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Sur la fonction de co-croissance des algèbres et son écart logarithmique

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Abstract. Let $A \cong k\langle X\rangle/I$ be an associative algebra. A finite word over alphabet X is I*-reducible* if its image in A is a k-linear combination of length-lexicographically lesser words. An *obstruction* is a subword-minimal I-reducible word. If the number of obstructions is finite then I has a finite Gröbner basis, and the word problem for the algebra is decidable. A *cogrowth* function is the number of obstructions of length $\leq n$. We show that the cogrowth function of a finitely presented algebra is either bounded or at least logarithmical. We also show that an uniformly recurrent word has at least logarithmical cogrowth.

Résumé. Soit $A \cong k\langle X\rangle/I$ une algèbre associative. Un mot fini sur l'alphabet X est I*-réductible* si son image dans A est une combinaison linéaire k de mots de longueur lexicographiquement moindre. Une *obstruction* dans un mot minimal I -réductible. Si le nombre d'obstructions est fini, alors I a une base finie Gröbner, et le mot problème pour l'algèbre est décidable. Une fonction *co-croissance* est le nombre d'obstructions de longueur $\leq n$. Nous montrons que la fonction de co-croissance d'une algèbre finement présentée est soit bornée, soit au moins logarithmique. Nous montrons également qu'un mot uniformément récurrent a au moins une co-croissance logarithmique.

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1. Cogrowth of associative algebras

Let A be a finitely generated associative algebra over a field k. Then $A \cong k\langle X\rangle/I$, where $k\langle X\rangle$ is a free algebra with generating set $X = \{x_1, \dots, x_s\}$ and I is a two-sided *ideal of relations*. Further we assume the generating set is fixed. Let "$\prec$" be a well-ordering of X, $x_1 \prec \cdots \prec x_s$. This order can be extended to a linear order on the set $\langle X\rangle$ of monomials of $k\langle X\rangle$, i.e. finite words in alphabet X: $u_1 \prec u_2$ if $|u_1| < |u_2|$ or $|u_1| = |u_2|$ and $u_1 <_{lex} u_2$. Here $|\cdot|$ denotes the length of a word, i.e. the degree of a monomial, and $<_{lex}$ is the lexicographical order. For $f \in k\langle X\rangle$ we denote by $\widehat{f}$ its leading (with respect to $\prec$) monomial. An algebra $k\langle X\rangle/I$ is said to be *finitely presented* if I is a finitely generated ideal.

We call a monomial $w \in \langle X\rangle$ I*-reducible* if $w = \widehat{f}$ for some relation $f \in I$. In the opposite case, we call w I*-irreducible*. Denote the set of all monomials of degree at most n by $\langle X\rangle_{\le n}$. Let $A_n \subseteq A$ be the image of $\langle X\rangle_{\le n}$ under the canonical map. The *growth* $V_A(n)$ is the dimension of the linear span of A_n. It is easily shown that $V_A(n)$ is equal to the number of I-irreducible monomials in $\langle X\rangle_{\le n}$.

We call a monomial $w \in \langle X\rangle$ an *obstruction* for I if w is I-reducible, but any proper subword of w is I-irreducible. The *cogrowth* of algebra A is defined as the function $O_A(n)$, the number of obstructions of length $\leqslant n$.

The celebrated Bergman gap theorem says that the growth function $V_A(n)$ is either constant, linear of no less than $(n+1)(n+2)/2$ [2]. In this section we give a non-trivial bound on the cogrowth function for finitely presented algebras.

Theorem 1. *Let A be a finitely presented algebra. Then the cogrowth function $O_A(n)$ is either constant or no less than logarithmic: $O_A(n) \ge \log_2(n) - C$. The constant C depends only on the maximal length of a relation.*

Recall that a *Gröbner basis* of an ideal I is a subset $G \subseteq I$ such that for any $f \in I$ there exists $g \in G$ such that the leading monomial of f contains the leading monomial of g as a subword. One of Gröbner bases can be obtained by taking for each obstruction u a relation $f_u \in I$ such that $\widehat{f_u} = u$.

If f and g are two elements of $k\langle X\rangle$, $g \in I$ and the word $\widehat{g}$ is a subword of $\widehat{f}$, then f can be replaced by f' such that $f' - f \in I$ and $\widehat{f'} \prec \widehat{f}$. This operation is called a *reduction*.

Let f and g be two elements of $k\langle X\rangle$. If $u_1u_2 = \widehat{f}$ and $u_2u_3 = \widehat{g}$ for some $u_1, u_2, u_3 \in \langle X\rangle$, then the word $u_1u_2u_3$ is called a *composition* of f and g, and the normed element $fu_3 - u_1g$ is the *result of this composition*.

Lemma 2 (Diamond Lemma [3]). *Let two-sided ideal I be generated by a subset U of a free associative algebra $k\langle X\rangle$. Suppose that*

(i) *there are no $f, g \in U$ such that $\widehat{g}$ is a proper subword of $\widehat{f}$, and*

(ii) *for any two elements $f, g \in U$ the result of any their composition can be reduced to* 0 *after finitely many reductions with elements from U.*

Then the set U is a Gröbner basis of I.

Example. Consider the associative algebra $A \cong k\langle x, y\rangle/I$, where I is a two-sided ideal generated by $f = x^2 - yx$. It can be shown that the set $\{xy^ix - y^{i+1}x \mid i \ge 0\}$ is a Gröbner basis of I, so $O_A(n) = n - 1$ for $n \ge 2$. A monomial is I-irreducible if and only if it contains at most one letter x, hence $V_A(n) = (n+1)(n+2)/2$.

Theorem 1 directly follows from

Lemma 3. *Let $A \cong k\langle X\rangle/I$ be a finitely presented algebra and let N be greater than the maximal length of its defining relation. Suppose there are no obstructions of length from the interval $[N, 2N]$. Then I has a finite Gröbner basis.*

Proof. Let S be the set of all obstructions in $\langle X\rangle_{\leq N}$. Take for each monomial $w \in S$ a relation f_w such that $\widehat{f_w} = w$. Let us show that this set $\{f_w \mid s \in S\}$ forms a Gröbner basis for I. Indeed, I is generated by the set $\{f_w \mid w \in S\}$. The condition (i) of the Diamond Lemma holds automatically because no obstruction can be a proper subword of another obstruction. Let us check the condition (ii).

Let $u, v \in S$ and let h be the result of some composition of f_u and f_v. It is clear that the leading monomial of h has length less then $2N$. We start reducing h with elements from $\{f_w \mid w \in S\}$. After finally many steps we obtain either 0 or an element h' such that $\widehat{h}'$ does not contain subwords from S. But since there are no obstructions from $[N, 2N]$, the second case is impossible. □

The *word problem* for a finitely presented algebra, i.e. the question whether a given element $f \in k\langle X\rangle$ lies in I, is undecidable in the general case. But if I has a finite Gröebner basis G, then A has a decidable word problem. Note also that the problem whether a given element in a finitely presented associative algebra is a zero divisor (or is it nilpotent) is undecidable, even if we are given a finite Gröebner basis [6]. But if the ideal of relations is generated by monomials and has a finite Gröebner basis, the nilpotency problem is algorithmically decidable [2].

2. Colength of a period

A *monomial algebra* is a finitely generated associative algebra whose defining relations are monomials. Let u be a finite word in alphabet X and let A_u be the algebra $k\langle X\rangle/I$, where I is generated by the set of monomials that are not subwords of the periodic sequence u^∞. Such algebras A_u play important role in the study of monomial algebras [2].

Let W be a sequence on alphabet X, i.e. a map $X^{\mathbb{N}}$. A finite word v is an *obstruction* for W if v is not a subword of W but any proper subword v' of v is a subword of W. If u is a finite word, the number of obstructions for u^∞ is always finite. We call this number the *colength* of the period u. We say that the period is *defined by* the set of obstructions.

In [5], G. R. Chelnokov proved that a sequence of minimal period n cannot be defined by fewer than $\log_2 n + 1$ obstructions. G. R. Chelnokov also gave for infinitely many n_i an example of a binary sequence with minimal period n_i and colength of the period $\log_\varphi n_i$, where $\varphi = \frac{\sqrt{5}+1}{2}$. P. A. Lavrov found the precise lower estimation for colength of period.

Theorem 4 (cf. [7]). *Let $A = \{a, b\}$ be a binary alphabet. Let u be a word of length n and colength c, then $\varphi_c \geq n$, where φ_c is the c-th Fibonacci number ($\varphi_1 = 1$, $\varphi_2 = 2$, $\varphi_3 = 3$, $\varphi_4 = 5$ etc.).*

The case of an arbitrary alphabet was considered in [8] by P. A. Lavrov and independently in [4] by I. I. Bogdanov and G. R. Chelnokov.

3. Cogrowth function for an uniformly recurrent sequence

A sequence of letters W on a finite alphabet is called *uniformly recurrent* (u.r. for brevity) if for any finite subword u of W there exists a number $C(u, W)$ such that any subword of W having length $C(u, W)$ contains u as a subword. This property can be considered as a generalization of periodicity [9].

For a sequence of letters W denote by A_W the algebra $k\langle X\rangle/I_W$, where I_W is generated by the set of monomials that are not subwords of W. A monomial algebra A is called *almost simple* if each of its proper factor algebras $B = A/I$ is nilpotent. In [2] it was shown that almost simple monomial algebras are algebras of the form A_W, where W is an u.r. sequence.

Again, a finite word u is an *obstruction* for W if it is not a subword of W but any its proper subword is a subword of W. The *cogrowth function* $O_W(n)$ is the number of obstructions with length $\leq n$.

Theorem 5. *Let W be an u.r. non-periodic sequence on a binary alphabet. Then* $\overline{\lim}_{n\to\infty} O_W(n)/\log_3 n \geq 1$.

A *factorial language* is a set $\mathscr{U}$ of finite words such that for any $u \in \mathscr{U}$ all subwords of u also belong to $\mathscr{U}$. Denote by $\mathscr{U}_k$ the words of $\mathscr{U}$ having length k. A finite word u is called an *obstruction* for $\mathscr{U}$ if $u \notin \mathscr{U}$, but any proper subword belongs to $\mathscr{U}$. Denote the factorial language consisting of all subwords of a given sequence W by $\mathscr{L}(W)$. To prove Theorem 5 we will assume the contrary and construct an infinite factorial language that is a proper subset of $\mathscr{L}(W)$.

Let $\mathscr{U}$ be a factorial language and k be an integer. The *Rauzy graph* $R_k(\mathscr{U})$ of order k is the directed graph with vertex set $\mathscr{U}_k$ and edge set $\mathscr{U}_{k+1}$. Two vertices u_1 and u_2 of $R_k(\mathscr{U})$ are connected by an edge u_3 if and only if $u_3 \in \mathscr{U}$, u_1 is a prefix of u_3, and u_2 is a suffix of u_3.

For a sequence W we denote the graph $R_k(\mathscr{L}(W))$ by $R_k(W)$. Further the word *graph* will always mean a directed graph, the word *path* will always mean a *directed path* in a directed graph. The *length* $|p|$ of a path p is the number of its vertices, i.e. the number of edges plus one. If a path p_2 starts at the end of a path p_1, we denote their concatenation by p_1p_2. Recall that a directed graph is *strongly connected* if for every pair of vertices $\{v_1, v_2\}$ it contains a directed path from v_1 to v_2 and a directed path from v_2 to v_1. It is clear that any Rauzy graph of an u.r. non-periodic sequence is a strongly connected digraph and is not a cycle.

Given a directed graph H, its *directed line graph* $L(H)$ is a directed graph such that each vertex of $L(H)$ represents an edge of H, and two vertices of $L(H)$ that represent edges e_1 and e_2 of H are connected by an arrow from e_1 to e_2 if and only if the head of e_1 meets the tail of e_2. For any $k > 0$ there is one-to-one correspondence between paths of length k in $L(H)$ and paths of length $k+1$ in H.

Let $\mathscr{U}$ be a factorial language and let $m \geqslant n$. A word $a_1 \dots a_m \in \mathscr{U}_m$ corresponds to a path of length $m-n+1$ in $R_n(\mathscr{U})$, this path visits vertices $a_1 \dots a_n$, $a_2 \dots a_{n+1}$, $\dots$, $a_{m-n+1} \dots a_m$. The graph $R_m(\mathscr{U})$ can be considered as a subgraph of $L^{m-n}(R_n(\mathscr{U}))$. Moreover, the graph $R_{n+1}(\mathscr{U})$ is obtained from $L(R_n(\mathscr{U}))$ by removing edges that correspond to obstructions of length $n+1$.

We call a vertex v of a directed graph H *a fork* if v has out-degree more than one. Furthermore we assume that all forks have out-degrees exactly 2 (this is the case of a binary alphabet). For a directed graph H we define its *entropy regulator*: $er(H)$ is the minimal integer such that any directed path of length $\mathrm{er}(H)$ in H contains at least one vertex that is a fork in H.

Proposition 6. *Let H be a strongly connected digraph that is not a cycle. Then* $\mathrm{er}(H) < \infty$.

Proof. Assume the contrary. Let n be the total number of vertices in H. Consider a path of length $n+1$ in H that does not contain forks. Note that this path visits some vertex v at least twice. This means that starting from v it is possible to obtain only vertices of this cycle. Since the graph H is strongly connected, H coincides with this cycle. □

Lemma 7. *Let H be a strongly connected digraph,* $\mathrm{er}(H) = K$. *Then* $\mathrm{er}(L(H)) = K$.

Proof. The forks of the digraph $L(H)$ are edges in H that end at forks. Consider K vertices forming a path in $L(H)$. This path corresponds to a path of length $K+1$ in H. Since $\mathrm{er}(H) \leq K$, there exists an edge of this path that ends at a fork. □

Lemma 8. *Let H be a strongly connected digraph,* $\mathrm{er}(H) = K$, *let v be a fork in H, the edge e starts at v. Let the digraph H^* be obtained from H by removing the edge e. Let G be a subgraph of H^* that consists of all vertices and edges reachable from v. Then G is a strongly connected digraph. Also G is either a cycle of length at most K, or* $\mathrm{er}(G) \leq 2K$.

Proof. First we prove that the digraph G is strongly connected. Let v' be an arbitrary vertex of G, then there is a path in G from v to v'. Consider a path p of minimum length from v' to v in H. Such a path exists, for otherwise H is not strongly connected. The path p does not contain the edge e, for otherwise it could be shortened. This means that p connects v' to v in the digraph G. From any vertex of G we can reach the vertex v, hence G is strongly connected.

Consider an arbitrary path p of length $2K$ in the digraph G, suppose that p does not have forks. Since $\mathrm{er}(H) = K$, then in p there are two vertices v_1 and v_2 which are forks in H and there are no forks in p between v_1 and v_2. The out-degrees of all vertices except v coincide in H and G. If $v_1 \neq v$ or $v_2 \neq v$, then we find a vertex of p that is a fork in G. If $v_1 = v_2 = v$, then there is a cycle C in G such that $|C| \leq K$ and C does not contain forks of G. Since G is a strongly connected graph, it coincides with this cycle C. □

Corollary 9. *Let W be a binary u.r. non-periodic sequence, then for any n*

$$\mathrm{er}(R_{n-1}(W)) \leq 2^{O_W(n)}.$$

Proof. We prove this by induction on n. The base case $n = 0$ is obvious. Let $\mathrm{er}(R_{n-1}(W)) = K$ and suppose W has exactly a obstructions of length $n+1$. These obstructions correspond to paths of length 2 in the graph $R_{n-1}(W)$, i.e. edges of the graph $H := L(R_{n-1}(W))$. From Lemma 7 we have that $\mathrm{er}(H) = K$. The graph $R_n(W)$ is obtained from the graph H by removing some edges $e_1, e_2, \ldots, e_a$. Since W is a u.r. sequence, the digraphs H and $H - \{e_1, e_2, \ldots, e_a\}$ are strongly connected. This means that the edges $e_1, \ldots, e_a$ start at different forks of H. We also know that $R_n(W)$ is not a cycle. The graph $R_n(W)$ can be obtained by removing edges e_i from H one by one. Applying Lemma 8 a times, we show that $\mathrm{er}(R_n(W)) \leq 2^a K$, which completes the proof. □

Lemma 10. *Let H be a strongly connected digraph, $\mathrm{er}(H) = K$, $k \geq 3K$. Let u be an arbitrary edge of the graph $L^k(H)$. Then the digraph $L^k(H) - u$ contains a strongly connected subgraph B such that* $\mathrm{er}(B) \leq 3K$.

Proof. Consider in H the path p_u of length $k+2$, corresponding to u. Divide first k vertices of p_u into three subpaths of length at least K. Since $\mathrm{er}(H) = K$, each of these subpaths contains a fork (some of these forks can coincide). Next, we consider three cases.

Case 1. Assume that the path p_u visits at least two different forks of H. Then p_u contains a subpath of the form pe, where p is a path connecting two different forks v_1 and v_2 (and not containing other forks) and e is an edge starting at v_2. It is clear that the length of p_1 does not exceed $K+1$. Lemma 8 implies that there is a strongly connected subgraph G of H such that G contains the vertex v_2 but does not contain the edge e_2.

If G is not a cycle, then $\mathrm{er}(G) \leq 2K$. Hence, the graph $B := L^k(G)$ is a subgraph of $L^k(H)$, and from Lemma 7 we have $\mathrm{er}(B) \leq 2K$. It is also clear that the digraph B does not contain the edge u.

If G is a cycle, we denote it by p_1 and denote its first edge by e_1 (we assume that v_2 is the first and last vertex of p_1). The length of p_1 does not exceed K. Among the vertices of p_1 there are no forks of H besides v_2. Therefore, $v_1 \not\in p_1$. Call a path t in H *good*, if t does not contain the subpath pe. Let us show that for any good path s in H there are two different paths s_1 and s_2 starting at the end of s such that $|s_1| = |s_2| = 3K$ and the paths ss_1, ss_2 are also good.

It is clear that for any good path we can add an edge such that the new path is also good. There is a path t_1, $|t_1| < K$ such that st_1 is a good path and ends at some fork v. If $v \neq v_2$, then two edges e_i, e_j start at v, the paths st_1e_i and st_2e_j are good, and each of them can be prolonged further to a good path of arbitrary length. If $v = v_2$, then the paths st_1p_1e and $st_1p_1e_1$ are good and can be extended.

Consider in $L^k(H)$ a subgraph that consists of all vertices and edges that are good paths in H, let B be a strongly connected component of this subgraph. It is clear that $\mathrm{er}(B) \leq 3K$ and the digraph B does not contain the edge u.

Case 2. Assume that the path p_u visits exactly one fork v_1 (at least 3 times), but there are forks besides v_1 in H. There are two edges e_1 and e_2 that start at v_1. Starting with these edges and moving until forks, we obtain two paths p_1 and p_2. The edge e_1 is the first edge of p_1, the edge e_2 is the first of p_2, and $|p_1|, |p_2| \le K$. We can assume that p_1 is a subpath of p_u. Then p_1 ends at v_1 (and is a cycle) and p_2 ends at some fork $v_2 \ne v_1$ (if $v_1 = v_2$, then v_1 is the only fork reachable from v_1). We complete the proof as in the previous case: $p_1 e_1$ is a subpath of p_u. We call a path *good* if it does not contain $p_1 e_1$. As above, we can show that if s is a good path in H, then there are two different paths s_1 and s_2 such that $|s_1| = |s_2| = 3L$ and the paths ss_1, ss_2 are also good.

As above, B will be a strongly connected component in the subgraph of $L^k(H)$ that consists of vertices and edges corresponding to good paths in H.

Case 3. Assume that there is only one fork v in H. Then there are two cycles p_1 and p_2 of length $\le K$ that start and end at v. Let e_1 be the first edge of p_1 and let e_2 be the first edge of p_2. The path p_u contains one of the following subpaths: $p_1 e_1$, $p_2 e_2$, $p_1 p_1 e_2$ or $p_2 p_2 e_1$. Denote this path by t. Call a path *good* if it does not contain t. A simple check shows that we can complete the proof as in the previous cases. □

Proof of Theorem 5. Arrange all the obstructions u_i of the u.r. binary sequence W by their length in non-descending order. If $\underline{\lim}_{k\to\infty} \frac{\log_3 |u_k|}{k} \le 1$, then the statement of the Theorem holds. If $\underline{\lim}_{k\to\infty} \frac{\log_3 |u_k|}{k} > 1$ then the sequence $|u_k|/3^k$ tends to infinity. Hence, there exists n_0 such that $|u_{n_0}|/3^{n_0} > 10$ and $|u_n|/3^n > |u_{n_0}|/3^{n_0}$ for all $n > n_0$. In this situation, $|u_{n_0+k}| > |u_{n_0}| + 4\cdot 2^{n_0}\cdot 3^k$ for any $k > 0$.

Let $v_i = u_i$ if $1 \le i \le n_0$ and let v_i be a subword of u_i of length $|u_{n_0}| + 4\cdot 2^{n_0}\cdot 3^{i-n_0}$ if $i > n_0$. Denote by $\mathscr{U}$ the set of all finite binary words that do not contain subwords from $\{v_i\}$. It is clear that $\mathscr{U}$ is a proper subset of $\mathscr{L}(W)$. We get a contradiction with the uniform recurrence of W if we show that the language $\mathscr{U}$ is infinite. The Rauzy graph $R_{u_{n_0}-1}(\mathscr{U})$ is equal to $R_{u_{n_0}-1}(W)$, and from Corollary 9 we have $\operatorname{er}(R_{u_{n_0}-1}(\mathscr{L})) \le 2^{n_0}$.

By induction on n we show that for all $n \ge n_0$ the graph $R_{|v_n|-1}(\mathscr{U})$ contains a strongly connected subgraph H_n such that $\operatorname{er}(H_n) \le 3^{n-n_0}\cdot 2^{n_0}$. We already have the base case $n = n_0$. The graph $R_{|v_{n+1}|-1}(\mathscr{U})$ is obtained from $L^{|v_{n+1}|-|v_n|}(R_{|v_n|-1})$ by removing at most one edge. Note that $|v_{n+1}| - |v_n| > 3\cdot \operatorname{er}(H_n)$, so we can use Lemma 10 for the digraph H_n and $k = |v_{n+1}| - |v_n|$. This completes the inductive step.

All the graphs $R_{|v_n|-1}(\mathscr{U})$ are nonempty and, therefore, the language $\mathscr{U}$ is infinite. □

For a sequence W over an alphabet $A = \{a_1, \dots, a_k\}$ of size k, we replace in W each letter a_i by $0^i 1$ and obtain a binary sequence W'. If W is u.r. and non-periodic, then W' is also u.r. and non-periodic. It is clear that all long enough obstructions of W' correspond to some of the obstructions of W, so we obtain

Corollary 11. *Let W be an u.r. non-periodic sequence on a finite alphabet. Then* $\underline{\lim}_{n\to\infty} O_W(n)/\log_3 n \ge 1$.

Example. Consider a finite alphabet $\{0,1\}$ and the sequence of words u_i, defined recursively as $u_0 = 0$, $u_1 = 01$, $u_k = u_{k-1}u_{k-2}$ for $k \ge 2$. Since u_i is a prefix of u_{i+1}, the sequence (u_i) has a limit, called a *Fibonacci word* $F = 0100101001001\ldots$. In Example 25 of [1] the set $\{11, 000, 10101, 00100100, \ldots\}$ of obstructions of F is described. These words have lengths equal to Fibonacci numbers. Since the Fibonacci word is u.r., in Theorem 5 we cannot replace the constant 3 by a number smaller than $\frac{\sqrt{5}+1}{2}$.

References

[1] M.-P. Béal, F. Mignosi, A. Restivo, M. Sciortino, "Forbidden Words in Symbolic Dynamics", *Adv. Appl. Math.* **25** (2020), no. 2, p. 163-193.

[2] A. Y. Belov, V. V. Borisenko, V. N. Latyshev, "Monomial algebras", *J. Math. Sci., New York* **87** (1997), no. 3, p. 3463-3575.

[3] G. M. Bergman, "The diamond lemma for ring theory", *Adv. Math.* **29** (1978), no. 2, p. 178-218.

[4] I. I. Bogdanov, G. R. Chelnokov, "The maximal length of the period of a periodic word defined by restrictions", https://arxiv.org/abs/1305.0460, 2013.

[5] G. R. Chelnokov, "On the number of restrictions defining a periodic sequence", *Model and Analysis of Inform. Systems* **14** (2007), no. 2, p. 12-16, in Russian.

[6] I. Ivanov-Pogodaev, S. Malev, "Finite Gröbner basis algebras with unsolvable nilpotency problem and zero divisors problem", *J. Algebra* **508** (2018), p. 575-588.

[7] P. A. Lavrov, "Number of restrictions required for periodic word in the finite alphabet", https://arxiv.org/abs/1209.0220, 2012.

[8] ———, "Minimal number of restrictions defining a periodic word", https://arxiv.org/abs/1412.5201, 2014.

[9] A. A. Muchnik, Y. L. Pritykin, A. L. Semenov, "Sequences close to periodic", *Russ. Math. Surv.* **64** (2009), no. 5, p. 805-871.