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Exponential inequalities for the supremum of some counting processes and their square martingales

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Abstract. We establish exponential inequalities for the supremum of martingales and square martingales obtained from counting processes, as well as for the oscillation modulus of these processes. Our inequalities, that play a decisive role in the control of errors in statistical procedures, apply to general non-explosive counting processes including Poisson, Hawkes and Cox models. Some applications for U-statistics are discussed.

Résumé. Nous établissons ici des inégalités exponentielles pour le supremum de martingales et de martingales carrées issues de processus de comptage, ainsi que pour le processus d'oscillation de ces processus. Ces inégalités, qui jouent un rôle essentiel dans le contrôle d'erreur de certaines procédures statistiques, s'appliquent à des processus de comptage non-explosifs généraux, comme les processus de Poisson, de Hawkes ou encore les processsus de Cox. Quelques applications aux U-statistiques sont aussi abordées dans cet article.

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1. Introduction

Counting processes naturally arise in a lot of applied fields and the understanding of their evolution is the object of a lot of modelling problems. In this context, exponential inequalities are of great interest, especially for the control of errors in statistics. Exponential inequalities for the distribution of random variables have been of interest for many years (see [13] for one of the first result in this field), and they are still a very active research area for various types of processes, like sums of i.i.d. random variables, empirical processes, U-statistics, Poisson processes, martingales and self-normalised martingales, with discrete or continuous time. For example, for discrete time processes with i.i.d. random variables, exponential inequalities have been obtained for the empirical process or for U-statistics of order two in [1, 7, 11, 12, 16, 18, 19, 25] or [10] to cite a few.

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We may refer also to [20] or [4] for a wide review of exponential inequalities for discrete time martingales.

In this paper we focus on counting processes in continuous time. Our aim is to provide exponential inequalities with explicit constants for general counting processes, specifically for their associated local martingales, for their local square martingales and for their oscillation modulus. The first one is useful for statistical applications like density estimation as exploited for instance in [22]. The second one involving local square martingales allows to control U-statistics (see [14]) which have a long history. For instance, the estimation of a quadratic functional of a density ([17]), or in testing problems (see [8] for a goodness-of-fit test in density or [9] for an adaptive test of homogeneity of a Poisson process), the estimator, as well as the test statistics are naturally U-statistics of order two. As to the third contribution concerning the oscillation modulus, applications may concern multiple testing problems where some procedures are based on the oscillation modulus of counting processes, which entails the need to control the supremum and the whole oscillation modulus of these processes, and not only their marginals. In a non-asymptotic framework, it is necessary to obtain inequalities with explicit constants. The keystone for controlling the statistical error is then to use exponential bounds for the right model, but the results obtained on square martingales are generally not the simple consequence of those obtained for simple martingales. To achieve our goal, we first exhibit local martingale properties of the exponential of counting processes, then we state exponential inequalities for the supremum of those processes, leading to exponential bounds for the oscillation modulus.

A first exponential inequality for martingales of counting processes in continuous time can be found in [15, Theorem 23.17], that concerns semimartingales M under the restrictive assumption that $[M]_\infty \le 1$ almost surely. The specific case of the Poisson process is studied in [22]. More general counting processes are considered in [26] or [23]. In these contributions, the exponential bounds are derived from techniques adapted from the empirical process, with extensions of Bernstein's exponential inequality for general martingales. As a consequence of the results of [26], exponential inequalities with explicit constants have been established in [23] for the supremum of counting processes with absolutely continuous compensators, as well as for the supremum of a countable family of martingales associated with counting processes.

However the case of the square martingale is not addressed in these previous results. As to the existing results concerning exponential inequalities for U-statistics, many of them come from results on sequences of i.i.d. random variables. Indeed, in the specific case of the Poisson process, a sharp exponential inequality with explicit constants holds for U-statistics of order two and for double integrals of Poisson processes in [14]. The Poisson process is seen as a point process $(T_i)_{i\ge 1}$ on the real line, allowing to use the inequalities obtained for U-statistics of i.i.d. random variables like Rosenthal's inequality and Talagrand's inequality, after conditioning by the total random number of point. Unfortunately this approach is no longer valid when we consider more general counting processes than the Poisson process.

This contribution unifies the above approaches since our exponential inequalities apply to general counting processes, including for instance the Poisson process, non-explosive Cox processes or nonlinear Hawkes processes under mild assumptions like bounded intensities (see e.g. [6]), and we consider both martingales and square-martingales. Comparing to the existing literature, we do not make any assumption about the independence of the underlying point process and we use quite different proofs involving stochastic calculus instead of adapting previous techniques in discrete time. Moreover, we get sharper inequalities when applied to the setting of existing works. For instance concerning the exponential inequality for the martingale, we get a non-asympotic inequality with explicit constants and we obtain a tail of order $x\log(x)$ instead of x in [26]. As an application of our results, we obtain a control of the supremum of some U-statistics and double integrals based on other counting processes than the Poisson process.

We also provide an inequality for the oscillation modulus, which allows a fine control of quadratic statistics based on counting processes.

The remainder of this article is organized as follows: in the next section, we introduce some general notations, while Section 3 is devoted to the exponential martingales of counting processes. The exponential inequalities of our martingales and their associated square martingales are presented in Section 4, while Section 5 details applications to U-statistics and oscillation modulus. Finally, we have gathered all the proofs in Section 6.

2. Notations

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a filtered probability space where $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ is a complete right-continuous filtration, $N = (N_t)_{t\geq 0}$ be a non-explosive $\mathscr{F}$-adapted counting process whose jump times are totally inaccessible, and $\Lambda = (\Lambda_t)_{t\geq 0}$ be its $\mathscr{F}$-compensator. We consider $H = (H_s)_{s\geq 0}$, a bounded predictable process, bounded by the non-random real number $\|H\|_{\infty,[c,d]}$ on the interval $[c,d]$, that is $\sup_{s\in[c,d]}|H_s| \leq \|H\|_{\infty,[c,d]}$ almost surely. If $c = 0$ and $T \geq 0$, $\|H\|_{\infty,[0,T]}$ will be written $\|H\|_{\infty,T}$ for short. The non-random real number $\|H\|_{2,[c,d]}$ will stand for a bound of the L^2 norm of H in $L^2(\Lambda([c,d]))$, that is $\int_c^d |H_u|^2 d\Lambda_u \leq \|H\|^2_{2,[c,d]} < +\infty$ almost surely.

Recall that for a semimartingale X, we define $[X]_t$ by

$$[X]_t = \left\langle X^c \right\rangle_t + \sum_{0<s\leq t} |\Delta X_s|^2$$

where $\langle X^c\rangle$ is the quadratic variation of the continuous martingale part of X and $\Delta X_s = X_s - X_{s^-}$ is the jump of X at s. We will use the fact that if X is a local martingale with jumps bounded by 1 and H is a bounded predictable process, then $(\int_0^t H_s dX_s)_{t\geq 0}$ is a local martingale. If in addition $\mathbb{E}[\int_0^t H_s^2 d[X]_s] < +\infty$ for every $t \geq 0$, then $(\int_0^t H_s dX_s)_{t\geq 0}$ is a martingale with $\mathbb{E}[(\int_0^t H_s dX_s)^2] < +\infty$ for every $t \geq 0$ (see [21, Corollary 3 p. 73]). Recall also that for a $\mathscr{C}^2$ function f and a càdlàg semimartingale $(Y_t)_{t\geq 0}$, the Itô formula ([3, Theorem 17.10]) entails

$$f(Y_t) = f(Y_0) + \int_{0^+}^t f'(Y_{s^-}) dY_s + \frac{1}{2}\int_{0^+}^t f''(Y_{s^-}) d\left\langle Y^c\right\rangle_s + \sum_{0<s\leq t}\left[f(Y_s) - f(Y_{s^-}) - f'(Y_{s^-})\Delta Y_s\right].$$

For $f = \exp$ and a semimartingale Y satisfying $\langle Y^c\rangle \equiv 0$ and $Y_0 = 0$, this leads to

$$e^{Y_t} = 1 + \int_{0^+}^t e^{Y_{s^-}}\, dY_s + \sum_{0<s\leq t} e^{Y_{s^-}}\left[e^{\Delta Y_s} - 1 - \Delta Y_s\right]. \tag{1}$$

Finally we define for every $n \geq 1$

$$S_n(Y) = \inf\left\{t > 0, e^{Y_{t^-}} \geq n\right\}$$

with the convention $\inf\varnothing = +\infty$. If $(e^{Y_{t^-}})_{t\geq 0}$ is a finite process, then $S_n(Y)$ is a stopping time (see [2, Theorem 2.4]) satisfying $\lim_{n\to+\infty} S_n(Y) = +\infty$ almost surely.

3. Martingale properties

Let $T > 0$. We consider in this section the three processes $M = (M_t)_{t\leq T}$, $\widetilde{M} = (\widetilde{M}_t)_{t\leq T}$ and $\tilde{\tilde{M}} = (\tilde{\tilde{M}}_t)_{t\leq T}$ defined for $t \leq T$ by

$$M_t = \int_0^t H_s d(N_s - \Lambda_s),$$

and the two double integrals

$$\begin{aligned}\widetilde{M}_t &= \left(\int_0^t H_s d\left(N_s - \Lambda_s\right)\right)^2 - \int_0^t H_s^2 dN_s \\ &= M_t^2 - \int_0^t H_s^2 dN_s \\ &= \int_0^t 2M_{s^-} H_s d\left(N_s - \Lambda_s\right),\end{aligned} \tag{2}$$

and

$$\begin{aligned}\tilde{\tilde{M}}_t &= \left(\int_0^t H_s d\left(N_s - \Lambda_s\right)\right)^2 - \int_0^t H_s^2 d\Lambda_s \\ &= M_t^2 - \int_0^t H_s^2 d\Lambda_s \\ &= \int_0^t \left(2M_{s^-} H_s + H_s^2\right) d\left(N_s - \Lambda_s\right).\end{aligned} \tag{3}$$

By definition $N - \Lambda$ is a local martingale. Since the jumps of N are totally inaccessible, we know that Λ is continuous and $N - \Lambda$ has jumps bounded by 1. The local martingale $N - \Lambda$ is then a locally square integrable local martingale of finite variations. As a consequence, M is a local martingale of finite variations, and $\widetilde{M}$, as well as $\tilde{\tilde{M}}$, is a semimartingale of finite variations.

Our main goal is to establish in the next section some exponential inequalities for these three semimartingales. We will use Chernoff's bounds in order to do that, so we are first interested by some exponentials associated with the three processes M, $\widetilde{M}$ and $\tilde{\tilde{M}}$. We start first with the process M in the following lemma, proving that the exponential of M is a local martingale. We follow the proof of [5, Theorem VI.2] where the case of an absolutely continuous compensator Λ is treated. We may also refer to [24] to find in that case some conditions on the counting process and its intensity to obtain an exponential which is a martingale.

Lemma 1. *Let Z be the process defined for a fixed real number λ and all $t \le T$ by*

$$Z_t = \lambda M_t - \int_0^t \left(e^{\lambda H_s} - 1 - \lambda H_s\right) d\Lambda_s.$$

Then for every $n \ge 1$, the process $(\exp(Z_{t \wedge S_n(Z)}))_{t \le T}$ is a martingale.

Let us define now for $a > 0$

$$T_a = \inf\{t \ge 0 : |M_t| > a\} \wedge T.$$

Since the jumps of N are totally inaccessible, T_a is a stopping time ([3, Proposition 16.3]). As a consequence of Lemma 1, replacing H by the process $2M_{s^-} H_s \mathbf{1}_{s \le T_a}$ which is also a bounded predictable process and using (2), we obtain the next lemma which sets out a stopped martingale associated with the exponential of $\widetilde{M}$.

Lemma 2. *Let $\widetilde{Z}$ be the process defined for a fixed real number λ and all $t \ge 0$ by*

$$\widetilde{Z}_t = \lambda \widetilde{M}_t - \int_0^t \left(e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s\right) d\Lambda_s.$$

For every positive a and every $n \ge 1$, the process $(\exp(\widetilde{Z}_{t \wedge T_a \wedge Sn(\tilde{Z})}))_{t \ge 0}$ is a martingale.

Finally we present the analogue of Lemma 2 for the process $\tilde{\tilde{M}}$, which is a consequence of (3) and Lemma 1.

Lemma 3. *Let $\tilde{\tilde{Z}}$ be the process defined for a fixed real number λ and all $t \ge 0$ by*

$$\tilde{\tilde{Z}}_t = \lambda \tilde{\tilde{M}}_t - \int_0^t \left(e^{\lambda H_s (H_s + 2M_s)} - 1 - \lambda H_s (H_s + 2M_s)\right) d\Lambda_s.$$

For every positive a and every $n \ge 1$, the process $(\exp(\tilde{\tilde{Z}}_{t \wedge T_a \wedge Sn(\tilde{\tilde{Z}})}))_{t \ge 0}$ is a martingale.

4. Exponential inequalities

We have gathered in this section our main results, that is the exponential inequalities for the three processes M, $\widetilde{M}$ and $\tilde{\tilde{M}}$. The rates that appear in these inequalities are governed by the rate function I defined for $x \geq 0$ by

$$I(x) = (1+x)\log(1+x) - x.$$

We start with a technical lemma that provides useful properties for the proofs of the main theorems.

Lemma 4. *Let $I_t(H,\lambda)$ be defined for $t \geq 0$ by $\int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$. For $t \leq T$ and every real λ, we get the almost sure inequality*

$$|I_t(H,\lambda)| \leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g\left(|\lambda| \, \|H\|_{\infty,T}\right) \tag{4}$$

where $g(x) = e^x - 1 - x$. Moreover the function g satisfies for every positive A, B and x

$$\inf_{\lambda > 0} \left(Ag(B\lambda) - \lambda x\right) = -AI\left(\frac{x}{AB}\right). \tag{5}$$

We present now in Theorem 5 an exponential inequality for the local martingale M, with its two-sided version.

Theorem 5. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq x\right) \leq \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right) \tag{6}$$

and

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| \geq x\right) \leq 2\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right). \tag{7}$$

Such exponential inequalities have already been obtained for martingales with bounded jumps in [15, 26] and [23]. In [15], the bound is of the form $\exp(-\frac{Ax^2}{1+Bx})$ for some constants A and B, and is available for a semimartingale M such that $[M]_\infty \leq 1$ almost surely, which is not our case here. In [26], the bound is of the form $A\exp(-Bx)$ for some constants A and B and x large enough. Finally in [23], the inequality is of the form $\mathbb{P}(\sup_{t\in[0,T]} \sup_a M_t^a \geq A\sqrt{x} + Bx) \leq \exp(-x)$ for a countable family of martingales $(M_t^a)_{t \geq 0}$. Comparing to all these results, in the case of the large deviations, that is when x tends to infinity, (6) and (7) provide a sharper bound with a more accurate tail, namely in $x\log x$ instead of x. When x tends to zero, these bounds are similar (up to constants), taking the form $A\exp(-Bx^2)$.

The next Theorem deals with the square martingale $\widetilde{M}$. The same inequality is obtained for $-\widetilde{M}$, leading to a two-sided inequality.

Theorem 6. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \widetilde{M}_t \geq x\right) \leq 3\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right) \tag{8}$$

and

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} -\widetilde{M}_t \geq x\right) \leq 3\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right), \tag{9}$$

thereby we have the following two-sided exponential inequality:

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \left|\widetilde{M}_t\right| \geq x\right) \leq 6\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right). \tag{10}$$

If we compare (7) and (10), we can notice that the upper bound in (10) involves $\sqrt{x}$ instead of x in the inequality (7), leading to different bounds when x tends to zero, contrary to the case of the large deviations. Finally the next Theorem 7 is the analogue of Theorem 6 for the martingale $\tilde{\tilde{M}}$.

Theorem 7. *For every positive x and T, we have the following inequalities:*

$$\mathbb{P}\left(\sup_{0\le t\le T}\tilde{\tilde{M}}_t \ge x\right) \le 3\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4}\right)\right) \tag{11}$$

and

$$\mathbb{P}\left(\sup_{0\le t\le T}-\tilde{\tilde{M}}_t \ge x\right) \le 3\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4}\right)\right), \tag{12}$$

thereby we have the following two-sided exponential inequality:

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|\tilde{\tilde{M}}_t\right| \ge x\right) \le 6\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4}\right)\right). \tag{13}$$

Comparing now (7) and (13), we observe that M and $\tilde{\tilde{M}}$ are behaving in the same way for x tending to zero. When x tends to infinity, (13) provides a similar bound (up to a constant) to (10), which is quite surprising in view of the relationship $\tilde{\tilde{M}} = \widetilde{M} + \int H^2 d(N-\Lambda)$. Although (10) remains sharper than (13) because $\|H\|_{\infty,T}(\sqrt{1+8x/\|H\|_{\infty,T}^2}-1)/4 \le \sqrt{x/2}$, one may find a constant C_H (depending on H) such that

$$\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2}\frac{\sqrt{1+8x/\|H\|_{\infty,T}^2}-1}{4}\right)\right) \le C_H \exp\left(-\frac{\|H\|_{2,T}^2}{2\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2}\sqrt{\frac{x}{2}}\right)\right)$$

for $x \ge 1$, so that the difference between (13) and (10) when x tends to infinity relies on the constants. Moreover the relationship $\tilde{\tilde{M}} = \widetilde{M} + \int H^2 d(N-\Lambda)$, (7) with H^2 instead of H, all along with (10) and $\frac{x}{2}$, will also lead to an exponential inequality but less sharp than (13) because $\|H\|_{2,T}^4 \le \|H^2\|_{2,T}^2$.

5. Examples of applications

5.1. *U-statistics of order two*

The main hypothesis of the previous theorems is to suppose that the countable process is non-explosive with totally inaccessible jumping times. This allows us to consider for instance Poisson, Cox or Hawkes processes with a bounded intensity. If N is a Poisson process, some sharp exponential inequalities have already been obtained in [14] for double stochastic integrals of the form $Z_t = \int_0^t \int_0^{y^-} h(x,y)d(N_x-\Lambda_x)d(N_y-\Lambda_y)$ where h is a (non-random) bounded Borel function. The Poisson process N is viewed as a point process $(T_i)_{i\ge 1}$, so that Z_t is a U-statistic for the Poisson process: $Z_t = \sum_{0\le T_i<T_j\le t} g(T_i,T_j)$ for some function g. We may then use the inequalities obtained for U-statistics after conditioning by the total random number of points, leading to a similar inequality as the one in [10]. However, Z_t takes the form of a U-statistics for any counting process N, and not only for the Poisson process.

In the particular case where h is a stochastic kernel of the form $h(x,y) = H(x)H(y)$, $\widetilde{M}$ may be written $\widetilde{M}_t = 2Z_t$, i.e. it is a double stochastic integrals or a U-statistics of order two. Although we are not limited to the Poisson case, by the Meyer theorem (see [21, page 104]), the jumps of a Poisson process are totally inaccessible so that we may apply Theorem 6. Comparing to [10] or [14], where the supremum of $(Z_t)_{t\geq 0}$ is not considered and h is not random, the inequality (8) provides sharper bounds for the large deviations with an additional $\log x$ in our inequality. Indeed in [10] or [14], the bound is of the form $L\exp(-\frac{1}{L}\min(\frac{x^{1/2}}{A^{1/2}},\frac{x^{2/3}}{B^{2/3}},\frac{x}{C},\frac{x^2}{D^2}))$ for some explicit constants A, B, C, D and L.

Such exponential inequalities for U-statistics are very useful for statistical applications. For instance the estimation of the L^2 norm $\int f^2(x)dx$ of the density of i.i.d. random variables via selection model is considered in [17] and [8]. The estimator of a quadratic distance is naturally a U-statistics of order two and the exponential inequality of [14] is a main tool for the study of the property of the estimator. In the Poisson model too, as in [9] where the homogeneity is tested, the method is based on an approximation of the L^2-norm of the intensity of the underlying Poisson process. Since our theorems in Section 4 apply to more varied counting processes, these quadratic form estimation procedures can be generalized to more general contexts than the Poisson framework.

5.2. *Oscillation modulus control*

The main theorems of Section 4 provide also an upper bound for the oscillation modulus of the three processes $M, \widetilde{M}$ and $\tilde{\tilde{M}}$. We consider c, d and x three non-negative real numbers, and the counting process $N_c(t) = N_{t+c} - N_c$ whose compensator is $\Lambda_c(t) = \Lambda_{t+c} - \Lambda_c$. The following theorem gives upper bounds for the oscillation modulus of the processes M and $\widetilde{M}$. As far as we know, this is the first time such an exponential inequality is stated for counting processes. An analogous inequality can be obtained for $\tilde{\tilde{M}}$ by following the same way of proof.

Theorem 8. *For every non-negative x, c and d, we have the following inequality for the oscillation modulus of M:*

$$\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}|M_t - M_s| \geq x\right) \leq 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\frac{x}{2}\right)\right). \tag{14}$$

For the process $\widetilde{M}$, we get the following exponential upper bound

$$\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}\left|\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u\right| \geq x\right) \leq 10\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right), \tag{15}$$

leading to the exponential inequality for the oscillation modulus of $\widetilde{M}$:

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}\left|\widetilde{M}_t - \widetilde{M}_s\right| \geq x\right) &\leq 10\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\sqrt{\frac{x}{16}}\right)\right)\\ &+ 2\exp\left(-\frac{\|H\|^2_{2,d}}{\|H\|^2_{\infty,d}} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right)\\ &+ 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right).\end{aligned} \tag{16}$$

In view of Theorems 5 and 6, the previous inequalities show that considering the oscillation modulus instead of the processes M and $\widetilde{M}$ themselves does not affect the rates (in x) of the exponential bounds, but only changes the constants. We obtain in Theorem 8 explicit constants with respect to the integrand H as well as the interval $[c,d]$, which may be useful for applications.

6. Proofs

Proof of Lemma 1. The process Z is defined as $\lambda M_t - \int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$ where λ is a fixed real number. Z is a càdlàg semimartingale of bounded variations because H is bounded and Λ, as well as M, is of bounded variations. The continuity of Λ entails the equality $\Delta Z_s = \lambda H_s \Delta N_s$. We get then from (1) that

$$\begin{aligned} e^{Z_t} &= 1 + \int_0^t e^{Z_{s-}} dZ_s + \sum_{0<s\leq t} e^{Z_{s-}} \left[e^{\lambda H_s \Delta N_s} - 1 - \lambda H_s \Delta N_s \right] \\ &= 1 + \int_0^t e^{Z_{s-}} \left[\lambda dM_s - \left(e^{\lambda H_s} - 1 - \lambda H_s \right) d\Lambda_s \right] + \int_0^t e^{Z_{s-}} \left(e^{\lambda H_s} - 1 - \lambda H_s \right) dN_s \\ &= 1 + \int_0^t e^{Z_{s-}} \left(e^{\lambda H_s} - 1 \right) d\left(N_s - \Lambda_s\right). \end{aligned}$$

For $n \geq 1$, the stopping time $S_n(Z)$ is defined by $S_n(Z) = \inf\{t > 0, e^{Z_{t-}} \geq n\}$. Then for every $s \leq S_n(Z) \wedge T$, $e^{Z_{s-}} \leq n$. Moreover, for every $t \leq T$,

$$e^{Z_{t\wedge S_n(Z)}} = 1 + \int_0^t e^{Z_{s-}} \left(e^{\lambda H_s} - 1 \right) \mathbf{1}_{s \leq S_n(Z)\wedge T} d\left(N_s - \Lambda_s\right).$$

To conclude, the result follows from $[N-\Lambda] = N$ and the inequality

$$\mathbb{E}\left[\int_0^\infty e^{2Z_{s-}} \left(e^{\lambda H_s} - 1 \right)^2 \mathbf{1}_{s \leq S_n(Z)\wedge T} dN_s \right] \leq n^2 \left(e^{\lambda \|H\|_{\infty,T}} + 1 \right)^2 \mathbb{E}\left[N_T\right] < +\infty$$

since N is non-explosive (i.e. $\mathbb{E}[N_t] < +\infty$ for all $t \geq 0$). □

Proof of Lemma 4. Let $s \leq t \leq T$ and $\lambda \in \mathbb{R}$. We use the following inequality:

$$\begin{aligned} \left| e^{\lambda H_s} - 1 - \lambda H_s \right| &= \left| \sum_{j\geq 2} \frac{(\lambda H_s)^j}{j!} \right| \\ &= \left| \frac{(\lambda H_s)^2}{2!} + H_s^2 \sum_{j\geq 3} \frac{\lambda^j H_s^{j-2}}{j!} \right| \\ &\leq \frac{(|\lambda|\,|H_s|)^2}{2!} + |H_s|^2 \sum_{j\geq 3} \frac{|\lambda|^j |H_s|^{j-2}}{j!} \\ &\leq H_s^2 \left(\frac{\lambda^2}{2!} + \frac{1}{\|H\|^2_{\infty,T}} \sum_{j\geq 3} \frac{|\lambda|^j \|H\|^j_{\infty,T}}{j!} \right), \end{aligned} \tag{17}$$

that is

$$\left| e^{\lambda H_s} - 1 - \lambda H_s \right| \leq \frac{H_s^2}{\|H\|^2_{\infty,T}} \sum_{j\geq 2} \frac{|\lambda|^j \|H\|^j_{\infty,T}}{j!}.$$

Integrating with respect to $d\Lambda_s$ we obtain

$$|I_t(H,\lambda)| \leq \frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g\left(|\lambda| \|H\|_{\infty,T}\right)$$

where $g(x) = e^x - 1 - x$. For the proof of (5), consider the function h defined for $\lambda > 0$ by $h(\lambda) = Ag(B\lambda) - \lambda x$. Since $h'(\lambda) = AB(e^{B\lambda} - 1) - x$, we get that the minimum of h is reached for $\lambda = \frac{1}{B}\log(1 + \frac{x}{AB}) =: \lambda_0$ and $h(\lambda_0) = -AI(\frac{x}{AB})$ □

Proof of Theorem 5. Recall that $I_t(H,\lambda)$ is defined by $\int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$. We define the process Z as in Lemma 5 by $Z_t = \lambda M_t - I_t(H,\lambda)$ and the stopping time $S_n(Z)$ for $n \geq 1$ by $S_n(Z) = \inf\{t > 0, e^{Z_{t-}} \geq n\}$. Since $(S_n(Z))_{n\geq 1}$ is a non-decreasing sequence of stopping times with $\lim_{n\to+\infty} S_n(Z) = +\infty$ almost surely, the sequence $(\sup_{0\le t\le T\wedge S_n(Z)} M_t)_{n\ge1}$ is constant for n large enough. We then get by monotony

$$\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge x\right) = \lim_{n\to+\infty} \mathbb{P}\left(\sup_{0\le t\le T\wedge S_n(Z)} M_t \ge x\right) = \sup_{n\ge1} \mathbb{P}\left(\sup_{0\le t\le T} M_{t\wedge S_n(Z)} \ge x\right).$$

Using Lemma 4 (4), we obtain for all $\lambda > 0$, $x > 0$ and $n \ge 1$,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} M_{t\wedge S_n(Z)} \ge x\right) &= \mathbb{P}\left(\sup_{0\le t\le T} e^{\lambda M_{t\wedge S_n(Z)} - I_{t\wedge S_n(Z)}(H,\lambda) + I_{t\wedge S_n(Z)}(H,\lambda)} \ge e^{\lambda x}\right)\\ &\le \mathbb{P}\left(e^{\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda\|H\|_{\infty,T})} \sup_{0\le t\le T} e^{Z_{t\wedge S_n(Z)}} \ge e^{\lambda x}\right).\end{aligned}$$

Doob's maximal inequality and Lemma 1 then lead to

$$\mathbb{P}\left(\sup_{0\le t\le T} M_{t\wedge S_n(Z)} \ge x\right) \le \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g\left(\lambda\|H\|_{\infty,T}\right) - \lambda x\right)$$

for every $\lambda > 0$ with $g(x) = e^x - 1 - x$, so taking the limit in n and the infimum in λ, we get by (5)

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge x\right) &\le \inf_{\lambda>0} \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g\left(\lambda\|H\|_{\infty,T}\right) - \lambda x\right)\\ &= \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right)\end{aligned}$$

that is (6). Applying this inequality with $-H$ instead of H, we obtain also

$$\mathbb{P}\left(\sup_{0\le t\le T} -M_t \ge x\right) \le \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right).$$

Then (7) follows from the inequality

$$\mathbb{P}\left(\sup_{0\le t\le T} |M_t| \ge x\right) \le \mathbb{P}\left(\sup_{0\le t\le T} M_t \ge x\right) + \mathbb{P}\left(\sup_{0\le t\le T} -M_t \ge x\right)$$

□

Proof of Theorem 6. Let us begin with the proof of (8). We define $\widetilde{Z}$ as in Lemma 6 by $\widetilde{Z}_t = \lambda\widetilde{M}_t - I_t(2HM,\lambda)$, thereby $(S_n(\widetilde{Z}))_{n\ge1}$ is a sequence of non-decreasing stopping times such that $\lim_{n\to+\infty} S_n(\widetilde{Z}) = +\infty$ almost surely. We proceed then as in the proof of Theorem 1. For all positive λ, a and x

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} \widetilde{M}_t \ge x\right) &= \sup_{n\ge1} \mathbb{P}\left(\sup_{0\le t\le T} \widetilde{M}_{t\wedge S_n(\tilde{Z})} \ge x\right)\\ &\le \mathbb{P}(T_a < T) + \sup_{n\ge1} \mathbb{P}\left(\sup_{0\le t\le T} \widetilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})} \ge x \cap T_a = T\right) \qquad (18)\\ &\le \mathbb{P}\left(\sup_{0\le t\le T} |M_t| \ge a\right) + \sup_{n\ge1} \mathbb{P}\left(\sup_{0\le t\le T} e^{\lambda\tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \ge e^{\lambda x}\right).\end{aligned}$$

Using the inequality (17), we get for $t \le T$ and $\lambda > 0$

$$I_{t\wedge T_a\wedge S_n(\tilde{Z})}(2HM,\lambda) = \int_0^{t\wedge T_a\wedge S_n(\tilde{Z})} \left(e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s\right) d\Lambda_s \le \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g\left(2\lambda a\|H\|_{\infty,T}\right).$$

Then Lemma 2 and Doob's maximal inequality yield for every $\lambda > 0$ and $n \geq 1$

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} e^{\lambda \tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \geq e^{\lambda x}\right) &\leq \mathbb{P}\left(\sup_{0\le t\le T} e^{\tilde{Z}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \geq e^{\lambda x - \frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g(2\lambda a\|H\|_{\infty,T})}\right)\\ &\leq \exp\left(\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g\left(2\lambda a\|H\|_{\infty,T}\right) - \lambda x\right)\end{aligned}$$

whereby

$$\begin{aligned}\sup_{n\ge 1}\mathbb{P}\left(\sup_{0\le t\le T} e^{\lambda \tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \geq e^{\lambda x}\right) &\leq \inf_{\lambda>0} \exp\left(\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g\left(2\lambda a\|H\|_{\infty,T}\right) - \lambda x\right)\\ &= \exp\left(-\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} I\left(\frac{\|H\|_{\infty,T}}{2a\|H\|^2_{2,T}} x\right)\right)\end{aligned}$$

thanks to (5). Coming back to the inequality (18), Theorem 5 then entails for every $a > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T} \widetilde{M}_t \geq x\right) \leq 2e^{-\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} I\left(\frac{\|H\|_{\infty,T}}{\|H\|^2_{2,T}} a\right)} + e^{-\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} I\left(\frac{\|H\|_{\infty,T}}{2a\|H\|^2_{2,T}} x\right)}.$$

We choose $a = \sqrt{\frac{x}{2}}$ in order to obtain (8). For the proof of (9), we consider $\widetilde{Z}_t = -\lambda\widetilde{M}_t - I_t(2HM, -\lambda)$ for $\lambda > 0$. We get similarly, thanks to Lemma 4 and Lemma 2

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} e^{-\lambda \tilde{M}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \geq e^{\lambda x}\right) &\leq \mathbb{P}\left(\sup_{0\le t\le T} e^{\tilde{Z}_{t\wedge T_a\wedge S_n(\tilde{Z})}} \geq e^{\lambda x - \frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g(2\lambda a\|H\|_{\infty})}\right)\\ &\leq e^{\frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g(2\lambda a\|H\|_{\infty,T}) - \lambda x}\end{aligned}$$

and the end of the proof is similar to the one of (8). To conclude, (10) follows from the inequality

$$\mathbb{P}\left(\sup_{0\le t\le T} \left|\widetilde{M}_t\right| \geq x\right) \leq \mathbb{P}\left(\sup_{0\le t\le T} \widetilde{M}_t \geq x\right) + \mathbb{P}\left(\sup_{0\le t\le T} -\widetilde{M}_t \geq x\right) \qquad \square$$

Proof of Theorem 7. We follow the steps of the proof of Theorem 6, adapting the computations to this case. Let us begin showing the inequality (11). We introduce $\tilde{\tilde{Z}}$ as in Lemma 7 with $\tilde{\tilde{Z}}_t = \lambda\tilde{\tilde{M}}_t - I_t(H(H+2M), \lambda)$ and its associated sequence of stopping times $S_n(\tilde{\tilde{Z}})$ to obtain for all positive a, λ and x

$$\mathbb{P}\left(\sup_{0\le t\le T} \tilde{\tilde{M}}_t \geq x\right) \leq \mathbb{P}\left(\sup_{0\le t\le T} |M_t| \geq a\right) + \sup_{n\ge 1}\mathbb{P}\left(\sup_{0\le t\le T} \exp\left(\lambda \tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right) \geq e^{\lambda x}\right). \tag{19}$$

Using the inequality (17), we get for $t \leq T$ and $\lambda > 0$

$$\int_0^{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})} \left(e^{\lambda H_s(H_s+2M_s)} - 1 - \lambda H_s\,(H_s+2M_s)\right) d\Lambda_s \leq \frac{\|H\|^2_{2,T}}{\|H\|^2_{\infty,T}} g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T} + 2a\right)\right).$$

Then Lemma 3 and Doob's maximal inequality yield for every $\lambda > 0$ and $n \geq 1$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T}\exp\left(\lambda\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right)\ge e^{\lambda x}\right)\\
&\qquad\le \mathbb{P}\left(\sup_{0\le t\le T}\exp\left(\tilde{\tilde{Z}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right)\ge e^{\lambda x-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T}+2a\right)\right)}\right)\\
&\qquad\le \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T}+2a\right)\right)-\lambda x\right).
\end{aligned}$$

As a consequence

$$\begin{aligned}
\sup_{n\ge1}\mathbb{P}\left(\sup_{0\le t\le T}\exp\left(\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right)\ge e^{\lambda x}\right)&\le\inf_{\lambda>0}\exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T}+2a\right)\right)-\lambda x\right)\\
&=\exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2\left(2a+\|H\|_{\infty,T}\right)}x\right)\right)
\end{aligned}$$

thanks to (5). The inequality (19) and Theorem 5 then entail for every $a > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T}\tilde{\tilde{M}}_t\ge x\right)\le 2e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I\left(\frac{\|H\|_{\infty,T}}{\|H\|_2^2}a\right)}+e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2\left(2a+\|H\|_{\infty,T}\right)}x\right)}.$$

We choose

$$a=\frac{x}{2a+\|H\|_{\infty,T}}\ \text{ i.e. }\ a=\frac{-\|H\|_{\infty,T}+\sqrt{\|H\|_{\infty,T}^2+8x}}{4}$$

in order to get (11). For the proof of (12), let $\tilde{\tilde{Z}}$ be defined by $\tilde{\tilde{Z}}_t=-\lambda\tilde{\tilde{M}}_t-I_t(H(H+2M),-\lambda)$ for $\lambda>0$. We obtain similarly with Lemma 4 and Lemma 3

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T}\exp\left(-\lambda\tilde{\tilde{M}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right)\ge e^{\lambda x}\right)\\
&\qquad\le \mathbb{P}\left(\sup_{0\le t\le T}\exp\left(\tilde{\tilde{Z}}_{t\wedge T_a\wedge S_n(\tilde{\tilde{Z}})}\right)\ge e^{\lambda x-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T}+2a\right)\right)}\right)\\
&\qquad\le \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2}g\left(\lambda\|H\|_{\infty,T}\left(\|H\|_{\infty,T}+2a\right)\right)-\lambda x\right)
\end{aligned}$$

and the end of the proof is similar to the one of (11). To conclude, (13) also comes from the inequality

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|\tilde{\tilde{M}}_t\right|\ge x\right)\le\mathbb{P}\left(\sup_{0\le t\le T}\tilde{\tilde{M}}_t\ge x\right)+\mathbb{P}\left(\sup_{0\le t\le T}-\tilde{\tilde{M}}_t\ge x\right)\qquad\square$$

Proof of Theorem 8. Let us prove (14) first. We use the relationship $M_t-M_s=\int_c^t H_u(dN_u-\Lambda_u)-\int_c^s H_u(dN_u-\Lambda_u)$ to get

$$\begin{aligned}
\sup_{(s,t)\in[c,d]}|M_t-M_s|&\le 2\sup_{t\in[c,d]}\left|\int_c^t H_u\left(dN_u-\Lambda_u\right)\right|\\
&=2\sup_{t\in[0,d-c]}\left|\int_0^t H_{u+c}\left(dN_c(u)-d\Lambda_c(u)\right)\right|.
\end{aligned}$$

Since N_c satisfies the same assumptions than N, we may apply (7) with N_c, Λ_c and the process $u \mapsto H_{u+c}$ in order to obtain

$$\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2} |M_t - M_s| \geq x\right) \leq 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\frac{x}{2}\right)\right),$$

that is (14). Let us prove (15) now. We shall consider the following relationship

$$\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u = \widetilde{M}_c(t) - \widetilde{M}_c(s) - 2(M_t - M_s)\int_c^s H_u (N_u - \Lambda_u)$$

where

$$\begin{aligned}\widetilde{M}_c(t) &= \left(\int_c^t H_u d(N_u - \Lambda_u)\right)^2 - \int_c^t H_u^2 dN_u \\ &= \left(\int_0^{t-c} H_{u+c} d(N_c(u) - \Lambda_c(u))\right)^2 - \int_0^{t-c} H_{u+c}^2 dN_c(u).\end{aligned}$$

This yields for $a > 0$

$$\begin{aligned}&\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}\left|\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u\right| \geq x\right) \\ &\leq \mathbb{P}\left(2\sup_{t\in[c,d]}\left|\widetilde{M}_c(t)\right| \geq \frac{x}{2}\right) + \mathbb{P}\left(\sup_{s\in[c,d]}\left|\int_c^s H_u (N_u - \Lambda_u)\right| \geq a\right) + \mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}|M_t - M_s| \geq \frac{x}{4a}\right).\end{aligned}$$

We get then from (10), (7) and (14)

$$\mathbb{P}\left(2\sup_{t\in[c,d]}\left|\widetilde{M}_t^c\right| \geq \frac{x}{2}\right) \leq 6\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right),$$

$$\mathbb{P}\left(\sup_{s\in[c,d]}\left|\int_c^s H_u (N_u - \Lambda_u)\right| \geq a\right) \leq 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}} a\right)\right)$$

and

$$\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}|M_t - M_s| \geq \frac{x}{4a}\right) \leq 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\frac{x}{8a}\right)\right).$$

If we choose $a = \sqrt{\frac{x}{8}}$, we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}\left|\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u\right| \geq x\right) \\ &\qquad \leq 10\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right), \qquad (20)\end{aligned}$$

that is (15). To conclude with the oscillation modulus of $\widetilde{M}$, we may use similarly

$$\widetilde{M}_t - \widetilde{M}_s = \left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u + 2(M_t - M_s) M_s$$

and

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in[c,d]}\left|\widetilde{M}_t - \widetilde{M}_s\right| \geq x\right) &\leq \mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}\left|\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u\right| \geq \frac{x}{2}\right) \\ &\quad + \mathbb{P}\left(\sup_{s\in[0,d]}|M_s| \geq a\right) + \mathbb{P}\left(\sup_{(s,t)\in[c,d]^2}|M_t - M_s| \geq \frac{x}{4a}\right).\end{aligned}$$

Using (15), (7), (14) and choosing

$$a = \sqrt{\frac{x}{8} \frac{\|H\|_{\infty,[c,d]}}{\|H\|_{\infty,d}}} \frac{\|H\|_{2,d}}{\|H\|_{2,[c,d]}}$$

we get

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in[c,d]} \left|\widetilde{M}_t - \widetilde{M}_s\right| \geq x\right) \leq 10\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|^2_{2,[c,d]}}\sqrt{\frac{x}{16}}\right)\right)\\ + 2\exp\left(-\frac{\|H\|^2_{2,d}}{\|H\|^2_{\infty,d}} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right)\\ + 2\exp\left(-\frac{\|H\|^2_{2,[c,d]}}{\|H\|^2_{\infty,[c,d]}} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}}\sqrt{\frac{x}{8}}\right)\right) \qquad \square\end{aligned}$$

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