INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Tan Nhat Tran and Akiyoshi Tsuchiya

Worpitzky-compatible subarrangements of braid arrangements and cocomparability graphs

Volume 359, issue 6 (2021), p. 665-674

<https://doi.org/10.5802/crmath.210>

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Comptes Rendus Mathématique
2021, 359, nº 6, p. 665-674
https://doi.org/10.5802/crmath.210

Combinatorics / *Combinatoire*

Worpitzky-compatible subarrangements of braid arrangements and cocomparability graphs

Tan Nhat Tran[*, a] **and Akiyoshi Tsuchiya**[b]

[a] Tan Nhat Tran, Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo 060-0810, Japan

[b] Akiyoshi Tsuchiya, Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mails: trannhattan@math.sci.hokudai.ac.jp, akiyoshi@ms.u-tokyo.ac.jp

Abstract. The class of Worpitzky-compatible subarrangements of a Weyl arrangement together with an associated Eulerian polynomial was recently introduced by Ashraf, Yoshinaga and the first author, which brings the characteristic and Ehrhart quasi-polynomials into one formula. The subarrangements of the braid arrangement, the Weyl arrangement of type A, are known as the graphic arrangements. We prove that the Worpitzky-compatible graphic arrangements are characterized by cocomparability graphs. This can be regarded as a counterpart of the characterization by Stanley and Edelman–Reiner of free and supersolvable graphic arrangements in terms of chordal graphs. Our main result yields new formulas for the chromatic and graphic Eulerian polynomials of cocomparability graphs.

2020 Mathematics Subject Classification. 05C75, 17B22, 52C35, 05C31.

Funding. The first author is supported by JSPS Research Fellowship for Young Scientists Grant Number 19J12024. The second author was partially supported by JSPS KAKENHI 19J00312 and 19K14505.

Manuscript received 29th January 2021, accepted 5th April 2021.

1. Introduction

Let V be an ℓ-dimensional Euclidean vector space with the standard inner product $(\cdot,\cdot)$. Let Φ be an irreducible (crystallographic) root system in V, with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$. For $m \in \mathbb{Z}$ and $\alpha \in \Phi$, define the *affine hyperplane* $H_{\alpha,m}$ by $H_{\alpha,m} := \{x \in V \mid (\alpha, x) = m\}$. For $\Psi \subseteq \Phi^+$, the *Weyl subarrangement* $\mathscr{A}_\Psi$ is defined by $\mathscr{A}_\Psi := \{H_{\alpha,0} \mid \alpha \in \Psi\}$. In particular, $\mathscr{A}_{\Phi^+}$ is called the *Weyl arrangement*. Define the partial order $\geq$ on Φ^+ as follows: $\beta_1 \geq \beta_2$ if $\beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$. A subset $\Psi \subseteq \Phi^+$ is an *ideal* of

[*] Corresponding author.

Φ^+ if for $\beta_1, \beta_2 \in \Phi^+$, $\beta_1 \geq \beta_2, \beta_1 \in \Psi$ implies $\beta_2 \in \Psi$. For an ideal $I \subseteq \Phi^+$, the corresponding Weyl subarrangement $\mathscr{A}_I$ is called the *ideal subarrangement*.

We will be mainly interested in the case Φ of type $A_{\ell-1}$, in which $\mathscr{A}_{\Phi^+}$ is widely known as the *braid arrangement*, denoted $\mathrm{Br}(\ell)$. Denote $[\ell] := \{1, 2, \dots, \ell\}$. We recall a popular construction of type A root systems. Let $\{\epsilon_1, \dots, \epsilon_\ell\}$ be an orthonormal basis for V, and define $U := \{\sum_{i=1}^{\ell} r_i\epsilon_i \in V \mid \sum_{i=1}^{\ell} r_i = 0\} \simeq \mathbb{R}^{\ell-1}$. The set $\Phi(A_{\ell-1}) = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell\}$ is a root system of type $A_{\ell-1}$ in U, with a positive system $\Phi^+(A_{\ell-1}) = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell\}$ and the associated set of simple roots $\Delta(A_{\ell-1}) = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq \ell - 1\}$. Thus, a subarrangement $\mathscr{B}$ of $\mathrm{Br}(\ell)$ is completely defined by a simple graph $G = ([\ell], \mathscr{E})$, where $\{x_i - x_j = 0\} \in \mathscr{B}$ if and only if $\{i, j\} \in \mathscr{E}$. Given a graph G, let $\mathscr{A}(G)$ be the arrangement that it defines, or the corresponding *graphic arrangement*. It is a standard fact that $\mathscr{A}(G)$ is the product (e.g., [18, Definition 2.13]) of the one dimensional empty arrangement and the Weyl subarrangement $\mathscr{A}_{\Psi(G)}$, where $\Psi(G) := \{\epsilon_i - \epsilon_j \mid \{i, j\} \in \mathscr{E}\,(i < j)\} \subseteq \Phi^+(A_{\ell-1})$. Throughout the paper, for any property that $\mathscr{A}_{\Psi(G)}$ has, we will say $\mathscr{A}(G)$ has that property as well. For example, we say that $\mathscr{A}(G)$ is compatible (resp., ideal-graphic) if $\mathscr{A}_{\Psi(G)}$ is compatible (Definition 2) (resp., an ideal subarrangement).

Weyl arrangements are an important class of *free* arrangements in the sense of Terao. In other words, the arrangement's logarithmic derivation module is a free module [18, Sections 4 and 6]. There has been considerable interest in analyzing subarrangements of a Weyl arrangement from the perspective of freeness. A central hyperplane arrangement is *supersolvable* if its intersection lattice is supersolvable in the sense of Stanley [21]. Jambu–Terao proved that any supersolvable arrangement is free [11]. Various free subarrangements of a Weyl arrangement of type B were studied, e.g., [6, 13, 23]. Remarkably, a striking result of Abe–Barakat–Cuntz–Hoge–Terao [1] asserts that any ideal subarrangement is free.

Although characterizing free subarrangements of an arbitrary Weyl arrangement is still a challenging problem, in the case of braid arrangement, the free subarrangements can be completely analyzed using the connection to graphs. It follows from the works of Stanley [21] and Edelman–Reiner [6] that free and supersolvable graphic arrangements are synonyms, and they correspond to *chordal graphs* (every induced cycle in the graph has exactly 3 vertices). Chordal graphs are a superclass of *(unit) interval graphs* (each vertex can be associated with (a unit) an interval on the real line, and two vertices are adjacent if the associated intervals have a nonempty intersection). More strongly, a graph is an interval graph if and only if it is a *cocomparability* (Definition 1) and chordal graph [9] (see Figure 1 for an illustration). It should also be noted that the ideal subarrangements of a braid arrangement are parametrized by unit interval graphs (Theorem 16).

Definition 1. *A graph is called a* comparability graph *if its edges can be transitively oriented, i.e., if $u \to v$ and $v \to w$, then $u \to w$. A graph is called a* cocomparability graph *if its complement is a comparability graph.*

Thus, it is natural to ask which class of Weyl subarrangements generalizes the cocomparability graphs. Recently, the notion of *(Worpitzky-)compatible arrangements* was introduced by Ashraf, Yoshinaga and the first author in the study of characteristic quasi-polynomials of Weyl subarrangements and Ehrhart theory [2]. It is shown that any ideal subarrangement is compatible [2, Theorem 4.16]. In this paper we prove that, interestingly, the Worpitzky-compatible graphic arrangements are characterized by cocomparability graphs, which gives an answer to the aforementioned question.

To state the result formally, we first recall the concept of compatibility. A connected component of $V \setminus \bigcup_{\alpha\in\Phi^+, m\in\mathbb{Z}} H_{\alpha,m}$ is called an *alcove*. Let A be an alcove. A *wall* of A is a hyperplane that supports a facet of A. The *ceilings* of A are the walls which do not pass through the origin and have the origin on the same side as A. The *upper closure* $A^\diamondsuit$ of A is the union of A and its facets supported by the ceilings of A. Let $P^{\diamondsuit} := \{x \in V \mid 0 < (\alpha_i, x) \leq 1\,(1 \leq i \leq \ell)\}$ be the *fundamental parallelepiped* (of the coweight lattice) of Φ. Thus,

$$P^{\diamondsuit} = \bigsqcup_{A:\,\text{alcove},\,A\subseteq P^{\diamondsuit}} A^{\diamondsuit},$$

which is known as the *Worpitzky partition*, e.g., [26, Proposition 2.5], [10, Exercise 4.3].

Definition 2 (cf. [2, Definition 4.8]). *A subset $\Psi \subseteq \Phi^+$ is said to be* Worpitzky-compatible *(or* compatible *for short) if every nonempty intersection of the upper closure $A^{\diamondsuit}$ of an alcove $A \subseteq P^{\diamondsuit}$ and an affine hyperplane w.r.t. a root in Ψ can be lifted to a facet intersection. That is, $A^{\diamondsuit} \cap H_{\alpha,m_\alpha}$ for $\alpha \in \Psi$, $m_\alpha \in \mathbb{Z}$ either is empty, or is contained in a ceiling H_{β,m_β} of A with $\beta \in \Psi$, $m_\beta \in \mathbb{Z}$. If Ψ is compatible, the Weyl subarrangement $\mathscr{A}_\Psi$ is said to be* compatible *as well.*

Our main result is the following.

Theorem 3. *Let $G = (\mathscr{V}, \mathscr{E})$ be a graph with $|\mathscr{V}| = \ell$. Then G has a labeling using elements from $[\ell]$ so that $\mathscr{A}(G)$ is a compatible graphic arrangement if and only if G is a cocomparability graph.*

The cocomparability graph is a combinatorial object while Worpitzky-compatibility is a geometric property. Our main result connects the geometric property of a graphic arrangement with the combinatorial property of the underlying graph. In particular, our main result yields a correspondence between interval graphs and graphic arrangements that are compatible and free. We record the results in Table 1.

Table 1. Parallel concepts in type A.

Graph class	Weyl subarrangement class	Location
cocomparability	compatible (= strongly compatible)	Theorems 3, 9
chordal	free (= supersolvable)	[6, 21]
interval	compatible $\cap$ free	Corollary 15
unit interval	ideal	Theorem 16

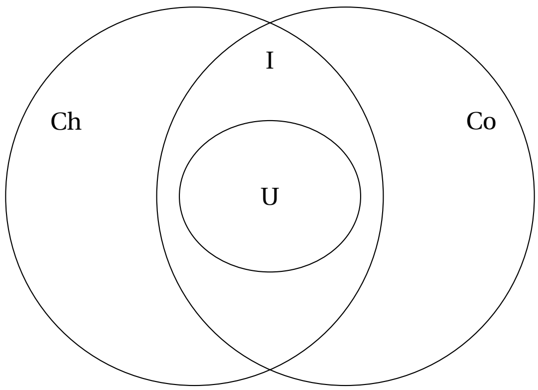

Figure 1. Relationship between graph classes: U $\subsetneq$ I = Ch $\cap$ Co. Co: cocomparability, Ch: chordal, I: interval, U: unit interval.

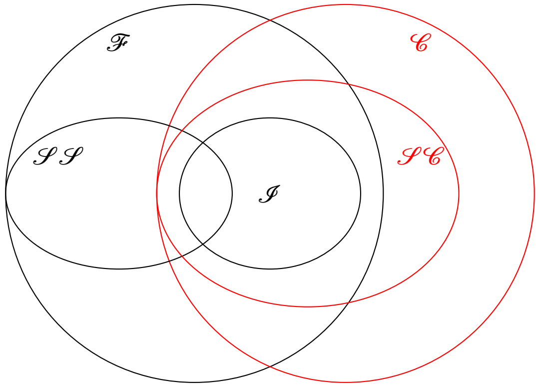

Figure 2. Relationship between Weyl subarrangement classes. $\mathscr{C}$: compatible, $\mathscr{S}\mathscr{C}$: strongly compatible, $\mathscr{F}$: free, $\mathscr{S}\mathscr{S}$: supersolvable, $\mathscr{I}$: ideal.

2. Proof of the main result

First, we introduce a new subclass of Worpitzky-compatible sets, which will play a key role in the proof of our main result.

Definition 4. *A subset $\Psi \subseteq \Phi^+$ is said to be* strongly (Worpitzky-)compatible[1] *if for any $\alpha \in \Psi$ and for every choice of positive roots $\beta_1, \dots, \beta_m \in \Phi^+$ such that $\alpha \in \sum_{i=1}^m \mathbb{Z}_{>0}\beta_i$, there exists k with $1 \le k \le m$ such that $\beta_k \in \Psi$.*

The strong Worpitzky-compatibility is a combinatorial property of the root system. This concept was made by inspiration of the following lemma, which is an important result in [2].

Lemma 5. *Let $A \subseteq P^\diamondsuit$ be an alcove. If there exist $\alpha \in \Phi^+, r_\alpha \in \mathbb{Z}$ so that $A^\diamondsuit \cap H_{\alpha, r_\alpha} = \bigcap_{j=1}^m H_{\beta_j, r_{\beta_j}} \cap A^\diamondsuit$ is a face of $A^\diamondsuit$, where $H_{\beta_j, r_{\beta_j}}$ are the ceilings of A, then $\alpha \in \sum_{j=1}^m \mathbb{Z}_{\ge 0}\beta_j$.*

Proof. See [2, Proof of Theorem 4.16]. □

Let $\mathscr{I}$ be the set of all ideals of Φ^+. In addition, let $\mathscr{C}$ (resp,. $\mathscr{S}\mathscr{C}$) be the set of all compatible (resp., strongly compatible) sets of Φ^+. We exhibit a relation between these sets (see also Figure 2 for an illustration).

Theorem 6. *If Φ is an irreducible root system, then*

$$\mathscr{I} \subseteq \mathscr{S}\mathscr{C} \subseteq \mathscr{C}.$$

Proof. The first inclusion is clear. The second inclusion follows from Lemma 5. □

Remark 7. Example 27 illustrates a strongly compatible set but not an ideal (w.r.t. any positive system of Φ). There exists a compatible set that is not strongly compatible when Φ is of type G_2 (or B_2) [2, Example 4.18 (d)].

For any alcove A and $\gamma \in \Phi^+$, there exists a unique integer r with $r-1 < (x, \gamma) < r$ for all $x \in A$. We denote this integer by $r(A, \gamma)$. The function $r(A, -) : \Phi^+ \to \mathbb{Z}$ determines the position of A, sometimes, it is called the *address* of A.

[1]We thank Christian Stump and Takuro Abe for pointing out that the notion of strongly compatible subsets is essentially equivalent to that of *coclosed* of the root system. Recall that a subset $\Psi \subseteq \Phi^+$ is said to be *coclosed* if for any $\alpha \in \Psi$ and for $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = \beta_1 + \beta_2$, either $\beta_1 \in \Psi$ or $\beta_2 \in \Psi$.

Lemma 8. *Suppose that for each $\gamma \in \Phi^+$ we are given a positive integer r_γ. There is an alcove A with $r(A,\gamma) = r_\gamma$ for all $\gamma \in \Phi^+$ (in other words, the function $r : \Phi^+ \to \mathbb{Z}$ with $r(\gamma) = r_\gamma$ is the address of an alcove A) if and only if $r_\gamma + r_{\gamma'} - 1 \le r_{\gamma+\gamma'} \le r_\gamma + r_{\gamma'}$ whenever $\gamma, \gamma', \gamma+\gamma' \in \Phi^+$.*

Proof. This was first proved by Shi in terms of coroots [20, Theorem 5.2]. The statement here is formulated in terms of roots, which can be found in, e.g., [3, Lemma 2.4]. □

Surprisingly, there is no difference between compatible and strongly compatible sets in the case of type A.

Theorem 9. *If Φ is of type A_ℓ, then every compatible set is strongly compatible, i.e.,*

$$\mathscr{SC} = \mathscr{C}.$$

Proof. Let $\Psi \subseteq \Phi^+$ be a compatible set. Let $\alpha \in \Psi$, and suppose that there are $\beta_1, \ldots, \beta_m \in \Phi^+$ such that $\alpha \in \sum_{i=1}^m \mathbb{Z}_{>0}\beta_i$. Thus, $\alpha = \sum_{i=1}^m \beta_i = \alpha_s + \alpha_{s+1} + \ldots + \alpha_t$ for some $1 \le s \le t \le \ell$ with $\alpha_j \in \Delta$ for all $s \le j \le t$. We want to show that there exists k with $1 \le k \le m$ such that $\beta_k \in \Psi$.

For $\beta = \sum_{i=1}^\ell d_i\alpha_i \in \Phi^+$, denote $\mathrm{supp}(\beta) := \{\alpha_i \mid d_i > 0\}$. The above expression of α induces a partition of Δ as follows:

$$\Delta = S_0 \sqcup S_1 \sqcup \cdots \sqcup S_m \sqcup S_{m+1},$$

where $S_0 := \{\alpha_1, \ldots, \alpha_{s-1}\}$, $S_{m+1} := \{\alpha_{t+1}, \ldots, \alpha_\ell\}$, $S_i := \mathrm{supp}(\beta_i)$ for $1 \le i \le m$. For each $\gamma \in \Phi^+$, define

$$r_\gamma := \#\{0 \le i \le m+1 \mid \mathrm{supp}(\gamma) \cap S_i \neq \emptyset\}.$$

For any $\gamma, \gamma' \in \Phi^+$ with $\gamma + \gamma' \in \Phi^+$, it is not hard to show the following facts:

(a) $r_{\gamma+\gamma'} = r_\gamma + r_{\gamma'}$ if there is no i with $0 \le i \le m+1$ such that $\mathrm{supp}(\gamma) \cap S_i \neq \emptyset$ and $\mathrm{supp}(\gamma') \cap S_i \neq \emptyset$,

(b) $r_{\gamma+\gamma'} = r_\gamma + r_{\gamma'} - 1$, otherwise.

By Lemma 8, there is an alcove A with $r(A,\gamma) = r_\gamma$ for all $\gamma \in \Phi^+$. In particular, $r(A,\alpha) = m$ and $r(A,\beta_i) = 1$ for $1 \le i \le m$. Clearly, $A \subseteq P^\diamondsuit$.

Fix β_i with $1 \le i \le m$. We will show that $H_{\beta_i,1}$ is a ceiling of A. The method used in [3, Theorem 3.11] or [24, Theorem 3.1] applies here as well. First, the above formulas imply that

(a) $r_{\beta_i+\gamma} = r_\gamma + r_{\beta_i}$ if $\gamma, \beta_i + \gamma \in \Phi^+$ and

(b) $r_{\beta_i} = r_\gamma + r_{\gamma'} - 1$ if $\beta_i = \gamma + \gamma'$ for $\gamma, \gamma' \in \Phi^+$.

Next, we again apply Lemma 8 to find another alcove, called B with $r(B,\beta_i) = r_{\beta_i} + 1 = 2$, $r(B,\gamma) = r_\gamma$ for all $\gamma \in \Phi^+ \setminus \{\beta_i\}$. Comparing the addresses of alcoves A and B implies that $H_{\beta_i,1}$ is a wall, hence a ceiling of A. Thus, the intersection $P := \bigcap_{i=1}^m H_{\beta_i,1} \cap A^\diamondsuit$ is a non-empty face, moreover, contained in the face $Q := H_{\alpha,m} \cap A^\diamondsuit$ of $A^\diamondsuit$. Since any proper face of a polytope is the intersection of all facets containing it, if $Q = \bigcap_{\delta\in D} H_{\delta,n_\delta} \cap A^\diamondsuit$ where each H_{δ,n_δ} is a wall of $A^\diamondsuit$ and D is a set of positive roots, then $D \subseteq \{\beta_1, \ldots, \beta_m\}$ and $n_\delta = 1$ for all $\delta \in D$. Applying Lemma 5 to the face $Q = H_{\alpha,m} \cap A^\diamondsuit = \bigcap_{\delta\in D} H_{\delta,1} \cap A^\diamondsuit$ implies that $\alpha = \sum_{\delta\in D'} \delta$ with $D' \subseteq D$. We must have $D' = D = \{\beta_1, \ldots, \beta_m\}$ and $P = Q$. Since Ψ is a compatible set, by Definition 2, there exists k with $1 \le k \le m$ such that $\beta_k \in \Psi$. This completes the proof. □

Thanks to Theorem 9, when Φ is of type A, the concept of compatibility originally defined by geometry of alcoves and affine hyperplanes can now be rephrased in terms of combinatorial property of graphs.

Corollary 10. *Let $G = ([\ell], \mathscr{E})$ be a graph. The following are equivalent.*

(i) *$\mathscr{A}(G)$ is a compatible graphic arrangement, i.e., $\Psi(G) \subseteq \Phi^+(A_{\ell-1})$ is a compatible set.*

(ii) *G has the following property: if $i < j$, $\{i,j\} \in \mathscr{E}$ and $(p_1, \ldots, p_m)$ is any sequence such that $i = p_1 < p_2 < \cdots < p_m = j$, then there exists p_a with $1 \le a < m$ such that $(p_a, p_{a+1}) \in \mathscr{E}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is clear. Now we prove (ii) $\Rightarrow$ (i). Let $\alpha = \epsilon_i - \epsilon_j \in \Psi(G)$. If there are $\beta_1, \dots, \beta_m \in \Phi^+$ such that $\alpha \in \sum_{i=1}^m \mathbb{Z}_{>0}\beta_i$, then $\alpha = \sum_{\beta \in M} \beta$ where $M = \{\beta_1, \dots, \beta_m\}$. Therefore, $\epsilon_i - \epsilon_j = \sum_{p \in H} \epsilon_p - \sum_{q \in T} \epsilon_q$ for $H, T \subseteq [\ell]$. Note that $\epsilon_i - \epsilon_j \geq \epsilon_p - \epsilon_q$ if and only if $i \leq p < q \leq j$. The independence of $\{\epsilon_1, \dots, \epsilon_\ell\}$ implies $H \setminus T = \{i\}$ and $T \setminus H = \{j\}$. Thus, there exist $p_1, \dots, p_{m+1} \in \mathbb{Z}_{>0}$ with $i = p_1 < p_2 < \cdots < p_{m+1} = j$ such that $M = \{\epsilon_{p_a} - \epsilon_{p_{a+1}} \mid 1 \leq a \leq m\}$. Therefore, there is p_a such that $(p_a, p_{a+1}) \in \mathscr{E}$ hence $\epsilon_{p_a} - \epsilon_{p_{a+1}} \in \Psi(G) \cap M$. This completes the proof. $\square$

Definition 11. *An ordering $v_1 < v_2 < \cdots < v_\ell$ of the vertices of a graph G is an* umbrella-free ordering *(or a* cocomparability ordering*) if $i < k < j$ and $\{v_i, v_j\} \in \mathscr{E}$, then either $\{v_i, v_k\} \in \mathscr{E}$ or $\{v_k, v_j\} \in \mathscr{E}$ or both.*

Lemma 12. *Let $G = (\mathscr{V}, \mathscr{E})$ be a graph with $|\mathscr{V}| = \ell$. An ordering $v_1 < v_2 < \cdots < v_\ell$ of the vertices of G is an umbrella-free ordering if and only if "$<$" satisfies the following condition: if $i < j$, $\{v_i, v_j\} \in \mathscr{E}$ and $(p_1, \dots, p_m)$ is any sequence such that $i = p_1 < p_2 < \cdots < p_m = j$, then there exists p_a with $1 \leq a < m$ such that $\{v_{p_a}, v_{p_{a+1}}\} \in \mathscr{E}$.*

Proof. The implication ($\Leftarrow$) is obvious. To prove ($\Rightarrow$), suppose to the contrary that $\{v_{p_a}, v_{p_{a+1}}\} \notin \mathscr{E}$ for all $1 \leq a < m$. Since $p_1 < p_{m-1} < p_m$, we must have $\{v_{p_1}, v_{p_{m-1}}\} \in \mathscr{E}$. Repeating yields $\{v_{p_1}, v_{p_3}\} \in \mathscr{E}$ with $p_1 < p_2 < p_3$, $\{v_{p_1}, v_{p_2}\} \notin \mathscr{E}$, $\{v_{p_2}, v_{p_3}\} \notin \mathscr{E}$. This is a contradiction. $\square$

We are now in position to prove our main result.

Proof of Theorem 3. It is known that a graph G is a cocomparability graph if and only if G has an umbrella-free ordering, e.g., [14, Section 2]. By Lemma 12, this is equivalent to saying that if we label vertex v_i of G by $i \in [\ell]$, then G has the property in Corollary 10. The rest follows from Corollary 10. $\square$

Remark 13. By [6, Theorem 3.3], a graph G is chordal if and only if $\mathscr{A}(G)$ is free (= supersolvable) for *any* labeling of G using $[\ell]$ (since if G and G' are isomorphic, then $\mathscr{A}(G)$ and $\mathscr{A}(G')$ have isomorphic intersection lattices). Given a cocomparability graph G, Theorem 3, however, can only tell the existence of a labeling that makes $\mathscr{A}(G)$ compatible (see Examples 25 and 26). We will see in Remark 24 that such a labeling is sufficient for computing some polynomial invariants of the graph.

Remark 14. Comparability and cocomparability graphs also have forbidden induced subgraph characterizations, see, e.g., [8, 25].

From the main result Theorem 3, we obtain a characterization of graphic arrangements that are compatible and free.

Corollary 15. *Let $G = (\mathscr{V}, \mathscr{E})$ be a graph with $|\mathscr{V}| = \ell$. The following are equivalent.*

(i) *G has a labeling using $[\ell]$ so that $\mathscr{A}(G)$ is a compatible and free graphic arrangement.*
(ii) *G has an ordering $v_1 < \cdots < v_\ell$ of its vertices such that if $i < k < j$ and $\{v_i, v_j\} \in \mathscr{E}$, then $\{v_i, v_k\} \in \mathscr{E}$.*
(iii) *G is an interval graph.*

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is well known, e.g., [17, Theorem 4]. The equivalence (i) $\Leftrightarrow$ (iii) follows from Theorem 3, Remark 13, and the fact that a graph is an interval graph if and only if it is a cocomparability and chordal graph [9, Theorem 2]. $\square$

We complete Table 1 by giving a graphic characterization of ideal subarrangements of braid arrangements, which was suggested to us by Shuhei Tsujie.

Theorem 16. *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with $|\mathcal{V}| = \ell$. The following are equivalent.*

(i) *G has a labeling using $[\ell]$ so that $\mathcal{A}(G)$ is an ideal-graphic arrangement, i.e., $\Psi(G) \subseteq \Phi^+(A_{\ell-1})$ is an ideal.*
(ii) *G has an ordering $v_1 < \cdots < v_\ell$ of its vertices such that if $i < k < j$ and $\{v_i, v_j\} \in \mathcal{E}$, then $\{v_i, v_k\} \in \mathcal{E}$ and $\{v_k, v_j\} \in \mathcal{E}$.*
(iii) *G is a unit interval graph.*

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is well known, e.g., [15, Theorem 1]. The equivalence (i) $\Leftrightarrow$ (ii) is not hard. The key observation is $\epsilon_i - \epsilon_j \geq \epsilon_p - \epsilon_q$ if and only if $i \leq p < q \leq j$. $\square$

3. Application to graph polynomials of Cocomparability graphs

Owing to Theorem 3, we will be able to give new formulas for the chromatic polynomial and the (reduced) graphic Eulerian polynomial of cocomparability graphs.

Let $G = ([\ell], \mathcal{E})$ be a simple graph, i.e., no loops and no multiple edges. Let $c_G(t)$ be the chromatic polynomial of G. The *graphic Eulerian polynomial* of G is the polynomial $W_G(t)$ defined by

$$\sum_{q \geq 0} c_G(q) t^q = \frac{W_G(t)}{(1-t)^{\ell+1}}.$$

The coefficients of $W_G(t)$ are proved to be nonnegative integers, and have various combinatorial interpretations, e.g., [4, 5, 22]. Let us recall one of its interpretations following the last two references. Denote by $\mathfrak{S}_\ell$ the symmetric group on $[\ell]$.

Definition 17. *Let $G = ([\ell], \mathcal{E})$ be a graph. For $\pi = \pi_1 \ldots \pi_\ell \in \mathfrak{S}_\ell$, the* rank $\rho(\pi_i)$ *of π_i $(1 \leq i \leq \ell)$ is defined to be the largest integer r so that there are values $1 \leq i_1 < i_2 < \cdots < i_r = i$ with $\{\pi_{i_j}, \pi_{i_{j+1}}\} \in \mathcal{E}$ for all $1 \leq j \leq r-1$. We say that $\pi \in \mathfrak{S}_\ell$ has a* graphic descent (w.r.t. G) at $i \in [\ell-1]$ *if either*

(1) *$\rho(\pi_i) > \rho(\pi_{i+1})$, or*
(2) *$\rho(\pi_i) = \rho(\pi_{i+1})$ and $\pi_i > \pi_{i+1}$.*

The graphic Eulerian numbers $w_k(G)$ $(1 \leq k \leq \ell)$ *are defined by*

$$w_k(G) = \#\left\{\pi \in \mathfrak{S}_\ell \,\middle|\, \pi \text{ has exactly } \ell - k \text{ graphic descents}\right\}.$$

Theorem 18. *We have $W_G(t) = \sum_{k=1}^{\ell} w_k(G) t^k$. Equivalently,*

$$c_G(t) = \sum_{k=1}^{\ell} w_k(G) \binom{t+\ell-k}{\ell}.$$

Proof. See, e.g., [5, Theorem 2], [22, Theorem 6]. The equivalence of the formulas holds true in a more general setting, e.g., [4, Theorem 2.1]. $\square$

Remark 19. If G is the empty graph, then the graphic descent is the ordinary descent, i.e., the index $i \in [\ell-1]$ such that $\pi_i > \pi_{i+1}$. In this case, $W_G(t)$ is known as the *classical ℓ-th Eulerian polynomial*, which first appeared in a work of Euler [7].

It is a standard fact that $c_G(t)$ is divisible by t. The *reduced graphic Eulerian polynomial* of G is the polynomial $Y_G(t)$ defined by

$$\sum_{q \geq 1} \frac{c_G(q)}{q} t^q = \frac{Y_G(t)}{(1-t)^\ell}.$$

The polynomial $Y_G(t)$ also has nonnegative integer coefficients and can be interpreted in terms of the h-polynomial of a certain *relative complex*, e.g., [12, Theorem 3.5]. However, unlike $W_G(t)$, less seems to be known about combinatorial interpretations of the coefficients of $Y_G(t)$. We will give a combinatorial interpretation for $Y_G(t)$ when G is a cocomparability graph (Theorem 23). One can readily compute $Y_G(t)$ from $W_G(t)$ and vice versa as we will see below. Write $Y_G(t) = \sum_{k=1}^{\ell} y_k(G) t^k$ (note that $Y_G(0) = 0$).

Proposition 20. *The polynomials $W_G(t)$ and $Y_G(t)$ satisfy the* Eulerian recurrence*, i.e.,*

$$W_G(t) = t(1-t)\frac{d}{dt}Y_G(t) + t\ell Y_G(t).$$

Equivalently, for every $1 \le k \le \ell$,

$$w_k(G) = k y_k(G) + (\ell - k + 1) y_{k-1}(G).$$

Conversely, if $F(t)$ is a polynomial such that $F(0) = 0$, and $W_G(t)$ and $F(t)$ satisfy the Eulerian recurrence, then $F(t) = Y_G(t)$.

Proof. Straightforward. □

Now to link the Eulerian polynomials above with the cocomparability graphs, we need to recall the notion of $\mathscr{A}$-Eulerian polynomial following [2, Section 4]. We will focus only on type A. Let $G^c = (\mathscr{V}, \mathscr{E}(G^c))$ denote the complement graph of a graph G.

Definition 21. *Let $G = ([\ell], \mathscr{E})$ be a graph. We say that $\pi = \pi_1 \dots \pi_\ell \in \mathfrak{S}_\ell$ has an $\mathscr{A}$-*descent (w.r.t. G) *at $i \in [\ell]$ if $\pi_i > \pi_{i+1}$ and $\{\pi_i, \pi_{i+1}\} \in \mathscr{E}(G^c)$ ($\pi_{\ell+1} = \pi_1$). For $0 \le k \le \ell$, define*

$$f_k(G) = \frac{1}{\ell}\#\left\{\pi \in \mathfrak{S}_\ell \,\middle|\, \pi \text{ has exactly } \ell - k\ \mathscr{A}\text{-descents}\right\}.$$

It is easily seen that $f_0(G) = 0$. Set $F_G(t) := \sum_{k=1}^{\ell} f_k(G) t^k$.

Recall the notation $\Psi(G) = \{\epsilon_i - \epsilon_j \mid \{i, j\} \in \mathscr{E}\ (i < j)\} \subseteq \Phi^+(A_{\ell-1})$ in Section 1. It is not hard to check that $F_G(t)$ equals the $\mathscr{A}$-Eulerian polynomial [2, Definition 4.2] of $\Psi(G)$. If G is the empty graph, then $F_G(t)$ is the classical $(\ell-1)$-th Eulerian polynomial. Unlike $W_G(t)$ or $Y_G(t)$, the polynomial $F_G(t)$ is in general not a graph invariant in the sense that it depends on the labeling of G (see Examples 25 and 26). The proposition below says computing $F_G(t)$ requires only $(\ell-1)!$ permutations.

Proposition 22. *If $1 \le k \le \ell$, then*

$$f_k(G) = \#\left\{\pi \in \mathfrak{S}_\ell \,\middle|\, \pi_1 = \ell \text{ and } \pi \text{ has exactly } \ell - k\ \mathscr{A}\text{-descents}\right\}.$$

Proof. When G is the empty graph on $[\ell]$, it is already proved in [19, Exercise 1.11]. The same argument applies to the general case. Let $\tau = 23\dots\ell 1 \in \mathfrak{S}_\ell$, then τ cyclically shifts the numbers 1 to ℓ. The action of τ on $\mathfrak{S}_\ell$ by right multiplication partitions $\mathfrak{S}_\ell$ into $(\ell-1)!$ orbits of size ℓ. The number of $\mathscr{A}$-descents is constant on each orbit. Also, we can choose the permutation π with $\pi_1 = \ell$ to be a representative of each orbit. □

The following is the main result of this section.

Theorem 23. *Let $G = ([\ell], \mathscr{E})$ be a graph. The following are equivalent.*

(i) *G is a cocomparability graph and the ordering $1 < 2 < \cdots < \ell$ is an umbrella-free ordering.*
(ii) *The ordering $1 < 2 < \cdots < \ell$ is an umbrella-free ordering.*
(iii) *$\Psi(G) \subseteq \Phi^+(A_{\ell-1})$ is compatible (= strongly compatible).*
(iv) *The chromatic polynomial of G is given by*

$$c_G(t) = t\sum_{k=1}^{n} f_k(G)\binom{t+\ell-1-k}{\ell-1}.$$

(v) *$F_G(t)$ equals the reduced graphic Eulerian polynomial G, i.e.,*

$$F_G(t) = Y_G(t) = (1-t)^\ell \sum_{q \ge 1} \frac{c_G(q)}{q} t^q,$$

Equivalently, $F_G(t) = Y_K(t)$, where K is any graph isomorphic to G.

(vi) *The polynomials $W_G(t)$ and $F_G(t)$ satisfy the Eulerian recurrence, i.e.,*

$$W_G(t) = t(1-t)\frac{d}{dt}F_G(t) + t\ell F_G(t)$$

Equivalently, for every $1 \le k \le \ell$,

$$w_k(G) = k f_k(G) + (\ell - k + 1) f_{k-1}(G).$$

Proof. The equivalences (i) ⇔ (ii) ⇔ (iii) follow from Corollary 10 and Lemma 12. The statements (iii), (iv) and (v) are simply those in [2, Theorems 4.11 and 4.24] when restricting the root system to type A, hence they are equivalent. The equivalence (v) ⇔ (vi) follows from Proposition 20 and the fact that $F_G(0) = 0$ (Definition 21). □

Remark 24. Thus, to compute $c_G(t)$, $Y_G(t)$ and $W_G(t)$ for a given cocomparability graph G, in principle, we can do as follows: find an umbrella-free ordering $v_1 < v_2 < \cdots < v_\ell$ of its vertices (which can be done in linear time [16]), label each vertex v_i by $i \in [\ell]$, and apply Theorem 23. In addition, Theorem 23 says that $F_G(t)$ is an invariant of cocomparability graphs if and only if the ordering $1 < 2 < \cdots < \ell$ is an umbrella-free ordering.

Example 25. Let G be a graph given in Figure 3. Thus, $\Psi(G) = \{\alpha_1, \alpha_2\} \subseteq \Phi^+(A_3)$, which is an ideal, hence compatible (Theorem 6). The computation of $F_G(t)$ and $W_G(t)$ is illustrated in Table 2. Let $\tau = 2341$. There are 8 permutations having 2 $\mathscr{A}$-descents and 16 having 1 $\mathscr{A}$-descent. Thus, $F_G(t) = 4t^3 + 2t^2$. There are 4 permutations having 2 graphic descents (cells in green), 4 having 0 graphic descent (cells in red), and the remaining 16 have 1 graphic descent. Thus, $W_G(t) = 4t^4 + 16t^3 + 2t^2$. The calculation is consistent with Theorem 23.

Table 2. Computation of $F_G(t)$ and $W_G(t)$ for the graph G in Figure 3.

#$\mathscr{A}$-descents	π	$\pi\tau$	$\pi\tau^2$	$\pi\tau^3$
2	4312	3124	1243	2431
	4231	2314	3142	1423
1	4213	2134	1342	3421
	4132	1324	3241	2413
	4321	3214	2143	1432
	4123	1234	2341	3412

Example 26. Let G' be a graph given in Figure 3. Note that G and G' have isomorphic underlying unlabeled graphs. But $\Psi(G')$ is not compatible, which can be checked either by Theorem 23(v) ($Y_{G'}(t) \neq F_{G'}(t)$), or by Theorem 23(ii) ($1 < 2 < 4$, $\{1,4\} \in \mathscr{E}$ but $\{1,2\} \notin \mathscr{E}$, $\{2,4\} \notin \mathscr{E}$).

Example 27. Let G be the claw $K_{1,3}$ given in Figure 3. By Theorem 23 (ii), $\Psi(K_{1,3})$ is compatible. By Theorem 16, $\Psi(K_{1,3})$ is not an ideal w.r.t. any positive system since $K_{1,3}$ is not an interval graph.

Acknowledgements

We would like to thank Ye Liu, Ahmed Umer Ashraf and Masahiko Yoshinaga for stimulating conversations on the graphic Eulerian polynomials. We would also like to thank Shuhei Tsujie for valuable discussion and for informing us of the characterization of ideal-graphic arrangements in terms of unit interval graphs (Theorem 16).

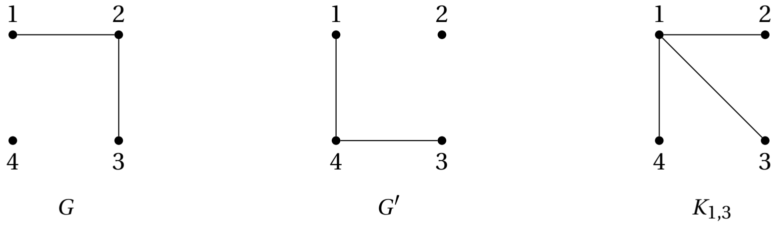

Figure 3. Three cocomparability graphs with 4 vertices.

References

[1] T. Abe, M. Barakat, M. Cuntz, T. Hoge, H. Terao, "The freeness of ideal subarrangements of Weyl arrangements", *J. Eur. Math. Soc.* **18** (2016), no. 6, p. 1339-1348.

[2] A. U. Ashraf, T. N. Tran, M. Yoshinaga, "Eulerian polynomials for subarrangements of Weyl arrangements", *Adv. Appl. Math.* **120** (2020), article no. 102064 (24 pages).

[3] C. A. Athanasiadis, "On a refinement of the generalized Catalan numbers for Weyl groups", *Trans. Am. Math. Soc.* **357** (2005), no. 1, p. 179-196.

[4] F. Brenti, "Expansions of chromatic polynomials and log-concavity", *Trans. Am. Math. Soc.* **332** (1992), no. 2, p. 729-756.

[5] F.-R. K. Chung, R. L. Graham, "On the cover polynomial of a digraph", *J. Comb. Theory, Ser. B* **65** (1995), no. 2, p. 273-290.

[6] P. H. Edelman, V. Reiner, "Free hyperplane arrangements between A_{n-1} and B_n", *Math. Z.* **215** (1994), no. 3, p. 347-365.

[7] L. Euler, "Methodus universalis series summandi ulterius promota", *Commentarii Acd. Scientiarum Imperialis Petropolitanae* **8** (1736), p. 147-158, Reprinted in his Opera Omnia, series 1, volume 14, 124-137, 1741.

[8] T. Gallai, "Transitiv orientierbare Graphen", *Acta Math. Acad. Sci. Hung.* **18** (1967), p. 25-66.

[9] P. C. Gilmore, A. J. Hoffman, "A characterization of comparability graphs and interval graphs", *Can. J. Math.* **16** (1964), p. 539-548.

[10] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.

[11] M. Jambu, H. Terao, "Free arrangements of hyperplanes and supersolvable lattices", *Adv. Math.* **52** (1984), no. 3, p. 248-258.

[12] J. Jonsson, "The Topology of the Coloring Complex", *J. Algebr. Comb.* **21** (2005), no. 3, p. 311-329.

[13] T. Józefiak, B. E. Sagan, "Basic derivations for subarrangements of Coxeter arrangements", *J. Algebr. Comb.* **2** (1993), no. 3, p. 291-320.

[14] D. Kratsch, L. Stewart, "Domination on cocomparability graphs", *SIAM J. Discrete Math.* **6** (1993), no. 3, p. 400-417.

[15] P. J. Looges, S. Olariu, "Optimal greedy algorithms for indifference graphs", *Comput. Math. Appl.* **25** (1993), no. 7, p. 15-25.

[16] R. M. McConnell, J. P. Spinrad, "Modular decomposition and transitive orientation", *Discrete Math.* **201** (1999), no. 1-3, p. 189-241.

[17] S. Olariu, "An optimal greedy heuristic to color interval graphs", *Inf. Process. Lett.* **37** (1991), no. 1, p. 21-25.

[18] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer, 1992.

[19] T. K. Petersen, *Eulerian numbers*, Birkhäuser Advanced Texts. Basler Lehrbücher, Birkhäuser, 2015.

[20] J.-Y. Shi, "Alcoves corresponding to an affine Weyl group", *J. Lond. Math. Soc.* **35** (1987), p. 42-55.

[21] R. P. Stanley, "Supersolvable lattices", *Algebra Univers.* **2** (1972), p. 197-217.

[22] E. Steingrímsson, "The coloring ideal and coloring complex of a graph", *J. Algebr. Comb.* **14** (2001), no. 1, p. 73-84.

[23] D. Suyama, M. Torielli, S. Tsujie, "Signed graphs and the freeness of the Weyl subarrangements of type B_ℓ", *Discrete Math.* **342** (2019), no. 1, p. 233-249.

[24] M. Thiel, "On floors and ceilings of the k-Catalan arrangement", *Electron. J. Comb.* **21** (2014), no. 4, article no. P4.36 (15 pages).

[25] W. T. Trotter, *Combinatorics and partially ordered sets: Dimension theory*, Johns Hopkins Series in the Mathematical Sciences, Johns Hopkins University Press, 1992.

[26] M. Yoshinaga, "Worpitzky partitions for root systems and characteristic quasi-polynomials", *Tôhoku Math. J.* **70** (2018), no. 1, p. 39-63.