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A new note on factored infinite series and trigonometric Fourier series

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Abstract. In this paper, we have proved two main theorems under more weaker conditions dealing with absolute weighted arithmetic mean summability factors of infinite series and trigonometric Fourier series. We have also obtained certain new results on the different absolute summability methods.

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1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By t_n^α we denote the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is (see [21])

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \ ({t_n}^1 = t_n) \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [23])

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad \left(P_{-i} = p_{-i} = 0, \quad i \geq 1\right). \tag{4}$$

The sequence-to-sequence transformation $(s_n) \to (v_n)$ with

$$v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (v_n) of weighted arithmetic mean or simply the $(\overline{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [24]). If we write $X_n = \sum_{v=0}^{n} \frac{p_v}{P_v}$, then (X_n) is a positive increasing sequence tending to infinity *as* $n \to \infty$. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |v_n - v_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all n (resp. $k = 1$), $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\overline{N}, p_n|$ (see [35]) summability. Also if we take $p_n = \frac{1}{n+1}$ and $k = 1$, then we obtain $|R, \log n, 1|$ summability (see [1]).

For any sequence (λ_n) we write that $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

2. The known result

Many works dealing with the absolute summability factors of infinite series and Fourier series have been done (see [2,4–20,25–34,36–40]). Among them, in [12], the following theorem has been proved.

Theorem 1. *Let (X_n) be a positive increasing sequence and let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{6}$$

If the conditions

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty \tag{8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

hold, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Remark. It should be noted that, in Theorem 1, there is a restriction on the sequence (p_n). Therefore, due to restriction (6) on (p_n) no result for $p_n = \frac{1}{n+1}$ can be deduced from Theorem 1.

3. The main result

The aim of this paper is to obtain a further generalization of Theorem 1 under weaker conditions. In this case, there is not any restriction on the sequence (p_n). It is clear that (6) and (9) imply the following formula

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty. \tag{10}$$

Also(6) implies that

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty. \tag{11}$$

It should be remarked that (6) implies (11) but the converse needs not be true (see [27]).

Now we shall prove the following general theorem.

Theorem 2. *If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions* (7)–(11)*, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

We need the following lemma for the proof of Theorem 2.

Lemma 3 (cf. [6]). *Under the conditions of Theorem 2, we get*

$$nX_n|\Delta\lambda_n| = O(1), \quad as \quad n \to \infty \tag{12}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty \tag{13}$$

$$X_n|\lambda_n| = O(1) \quad as \quad n \to \infty. \tag{14}$$

4. Proof of Theorem 2

Let (T_n) be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{15}$$

Then, for $n \geq 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v} va_v. \tag{16}$$

Applying Abel's transformation to the right-hand side of (16), we have

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}\lambda_v}{v}\right)\sum_{r=1}^{v} ra_r + \frac{p_n\lambda_n}{nP_n}\sum_{v=1}^{n} va_v \\
&= \frac{(n+1)p_nt_n\lambda_n}{nP_n} - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vt_v\lambda_v\frac{v+1}{v} \\
&\quad + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_vt_v\frac{v+1}{v} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\lambda_{v+1}t_v\frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,r}\right|^k < \infty, \quad \text{for} \quad r = 1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{p_n}{P_n}|t_n|^k = O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}\frac{|t_n|^k}{{X_n}^{k-1}} \\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{p_v}{P_v}\frac{|t_v|^k}{{X_v}^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{{X_n}^{k-1}} \\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by the hypotheses of Theorem 2 and Lemma 3. Also, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{{X_v}^{k-1}} = O(1) \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Again, by using (11), we obtain that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}v|\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}(v|\Delta\lambda_v|)^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{|t_r|^k}{r{X_r}^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by the hypotheses of Theorem 2 and Lemma 3. Finally by using (11), as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}|^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{v{X_v}^{k-1}}=O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}$$

This completes the proof of Theorem 2.

If we take $p_n = 1$ for all n, then we obtain a new result dealing with $|C,1|_k$ summability factors of infinite series. Also if we set $k = 1$, then we obtain a new result concerning the $|\overline{N},p_n|$ summability factors of infinite series. Finally, if we take $p_n = \frac{1}{n+1}$ and $k = 1$, then we obtain a new result for $|R,\log n,1|$ summability of factored infinite series.

5. An application to trigonometric Fourier series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x),$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\mathrm{d}x,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\mathrm{d}x,\quad\text{and}\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\mathrm{d}x.$$

Write $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$, and $\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\mathrm{d}u$, $(\alpha > 0)$.

It is known that if $\phi_1(t) \in \mathscr{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nA_n(x))$ (see [22]). Using this fact, we have obtained the following theorem dealing with trigonometric Fourier series.

Theorem 4 (cf. [12]). *If $\phi_1(t) \in \mathscr{BV}(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of the Theorem 1, then the series $\sum A_n(x)\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

Now, we can generalize Theorem 4 under weaker conditions in the following form.

Theorem 5. *If $\phi_1(t) \in \mathscr{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2, then the series $\sum A_n(x)\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

In the special cases of (p_n) and k as in Theorem 2, we can obtain similar results from Theorem 5 for the trigonometric Fourier series.

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