of a symmetric matrix G in decreasing order by $\lambda_1(G) \geq \lambda_2(G) \geq \dots$. Recall that $\lambda_1(G) \approx 2$ holds with high probability when $G \sim GOE(N)$. In fact the following fundamental result states that many eigenvalues are at least $2 - \delta$ with extremely high $1 - e^{-\Omega_\delta(N^2)}$ probability. It follows from [3, Theorem 1.1] and is also used in the proof of [24, Lemma 3]. See also [2, Theorem 2.6.1].

Proposition 3. *For any $\delta > 0$ and fixed positive integer k , if $G \sim GOE(N)$ then*

$$\mathbb{P}[\lambda_k(G) \geq 2 - \delta] \geq 1 - e^{-\Omega_{\delta,k}(N^2)}.$$

We also require the following a priori uniform bound on the derivatives of H_N taken from [5]. Below \mathbb{B}^N denotes the unit ball $\{\sigma \in \mathbb{R}^N : |\sigma|_2 \leq 1\}$ while \mathbb{S}^{N-1} denotes the unit sphere $\{v \in \mathbb{R}^N : |v|_2 = 1\}$. Here again we use the rescaled norm in which

$$|v|_2 = \sqrt{\frac{\sum_i v_i^2}{N}};$$

moreover our definition of H_N differs from that of [5] by a factor N . This is why the derivative estimates below are of constant order unlike in [5].

Lemma 4 ([5, Corollary 59]). *Let H_N be the Hamiltonian for a mixed p -spin model with fixed mixture ξ satisfying $\lim_{p \rightarrow \infty} \frac{\log \gamma_p}{p} < 0$. For appropriate $C > 0$ and $i = 1, 2, 3$ we have:*

$$\mathbb{P} \left[\forall \sigma \in \mathbb{B}^N, \forall v \in \mathbb{S}^{N-1} : \left| \partial_v^i H_N(\sigma) \right| < C \right] \geq 1 - e^{-\Omega(N)}.$$

$$\mathbb{P} \left[\forall \sigma, \sigma' \in \mathbb{B}^N : \left\| \nabla^2 H_N(\sigma) - \nabla^2 H_N(\sigma') \right\|_{op} < C \left\| \sigma - \sigma' \right\| \right] \geq 1 - e^{-\Omega(N)}.$$

We next define the class of axis-aligned subspaces W_S for $S \subseteq [N]$. The key to our proof is to obtain uniform control on the Hessians $H_N(x)|_{W_S}$ over all $x \in [-1, 1]^N$ and large S .

Definition 5. *Given a subset $S \subseteq [N]$ we denote by W_S the $|S|$ dimensional subspace spanned by elementary basis vectors e_s for $s \in S$. We set $W_S(x) = W_S \cap x^\perp$ so that $\dim(W_S(x)) \in \{|S| - 1, |S|\}$.*

Definition 6. *The Hamiltonian $H_N : [-1, 1]^N \rightarrow \mathbb{R}$ is (ϵ, δ) -good at $x \in [-1, 1]^N$ if for every subset $S \subseteq [N]$ of size $|S| \geq \epsilon N$,*

$$\lambda_1 \left(\nabla^2 H_N(x) \Big|_{W_S(x)} \right) \geq 2\zeta \left(|x|_2^2 \right) \sqrt{\epsilon} - \delta.$$

H_N is (ϵ, δ) -good if it is (ϵ, δ) -good at all $x \in [-1, 1]^N$, and is δ -good if it is (ϵ, δ) -good for all $\epsilon \geq \delta$.

Roughly speaking, H_N is δ -good if its Hessian has a maximum eigenvalue of typical size or larger on all high-dimensional axis-aligned affine subspaces. We next show this condition occurs with exponentially high probability.

Lemma 7. *Fix $\delta > 0$ and $\xi(t) = \sum_{p \geq 1} \gamma_p^2 t^p$. Then $H_N(\cdot)$ is δ -good with probability $1 - e^{-\Omega_\delta(N)}$.*

Proof. We follow the proof of the case $S = [N]$ in [24, Lemma 3], extending the union bound to be over subsets S as well as points x . First, it suffices to show H_N is $(\epsilon, \delta/2)$ -good with the claimed probability for all fixed (ϵ, δ) since one can then union bound over $O_\delta(1)$ values of ϵ using uniform continuity of ζ . Replacing $\delta/2$ by δ , we will show that H_N is (ϵ, δ) -good with probability $1 - e^{-\Omega_{\epsilon, \delta}(N)}$ which implies the conclusion.

For any fixed $x \in [-1, 1]^N$ and $S \subseteq [N]$, because $W_S(x) \subseteq x^\perp$, we obtain from Proposition 2 that the restricted Hessian $\nabla^2 H_N(x)|_{W_S(x)}$ has the law of

$$\zeta \left(|x|_2^2 \right) \sqrt{\frac{\dim W_S(x)}{N}} \cdot GOE \left(\dim W_S(x) \right).$$

As $\dim W_S(x) \geq |S| - 1$, Proposition 3 implies:

$$\mathbb{P} \left[\lambda_2 \left(\nabla^2 H_N(x) \Big|_{W_S(x)} \right) \geq 2\zeta \left(|x|_2^2 \right) \sqrt{\frac{|S| - 1}{N}} - \frac{\delta}{2} \right] \geq 1 - e^{-\Omega_\delta(|S|^2)}. \tag{1}$$

Restricting to $|S| \geq \epsilon N$ and observing there are at most 2^N possibilities for S , we conclude that for any fixed $x \in [-1, 1]^N$,

$$\mathbb{P} \left[\forall S \subseteq [N], |S| \geq \epsilon N : \lambda_2 \left(\nabla^2 H_N(x) \Big|_{W_S(x)} \right) \geq 2\zeta \left(|x|_2^2 \right) \sqrt{\frac{|S| - 1}{N}} - \frac{\delta}{2} \right] \geq 1 - e^{-\Omega_{\epsilon, \delta}(N^2)}. \tag{2}$$

Next choose a δ' -net $\mathcal{N}_{\delta'}$ for $[-1, 1]^N$ of size $|\mathcal{N}_{\delta'}| = e^{O_{\delta'}(N)}$. Union bounding over $y \in \mathcal{N}_{\delta'}$, it follows that (2) holds for all $y \in \mathcal{N}_{\delta'}$ simultaneously with the same high probability $1 - e^{-\Omega_{\epsilon, \delta}(N^2)}$. Assume additionally that the conclusions of Lemma 4 hold, which is with probability $1 - e^{-\Omega(N)}$. Under these conditions we now show that Equation (2) holds simultaneously for all $x \in [-1, 1]^N$ and $|S| \geq \epsilon N$. For such an x choose $y = y(x) \in \mathcal{N}_{\delta'}$ with $|x - y|_2 \leq \delta'$. Difference of squares and the triangle inequality imply

$$|x|_2^2 - |y|_2^2 \leq 2|x - y|_2 \leq 2\delta'.$$

From Lemma 4 and the fact (which follows from the Courant–Fisher characterization) that $|\lambda_k(M) - \lambda_k(M')| \leq |M - M'|_{op}$ for any symmetric matrices M, M' and any integer k ,

$$\begin{aligned} |\lambda_2(\nabla^2 H_N(x)|_{W_S(y)}) - \lambda_2(\nabla^2 H_N(y)|_{W_S(y)})| &\leq |(\nabla^2 H_N(x) - \nabla^2 H_N(y))|_{W_S(y)}|_{op} \\ &\leq |\nabla^2 H_N(x) - \nabla^2 H_N(y)|_{op} \\ &\leq 2C\delta'. \end{aligned}$$

From here we derive the eigenvalue lower bound

$$\begin{aligned} \lambda_1(\nabla^2 H_N(x)|_{W_S(x)}) &\geq \lambda_1(\nabla^2 H_N(x)|_{W_S(x) \cap y^\perp}) \\ &\geq \lambda_2(\nabla^2 H_N(x)|_{W_S(y)}) \\ &\geq \lambda_2(\nabla^2 H_N(y)|_{W_S(y)}) - 2C\delta' \\ &\geq 2\zeta(|y|_2^2) \sqrt{\varepsilon} - \frac{\delta}{2} - 2C\delta' \\ &\geq 2\zeta(\max(0, |x|_2^2 - 2\delta')) \sqrt{\varepsilon} - \frac{\delta}{2} - 2C\delta'. \end{aligned}$$

As ζ is uniformly continuous on $[0, 1]$, taking δ' sufficiently small gives the conclusion

$$\lambda_1(\nabla^2 H_N(x)|_{W_S(x)}) \geq 2\zeta(|x|_2^2) \sqrt{\varepsilon} - \delta.$$

Because $x \in [-1, 1]^N$ and $|S| \geq \varepsilon N$ were arbitrary, we conclude that H_N is (ε, δ) -good with probability $1 - e^{-\Omega(N)} - e^{-\Omega_{\varepsilon, \delta}(N^2)} = 1 - e^{-\Omega_{\varepsilon, \delta}(N)}$. Recalling the discussion at the beginning of the proof, it follows that H_N is δ -good with probability $1 - e^{-\Omega_\delta(N)}$ as claimed. \square

The next lemma shows how to use Lemma 7 to obtain local improvements to $H_N(\cdot)$ from any point $x \in [-1, 1]^N$ which is far from a corner.

Lemma 8. *Suppose the Hamiltonian H_N is δ -good and satisfies the guarantee of Lemma 4. Then for any $x \in [-1, 1]^N$ with $|x|_2^2 \leq 1 - \delta$ there is a non-zero vector v orthogonal to x such that:*

- (1) $x + v \in [-1, 1]^N$
- (2) If $|x_i| = 1$ then $v_i = 0$.
- (3)

$$H_N(x + v) - H_N(x) \geq \left(\zeta(|x|_2^2) \sqrt{1 - |x|_2^2 - \delta} \right) |v|_2^2.$$

- (4) $|v|_2 \leq \frac{\delta}{10C}$.
- (5) Either $|v|_2 = \frac{\delta}{10C}$ or $x + v$ has strictly more ± 1 -valued coordinates than x .

Proof. By a simple Markov inequality we know that x has a set S of at least $(1 - |x|_2^2)N$ coordinates not equal to ± 1 . Because H_N is δ -good the restriction $\nabla^2 H_N(x)|_{W_S(x)}$ has an eigenvalue at least

$$2\zeta(|x|_2^2) \sqrt{1 - |x|_2^2 - \delta}$$

with corresponding eigenvector v . Since $v \in W_S \subseteq \mathbb{R}^N$, we may by slight abuse of notation treat v as a vector in \mathbb{R}^N . Of course this $v \in \mathbb{R}^N$ need not be an eigenvector of $\nabla^2 H_N(x)$ but we retain the Rayleigh quotient lower bound

$$\langle v, \nabla^2 H_N(x)v \rangle \geq \left(2\zeta(|x|_2^2) \sqrt{1 - |x|_2^2 - \delta} \right) |v|_2^2.$$

Since $v, -v$ play symmetric roles we may assume by symmetry that $\langle \nabla H_N(x), v \rangle \geq 0$. By scaling v to be sufficiently small we may assume that $x + v \in [-1, 1]^N$ and that $|v|_2 \leq \frac{\delta}{10C}$. Using the guarantee of Lemma 4 with $i = 3$, it follows that along the line segment $x + [0, 1]v$ the Hessian

of H_N varies in operator norm by at most $\frac{\delta}{5}$. This combined with $\langle \nabla H_N(x), v \rangle \geq 0$ easily implies that

$$H_N(x + v) \geq H_N(x) + \left(\zeta(|x|_2^2) \sqrt{1 - |x|_2^2} - \delta \right) |v|_2^2.$$

Hence v satisfies the first 4 claimed conditions. By scaling v to be as long as possible given the constraints $x + v \in [-1, 1]^N$ and $|v|_2 \leq \frac{\delta}{10C}$ we ensure that item 5 is satisfied. \square

Proof of Theorem 1. We take δ small depending on ε and assume H_N is δ -good and that the conclusion of Lemma 4 holds. For any point $x^0 \in [-1, 1]^N$ with $|x^0|_2^2 \leq 1 - \varepsilon$ we choose v^0 as guaranteed by Lemma 8 and set $x^1 = x^0 + v^0$. We continue producing iterates $x^{i+1} = x^i + v^i$ via Lemma 8 with increasing energies until we reach an x^m with $|x^m|_2^2 \geq 1 - \delta$. By Lemma 8(5), this occurs for some finite m .

Since v^i is orthogonal to x^i , we find

$$H_N(x^m) - H_N(x^0) = \sum_{i < m} H_N(x^{i+1}) - H_N(x^i) \tag{3}$$

$$\geq \sum_{i < m} \left(\zeta(|x^i|_2^2) \sqrt{1 - |x^i|_2^2} - \delta \right) |v^i|_2^2 \tag{4}$$

$$= \sum_{i < m} \left(\zeta(|x^i|_2^2) \sqrt{1 - |x^i|_2^2} - \delta \right) (|x^{i+1}|_2^2 - |x^i|_2^2). \tag{5}$$

Up to the error

$$\sum_{i < m} \delta (|x^{i+1}|_2^2 - |x^i|_2^2) \leq \delta,$$

this is exactly a Riemann sum for the integral

$$\int_{|x^0|_2^2}^{|x^m|_2^2} \zeta(t) \sqrt{1 - t} dt.$$

Because $|v^i|_2^2 \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in i , and $|x^0|_2^2 \leq 1 - \varepsilon, |x^m|_2^2 \geq 1 - \delta$, these Riemann sums have limit infimum at least the integral $\int_{1-\varepsilon}^1 \zeta(t) \sqrt{1 - t} dt$. Hence for fixed ε , and $\delta \rightarrow 0$, we obtain

$$H_N(x^m) - H_N(x^0) \geq \int_{1-\varepsilon}^1 \zeta(t) \sqrt{1 - t} dt - o_{\delta \rightarrow 0}(1).$$

Here $o_{\delta \rightarrow 0}(1)$ indicates a term tending to 0 as $\delta \rightarrow 0$, uniformly in N . Since x^0 was arbitrary given the constraint $|x^0|_2^2 \leq 1 - \varepsilon$ and $H_N(x^m) \leq \max_{y \in [-1, 1]^N} H_N(y)$, taking δ small enough depending on (ε, η) concludes the proof. \square

3. Extension to General Polytopes

Theorem 1 extends to more general polytopes than cubes. In particular we show that for bounded polytopes with $e^{o(N^2)}$ total faces, all near-maxima of H_N over the polytope occur near a point at which $(1 - \varepsilon)N$ faces are incident. We remark that the condition of $e^{o(N^2)}$ total faces is implied by having either $e^{o(N)}$ vertices or $e^{o(N)}$ maximal (i.e. codimension 1) faces. It also holds for any product of $O(N)$ polytopes of constant dimension.

Definition 9. A sequence of polytopes $\mathcal{P}_N \subseteq \mathbb{R}^N$ is said to be regular if:

- (1) \mathcal{P}_N has at most $e^{o(N^2)}$ faces of all dimensions.
- (2) $\mathcal{P}_N \subseteq \mathbb{B}_N$.

The second condition ensures that Lemma 4 continues to hold over \mathcal{P}_N . We again remind the reader that we use the normalization $|x|_2^2 = \frac{\sum_{i=1}^N x_i^2}{N}$

Definition 10. Given a polytope $\mathcal{P}_N \subseteq \mathbb{R}^N$ an ε -corner is a point on the boundary $\partial\mathcal{P}_N$ at which at least $(1 - \varepsilon)N$ faces meet.

Theorem 11. Let ξ define a mixed p -spin model and fix $\varepsilon, \varepsilon', \eta > 0$. Let $\mathcal{P}_N \subseteq \mathbb{R}^N$ be a regular sequence of polytopes. Then for N sufficiently large, with probability $1 - e^{-\Omega_{\varepsilon, \varepsilon', \eta}(N)}$, for any $x \in \mathcal{P}_N$ satisfying

$$H_N(x) \geq \max_{y \in \mathcal{P}_N} H_N(y) - \sqrt{\varepsilon} \int_{|x|_2^2}^{|x|_2^2 + \varepsilon'} \zeta(t) dt + \eta$$

there exists an ε -corner \hat{x} of \mathcal{P}_N with $|x - \hat{x}|_2^2 \leq \varepsilon'$.

Note that because ζ is increasing, we have

$$\int_{|x|_2^2}^{|x|_2^2 + \varepsilon'} \zeta(t) dt \geq \int_0^{\varepsilon'} \zeta(t) dt$$

which is positive and independent of x . The proof is almost the same as the cubical case. The subspaces W_S are replaced by the family of all \mathcal{P}_N -face-aligned subspaces in \mathbb{R}^N . The main difference is that to prove Theorem 11 it does not suffice to track the distance $|x^i|_2$ to the origin, as being a near-corner is no longer equivalent to having near-maximal distance from the origin. Because of this we additionally track the distances $|x^i - x^0|_2$ of our sequence x^0, x^1, x^2, \dots from the starting point x^0 . This leads to an additional linear constraint on the increment vectors v^i and hence requires one more large eigenvalue of the restricted Hessians.

Definition 12. We say a subspace $U \subseteq \mathbb{R}^N$ is \mathcal{P}_N -face-aligned if \mathcal{P}_N has a face whose tangent space is exactly (a translate of) U .

Definition 13. For a mixture ξ and polytope $\mathcal{P}_N \subseteq \mathbb{R}^N$, the Hamiltonian H_N is (ε, δ) -superb if for all $x \in \mathcal{P}_N$ and all \mathcal{P}_N -face-aligned subspaces U with $\dim(U) \geq \varepsilon N$,

$$\lambda_2(\nabla^2 H_N(x)|_{U \cap x^\perp}) \geq 2\zeta(|x|_2^2) \sqrt{\varepsilon} - \delta.$$

Lemma 14. Fix $\varepsilon, \delta > 0$ a mixture ξ , and a regular sequence \mathcal{P}_N of polytopes. Then with probability $1 - e^{-\Omega_{\varepsilon, \delta}(N)}$ the random function H_N is (ε, δ) -superb.

Proof. The proof is almost identical to that of Lemma 7 - note that $U \cap x^\perp$ is exactly the same as $W_S(x)$ for the case of the cube. To obtain a lower bound on λ_2 rather than λ_1 , we simply change all instances of λ_k to λ_{k+1} in the proof of Lemma 7. Regularity of \mathcal{P}_N ensures that when we take a union bound over pairs (y, U) for y in a δ' -net $\mathcal{N}_{\delta'} \subseteq \mathcal{P}_N$ and all \mathcal{P}_N -face-aligned subspaces U , we only consider $e^{o(N^2)}$ distinct pairs. Hence the N^2 large deviation rate of Proposition 3 ensures uniform eigenvalue lower bounds across all such pairs with exponentially high probability. As remarked previously, Lemma 4 continues to apply to \mathcal{P}_N , so that by again taking δ' small we extend from a δ' net to all of \mathcal{P}_N just as in Lemma 7. \square

We next give the analog of Lemma 8 with the new constraint $\langle v, x^0 \rangle = 0$.

Lemma 15. Suppose the Hamiltonian H_N is (ε, δ) -superb and satisfies the guarantee of Lemma 4. Then for any $x \in \mathcal{P}_N$ not a ε -corner and for any $x^0 \in \mathcal{P}_N$ there is a non-zero vector v orthogonal to both x and x^0 such that:

- (1) $x + v \in \mathcal{P}_N$
- (2) If x is contained in a boundary face of \mathcal{P}_N , then $x + v$ is in the same face.
- (3)

$$H_N(x + v) - H_N(x) \geq (\zeta(|x|_2^2) \sqrt{\varepsilon} - \delta) |v|_2^2.$$

- (4) $|v|_2 \leq \frac{\delta}{10C}$.
- (5) Either $|v|_2 = \frac{\delta}{10C}$ or $x + v$ is contained in a face of dimension strictly smaller than that of any face containing x .

Proof. Let U be the subspace corresponding to the minimal face containing x . As x is not an ε -corner we know that $|U| \geq \varepsilon N$. By Cauchy interlacing,

$$\lambda_1(\nabla^2 H_N(x)|_{U \cap x^\perp \cap x^{0\perp}}) \geq \lambda_2(\nabla^2 H_N(x)|_{U \cap x^\perp}) \quad (6)$$

$$\geq 2\zeta(|x|_2^2)\sqrt{\varepsilon} - \delta. \quad (7)$$

Let $v \in U \cap x^\perp \cap x^{0\perp}$ be the top eigenvector of $\nabla^2 H_N(x)$. As before since $U \cap x^\perp \cap x^{0\perp} \subseteq \mathbb{R}^N$ we may treat v as a vector in \mathbb{R}^N . The remainder of the proof is identical to that of Lemma 8. \square

Proof of Theorem 11. We begin with a point x^0 and repeatedly use Lemma 15, whose assumptions hold with probability $1 - e^{-\Omega_{\varepsilon, \delta}(N)}$. We obtain a sequence $x^0, x^1 = x^0 + v^i, x^2 = x^1 + v^i, \dots$ of points in our polytope. We continue until reaching an ε -corner x^m . We have for each i :

$$H_N(x^{i+1}) - H_N(x^i) \geq (\zeta(|x^i|_2^2)\sqrt{\varepsilon} - \delta)|v^i|_2^2.$$

From the orthogonality conditions on v^i we have

$$|x^{i+1}|_2^2 - |x^i|_2^2 = |v^i|_2^2 = |x^{i+1} - x^0|_2^2 - |x^i - x^0|_2^2 \quad (8)$$

which implies $\delta \sum_{i=0}^{m-1} |v^i|_2^2 = O(\delta) = o_{\delta \rightarrow 0}(1)$. As in the proof of Theorem 1, the fact that $|v^i|_2^2 \rightarrow 0$ uniformly as $\delta \rightarrow 0$ gives the Riemann sum convergence

$$\sum_{i=0}^{m-1} \zeta(|x^i|_2^2)\sqrt{\varepsilon}|v^i|_2^2 = \sum_{i=0}^{m-1} \zeta(|x^i|_2^2)\sqrt{\varepsilon}(|x^{i+1}|_2^2 - |x^i|_2^2) \rightarrow \sqrt{\varepsilon} \int_{|x^0|_2^2}^{|x^m|_2^2} \zeta(t) dt.$$

It follows that for δ sufficiently small as a function of ε , with probability $1 - e^{-\Omega_{\varepsilon, \delta}(N)}$:

$$\begin{aligned} H_N(x^m) - H_N(x^0) &\geq \sqrt{\varepsilon} \int_{|x^0|_2^2}^{|x^m|_2^2} \zeta(t) dt - o_{\delta \rightarrow 0}(1) \\ &= \sqrt{\varepsilon} \int_{|x^0|_2^2}^{|x^0|_2^2 + |x^m - x^0|_2^2} \zeta(t) dt - o_{\delta \rightarrow 0}(1). \end{aligned}$$

Here the latter equality follows from (8). Since x^0 was arbitrary and x^m is an ε -corner, taking δ small enough depending on $(\varepsilon, \varepsilon', \eta)$ completes the proof of Theorem 11. \square

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