



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique


Indranil Biswas, Michi-aki Inaba, Arata Komyo and Masa-Hiko Saito

On the moduli spaces of framed logarithmic connections on a Riemann surface

Volume 359, issue 5 (2021), p. 617-624

<<https://doi.org/10.5802/crmath.199>>

© Académie des sciences, Paris and the authors, 2021.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org



Complex analysis and geometry / *Analyse et géométrie complexes*

On the moduli spaces of framed logarithmic connections on a Riemann surface

Indranil Biswas^a, Michi-aki Inaba^b, Arata Komyo^c and Masa-Hiko Saito^d

^a School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

^b Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

^c Center for Mathematical and Data Sciences, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501, Japan

^d Department of Mathematics, Graduate School of Science, Kobe University, Kobe, Rokko, 657-8501, Japan

E-mails: indranil@math.tifr.res.in, inaba@math.kyoto-u.ac.jp, akomyo@math.kobe-u.ac.jp, mhsaito@math.kobe-u.ac.jp

Abstract. We describe some results on moduli space of logarithmic connections equipped with framings on a n -pointed compact Riemann surface.

Mathematical subject classification (2010). 53D30, 14D20, 53B15.

Funding. This work was partly supported by JSPS KAKENHI: Grant Numbers JP17H06127, JP19K03422, JP19K14506. IB is partially supported by a J. C. Bose Fellowship.

Manuscript received 28th December 2020, accepted 22nd March 2021.

1. Introduction

Our aim here is to initiate the study of logarithmic connections, on a Riemann surface, equipped with a framing. To describe these objects, let X be a compact connected Riemann surface and $D \subset X$ a finite subset. Given a holomorphic vector bundle E on X of rank r , a framing of it is an isomorphism $\phi_x : \mathbb{C}^r \rightarrow E_x$ for every $x \in D$ (see [7]). A framed logarithmic connection of rank r on (X, D) is a triple of the form (E, ∇, ϕ) , where E is a holomorphic vector bundle on X of rank r , ∇ is a logarithmic connection on E whose polar part is contained in D and ϕ is a framing of E .

It can be shown that a moduli space of framed logarithmic connections exists as a Deligne–Mumford stack. Furthermore, this moduli stack is equipped with a natural algebraic symplectic structure. We note that the moduli space of logarithmic connections has a natural Poisson structure. The forgetful map, from the moduli space of framed logarithmic connections to the moduli space of logarithmic connections that simply forgets the framing, is in fact Poisson.

Actually, the above results holds in a more general setting; see Section 2. Also, the above results can be extended to the set-up of holomorphic principal bundles with a logarithmic connection.

It is known that the moduli space of connections is not an affine variety. This should be contrasted with the fact that moduli space of connections is canonically biholomorphic to an affine scheme. Indeed, the character variety is affine, and the Riemann–Hilbert correspondence produces a biholomorphism of it with the moduli space of connections. This result is extended to the context of vector bundles equipped with parabolic structure (see Theorem 10).

The full details of the proofs will appear in [6].

2. Symplectic form on the moduli space of framed connections

Let X be a compact connected Riemann surface of genus g , and let D be a reduced effective divisor on X . The holomorphic line bundle $K_X \otimes \mathcal{O}_X(D)$, where K_X is the holomorphic cotangent bundle of X , will be denoted by $K_X(D)$. Fix a connected complex algebraic proper subgroup $H_x \subsetneq \mathrm{GL}(r, \mathbb{C})$ for each $x \in D$, and set $H = \{H_x\}_{x \in D}$.

Let E be a holomorphic vector bundle on X of rank r . A *framing* of E is an \mathcal{O}_D -linear isomorphism $\phi: E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$. Two framings ϕ and ϕ' are called equivalent if $\phi' \circ \phi^{-1}|_x \in H_x$ for all $x \in D$. A framing of E with respect to H is an equivalence class of framings of E .

Let \mathfrak{h}_x be the Lie algebra of H_x . The orthogonal complement of \mathfrak{h}_x , with respect to the trace pairing $(A, B) \mapsto \mathrm{trace}(AB)$, will be denoted by \mathfrak{h}_x^\perp .

Let $F(E) \rightarrow X$ be the frame bundle for E ; it is also the principal $\mathrm{GL}(r, \mathbb{C})$ -bundle associated to E . Giving a framing of E with respect to H is equivalent to giving a reduction of structure group of $F(E)_x \rightarrow x$ to H_x for every $x \in D$. Given a framing of E with respect to H , choose a framing ϕ in the equivalence class, and consider the corresponding isomorphism $\mathrm{Lie}(\mathrm{GL}(r, \mathbb{C})) = \mathrm{M}(r, \mathbb{C}) \rightarrow \mathrm{End}(E)_x$ for every $x \in D$. The restriction of this homomorphism to $\mathfrak{h}_x^\perp \subset \mathrm{M}(r, \mathbb{C})$ depends on the choice of ϕ in the equivalence class, but the subspace $\mathrm{image}(\mathfrak{h}_x^\perp) \subset \mathrm{End}(E)_x$ is independent of the choice of ϕ . This subspace of $\mathrm{End}(E)_x$ will be denoted by $[\phi](\mathfrak{h}_x^\perp)$. Similarly, the subspace $\mathrm{image}(\mathfrak{h}_x) \subset \mathrm{End}(E)_x$ is independent of the choice of ϕ . This subspace of $\mathrm{End}(E)_x$ will be denoted by $[\phi](\mathfrak{h}_x)$.

Definition 1. A framed connection with respect to H is a triple $(E, [\phi], \nabla)$, where

- E is a holomorphic vector bundle on X of rank r equipped with a framing $[\phi]$ with respect to H , and
- $\nabla: E \rightarrow E \otimes K_X(D)$ is a logarithmic connection on E such that $\mathrm{res}_x(\nabla) \in [\phi](\mathfrak{h}_x^\perp)$ for every $x \in D$.

The group of all automorphisms T of E preserving $[\phi]$ such that $(T \otimes \mathrm{Id}_{K_X(D)}) \circ \nabla = \nabla \circ T$ will be denoted by $\mathrm{Aut}(E, [\phi], \nabla)$.

Definition 2. A framed connection $(E, [\phi], \nabla)$ with respect to H is simple if the quotient group $\mathrm{Aut}(E, [\phi], \nabla) / (\mathbb{C}^* \mathrm{id}_E \cap \mathrm{Aut}(E, [\phi], \nabla))$ is a finite group.

Let $\mathcal{M}_{\mathrm{FC}}^H(d)$ denote the moduli space of all framed connections $(E, [\phi], \nabla)$ with respect to H such that $\mathrm{deg} E = d$.

Theorem 3. The moduli space $\mathcal{M}_{\mathrm{FC}}^H(d)$ exists as a Deligne–Mumford stack.

Given a framed connection $(E, [\phi], \nabla)$ with respect to H , we construct a complex of sheaves \mathcal{D}^\bullet as follows:

$$\begin{aligned} \mathcal{D}^0 &= \{u \in \mathcal{E}\mathrm{nd}(E) \mid u(x) \in [\phi](\mathfrak{h}_x), \forall x \in D\} \\ \mathcal{D}^1 &= \{v \in \mathcal{E}\mathrm{nd}(E) \otimes K_X(D) \mid \mathrm{res}_x(v) \in [\phi](\mathfrak{h}_x^\perp), \forall x \in D\} \\ \mathcal{D}^0 \ni u &\mapsto \nabla \circ u - (u \otimes \mathrm{id}) \circ \nabla \in \mathcal{D}^1. \end{aligned}$$

Theorem 4. *The tangent space of $\mathcal{M}_{\text{FC}}^H(d)$ at any point $(E, [\phi], \nabla) \in \mathcal{M}_{\text{FC}}^H(d)$ is identified with the hypercohomology $\mathbb{H}^1(\mathcal{D}^\bullet)$ of the above complex \mathcal{D}^\bullet .*

Consider the following homomorphism of complexes $\mathcal{D}^\bullet \otimes \mathcal{D}^\bullet \rightarrow \Omega_X^\bullet$:

$$\begin{aligned} (\mathcal{D}^\bullet \otimes \mathcal{D}^\bullet)^0 &= \mathcal{D}^0 \otimes \mathcal{D}^0 \ni f \otimes g \mapsto \text{trace}(fg) \in \mathcal{O}_X \\ (\mathcal{D}^\bullet \otimes \mathcal{D}^\bullet)^1 &= \mathcal{D}^0 \otimes \mathcal{D}^1 \oplus \mathcal{D}^1 \otimes \mathcal{D}^0 \ni (f \otimes \omega, \eta \otimes g) \mapsto \text{trace}(f\omega - g\eta) \in \Omega_X^1 \end{aligned}$$

It produces a pairing

$$\Theta^H: \mathbb{H}^1(\mathcal{D}^\bullet) \otimes \mathbb{H}^1(\mathcal{D}^\bullet) \rightarrow \mathbb{H}^2(\mathcal{D}^\bullet \otimes \mathcal{D}^\bullet) \rightarrow \mathbb{H}^2(\Omega_X^\bullet) = \mathbb{C}. \tag{1}$$

We have the following theorem.

Theorem 5. *The pairing Θ^H in (1) defines a symplectic form on $\mathcal{M}_{\text{FC}}^H(d)$.*

We give an outline of the proof of Theorem 5. We can check that Θ^H is the skew-symmetric by a calculation in Čech cohomology. The homomorphism $\mathbb{H}^1(\mathcal{D}^\bullet) \rightarrow \mathbb{H}^1(\mathcal{D}^\bullet)^*$ induced by Θ^H is isomorphic, because the vertical arrows except the middle one in the exact commutative diagram

$$\begin{array}{ccccccccc} H^0(\mathcal{D}^0) & \longrightarrow & H^0(\mathcal{D}^1) & \longrightarrow & \mathbb{H}^1(\mathcal{D}^\bullet) & \longrightarrow & H^1(\mathcal{D}^0) & \longrightarrow & H^1(\mathcal{D}^1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\mathcal{D}^1)^* & \longrightarrow & H^1(\mathcal{D}^0)^* & \longrightarrow & \mathbb{H}^1(\mathcal{D}^\bullet)^* & \longrightarrow & H^0(\mathcal{D}^1)^* & \longrightarrow & H^0(\mathcal{D}^0)^* \end{array}$$

are Serre duality isomorphisms. In other words, the 2-form Θ^H is nondegenerate.

It remains to prove that the 2-form Θ^H is d -closed. Let us consider the moduli space $\mathcal{M}_{\text{FC}}^e(d)$ of framed connections with respect to the trivial group. When $H_x = \{e\}$, then $\mathfrak{h}_x^\perp = \text{M}(r, \mathbb{C})$. Let $\mathcal{M}_{\text{FC}}^e(d)^{\mathfrak{h}^\perp} \subset \mathcal{M}_{\text{FC}}^e(d)$ be the locus of all (E, ϕ, ∇) such that $\text{res}_x(\nabla)$ lies in the image of \mathfrak{h}_x^\perp for all $x \in D$. Then we have the following diagram

$$\begin{array}{ccc} \mathcal{M}_{\text{FC}}^e(d)^{\mathfrak{h}^\perp} & \xhookrightarrow{\iota} & \mathcal{M}_{\text{FC}}^e(d) \\ \pi \downarrow & & \\ \mathcal{M}_{\text{FC}}^H(d) & & \end{array}$$

The above morphism π sends any (E, ϕ, ∇) to $(E, [\phi], \nabla)$, and it is a smooth morphism. So Θ^H is d -closed if $\pi^* \Theta^H$ is d -closed. On the other hand, we can check the equality $\pi^* \Theta^H = \iota^* \Theta^e$. So it suffices to prove the theorem for the case $H = \{e\}$. Since the moduli space $\mathcal{M}_{\text{FC}}^e(d)$ is irreducible, it is enough to prove that the restriction of Θ^e to some non-empty open subset of $\mathcal{M}_{\text{FC}}^e(d)$ is d -closed.

For simplicity, we consider the case where $g \geq 2$. Let us consider the following moduli spaces

$$\begin{aligned} \mathcal{N}(d) &= \{E \mid E \text{ is a stable bundle of rank } r \text{ and degree } d\} \\ \mathcal{N}^e(d) &= \left\{ (E, \phi) \mid \begin{array}{l} E \text{ is a stable bundle of rank } r \text{ and degree } d \\ \text{and } \phi: E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r} \text{ is an isomorphism} \end{array} \right\} \\ \mathcal{M}(d)_0 &= \left\{ (E, \nabla) \mid \begin{array}{l} E \text{ is a stable bundle of rank } r \text{ and degree } d, \\ \nabla: E \rightarrow E \otimes K_X(D) \text{ is a logarithmic connection such that} \\ \text{res}_{x_i}(\nabla) = 0 \text{ for } 1 \leq i \leq n-1 \text{ and } \text{res}_{x_n}(\nabla) = -\frac{d}{r} \text{id} \end{array} \right\} \\ \mathcal{M}_{\text{FC}}^e(d)_0 &= \{(E, \phi, \nabla) \in \mathcal{M}_{\text{FC}}^e(d) \mid (E, \phi) \in \mathcal{N}^e(d) \text{ and } (E, \nabla) \in \mathcal{M}(d)_0\}. \end{aligned}$$

Then we have the following diagram

$$\begin{array}{ccccc}
 \mathcal{M}(d)_0 & \xleftarrow{\tilde{q}} & \mathcal{M}_{\text{FC}}^e(d)_0 & \xrightarrow{\iota} & \mathcal{M}_{\text{FC}}^e(d) \\
 \uparrow s & \downarrow p_0 & \downarrow p_0^e & \swarrow p^e & \\
 \mathcal{N}(d) & \xleftarrow{q} & \mathcal{N}^e(d) & &
 \end{array}$$

whose left square is a Cartesian diagram. We can see that p_0, p_0^e are smooth morphisms and ι is a locally closed immersion. So we can take a non-empty analytic open subset (or an étale neighborhood) $U \subset \mathcal{N}(d)$ with a section $s: U \rightarrow p_0^{-1}(U)$ of p_0 . Let U^e be the pullback of U to $\mathcal{N}^e(d)$. Then s induces a section

$$s^e: U^e \rightarrow (p_0^e)^{-1}(U^e) \hookrightarrow \mathcal{M}_{\text{FC}}^e(d) \tag{2}$$

of p_0^e . Using the section s^e , we obtain the following isomorphism

$$P_1: T^*U^e \xrightarrow{\cong} (p^e)^{-1}(U^e), (y, v) \mapsto s^e(y) + v.$$

Lemma 6. *Let Φ_U be the Liouville 2-form on the cotangent bundle T^*U^e . Then,*

$$\Theta^e - (P_1^{-1})^*\Phi_U = (p^e)^*((s^e)^*\Theta^e).$$

In view of Lemma 6 it is enough to prove that $(s^e)^*\Theta^e$ is d -closed, because the Liouville form is d -closed. We can construct a two form Θ_0 on the moduli space $\mathcal{M}(d)_0$ in the same way as done in (1). By construction, it is just the pullback of the Goldman symplectic form (cf. [3, 10]) via the Riemann–Hilbert morphism. So we have $d\Theta_0 = 0$.

Lemma 7. *The equality $\iota^*\Theta^e = \tilde{q}^*\Theta_0$ holds.*

From the above lemma we can see that $\iota^*\Theta^e$ is d -closed. Therefore, so is $(s^e)^*\iota^*\Theta^e = (s^e)^*\Theta^e$, and we are done.

2.1. A Poisson map

In this subsection we assume that $H_x = \{e\}$ for all $x \in D$. Let $\mathcal{M}_C^H(d)$ be the moduli space of pairs (E, ∇) , where E is a holomorphic vector bundle on X of rank r and degree d , and ∇ is a logarithmic connection on E whose singular part is contained in D . Given any $(E, \nabla) \in \mathcal{M}_C^H(d)$, construct a complex $\tilde{\mathcal{D}}^\bullet$ as follows:

$$\tilde{\mathcal{D}}^0 = \mathcal{E}nd(E), \quad \tilde{\mathcal{D}}^1 = \mathcal{E}nd(E) \otimes K_X(D),$$

and $\tilde{\mathcal{D}}^0 \ni u \mapsto \nabla \circ u - (u \otimes \text{id}) \circ \nabla \in \tilde{\mathcal{D}}^1$. Then we have

$$T_{(E, \nabla)}\mathcal{M}_C^H(d) = \mathbb{H}^1(\tilde{\mathcal{D}}^\bullet).$$

The homomorphism of hypercohomologies corresponding to the natural inclusion of the Serre dual complex $\tilde{\mathcal{D}}^\bullet$ in $\tilde{\mathcal{D}}^\bullet$ produces a Poisson structure on the moduli space $\mathcal{M}_C^H(d)$.

For $(E, \nabla, \phi) \in \mathcal{M}_{\text{FC}}^H(d)$, the differential $T_{(E, \nabla, \phi)}\mathcal{M}_{\text{FC}}^H(d) \rightarrow T_{(E, \nabla)}\mathcal{M}_C^H(d)$ of the natural forgetful map

$$\mathcal{M}_{\text{FC}}^H(d) \rightarrow \mathcal{M}_C^H(d) \tag{3}$$

that forgets the framing coincides with the homomorphism of hypercohomologies given by the inclusion map of the complex \mathcal{D}^\bullet in $\tilde{\mathcal{D}}^\bullet$. Using this it follows that the forgetful map in (3) is Poisson.

3. Moduli space of parabolic connections

It is known that there is no non-constant global algebraic function on the moduli space of logarithmic connections with central residues on a curve of genus at least 3 [8]. On the other hand, the character variety, which is a moduli space of representations of a fundamental group, is affine. So we can see that the Riemann–Hilbert morphism, from the moduli space of connections to the character variety, is not algebraic. This non-algebraic map preserves the algebraic symplectic forms on these two moduli spaces [4].

We replace the moduli space \mathcal{M}_{FC}^B with the moduli space of the more general objects of parabolic connections. Let X be a compact Riemann surface and D be a reduced effective divisor on X . We fix data $\mathbf{v} = \{v_j^x \in \mathbb{C}\}_{1 \leq j \leq r, x \in D}$ satisfying

$$\sum_{x \in D} \sum_{j=1}^r v_j^x = 0.$$

Definition 8. We say that a triple (E, l, ∇) is a \mathbf{v} -parabolic connection if

- (i) E is a vector bundle of rank r and degree 0,
- (ii) $\nabla: E \rightarrow E \otimes K_X(D)$ is a connection admitting poles along D and
- (iii) l is a filtration $E|_D = l_1 \supset l_2 \supset \dots \supset l_r \supset l_{r+1} = 0$ satisfying the condition $(\text{res}_D(\nabla) - v_j \text{id})(l_j) \subset l_{j+1}$ for all $j = 1, \dots, r$.

If we take $\mathbf{v} = \mathbf{0}$ so that all $v_j = 0$, then a $\mathbf{0}$ -parabolic connection is equivalent to a framed connection with respect to the Borel subgroup B .

For simplicity we adopt the following genericity assumption on \mathbf{v} .

Assumption 9. For any integer $1 \leq s < r$ and for any choice of s elements $\{j_1^x, \dots, j_s^x\}$ in $\{1, \dots, r\}$ for each $x \in D$, the following holds:

$$\sum_{x \in D} \sum_{k=1}^s v_{j_k^x} \notin \mathbb{Z}.$$

Let $\mathcal{M}(\mathbf{v})$ be the moduli space of \mathbf{v} -parabolic connections. Assumption 9 ensures that any \mathbf{v} -parabolic connection is irreducible. So it is stable with respect to any parabolic weight. Let $\mathcal{M}_{SL}(\mathbf{v})$ be the moduli space of \mathbf{v} -parabolic connections (E, ∇, l) with $\det(E, \nabla) \cong (\mathcal{O}_X, d)$. There is a closed immersion $\iota: \mathcal{M}_{SL}(\mathbf{v}) \rightarrow \mathcal{M}(\mathbf{v})$, and also $\mathcal{M}_{SL}(\mathbf{v})$ is smooth.

Theorem 10. Assume that $r \geq 2, n \geq 1$ and $g \geq 2$. Then the moduli space $\mathcal{M}_{SL}(\mathbf{v})$ of \mathbf{v} -connections with the trivial determinant is not affine.

Remark 11.

- (1) There is a Riemann–Hilbert morphism from the moduli space of \mathbf{v} -parabolic connections to the character variety parameterizing the representations of the fundamental group $\pi_1(X \setminus D)$ with a fixed local monodromy data. If we assume, in addition to Assumption 9, that $v_j^x - v_k^x \notin \mathbb{Z}$ for $j \neq k$, then the Riemann–Hilbert morphism is an analytic isomorphism. Since the character variety is affine, the property that $\mathcal{M}_{SL}(\mathbf{v})$ is not affine implies that the Riemann–Hilbert morphism is not algebraic.
- (2) For a moduli space of framed connections with respect to the Borel subgroup B it is better to consider a stability with respect to a parabolic weight. In that case, Assumption 9 does not hold nor the assumption in (1) above. The moduli space of \mathbf{v} -parabolic connections in the case of non-generic exponent \mathbf{v} , involving the case of $g = 0$ is also interesting. Indeed it contains loci of some special solutions of the isomonodromy equations, which is like Riccati loci in Painlevé VI equations (see [12, 16]). The property of global algebraic functions on the moduli space of connections in a special case of $g = 0$ is also used in [1].

- (3) The formulation of the isomonodromy equation is given in [11], [12] and [13], but it is also formulated in [5] and [9]. Classically the isomonodromy equation is known to be characterized by the isomonodromy 2-form (cf. [14, 15]). Since the isomonodromy equation gives an algebraic splitting of the tangent bundle of the moduli space of connections, we can construct the isomonodromy 2-form from the symplectic form constructed in Section 2.

We will give an outline of the proof of Theorem 10, using the following proposition.

Proposition 12. *Under the assumption in Theorem 10, consider the locus Z in $\mathcal{M}_{SL}(\mathbf{v})$ consisting of \mathbf{v} -parabolic connections (E, ∇, l) whose underlying quasi-parabolic vector bundle (E, l) is not simple. Then the codimension of Z in $\mathcal{M}_{SL}(\mathbf{v})$ is at least 2.*

We denote by $\mathcal{N}_{\text{par}}^{\text{spl}}$ the moduli space of simple quasi-parabolic bundles (E, l) such that $\det E = \mathcal{O}_X$. Let $\mathcal{M}_{SL}^{\mathcal{N}\text{-spl}}(\mathbf{v})$ be the moduli space of \mathbf{v} -parabolic connections with the trivial determinant such that the underlying quasi-parabolic bundle (E, l) is simple. Then there is a morphism $\mathcal{M}_{SL}^{\mathcal{N}\text{-spl}}(\mathbf{v}) \rightarrow \mathcal{N}_{\text{par}}^{\text{spl}}$ which is an affine space bundle. There is a universal family (\tilde{E}, \tilde{l}) on $X \times \mathcal{N}_{\text{par}}^{\text{spl}}$. Let $\text{At}(\tilde{E})$ be the Atiyah bundle of \tilde{E} introduced in [2], which fits in the short exact sequence

$$0 \rightarrow \mathcal{E}nd(\tilde{E}) \rightarrow \text{At}(\tilde{E}) \rightarrow T_{X \times \mathcal{N}_{\text{par}}^{\text{spl}} / \mathcal{N}_{\text{par}}^{\text{spl}}} \rightarrow 0.$$

We set

$$\begin{aligned} \mathcal{E}nd_{\text{sl}}(\tilde{E}, \tilde{l}) &:= \left\{ a \in \mathcal{E}nd(\tilde{E}) \mid \text{Tr}(a) = 0 \text{ and } a|_{D \times \mathcal{N}_{\text{par}}^{\text{spl}}}(\tilde{l}_j) \subset \tilde{l}_j \text{ for any } j \right\} \\ \mathcal{E}nd_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) &:= \left\{ a \in \mathcal{E}nd(\tilde{E}) \mid \text{Tr}(a) = 0 \text{ and } a|_{D \times \mathcal{N}_{\text{par}}^{\text{spl}}}(\tilde{l}_j) \subset \tilde{l}_{j+1} \text{ for any } j \right\}. \end{aligned}$$

We can define a subsheaf $\text{At}(\tilde{E}, \tilde{l}) \subset \text{At}(\tilde{E})$ fitting in the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}nd_{\text{sl}}(\tilde{E}, \tilde{l}) & \longrightarrow & \text{At}(\tilde{E}, \tilde{l}) & \longrightarrow & T_{X \times \mathcal{N}_{\text{par}}^{\text{spl}} / \mathcal{N}_{\text{par}}^{\text{spl}}}(-D \times \mathcal{N}_{\text{par}}^{\text{spl}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}nd(\tilde{E}) & \longrightarrow & \text{At}(\tilde{E}) & \longrightarrow & T_{X \times \mathcal{N}_{\text{par}}^{\text{spl}} / \mathcal{N}_{\text{par}}^{\text{spl}}} \longrightarrow 0 \end{array}$$

Tensoring $K_X(D)$, we get an exact sequence

$$0 \rightarrow \mathcal{E}nd_{\text{sl}}(\tilde{E}, \tilde{l}) \otimes K_X(D) \rightarrow \text{At}(\tilde{E}, \tilde{l}) \otimes K_X(D) \rightarrow \mathcal{O}_{X \times \mathcal{N}_{\text{par}}^{\text{spl}}} \rightarrow 0,$$

from which we get a short exact sequence

$$0 \rightarrow \pi_* (\mathcal{E}nd_{\text{sl}}(\tilde{E}, \tilde{l}) \otimes K_X(D)) \rightarrow \pi_* (\text{At}(\tilde{E}, \tilde{l}) \otimes K_X(D)) \rightarrow \pi_* (\mathcal{O}_{X \times \mathcal{N}_{\text{par}}^{\text{spl}}}) \rightarrow 0.$$

We put $\mathcal{Q} := \pi_* (\text{At}(\tilde{E}, \tilde{l}) \otimes K_X(D))$, and consider the projective bundle

$$\mathbb{P}_*(\mathcal{Q}) = \text{Proj}(\text{Sym}^*(\mathcal{Q}^\vee))$$

where $\text{Sym}^*(\mathcal{Q}^\vee)$ is the symmetric algebra of \mathcal{Q}^\vee over $\mathcal{O}_{\mathcal{N}_{\text{par}}^{\text{spl}}}$. There is a tautological sub line bundle

$$\mathcal{O}_{\mathbb{P}_*(\mathcal{Q})}(-1) \hookrightarrow Q \otimes \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})}.$$

There are induced sections

$$\begin{aligned} \epsilon : \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})}(-1) &\hookrightarrow Q \otimes \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})} \rightarrow \pi_* \left(\mathcal{O}_{X \times \mathcal{N}_{\text{par}}^{\text{spl}}} \right) \otimes \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})} = \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})} \\ \tilde{\nu}_j : \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})}(-1) &\hookrightarrow Q \otimes \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})} \xrightarrow{\text{res}_D} \mathcal{E}nd_{\text{sl}}(\tilde{E}, \tilde{l})|_{D \times \mathbb{P}_*(\mathcal{Q})} \rightarrow \mathcal{E}nd(\tilde{l}_j / \tilde{l}_{j+1}) \otimes \mathcal{O}_{\mathbb{P}_*(\mathcal{Q})} = \mathcal{O}_{D \times \mathbb{P}_*(\mathcal{Q})} \end{aligned}$$

Let \mathcal{I} be the ideal sheaf of the graded $\mathcal{O}_{\mathcal{N}_{\text{par}}^{\text{spl}}}$ -algebra $\text{Sym}^*(\mathcal{Q}^\vee)$ generated by

$$\{\tilde{v}_j|_x - v_j|_x \epsilon \mid x \in D, 1 \leq j \leq r\}.$$

Then there is a short exact sequence

$$0 \longrightarrow \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I} \xrightarrow{\epsilon} \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I} \longrightarrow \text{Sym}^*\left(\pi_*\left(\mathcal{E}\text{nd}_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) \otimes K_X(D)\right)\right) \longrightarrow 0. \tag{4}$$

We put

$$\overline{\mathcal{M}'} := \text{Proj}(\text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}) \subset \mathbb{P}_*(\mathcal{Q}).$$

Let $Y \subset \overline{\mathcal{M}'}$ be the effective divisor defined by $\epsilon = 0$. There is a canonical isomorphism $\mathcal{M}_{\text{SL}}^{\mathcal{N}\text{-spl}}(\mathbf{v}) \cong \overline{\mathcal{M}'} \setminus Y \cong \text{Spec}(\text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I})_{(\epsilon)}$, where $(\text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I})_{(\epsilon)}$ is the degree zero component of the localized graded ring $(\text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I})_\epsilon$.

Suppose that $\mathcal{M}_{\text{SL}}(\mathbf{v})$ is affine. Then the ring of global algebraic functions

$$A = \Gamma(\mathcal{M}_{\text{SL}}(\mathbf{v}), \mathcal{O}_{\mathcal{M}_{\text{SL}}(\mathbf{v})})$$

is a finitely generated \mathbb{C} -algebra of Krull dimension $2(r^2 - 1)(g - 1) + r(r - 1)n$. By Proposition 12,

$$A = \Gamma\left(\mathcal{M}_{\text{SL}}^{\mathcal{N}\text{-spl}}(\mathbf{v}), \mathcal{O}_{\mathcal{M}_{\text{SL}}^{\mathcal{N}\text{-spl}}(\mathbf{v})}\right) = \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, (\text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I})_{(\epsilon)}\right) = \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right)_{(\epsilon)}.$$

Since A is a finitely generated \mathbb{C} -algebra, it can be proved that $\Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right)$ is a finitely generated graded \mathbb{C} -algebra. So the Krull dimension of $\Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right)$ should be $\text{Krull-dim}(A) + 1$. From the exact sequence (4), we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right) &\xrightarrow{\epsilon} \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right) \\ &\longrightarrow \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*\left(\pi_*\left(\mathcal{E}\text{nd}_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) \otimes \Omega_X^1(D)\right)\right)\right). \end{aligned}$$

So we have

$$\begin{aligned} \text{Krull-dim}(A) &= \text{Krull-dim}\left(\Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right) / \epsilon \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right)\right) \\ &= \text{tr. deg}_{\mathbb{C}}\left(\Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right) / \epsilon \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*(\mathcal{Q}^\vee)/\mathcal{I}\right)\right) \\ &\leq \text{tr. deg}_{\mathbb{C}} \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*\left(\pi_*\left(\mathcal{E}\text{nd}_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) \otimes \Omega_X^1(D)\right)\right)\right) \end{aligned} \tag{5}$$

On the other hand, let $\mathcal{M}_{\text{SL,Higgs}}^{\text{spl}}$ be the moduli space of simple parabolic Higgs bundles (E, Φ, l) such that $\Phi|_D(l_j) \subset l_{j+1}$ for $1 \leq j \leq r$, $\det(E) = \mathcal{O}_X$ and $\text{trace}(\Phi) = 0$. It contains the moduli space $\mathcal{M}_{\text{SL,Higgs}}^\alpha$ of α -stable parabolic Higgs bundles for any choice of parabolic weight α . Let $\mathcal{M}_{\text{SL,Higgs}}^{\mathcal{N}\text{-spl}}$ be the open subspace of $\mathcal{M}_{\text{SL,Higgs}}^{\text{spl}}$ consisting of those whose underlying quasiparabolic bundle (E, l) is simple. Then $\mathcal{M}_{\text{SL,Higgs}}^{\mathcal{N}\text{-spl}}$ is isomorphic to the cotangent bundle of $\mathcal{N}_{\text{par}}^{\text{spl}}$, which means that

$$\mathcal{M}_{\text{SL,Higgs}}^{\mathcal{N}\text{-spl}} \cong \text{Spec}\left(\text{Sym}^*\left(\pi_*\left(\mathcal{E}\text{nd}_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) \otimes \Omega_X^1(D)\right)\right)\right).$$

We have the codimension estimation for the moduli space of parabolic Higgs bundles which is similar to Proposition 12. So we have

$$\begin{aligned} \Gamma\left(\mathcal{N}_{\text{par}}^{\text{spl}}, \text{Sym}^*\left(\pi_*\left(\mathcal{E}\text{nd}_{\text{sl}}^{\text{nil}}(\tilde{E}, \tilde{l}) \otimes \Omega_X^1(D)\right)\right)\right) &\cong \Gamma\left(\mathcal{M}_{\text{SL,Higgs}}^{\mathcal{N}\text{-spl}}, \mathcal{O}_{\mathcal{M}_{\text{SL,Higgs}}^{\mathcal{N}\text{-spl}}}\right) \\ &= \Gamma\left(\mathcal{M}_{\text{SL,Higgs}}^{\text{spl}}, \mathcal{O}_{\mathcal{M}_{\text{SL,Higgs}}^{\text{spl}}}\right) \\ &\subseteq \Gamma\left(\mathcal{M}_{\text{SL,Higgs}}^\alpha, \mathcal{O}_{\mathcal{M}_{\text{SL,Higgs}}^\alpha}\right). \end{aligned} \tag{6}$$

Choosing α generically, we may assume that the α -semistability implies α -stability. So the Hitchin map

$$\mathcal{M}_{\text{SL}, \text{Higgs}}^{\alpha} \longrightarrow \bigoplus_{k=2}^r H^0\left(X, K_X^{\otimes k}((k-1)D)\right)$$

is a proper morphism. Hence the \mathbb{C} -algebra $\Gamma\left(\mathcal{M}_{\text{SL}, \text{Higgs}}^{\alpha}, \mathcal{O}_{\mathcal{M}_{\text{SL}, \text{Higgs}}^{\alpha}}\right)$ is a finite algebra over

$$\Gamma\left(\bigoplus_{k=2}^r H^0\left(X, K_X^{\otimes k}((k-1)D)\right), \mathcal{O}_{\bigoplus_{k=2}^r H^0\left(X, K_X^{\otimes k}((k-1)D)\right)}\right),$$

whose transcendence degree is the same as its Krull dimension $r^2(g-1) - g + 1 + nr(r-1)/2$. Combining (5) and (6) we get the inequality

$$\begin{aligned} 2(r^2 - 1)(g - 1) + r(r - 1)n &= \text{Krull-dim}(A) \leq \text{tr. deg}_{\mathbb{C}} \Gamma\left(\mathcal{M}_{\text{SL}, \text{Higgs}}^{\alpha}, \mathcal{O}_{\mathcal{M}_{\text{SL}, \text{Higgs}}^{\alpha}}\right) \\ &\leq (r^2 - 1)(g - 1) + \frac{nr(r - 1)}{2}, \end{aligned}$$

which is a contradiction.

References

- [1] D. Arinkin, “Orthogonality of natural sheaves on moduli stacks of $SL(2)$ -bundles with connections on \mathbb{P}^1 minus 4 points”, *Sel. Math., New Ser.* **7** (2001), no. 2, p. 213-239.
- [2] M. F. Atiyah, “Complex analytic connections in fibre bundles”, *Trans. Am. Math. Soc.* **85** (1957), p. 181-207.
- [3] M. F. Atiyah, R. Bott, “The Yang–Mills equations over Riemann surfaces”, *Philos. Trans. R. Soc. Lond., Ser. A* **308** (1983), p. 523-615.
- [4] I. Biswas, “On the moduli space of holomorphic G -connections on a compact Riemann surface”, *Eur. J. Math.* **6** (2020), no. 2, p. 321-335.
- [5] I. Biswas, V. Heu, J. Hurtubise, “Isomonodromic deformations of logarithmic connections and stability”, *Math. Ann.* **366** (2016), no. 1-2, p. 121-140.
- [6] I. Biswas, M.-A. Inaba, A. Komyo, M.-H. Saito, in preparation.
- [7] I. Biswas, M. Logares, A. Peón-Nieto, “Moduli spaces of framed G -Higgs bundles and symplectic geometry”, *Commun. Math. Phys.* **376** (2020), no. 3, p. 1875-1908.
- [8] I. Biswas, N. Raghavendra, “Line bundles over a moduli space of logarithmic connections on a Riemann surface”, *Geom. Funct. Anal.* **15** (2005), no. 4, p. 780-808.
- [9] T. Chen, “The associated map of the nonabelian Gauss–Manin connection”, *Cent. Eur. J. Math.* **10** (2012), no. 4, p. 1407-1421.
- [10] W. M. Goldman, “The symplectic nature of fundamental groups of surfaces”, *Adv. Math.* **54** (1984), p. 200-225.
- [11] M.-A. Inaba, “Moduli of parabolic connections on a curve and Riemann–Hilbert correspondence”, *J. Algebr. Geom.* **22** (2013), p. 407-480.
- [12] M.-A. Inaba, K. Iwasaki, M.-H. Saito, “Dynamics of the sixth Painlevé equation”, in *Asymptotic theories and Painlevé equations*, Séminaires et Congrès, vol. 14, Société Mathématique de France, 2006, p. 103-167.
- [13] ———, “Moduli of stable parabolic connections, Riemann–Hilbert correspondence and geometry of Painlevé equation of type VI. I.”, *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 4, p. 987-1089.
- [14] K. Iwasaki, “Fuchsian moduli on a Riemann surface – Its Poisson structure and Poincaré–Lefschetz duality”, *Pac. J. Math.* **155** (1992), no. 2, p. 319-340.
- [15] M. Jimbo, T. Miwa, K. Ueno, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ -function”, *Physica D* **2** (1981), no. 2, p. 306-352.
- [16] M.-H. Saito, H. Terajima, “Nodal curves and Riccati solutions of Painlevé equations”, *J. Math. Kyoto Univ.* **44** (2004), no. 3, p. 529-568.