INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Salah El Ouadih and Radouan Daher

Lipschitz Conditions in Damek–Ricci Spaces

Volume 359, issue 6 (2021), p. 675-685

<https://doi.org/10.5802/crmath.211>

CENTRE MERSENNE

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org

Comptes Rendus
Mathématique
2021, 359, nº 6, p. 675-685
https://doi.org/10.5802/crmath.211

Harmonic analysis / *Analyse harmonique*

Lipschitz Conditions in Damek–Ricci Spaces

Salah El Ouadih [a] **and Radouan Daher** [b]

[a] Laboratory MC, Polydisciplinary Faculty of Safi, Cadi Ayyad University, Marrakech, Morocco

[b] Laboratory TAGMD, Faculty of Sciences Aïn Chock, Hassan II University, Casablanca, Morocco

E-mails: salahwadih@gmail.com, rjdaher024@gmail.com

Abstract. In this paper we extend classical Titchmarsh theorems on the Fourier–Helgason transform of Lipschitz functions to the setting of L^p-space on Damek–Ricci spaces. As consequences, quantitative Riemann–Lebesgue estimates are obtained and an integrability result for the Fourier–Helgason transform is developed extending ideas used by Titchmarsh in the one dimensional setting.

2020 Mathematics Subject Classification. 43A30, 42B10.

Manuscript received 19th January 2021, revised and accepted 7th April 2021.

1. Introduction

The studies of the convergence and of the rate of decay of Fourier transform/ coefficients are among the most classical problems in Fourier analysis. Starting from the Riemann–Lebesgue theorem relating the integrability of a function on the torus $\mathbb{T}^1$ and the convergence of its Fourier coefficients, through the Hausdorff–Young inequality relating the integrability of a function and of its Fourier transform. In this vein, Titchmarsh showed that the decay of Fourier transform can be improved for univariate functions satisfying a Lipschitz condition defined by smoothness. His result reads as follows.

Theorem 1 (cf. [26, Theorem 84]). *If f belongs to the Lipschitz class $\mathrm{Lip}(\eta, p)$ in the L^p norm on the real line, that is*

$$\omega_p(f,t) = \left\| f(\cdot + t) - f(\cdot)\right\|_p = O(|t|^{\eta}), \quad t \to 0,$$

then its Fourier transform $\widehat{f}$ belongs to $L^{\delta}(\mathbb{R})$ for

$$\frac{p}{p+\eta p-1} \le \delta \le \frac{p}{p-1}, \quad 0<\eta\le 1, 1<p\le 2.$$

He also proved in [26, Theorem 85] another reversible form in the L^2 case, namely:

Theorem 2. *Let $0<\eta\le 1$ and $f \in L^2(\mathbb{R})$. Then $f \in \mathrm{Lip}(\eta,2)$ if and only if*

$$\int_{|\lambda|\ge r} \left|\widehat{f}(\lambda)\right|^2 \mathrm{d}\lambda = O(r^{-2\eta}), \quad r \to \infty.$$

An extension of these theorems to functions of several variables on $\mathbb{R}^n$ and on the torus group $\mathbb{T}^n$ was studied by Younis [28, 29]. Later, analogous results were given, where considering generalized Fourier transforms: Bessel, Dunkl, Jacobi,... One can cite [8–10, 16]. On the other hand, Younis (in [30, Theorem 5.2]) recently has extended Titchmarsh results to functions on $\mathbb{R}^d$, replacing Lipschitz condition $|t|^\eta$ with Dini–Lipschitz condition $|t|^\eta \left(\log \frac{1}{|t|}\right)^{-\gamma}$. These were inspired from Weiss and Zygmund [27].

A continuous version was studied by Bray and Pinsky [5]. They proved the following estimate:

$$\left(\int_{\mathbb{R}} \min\left\{1, (\lambda t)^{2p'}\right\} \left|\widehat{f}(\lambda)\right|^{p'} \mathrm{d}\lambda\right)^{1/p'} \leq c_p \Omega_p(f, t), \tag{1}$$

where $1 < p \leq 2$, $p' = p/p-1$ and the modulus of smoothness $\Omega_p(f, t)$ of a function $f \in L^p(\mathbb{R})$ is defined by

$$\Omega_p(f, t) = \sup_{0<h<t} \left\| f(\cdot + h) + f(\cdot - h) - 2f(\cdot) \right\|_p.$$

The significance of this inequality stems from the presence of the minimum function that gives control over the Fourier transform for small and large λ. Indeed, for $1 < p \leq 2$, the inequality may be rewritten

$$\underbrace{\int_{|\lambda|\geq 1/t} \left|\widehat{f}(\lambda)\right|^{p'} \mathrm{d}\lambda}_{\text{large } \lambda} + \underbrace{t^{2p'} \int_{|\lambda|<1/t} \lambda^{2p'} \left|\widehat{f}(\lambda)\right|^{p'} \mathrm{d}\lambda}_{\text{small } \lambda} \leq c_p^{p'} \Omega_p^{p'}(f, t), \tag{2}$$

As shown in [5], the estimate for large λ yields a qualitative Riemann–Lebesgue lemma (i.e. a result of the type Titchmarsh Theorem 2, with Lipschitz or Dini–Lipschitz conditions). On the other hand, from the estimate for small λ, an integrability result can be achieved as done by Titchmarsh in Theorem 1 (see also [4, Theorem 3.4]).

In our present paper, we investigate among other things the validity of classical Titchmarsh theorems in case of functions of the wider Lipschitz and Dini–Lipschitz class in the context of Damek–Ricci spaces, also known as harmonic NA groups. This generalizes the corresponding result for noncompact rank one symmetric spaces (see [14]). Our current interest in this theme stems from a result of Kumar and al. [20] which is based on the work of Bray and Pinsky [5].

2. Preliminaries on Damek–Ricci spaces

A Damek–Ricci space is a one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type. It is a solvable Lie group with a left invariant metric, and is a Riemannian manifolds which includes all rank-one symmetric spaces of the noncompact type; except from these, Damek–Ricci spaces are harmonic manifolds in general non symetric [12]. One of the interesting features of these spaces is that the radial analysis on these spaces behaves similar to the hyperbolic spaces as observed in [1] and therefore it fits into the perfect setting of Jacobi analysis developed by Flensted-Jensen and Koornwinder [18, 19].

In this section, we will explain the notation and gather relevant results on Damek–Ricci spaces. Most of these results can be found in [1, 7, 12, 25]. Relevant results for the spherical and Fourier transforms on these spaces can be found in [1–3].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$ and $\mathfrak{a}$ its orthogonal complement. We say that $\mathfrak{n}$ is an H-type algebra if for every $Z \in \mathfrak{z}$ the map $J_Z : \mathfrak{a} \to \mathfrak{a}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle, \quad X, Y \in \mathfrak{a},$$

satisfies the condition $J_Z^2 = -\|Z\|^2 I_{\mathfrak{a}}$, $I_{\mathfrak{a}}$ being the identity operator on $\mathfrak{a}$. A connected and simply connected Lie group N is called an H-type group if its Lie algebra is H-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a dffeomorphism and hence we can parametrize the elements in $N = \exp \mathfrak{n}$ by (X, Y), for $X \in \mathfrak{a}$, $Z \in \mathfrak{z}$. It follows from the Campbell–Baker–Hausdorff formula that the group law in N is given by

$$(X,Z)(X',Z') = \left(X + X', Z + Z' + \frac{1}{2}[X,X']\right), \quad X, X' \in \mathfrak{a}, \quad Z, Z' \in \mathfrak{z}.$$

The group $A = \mathbb{R}_+^*$ acts on an H-type group N by nonisotropic dilation: $(X, Y) \mapsto (a^{\frac{1}{2}} X, aZ)$. Let $S = NA$ be the semidirect product of N and A under the above action. Thus the multiplication in S is given by

$$(X,Z,a)(X',Z',a') = \left(X + a^{\frac{1}{2}} X', Z + aZ' + \frac{1}{2} a^{\frac{1}{2}} [X,X'], aa'\right),$$

for $X, X' \in \mathfrak{a}$, $Z, Z' \in \mathfrak{z}$, $a, a' \in \mathbb{R}_+^*$. Then S is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket

$$[(X,Z,k),(X',Z',k')] = \left(\frac{1}{2}kX' - \frac{1}{2}k'X, kZ' - k'Z + [X,X'], 0\right).$$

We suppose $\dim \mathfrak{a} = m$ and $\dim \mathfrak{z} = l$. Then $Q = \frac{m}{2} + l$ is called the homogenous dimension of S. For convenience we will use the symbol ρ for $\frac{Q}{2}$ and d for $m + l + 1 = \dim(\mathfrak{s})$.

The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X,Z,k),(X',Z',k')\rangle = \langle X, X'\rangle + \langle Z, Z'\rangle + kk'$$

on $\mathfrak{s}$. The associated left Haar measure on S is given by $a^{-Q-1}\mathrm{d}X\mathrm{d}Z\mathrm{d}a$, where $\mathrm{d}X$, $\mathrm{d}Z$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathfrak{a}$, $\mathfrak{z}$ and $\mathbb{R}_+^*$ respectively.

To define the Fourier–Helgason transform on S we need to introduce the notion of Poisson kernel [2]. The Poisson kernel $\mathscr{P} : S \times N \to \mathbb{R}$ is given by

$$\mathscr{P}(na_t, n') = P_{a_t}(n'^{-1}n),$$

where

$$P_{a_t}(n) = P_{a_t}(X,Z) = C a_t^Q \left(\left(a_t + \frac{|X|^2}{4}\right)^2 + |Z|^2\right)^{-Q},$$

and $a_t = e^t$, $t \in \mathbb{R}$; $n = (X, Z) \in N$. The value of C is suitably adjusted so that $\int_N P_a(n)\mathrm{d}n = 1$ and $P_1(n) \le 1$. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined by

$$\mathscr{P}_\lambda(x,n) = \mathscr{P}(x,n)^{\frac{1}{2} - \frac{i\lambda}{Q}}.$$

It is known [2, 24] that for each fixed $x \in S$, $\mathscr{P}_\lambda(x, .) \in L^p(N)$ for $1 \le p \le \infty$ if $\lambda = i\gamma_p\rho$, where $\gamma_p = \frac{2}{p} - 1$. A very special feature of $\mathscr{P}_\lambda(x,n)$ is that it is constant on the hypersurfaces $H_{n,a_t} = \{n\sigma(a_t n') : n' \in N\}$, where σ stands for the geodesic inversion [24].

Let Δ_S be the Laplace–Beltrami operator on S. Then for every fixed $n \in N$, $\mathscr{P}_\lambda(x,n)$ is an eigenfunction of Δ_S with eigenvalue $-\left(\lambda^2 + \frac{Q^2}{4}\right)$ (see [2]). For a measurable function f on S, the Fourier–Helgason transform is defined as

$$\widetilde{f}(\lambda,n) = \int_S f(x)\mathscr{P}_\lambda(x,n)\mathrm{d}x,$$

whenever the integral converge.

It is known that for $f \in C_c^\infty(S)$ the following Fourier inversion and the Plancherel formula holds [2]:

(1) For $f \in C_c^\infty(S)$,

$$f(x) = C\int_{\mathbb{R}}\int_N \widetilde{f}(\lambda,n)\mathscr{P}_{-\lambda}(x,n)\,|c(\lambda)|^{-2}\,\mathrm{d}\lambda\mathrm{d}n, \quad \forall\, x \in S,$$

where

$$c(\lambda)=\frac{2^{Q-2i\lambda}\Gamma(2i\lambda)\Gamma\left(\frac{2m+l+1}{2}\right)}{\Gamma\left(\frac{Q}{2}+i\lambda\right)\Gamma\left(\frac{m+1}{2}+i\lambda\right)}.$$

(2) The Fourier transform extends from $C_c^\infty(S)$ to an isometry from $L^2(S)$ onto the space $L^2\left(\mathbb{R}_+\times N, C|c(\lambda)|^{-2}\,\mathrm{d}\lambda\mathrm{d}n\right)$.

The precise value of the constants C are given in [2]. The following estimates for the function $|c(\lambda)|$ holds:

$$c'\,|\lambda|^{d-1}\le|c(\lambda)|^{-2}\le(1+|\lambda|)^{d-1}, \tag{3}$$

for all $\lambda\in\mathbb{R}$ (e. g. see [24]).

A function f on S is called radial if for all $x,y\in S$, $f(x)=f(y)$ if $\mu(x,e)=\mu(y,e)$ where μ is the metric induced by the canonical left invariant Riemannian structure on S and e is the identity element of S. Note that radial functions on S can be identified with the functions $f=f(r)$ of the geodesic distance $r=\mu(x,e)\in[0,\infty)$ to the identity. It is clear that $\mu(a_t,e)=|t|$ for $t\in\mathbb{R}$. At times, for any radial function f we use the notation $f(a_t)=f(t)$. For any function space $\mathscr{F}(S)$ on S, the subspace of radial functions will be denoted by $\mathscr{F}(S)^\sharp$. The elementary spherical function $\phi_\lambda(x)$ is defined by

$$\phi_\lambda(x):=\int_N\mathscr{P}_\lambda(x,n)\mathscr{P}_{-\lambda}(x,n)\mathrm{d}n.$$

It follows [1, 2] that ϕ_λ is a radial eigenfunction of the Laplace–Beltrami operator Δ_S of S with eigenvalue $-\left(\lambda^2+\frac{Q^2}{2}\right)$ such that $\phi_\lambda(x)=\phi_{-\lambda}(x)$, $\phi_\lambda(x)=\phi_\lambda(x^{-1})$ and $\phi_\lambda(e)=1$. It is also evident from the fact that, for every fixed $n\in N$, $\mathscr{P}_\lambda(x,n)$ is an eigenfunction of Δ_S with eigenvalue $-\left(\lambda^2+\frac{Q^2}{2}\right)$, that, for suitable function f on S, we have

$$\widetilde{\Delta_S^l f}(\lambda,n)=-\left(\lambda^2+\frac{Q^2}{2}\right)^l\widetilde{f}(\lambda,n),$$

for every natural number l (cf. [2, p. 416]). In [1], the authors showed that the radial part (in geodesic polar coordinates) of the Laplace–Beltrami operator Δ_S given by

$$\operatorname{rad}\Delta_S=\frac{\partial^2}{\partial t}+\left(\frac{m+l}{2}\coth\frac{t}{2}+\frac{l}{2}\tanh\frac{t}{2}\right)\frac{\partial}{\partial t},$$

is (by substituting $r=\frac{t}{2}$) equal to $\frac14\mathscr{L}_{\alpha,\beta}$ with indices $\alpha=\frac{m+l+2}{2}$ and $\beta=\frac{l-1}{2}$, where $\mathscr{L}_{\alpha,\beta}$ is the Jacobi operator studied by Koornwinder [19] in detail. It is worth noting that we are in the ideal situation of Jacobi analysis with $\alpha>\beta>-\frac12$. In fact, the Jacobi functions $\phi_\lambda^{\alpha,\beta}$ and elementary spherical functions ϕ_λ are related as [1]: $\phi_\lambda(t)=\phi_{2\lambda}^{\alpha,\beta}(\frac{t}{2})$. As consequence of this relation, the following estimates for the elementary spherical functions hold true:

Lemma 3 (cf. [22]). *The following inequalities are valid for the spherical functions* $\phi_\lambda(t)$ $(\lambda,t\in\mathbb{R}_+)$

(i) $\left|\phi_\lambda(t)\right|\le1$.

(ii) $\left|1-\phi_\lambda(t)\right|\le\frac{t^2}{2}\left(\lambda^2+\frac{Q^2}{4}\right)$.

(iii) *There exists a constant* $c>0$*, depending only on* λ*, such that*

$$\left|1-\phi_\lambda(t)\right|\ge c,$$

for $\lambda t\ge1$.

Lemma 4 (cf. [6]). *Let* $\alpha>-1/2$, $-1/2\le\beta\le\alpha$*, and let* $0<\gamma_0<\rho$*, there exists a positive constant* $c_1=C(\alpha,\beta,\rho)$ *such that*

$$|1-\phi_{\lambda+i\gamma}(t)|\ge c_1\min\left\{1,(\lambda t)^2\right\},$$

for all $\left|\gamma\right|\le\gamma_0$, $\lambda\in\mathbb{R}$*, and* $t>0$.

Let σ_t be the normalized surface measure of the geodesic sphere of radius t. Then σ_t is a nonnegative radial measure. The spherical mean operator M_t on a suitable function space on S is defined by $M_t f := f * \sigma_t$. It can be noted that $M_t f(x) = \mathscr{R}(f^x)(t)$, where f^x denotes the right translation of function f by x and $\mathscr{R}$ is the radialization operator defined, for suitable function f, by

$$\mathscr{R} f(x) = \int_{S_\nu} f(y)\,\mathrm{d}\sigma_\nu(y),$$

where $\nu = r(x) = \mu(C(x), 0)$, here C is the Cayley transform, and $\mathrm{d}\sigma_\nu$ is the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S : \mu(y, e) = \nu\}$. It is easy to see that $\mathscr{R} f$ is a radial function and for any radial function f, $\mathscr{R} f = f$. Consequently, for a radial function f, $M_t f$ is the usual translation of f by t. In [20], the authors proved that, for a suitable function f on S, $\widetilde{M_t f}(\lambda, n) = \phi_\lambda(a_t)\widetilde{f}(\lambda, n)$ whenever both make sense. Also, $M_t f$ converges to f as $t \to 0$ i.e., $\mu(a_t, e) \to 0$. It is also known that M_t is a bounded operator on $L^2(S)$ with operator norm equal to $\phi_0(t)$. In particular, for $f \in L^2(S)$, we have $\|M_t f\|_2 \le \phi_0(t)\|f\|_2$. In [20, Theorem 4], the authors proved the following inequality: For $1 < p \le 2$, $p \le q \le p' = p/(p-1)$ and $f \in L^p(S)$ we have

$$\int_{\mathbb{R}} \min\left\{1, (\lambda t)^{2p'}\right\} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q \rho, n)|^q \mathrm{d}n\right)^{p'/q} \mathrm{d}\mu(\lambda) \le C_{p,q}^{p'} \|M_t f - f\|_p^{p'}, \tag{4}$$

where $\mathrm{d}\mu(\lambda) = |c(\lambda)|^{-2}\mathrm{d}\lambda$.

3. Lipschitz conditions in Damek–Ricci spaces

In this section, we give the main result of the paper but first we need to define the Lipschitz class.

Definition 5. *Let* $0 < \eta \le 1$. *A function* $f \in L^p(S)$ *is said to be in the Damek–Ricci–Lipschitz class, denoted by* $\mathrm{Lip}(\eta, p)$, *if it satisfies*

$$\|M_t f - f\|_p = O(|t|^\eta), \quad t \to 0.$$

The following Theorem represents a quantified Riemann–Lebesgue lemma (item (1)), and is an extension of results in one dimension given in Titchmarsh [26].

Theorem 6. *Let* $1 < p \le 2$ *and* $p' = p/(p-1)$.

(1) *If* $f \in \mathrm{Lip}(\eta, p)$, $0 < \eta \le 1$, *then*

$$\int_{|\lambda| \ge r} \int_N \left|\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)\right|^{p'} \mathrm{d}n\mathrm{d}\lambda = O\left(r^{-p'\eta - d + 1}\right), \quad as \quad r \to \infty;$$

(2) *when* $p = 2$ *and* $0 < \eta < 1$, *the converse statement holds as well.*

Proof. (1). The proof of this result is immediate from the estimate (4). Indeed, for $q = p'$ we obtain,

$$\int_{|\lambda| \ge 1/t} \int_N \left|\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)\right|^{p'} \mathrm{d}n\mathrm{d}\mu(\lambda) \le C_{p,p'}^{p'} \|M_t f - f\|_p^{p'},$$

then

$$\int_{|\lambda| \ge 1/t} \int_N \left|\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)\right|^{p'} \mathrm{d}n\mathrm{d}\mu(\lambda) = O(|t|^{p'\eta}),$$

and by $|c(\lambda)|^{-2} \asymp |\lambda|^{d-1}$, we get

$$\int_{|\lambda| \ge r} \int_N \left|\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)\right|^{p'} \mathrm{d}n\mathrm{d}\lambda = O\left(r^{-p'\eta - d + 1}\right), \quad as \quad r \to \infty.$$

(2). For the converse, when $p = 2$, the same proof presented in [14] for noncompact rank one symmetric spaces can be rewritten with minor adjustments as follows. Suppose that

$$\int_{|\lambda|\geq r}\int_N \left|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)\right|^2 \mathrm{d}n\mathrm{d}\lambda = O\left(r^{-2\eta-d+1}\right), \quad as \quad r\to\infty,$$

and

$$F(\lambda)=\int_N \left|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)\right|^2 \mathrm{d}n.$$

Then, we have

$$\begin{aligned}\int_{r\leq|\lambda|\leq 2r} F(\lambda)|\lambda|^{d-1}\mathrm{d}\lambda &\leq (2r)^{d-1}\int_{r\leq|\lambda|\leq 2r} F(\lambda)\mathrm{d}\lambda\\ &\leq 2^{d-1}r^{d-1}\int_{|\lambda|\geq r} F(\lambda)\mathrm{d}\lambda\\ &\leq c_2 r^{-2\eta}.\end{aligned}$$

Now,

$$\begin{aligned}\int_{|\lambda|\geq r} F(\lambda)|\lambda|^{d-1}\mathrm{d}\lambda &= \sum_{k=0}^{\infty}\int_{2^k r\leq|\lambda|\leq 2^{k+1}r} F(\lambda)|\lambda|^{d-1}\mathrm{d}\lambda\\ &\leq c_2\sum_{k=0}^{\infty} 2^{-2k\eta}r^{-2\eta}.\end{aligned}$$

Consequently,

$$\int_{|\lambda|\geq r} F(\lambda)|\lambda|^{d-1}\mathrm{d}\lambda = O\left(r^{-2\eta}\right),$$

and, by $|c(\lambda)|^{-2}\asymp|\lambda|^{d-1}$,

$$\int_{|\lambda|\geq r} F(\lambda)\mathrm{d}\mu(\lambda) = O\left(r^{-2\eta}\right).$$

According to the Plancherel formula, one has $\|M_t f - f\|_2^2 = I_1 + I_2$, where

$$I_1=\int_0^{1/t}|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)|^2F(\lambda)\mathrm{d}\mu(\lambda) \quad\text{and}\quad I_2=\int_{\frac{1}{t}}^{+\infty}|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)|^2F(\lambda)\mathrm{d}\mu(\lambda),$$

estimate the summands I_1 and I_2 from above. Firstly, it follows from the inequality $|\phi_{\lambda+i\gamma_{p'}\rho}(a_t)|\leq 1$ that

$$I_2\leq 4\int_{\frac{1}{t}}^{+\infty}F(\lambda)\mathrm{d}\mu(\lambda)=O\left(t^{2\eta}\right), \quad as \quad t\to 0.$$

To estimate I_1, we use the inequalities (i) and (ii) of Lemma 3

$$\begin{aligned}I_1 &= \int_0^{1/t}\left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right||1-\phi_\lambda(a_t)|F(\lambda)\mathrm{d}\mu(\lambda)\\ &\leq 2\int_0^{1/t}\left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right|F(\lambda)\mathrm{d}\mu(\lambda)\\ &\leq t^2\int_0^{1/t}\left(\lambda^2+\frac{Q^2}{2}\right)F(\lambda)\mathrm{d}\mu(\lambda).\end{aligned}$$

Consider the function $\varphi(r) = \int_r^\infty F(\lambda)\mathrm{d}\mu(\lambda)$. An integration by parts gives:

$$\begin{aligned}\int_0^{1/t}\left(\lambda^2+\frac{Q^2}{2}\right)F(\lambda)\mathrm{d}\mu(\lambda) &= \int_0^{1/t} -\left(r^2+\frac{Q^2}{2}\right)\varphi'(r)\mathrm{d}r\\ &\le \int_0^{1/t} -r^2\varphi'(r)\mathrm{d}r\\ &= -\frac{1}{t^2}\varphi\left(\frac{1}{t}\right)+2\int_0^{1/t} r\varphi(r)\mathrm{d}r\\ &\le 2\int_0^{1/t} r\varphi(r)\mathrm{d}r.\end{aligned}$$

Since $\varphi(r) = O\left(r^{-2\eta}\right)$, we have $r\varphi(r) = O\left(r^{1-2\eta}\right)$ and

$$\int_0^{1/t} r\varphi(r)\mathrm{d}r = O\left(\int_0^{1/t} r^{1-2\eta}\mathrm{d}r\right) = O\left(t^{2\eta-2}\right), \text{(the integral exists since } \eta<1\text{)},$$

so that $I_1 = O\left(t^{2\eta}\right)$. Combining the estimates for I_1 and I_2 gives

$$\|M_t f - f\|_2 = O\left(t^\eta\right) \quad \text{as} \quad t\to 0,$$

and this ends the proof of the theorem. □

For $f\in L^p(S)$, we define the finite differences of first and higher order as follows:

$$\begin{aligned}&\Delta_t^1 f = \Delta_t f = (I-M_t)f,\\ &\Delta_t^k f = \Delta_t(\Delta_t^{k-1} f) = (I-M_t)^k f, \quad k=2,3,\dots,\end{aligned}$$

where I is the unit operator in the space $L^p(S)$.

Consequently, for each $f\in L^p(S)$,

$$\widetilde{\Delta_t^k f}(\lambda+i\gamma_{p'}\rho,n) = (1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t))^k\widetilde{f}(\lambda+i\gamma_{p'}\rho,n),$$

and, by Plancherel formula, we have

$$\left\|\Delta_t^k f\right\|_2^2 = \int_0^{+\infty}\int_N \left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right|^{2k}|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)|^2\,|c(\lambda)|^{-2}\,\mathrm{d}\lambda\mathrm{d}n, \tag{5}$$

By analogy with the proof of Theorem 6, we can establish from formula (5) the following result:

Theorem 7. *Let $1<p\le 2$ and $p' = p/(p-1)$.*

(1) *If $\left\|\Delta_t^k f\right\|_2 = O(|t|^\eta)$, $0<\eta\le 1$, then*

$$\int_{|\lambda|\ge r}\int_N \left|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)\right|^{p'}\mathrm{d}n\mathrm{d}\lambda = O\left(r^{-p'\eta-d+1}\right), \quad \textit{as} \quad r\to\infty;$$

(2) *when $p=2$, $0<\eta<1$ and $k=1,2,\dots$, the converse statement holds as well.*

We now state our second main result which extends the integrability Theorem 1 to Damek–Ricci spaces.

Theorem 8. *Let $1<p\le 2$, $p' = p/(p-1)$, $0<\eta\le 1$ and $f\in \mathrm{Lip}(\eta,p)$. Then its transform $\widetilde{f}(\cdot+i\gamma_{p'}\rho,\cdot)$ is in $L^\delta(\mathbb{R}\times N)$ with respect to the Plancherel measure $\mathrm{d}n\mathrm{d}\mu(\lambda)$ for every δ,*

$$\frac{pd}{d(p-1)+p\eta} < \delta \le p'.$$

Proof. Using formula (4), we see that

$$\int_{|\lambda|\le 1/t}\lambda^{2p'}G(\lambda)\mathrm{d}\mu(\lambda) = O\left(|t|^{(\eta-2)p'}\right), \tag{6}$$

where

$$G(\lambda) = \int_N \left|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)\right|^{p'}\mathrm{d}n.$$

Now, let
$$\varphi(X)=\int_{1\le|\lambda|\le X}\lambda^{2\delta}G(\lambda)\mathrm{d}\mu(\lambda).$$
Applying Hölder's inequality with $\delta\le p'$ for the last estimate one arrives at
$$\varphi(X)\le\left(\int_{1\le|\lambda|\le X}\lambda^{2p'}G(\lambda)\mathrm{d}\mu(\lambda)\right)^{\frac{\delta}{p'}}\left(\int_{1\le|\lambda|\le X}1\mathrm{d}\mu(\lambda)\right)^{1-\frac{\delta}{p'}}.$$
Hence, by using relations (3) and (6), we obtain
$$\varphi(X)=O\left(X^{(2-\eta)\delta+d\left(1-\frac{\delta}{p'}\right)}\right). \tag{7}$$
Remark that
$$\int_{1\le|\lambda|\le X}G(\lambda)\mathrm{d}\mu(\lambda)=\int_{1\le|\lambda|\le X}\lambda^{-2\delta}\varphi'(\lambda)\mathrm{d}\lambda.$$
Making an integration by parts, we get
$$\int_{1\le|\lambda|\le X}G(\lambda)\mathrm{d}\mu(\lambda)=X^{-2\delta}\varphi(X)+2\delta\int_{1\le|\lambda|\le X}t^{-2\delta-1}\varphi(t)\mathrm{d}t, \tag{8}$$
From relation (7), we have
$$\int_{1\le|\lambda|\le X}G(\lambda)\mathrm{d}\mu(\lambda)=O\left(X^{-2\delta+(2-\eta)\delta+d\left(1-\frac{\delta}{p'}\right)}\right)+O\left(\int_{1\le|\lambda|\le X}t^{-2\delta-1+(2-\eta)\delta+d\left(1-\frac{\delta}{p'}\right)}\mathrm{d}t\right),$$
and this is bounded as $X\to\infty$ if $-\delta\left(\eta+\frac{d}{p'}\right)+d<0$, which gives
$$\delta>\frac{pd}{d(p-1)+p\eta}. \qquad \square$$

4. Dini–Lipschitz conditions in Damek–Ricci spaces

The reader can find analogous results of this section in the references [11, 13–15, 17, 21, 23].

Definition 9. *Let* $0<\eta\le 1$ *and* $\gamma\ge 0$. *A function* $f\in L^p(S)$ *is said to be in the Damek–Ricci–Dini–Lipschitz class, denoted by* $\mathrm{DLip}(\eta,\gamma,p)$, *if*
$$\|M_tf-f\|_p=O\left(|t|^{\eta}\left(\log\frac{1}{|t|}\right)^{-\gamma}\right)\quad \textit{as}\quad |t|\to 0.$$

By using the same tricks of calculation that we have already used to show the previous theorems, we prove the following theorems.

Theorem 10. *Let* $1<p\le 2$ *and* $p'=p/(p-1)$.

(1) *If* $f\in\mathrm{DLip}(\eta,\gamma,p)$, $0<\eta\le 1$, $\gamma\ge 0$, *then*
$$\int_{|\lambda|\ge r}\int_N|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)|^{p'}\mathrm{d}n\mathrm{d}\lambda=O\left(r^{-p'\eta-d+1}(\log r)^{-p'\gamma}\right),\quad \textit{as}\quad r\to\infty;$$
(2) *when* $p=2$, $\gamma\ge 0$ *and* $0<\eta<1$, *the converse statement holds as well.*

Proof. By proceeding similarly to Theorem 3.2, item (1), we have,
$$\int_{|\lambda|\ge 1/t}\int_N|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)|^{p'}\mathrm{d}n\mathrm{d}\mu(\lambda)=O\left(|t|^{p'\eta}\left(\log\frac{1}{|t|}\right)^{-p'\gamma}\right),$$
Thus,
$$\int_{|\lambda|\ge r}\int_N|\widetilde{f}(\lambda+i\gamma_{p'}\rho,n)|^{p'}\mathrm{d}n\mathrm{d}\lambda=O\left(r^{-p'\eta-d+1}(\log r)^{-p'\gamma}\right),\quad \textit{as}\quad r\to\infty.$$

The converse can be done in the same way as in Theorem 6 above and Theorem 8 in [14] for noncompact rank one symmetric spaces. Consider the same notation $\|M_t f - f\|_2^2 = I_1 + I_2$ and $\varphi(r) = O\left(r^{-2\eta}\left(\log r\right)^{-2\gamma}\right)$. Then, we get

$$I_2 = O\left(t^{2\eta}\left(\log\frac{1}{t}\right)^{-2\gamma}\right), \quad as \quad t \to 0,$$

and,

$$I_1 = O\left(t^2\int_0^{1/t} r\varphi(r)\mathrm{d}r\right) = O\left(t^{2\eta}\left(\log\frac{1}{t}\right)^{-2\gamma}\right), \quad as \quad t \to 0, \qquad \square$$

Theorem 11. *Let* $1 < p \le 2$, $p' = p/(p-1)$, $0 < \eta \le 1$, $\gamma \ge 0$ *and* $f \in \mathrm{DLip}(\eta,\gamma,p)$. *Then its transform* $\widetilde{f}(\cdot + i\gamma_{p'}\rho, \cdot)$ *is in* $L^\delta(\mathbb{R}\times N)$ *with respect to the Plancherel measure* $\mathrm{d}n\mathrm{d}\mu(\lambda)$ *for every* δ,

$$\frac{pd}{d(p-1)+p\eta} < \delta \le p'.$$

Proof. As in Theorem 8, we have

$$\int_{|\lambda|\le 1/t} \lambda^{2p'} G(\lambda)\mathrm{d}\mu(\lambda) = O\left(|t|^{(\eta-2)p'}\left(\log\frac{1}{|t|}\right)^{-p'\gamma}\right).$$

For $\delta \le p'$, this implies via Hölder's inequality

$$\varphi(X) = O\left(X^{(2-\eta)\delta + d\left(1-\frac{\delta}{p'}\right)}\left(\log X\right)^{-\delta\gamma}\right).$$

This allows us to deduce, by relation (8), that

$$\int_{1\le|\lambda|\le X} G(\lambda)\mathrm{d}\mu(\lambda) = O\left(X^{-\eta\delta + d\left(1-\frac{\delta}{p'}\right)}\left(\log X\right)^{-\delta\gamma}\right) + O\left(\int_{1\le|\lambda|\le X} t^{-1-\eta\delta + d\left(1-\frac{\delta}{p'}\right)}\left(\log t\right)^{-\delta\gamma}\mathrm{d}t\right).$$

For the right hand side of the last estimate to be bounded as X goes to ∞ we must have $-\delta\left(\eta + \frac{d}{p'}\right) + d < 0$, which gives $\delta > \frac{pd}{d(p-1)+p\eta}$. $\square$

This section concludes with the following result:

Theorem 12. *Let* $\eta > 2$, $\gamma \ge 0$ *and* $f \in \mathrm{DLip}(\eta,\gamma,2)$, *then* $f = 0$ *a.e.*

Proof. Assume that $f \in \mathrm{DLip}(\eta,\gamma,2)$. Then

$$\|M_t f - f\|_2 \le c_3|t|^\eta\left(\log\frac{1}{|t|}\right)^{-\gamma}.$$

In view of formula (5), we conclude that

$$\int_0^{+\infty} \left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right|^2 F(\lambda)\mathrm{d}\mu(\lambda) \le c_3^2|t|^{2\eta}\left(\log\frac{1}{|t|}\right)^{-2\gamma}.$$

Thus,

$$\frac{\int_0^{+\infty} \left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right|^2 F(\lambda)\mathrm{d}\mu(\lambda)}{|t|^4} \le c_3^2|t|^{2\eta-4}\left(\log\frac{1}{|t|}\right)^{-2\gamma}.$$

Since $\eta > 2$, then

$$\lim_{t\to 0}|t|^{2\eta-4}\left(\log\frac{1}{|t|}\right)^{-2\gamma} = 0.$$

Hence,

$$\lim_{t\to 0}\int_0^{+\infty}\left(\frac{\left|1-\phi_{\lambda+i\gamma_{p'}\rho}(a_t)\right|}{\lambda^2 t^2}\right)^2 \lambda^4 F(\lambda)\mathrm{d}\mu(\lambda) = 0,$$

and also from Lemma 4 and Fatou theorem, we obtain

$$\left\|\lambda^2\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)\right\|_{L^2(\mathbb{R}_+\times N)} = 0.$$

Thereby for all $(\lambda, n) \in \mathbb{R}_+ \times N$, $\lambda^2 \widetilde{f}(\lambda + i\gamma_{p'}\rho, n) = 0$. The injectivity of the Fourier–Helgason transform yields to the wanted result. □

Acknowledgements

The authors would like to thank the referee for several valuable suggestions and criticisms which improved the paper.

References

[1] J.-P. Anker, E. Damek, C. Yacoub, "Spherical analysis on harmonic AN groups", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **23** (1996), no. 4, p. 643-679.

[2] F. Astengo, R. Camporesi, B. Di Blasio, "The Helgason Fourier transform on a class of nonsymmetric harmonic spaces", *Bull. Aust. Math. Soc.* **55** (1997), no. 3, p. 405-424.

[3] F. Astengo, B. Di Blasio, "A Paley–Wiener theorem on NA harmonic spaces", *Colloq. Math.* **80** (1999), no. 2, p. 211-233.

[4] W. O. Bray, "Growth and integrability of Fourier transforms on Euclidean space", *J. Fourier Anal. Appl.* **20** (2014), no. 6, p. 1234-1256.

[5] W. O. Bray, M. A. Pinsky, "Growth properties of Fourier transforms via moduli of continuity", *J. Funct. Anal.* **255** (2008), no. 9, p. 2265-2285.

[6] ———, "Growth properties of the Fourier transform", *Filomat* **26** (2012), no. 4, p. 755-760.

[7] M. Cowling, A. Dooley, A. Korányi, F. Ricci, "An approach to symmetric spaces of rank one via groups of Heisenberg type", *J. Geom. Anal.* **8** (1998), no. 2, p. 199-237.

[8] R. Daher, J. Delgado, M. Ruzhansky, "Titchmarsh theorems for Fourier transforms of Hölder-Lipschitz functions on compact homogeneous manifolds", *Monatsh. Math.* **189** (2019), no. 1, p. 23-49.

[9] R. Daher, M. El Hamma, "An analog of Titchmarsh's theorem for the generalized Dunkl transform", *J. Pseudo-Differ. Oper. Appl.* **7** (2016), no. 1, p. 59-65.

[10] R. Daher, M. El Hamma, S. El Ouadih, "An analog of Titchmarsh's theorem for the generalized Fourier-Bessel Transform", *Lobachevskii J. Math.* **37** (2016), no. 2, p. 114-119.

[11] R. Daher, S. El Ouadih, "Best trigonometric approximation and Dini-Lipschitz classes", *J. Pseudo-Differ. Oper. Appl.* **9** (2018), no. 4, p. 903-912.

[12] E. Damek, F. Ricci, "Harmonic analysis on solvable extensions of H-type groups", *J. Geom. Anal.* **2** (1992), no. 3, p. 213-248.

[13] M. El Hamma, R. Daher, "Dini Lipschitz functions for the Dunkl transform in the space $L^2(\mathbb{R}^d, w_k(x)dx)$", *Rend. Circ. Mat. Palermo* **64** (2015), no. 2, p. 241-249.

[14] S. El Ouadih, R. Daher, "Characterization of Dini-Lipschitz functions for the Helgason Fourier transform on rank one symmetric spaces", *Adv. Pure Appl. Math.* **7** (2016), no. 4, p. 223-230.

[15] ———, "Jacobi–Dunkl Dini Lipschitz functions in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$", *Appl. Math. E-Notes* **16** (2016), p. 88-98.

[16] ———, "Lipschitz conditions for the generalized discrete Fourier transform associated with the Jacobi operator on $[0,\pi]$", *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 3, p. 318-324.

[17] S. Fahlaoui, M. Boujeddaine, M. El Kassimi, "Fourier transforms of Dini-Lipschitz functions on rank 1 symmetric spaces", *Mediterr. J. Math.* **13** (2016), no. 6, p. 4401-4411.

[18] M. Flensted-Jensen, T. H. Koornwinder, "Jacobi functions: the addition formula and the positivity of the dual convolution structure", *Ark. Mat.* **17** (1979), p. 139-151.

[19] T. H. Koornwinder, "Jacobi functions and analysis on noncompact semisimple Lie groups", in *Special functions: Group theoretical aspects and applications*, Mathematics and its Applications, vol. 18, Reidel Publishing Company, 1984, p. 1-85.

[20] P. Kumar, S. K. Ray, R. P. Sarkar, "The role of restriction theorems in harmonic analysis on harmonic NA groups", *J. Funct. Anal.* **258** (2010), no. 7, p. 2453-2482.

[21] S. Negzaoui, "Lipschitz conditions in Laguerre hypergroup", *Mediterr. J. Math.* **14** (2017), no. 5, article no. 191 (12 pages).

[22] S. S. Platonov, "Approximation of functions in the L^2 Metric on noncompact rank 1 symmetric spaces", *Algebra Anal.* **11** (1999), no. 1, p. 244-270.

[23] ———, "The Fourier transform of functions satisfying the Lipschitz condition on rank 1 symmetric spaces", *Sib. Math. J.* **46** (2005), no. 6, p. 1108-1118.

[24] S. K. Ray, R. P. Sarkar, "Fourier and Radon transform on harmonic NA groups", *Trans. Am. Math. Soc.* **361** (2009), no. 8, p. 4269-4297.

[25] F. Rouvière, "Espaces de Damek-Ricci, géométrie et analyse", in *Analyse sur les groupes de Lie et théorie des représentations*, Séminaires et Congrès, vol. 7, Société Mathématique de France, 2003, p. 45-100.
[26] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press, 1937.
[27] M. Weiss, A. Zygmund, "A note on smooth functions", *Indag. Math.* **62** (1959), p. 52-58.
[28] M. S. Younis, "Fourier transforms in L^p spaces", PhD Thesis, Chelsea College (UK), 1970.
[29] ———, "Fourier transforms of Lipschitz functions on compact groups", PhD Thesis, McMaster University (Canada), 1974.
[30] ———, "Fourier transforms of Dini-Lipschitz functions", *Int. J. Math. Math. Sci.* **9** (1986), no. 2, p. 301-312.