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Abstract. We study the existence of ground states of a Schrödinger–Poisson–Slater type equation with pure power nonlinearity. By carrying out the constrained minimization on a special manifold, which is a combination of the Pohozaev manifold and Nehari manifold, we obtain the existence of ground state solutions of this system.

Keywords. Schrödinger–Poisson–Slater type equation, ground state, Coulomb–Sobolev inequality.

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1. Introduction and main result

In [2] and [3], it is showed that the following minimization problem

$$\inf_{\phi \in X^{1,\alpha}\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^n} |\nabla\phi|^2 \mathrm{d}x\right)^{\frac{p(n+\alpha)-4n}{2(4+\alpha-n)}} [L(\phi)]^{\frac{2n-p(n-2)}{2(4+\alpha-n)}}}{\int_{\mathbb{R}^n} |\phi|^p \mathrm{d}x}, \tag{1}$$

is attained if and only if $n \neq 4 + \alpha$ and

$$\begin{cases} p \in \left[\frac{2(4+\alpha)}{2+\alpha}, \infty\right), & n = 2; \\ p \in \left[\frac{2(4+\alpha)}{2+\alpha}, \frac{2n}{n-2}\right], & 3 \leq n < 4 + \alpha; \\ p \in \left[\frac{2n}{n-2}, \frac{2(4+\alpha)}{2+\alpha}\right], & n > 4 + \alpha. \end{cases} \tag{2}$$

Here $X^{1,\alpha} := \{v \in \mathscr{D}^{1,2}(\mathbb{R}^n); L(v) < \infty\}$ with $n \geq 2$, and $L(v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{v^2(x)v^2(y)}{|x-y|^{n-\alpha}} \mathrm{d}x\mathrm{d}y$. The functional L is the so-called Coulomb energy of the wave. We endow the space $X^{1,\alpha}$ with the norm $\|u\|^2_{X^{1,\alpha}} = \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + [L(u)]^{\frac{1}{2}}$.

The minimization problem (1) is associated with the best constant of the Coulomb–Sobolev inequality

$$\int_{\mathbb{R}^n} |\phi|^p \mathrm{d}x \le C \left(\int_{\mathbb{R}^n} |\nabla\phi|^2 \mathrm{d}x \right)^{\frac{p(n+\alpha)-4n}{2(4+\alpha-n)}} [L(\phi)]^{\frac{2n-p(n-2)}{2(4+\alpha-n)}}, \quad \forall\, \phi \in X^{1,\alpha}, \tag{3}$$

which is helpful to well understand the lower bound of the Coulomb energy. The Euler–Lagrange equation of (1) is the following Schrödinger–Poisson–Slater type equation

$$-\Delta u + (|x|^{\alpha-n} * u^2)u = \mu|u|^{p-2}u, \quad u > 0 \text{ in } \mathbb{R}^n, \tag{4}$$

where $n \ge 2$, $\alpha \in (0, n)$, $\mu > 0$ is the so-called Slater constant of the equation, and p belongs to the intervals in (2). Equation (4) appears in various physical frameworks, such as plasma physics or semiconductor theory. In equation (4), $|x|^{\alpha-n} * u^2$ is known as the repulsive Coulomb potential, which makes the usual Sobolev space $H^1(\mathbb{R}^n)$ not to be a good framework for posing the equation (4). In this paper, we are interested in studying the ground state solutions of (4) in $X^{1,\alpha}$.

There are a series of analytical results on equation (4) in literatures, see [8, 10] and the references therein. In particular, Ianni and Ruiz [6] studied the following version of the Schrödinger–Poisson–Slater system

$$-\Delta u + \left(u^2 * \frac{1}{4\pi|x|} \right)u = \mu|u|^{p-2}u, \text{ in } \mathbb{R}^3. \tag{5}$$

We also note that there are many results on other Schrödinger–Poisson systems in literatures; see [1, 4, 5, 7, 11] and many others.

This paper is concerned with the variational problem of (4). It is known that every solution to equation (4) is a critical point of the energy functional $J : X^{1,\alpha} \to \mathbb{R}$, which is given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u^2(x)u^2(y)}{|x-y|^{n-\alpha}} \mathrm{d}x\mathrm{d}y - \frac{\mu}{p} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x.$$

Clearly, the trivial solution 0 is a local minimum of J. So J has a mountain pass geometry structure. However, it seems difficult to verify the (PS) condition when $p \in (2, 4)$. It is natural that the critical point can be searched for in some constrained classes. We point out that the usual Nehari manifold is not suitable because it is difficult to prove the boundedness of the minimizing sequences. Neither the Pohozaev manifold is a suitable constrained class. In fact, when $n = 2$, the minimization problem $\inf_{u\in\mathcal{N}} J(u)$ has no solution when $p = \frac{2\alpha+8}{2+\alpha}$, where $\mathcal{N} := \{u \in X^{1,\alpha}\backslash\{0\} : \frac{2+\alpha}{4}L(u) - \frac{2}{p}\int_{\mathbb{R}^2} |u|^p \mathrm{d}x = 0\}$. In addition, J is unbounded from below on $\mathcal{N}$ when $p > \frac{2\alpha+8}{2+\alpha}$.

Now we try to minimize J on another suitable manifold. In Section 2, we carry out the constrained minimization on a new manifold $\mathcal{M}$. Here,

$$\mathcal{M} := \{u \in X^{1,\alpha}\backslash\{0\} : I(u) = 0\}, \tag{6}$$

where

$$I(u) := \frac{4+\alpha-n}{2+\alpha} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + \frac{4+\alpha-n}{2(2+\alpha)} L(u) - \left(p - \frac{2n}{2+\alpha} \right) \frac{\mu}{p} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x.$$

This new manifold $\mathcal{M}$ can be viewed as the combination of the Nehari manifold and the Pohozaev manifold which was introduced in [9, 12]. A function u is called the Nehari–Pohozaev type ground state solution of (4), if u is a solution of the least energy problem

$$\min\{J(u) : u \in \mathcal{M}\}.$$

We summarize our main existence result in the following statement.

Theorem 1. *Assume that* $\frac{2(\alpha+4)}{2+\alpha} < p < \frac{2n}{n-2}$ *when* $n < 4 + \alpha$, *or* $\frac{2n}{n-2} < p < \frac{2(\alpha+4)}{2+\alpha}$ *when* $n > 4 + \alpha$. *Then problem* (4) *has a ground state solution of the Nehari–Pohozaev type.*

2. Proof of Theorem 1

Let us define $\mathscr{F} : X^{1,\alpha} \to \mathbb{R}$ as

$$\mathscr{F}(u) := \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u^2(x) u^2(y)}{|x-y|^{n-\alpha}} \mathrm{d}x \mathrm{d}y.$$

Similar to Proposition 3.6 in [6], by the concentration-compactness principle and the non-local Brezis–Lieb lemma (cf. [2, Lemma 2.2]), we also have the following result.

Lemma 2. *Assume that $\{u_m\} \subset X^{1,\alpha}$ is a bounded (PS) sequence of J at a certain level $c > 0$. Then, up to a subsequence, there exists $k \in \mathbb{N} \cup \{0\}$ and a finite sequence*

$$(v_0, v_1, \dots, v_k) \subset X^{1,\alpha}, v_i \not\equiv 0, \; for \; i > 0$$

of solutions of equation (4) *and k sequences $\{\xi_m^1\}, \dots, \{\xi_m^k\} \subset \mathbb{R}^n$, such that as $m \to +\infty$,*

$$\left\| u_m - v_0 - \sum_{i=1}^k v_i(\cdot - \xi_m^i) \right\| \to 0; \quad |\xi_m^i| \to +\infty, \; |\xi_m^i - \xi_m^j| \to +\infty, \; i \neq j;$$

and

$$\sum_{i=0}^k J(v_i) = c, \; \mathscr{F}(u_m) \to \sum_{i=0}^k \mathscr{F}(v_i).$$

Lemma 3. *Assume that $\frac{2(\alpha+4)}{2+\alpha} < p < \frac{2n}{n-2}$ when $n < 4 + \alpha$ or $\frac{2n}{n-2} < p < \frac{2(\alpha+4)}{2+\alpha}$ when $n > 4 + \alpha$. Then J is unbounded from below.*

Proof. (i). If $\frac{2(\alpha+4)}{2+\alpha} < p < \frac{2n}{n-2}$ with $n < 4 + \alpha$. Let $u \in X^{1,\alpha}$, and $u_t = t u(t^b x)$, $b = \frac{2}{2+\alpha}$, $t > 0$. By the standard scaling we have $\int_{\mathbb{R}^n} |\nabla u_t|^2 \mathrm{d}x = t^{\frac{2(4+\alpha-n)}{2+\alpha}} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x$, $L(u_t) = t^{\frac{2(4+\alpha-n)}{2+\alpha}} L(u)$, and $\int_{\mathbb{R}^n} |u_t|^p \mathrm{d}x = t^{p-nb} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x$. Hence,

$$J(u_t) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_t|^2 \mathrm{d}x + \frac{1}{4} L(u_t) - \frac{\mu}{p} \int_{\mathbb{R}^n} |u_t|^p \mathrm{d}x$$

$$= \frac{t^{\frac{2(4+\alpha-n)}{2+\alpha}}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{t^{\frac{2(4+\alpha-n)}{2+\alpha}}}{4} L(u) - \frac{\mu t^{p-nb}}{p} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x.$$

We see that $J(u_t) \to -\infty$ as $t \to +\infty$ provided $p > \frac{8+2\alpha}{2+\alpha}$ and $\mu > 0$.

(ii). For the case $\frac{2n}{n-2} < p < \frac{2(\alpha+4)}{2+\alpha}$ when $n > 4 + \alpha$. Letting $u \in X^{1,\alpha}$, and $u_t = t^{-1} u(t^{\overline{b}} x)$ with $\overline{b} = -\frac{2}{2+\alpha}$ and $t > 0$, we also have $\int_{\mathbb{R}^n} |\nabla u_t|^2 \mathrm{d}x = t^{\frac{2(n-4-\alpha)}{2+\alpha}} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x$, $L(u_t) = t^{\frac{2(n-4-\alpha)}{2+\alpha}} L(u)$, and $\int_{\mathbb{R}^n} |u_t|^p \mathrm{d}x = t^{-p-n\overline{b}} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x$. Therefore,

$$J(u_t) = \frac{t^{\frac{2(n-4-\alpha)}{2+\alpha}}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{t^{\frac{2(n-4-\alpha)}{2+\alpha}}}{4} L(u) - \frac{\mu t^{-p-n\overline{b}}}{p} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x.$$

It is easily see that $J(u_t) \to -\infty$ as $t \to +\infty$ if and only if $p < \frac{8+2\alpha}{2+\alpha}$ and $\mu > 0$. The proof is complete. $\square$

2.1. Case 1. $\frac{2(\alpha+4)}{2+\alpha} < p < \frac{2n}{n-2}$ with $n < 4 + \alpha$

By calculations, we can easily get the following lemma.

Lemma 4. *Let a_1, a_2 be positive constants, and $f(t) = a_1 t^{\frac{2(4+\alpha-n)}{2+\alpha}} - a_2 t^{p-nb}$ for $t \geq 0$. When $\frac{2(\alpha+4)}{2+\alpha} < p < \frac{2n}{n-2}$ with $n < 4 + \alpha$, f has a unique critical point, corresponding to its maximum.*

Proof. For $t \geq 0$, and $p > \frac{8+2\alpha}{2+\alpha}$, $f'(t) = t^{\frac{6+\alpha-2n}{2+\alpha}} a_1 \frac{2(4+\alpha-n)}{2+\alpha} - a_2(p-nb) t^{p-\frac{8+2\alpha}{2+\alpha}}$. Clearly, $f'(t)$ has a unique positive real zero point t_0. Noting that $f(t) \to 0^+$ as $t \to 0^+$ and $f(t) \to -\infty$ as $t \to +\infty$, thus f achieves its maximum at the point t_0. The proof is complete. $\square$

Assume that u is a critical point of J. Write $u_t = tu(t^b x)$ with $b = \frac{2}{2+\alpha}$ and $t > 0$. Clearly, $\varphi(t) := J(u_t)$ is positive for small t and tends to $-\infty$ as $t \to +\infty$. By Lemma 4, φ has a unique critical point which corresponds to its maximum. Since u is a critical point of J, the maximum of $\varphi(t)$ should be achieved at $t = 1$ and $\varphi'(1) = 0$. Clearly, $I(u) = \varphi'(1)$, where I is defined in (6). This is a reason why we choose $\mathcal{M}$ as the constrained manifold.

Obviously, $\mathcal{M} \neq \emptyset$. Indeed, for given any $v \neq 0$, Lemma 3 shows that there exists $t > 0$ such that $u_t^v \in \mathcal{M}$. Moreover, the curve $\Gamma = \{u_t\}_{t \in \mathbb{R}}$ intersects the manifold $\mathcal{M}$ and $J|_\Gamma$ attains its maximum along Γ at the point u. If u is a mountain pass type solution of problem (4), it is natural to look for the minima of J on $\mathcal{M}$. In addition, For any nontrivial critical point u of J, it is standard to prove the following Pohozaev identity

$$\mathscr{P}(u) := \frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + \frac{n+\alpha}{4} L(u) - \frac{n\mu}{p} \int_{\mathbb{R}^n} |u|^p \mathrm{d}x = 0.$$

It is clear that $I(u) = \langle J'(u), u \rangle - b\mathscr{P}(u)$ with $b = \frac{2}{\alpha+2}$. If u is a nontrivial solution of (4), then $u \in \mathcal{M}$. Moreover, we have the following result.

Lemma 5. *If $\frac{8+2\alpha}{2+\alpha} < p < \frac{2n}{n-2}$ with $n < 4 + \alpha$. Then $\mathcal{M}$ is a C^1-manifold and every critical point of J in $\mathcal{M}$ is a critical point of J.*

Proof. We proceed in four steps.

Step 1. We claim $0 \notin \partial \mathcal{M}$.

By (3), there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} |u|^p \mathrm{d}x \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \right)^{\frac{p(n+\alpha)-4n}{2(4+\alpha-n)}} (L(u))^{\frac{2n-p(n-2)}{2(4-n+\alpha)}}.$$

Since $p > \frac{2(\alpha+4)}{2+\alpha}$ with $n < 4 + \alpha$, we deduce $\frac{2n-p(n-2)}{2(4-n+\alpha)} < 1$. Using Young's inequality, there holds

$$\begin{aligned} I(u) &= \frac{4+\alpha-n}{2+\alpha} \|\nabla u\|_2^2 + \frac{4+\alpha-n}{2(2+\alpha)} L(u) - \frac{p-nb}{p} \mu \int_{\mathbb{R}^n} |u|^p \mathrm{d}x \\ &\geq \frac{4+\alpha-n}{2+\alpha} \|\nabla u\|_2^2 + \frac{4+\alpha-n}{2(2+\alpha)} L(u) - C\|\nabla u\|_2^{\frac{p(n+\alpha)-4n}{4+\alpha-n}} (L(u))^{\frac{2n-p(n-2)}{2(4-n+\alpha)}} \\ &= \frac{4+\alpha-n}{2+\alpha} \|\nabla u\|_2^2 - C_1 \|\nabla u\|_2^{\frac{2p(n+\alpha)-8n}{8+2\alpha-4n+p(n-2)}}, \end{aligned}$$

where C_1 is a positive constant. In view of $p > \frac{8+2\alpha}{2+\alpha}$, there holds $\frac{2p(n+\alpha)-8n}{8+2\alpha-4n+p(n-2)} > 2$. Therefore, we can find suitably small $r > 0$ such that when $\|\nabla u\|_2 < r$, there holds $I(u) > \rho$ for some $\rho > 0$. Then $0 \notin \partial \mathcal{M}$.

Step 2. We claim $\inf_\mathcal{M} J > 0$.

For any $u \in \mathcal{M}$, let $\mathscr{A} = \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x$, $\mathscr{B} = L(u)$, $\mathscr{C} = \mu \int_{\mathbb{R}^n} |u|^p \mathrm{d}x$. Then $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are positive and $I(u) = \frac{4+\alpha-n}{2+\alpha} \mathscr{A} + \frac{4+\alpha-n}{2(2+\alpha)} \mathscr{B} - \frac{p-nb}{p} \mathscr{C} = 0$. Therefore, by $p > \frac{8+2\alpha}{2+\alpha}$,

$$\begin{aligned} J(u) &= \frac{1}{2} \mathscr{A} + \frac{1}{4} \mathscr{B} - \frac{1}{p} \mathscr{C} \\ &= \frac{p(2+\alpha) - (8+2\alpha)}{2(p-nb)(2+\alpha)} \mathscr{A} + \frac{p(2+\alpha) - (8+2\alpha)}{4(p-nb)(2+\alpha)} \mathscr{B} > 0. \end{aligned} \tag{7}$$

Step 3. We claim that $\mathcal{M}$ is a C^1-manifold.

By the implicit function theorem, it only need $I'(u) \neq 0$ for any $u \in \mathcal{M}$. We prove it by argument of contradiction. Namely, suppose that $I'(u) = 0$ for some $u \in \mathcal{M}$. Thus, in a weak sense there holds

$$-\frac{2(4+\alpha-n)}{2+\alpha} \Delta u + \frac{2(4+\alpha-n)}{2+\alpha} (|x|^{\alpha-n} * u^2) u = (p-nb)\mu u^{p-1}. \tag{8}$$

Multiplying (8) by u and integrating, we have

$$\frac{2(4+\alpha-n)}{2+\alpha}\mathscr{A} + \frac{2(4+\alpha-n)}{2+\alpha}\mathscr{B} - (p-nb)\mathscr{C} = 0. \tag{9}$$

The Pohozaev identity corresponding to equation (8) is

$$\frac{2(4+\alpha-n)}{2+\alpha}\frac{n-2}{2}\mathscr{A} + \frac{2(4+\alpha-n)}{2+\alpha}\frac{n+\alpha}{4}\mathscr{B} - \frac{p-nb}{p}\mathscr{C} = 0. \tag{10}$$

(i). When $n = 2$, it follows from (10) that $(p-2b)\mathscr{C} = \frac{p(2+\alpha)}{4}\mathscr{B}$. Inserting into $I(u) = 0$ and (9), we get $\mathscr{A} - \frac{\alpha}{4}\mathscr{B} = 0$ and $\mathscr{A} - \frac{p(2+\alpha)-8}{8}\mathscr{B} = 0$, which implies $\frac{2\alpha-p(2+\alpha)+8}{8}\mathscr{B} = 0$. This is impossible since $p > \frac{8+2\alpha}{2+\alpha}$ and $\mathscr{B} > 0$.

(ii). When $n \neq 2$, it follows from $I(u) = 0$ and (9) that $\frac{4+\alpha-n}{2+\alpha}\mathscr{B} = \frac{p-2}{p}(p-nb)\mathscr{C}$. Applying $(n-2)I(u) = 0$ and (10), we have $\frac{4+\alpha-n}{2}\mathscr{B} = \frac{2}{p}(p-nb)\mathscr{C}$. Therefore, $\frac{[(p-2)(2+\alpha)-4](p-nb)}{p}\mathscr{C} = 0$. Since $\mathscr{C} > 0$ and $p - nb \neq 0$, we have $p = \frac{8+2\alpha}{2+\alpha}$. We reach a contradiction. Thus, $\mathscr{M}$ is a C^1-manifold.

Step 4. We claim that every critical point of J on $\mathscr{M}$ is a critical point of J in $X^{1,\alpha}$.

Assume that u is a critical point of J on $\mathscr{M}$, there exists a Lagrange multiplier λ such that $J'(u) = \lambda I'(u)$. It can be written, in a weak sense, as

$$-\left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\Delta u + \left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right](|x|^{\alpha-n} * u^2)u = [1 - (p-nb)\lambda]\mu u^{p-1}. \tag{11}$$

It remains now to prove that $\lambda = 0$.

Recalling the definitions of $\mathscr{A}, \mathscr{B}, \mathscr{C}$, arguing as Step 3, we can establish the following equations

$$\begin{cases} J(u) = \frac{1}{2}\mathscr{A} + \frac{1}{4}\mathscr{B} - \frac{1}{p}\mathscr{C}, \\ I(u) = \frac{4+\alpha-n}{2+\alpha}\mathscr{A} + \frac{4+\alpha-n}{2(2+\alpha)}\mathscr{B} - \frac{p-nb}{p}\mathscr{C} = 0, \\ \left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{A} + \left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{B} - [1 - (p-nb)\lambda]\mathscr{C} = 0, \\ \frac{n-2}{2}\left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{A} + \frac{n+\alpha}{4}\left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{B} - [1 - (p-nb)\lambda]\frac{n}{p}\mathscr{C} = 0, \end{cases} \tag{12}$$

where the third equation follows by multiplying (11) by u and integrating, and the fourth equality is the Pohozaev identity corresponding to equation (11).

(i). When $n = 2$, from the fourth equation in (12), we have $[1 - (p-2b)\lambda]\mathscr{C} = \frac{p(1-2\lambda)(2+\alpha)}{8}\mathscr{B}$. Inserting into (12), we have

$$\begin{cases} \mathscr{A} + \frac{4[1-(p-2b)\lambda]-(2+\alpha)(p-2b)(1-2\lambda)}{8[1-(p-2b)\lambda]}\mathscr{B} = 0, \\ (1-2\lambda)\mathscr{A} - \frac{p(2+\alpha)-8}{8}(1-2\lambda)\mathscr{B} = 0, \\ [1 - (p-2b)\lambda]\mathscr{C} = \frac{p(1-2\lambda)(2+\alpha)}{8}\mathscr{B}. \end{cases} \tag{13}$$

If $\lambda = \frac{1}{2}$, from the third equation in (13), we obtain $p = \frac{8+2\alpha}{2+\alpha}$. This is impossible, and hence $\lambda \neq \frac{1}{2}$. Therefore, from the second equation in (13), we have $\mathscr{A} = \frac{p(2+\alpha)-8}{8}\mathscr{B}$. Inserting into the first equation in (13), and noting that $\mathscr{B} \neq 0$, we obtain that $(p-2b)[8+2\alpha-p(2+\alpha)]\lambda = 0$. This implies that $\lambda \equiv 0$.

(ii). When $n \neq 2$, the second equation is multiplied by 2λ, and add to the third equation in (12), we have

$$\mathscr{A} + \left[1 - \frac{4+\alpha-n}{2+\alpha}\lambda\right]\mathscr{B} = \left[1 - \frac{p-2}{p}(p-nb)\lambda\right]\mathscr{C}. \tag{14}$$

The fourth equation in (12) can be rewritten as

$$\left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{A} + \frac{n+\alpha}{2(n-2)}\left[1 - \frac{2(4+\alpha-n)}{2+\alpha}\lambda\right]\mathscr{B} = [1 - (p-nb)\lambda]\frac{2n}{p(n-2)}\mathscr{C}. \tag{15}$$

It follows from (15) and the third equation in (12) that

$$\frac{2+\alpha-2(4+\alpha-n)\lambda}{2+\alpha}\mathscr{B} = [1-(p-nb)\lambda]\frac{2[2n-(n-2)p]}{p(4+\alpha-n)}\mathscr{C}. \tag{16}$$

The second equation in (12) can be rewritten as $\mathscr{A} = \frac{(2+\alpha)(p-nb)}{p(4+\alpha-n)}\mathscr{C} - \frac{1}{2}\mathscr{B}$. Inserting into equation (14), one obtains

$$\frac{2+\alpha-2(4+\alpha-n)\lambda}{2+\alpha}\mathscr{B} = 2\left[1-\frac{(p-nb)(p-2)}{p}\lambda - \frac{(p-nb)(2+\alpha)}{p(4+\alpha-n)}\right]\mathscr{C}. \tag{17}$$

By $\mathscr{C} > 0$, it follows from (16) and (17) that

$$2n-(n-2)p - p(4+\alpha-n)+(p-nb)(2+\alpha) = (p-nb)[2n-(n-2)p-(p-2)(4+\alpha-n)]\lambda.$$

By computing, we have $p-nb>0$, $2n-(n-2)p-(p-2)(4+\alpha-n) = 8+2\alpha-p(2+\alpha)<0$, and $2n-(n-2)p-p(4+\alpha-n)+(p-nb)(2+\alpha) = 0$. Consequently, $\lambda \equiv 0$. We conclude that $J'(u) = 0$ for $n \geq 2$, i.e., u is a critical point of J. The proof is complete. $\qquad\square$

Proof of Theorem 1 in Case 1. We prove that J attains its minimum in $\mathscr{M}$.

Let $\{u_m\} \subset \mathscr{M}$ be a minimizing sequence of J in $\mathscr{M}$, that is, $J(u_m) \to \inf_{\mathscr{M}} J$ as $m \to \infty$. We follow the ideas of Theorems 8.6 and 8.7 in [13] to claim that $\{u_m\}$ is also a (PS) sequence of J. In fact, Step 3 in the proof of Lemma 5 shows $I'(u_m) \neq 0$. Therefore, by the Ekeland variational principle (see [13, Theorem 8.5]), there exists $\{\lambda_m\} \subset \mathbb{R}$ such that

$$J'(u_m) - \lambda_m I'(u_m) \to 0 \text{ as } m \to \infty. \tag{18}$$

Therefore, when $m \to \infty$, we have

$$\begin{cases} I(u_m) = \frac{4+\alpha-n}{2+\alpha}\mathscr{A}_m + \frac{4+\alpha-n}{2(2+\alpha)}\mathscr{B}_m - \frac{p-nb}{p}\mathscr{C}_m = 0, \\ \left[1-\frac{2(4+\alpha-n)}{2+\alpha}\lambda_m\right]\mathscr{A}_m + \left[1-\frac{2(4+\alpha-n)}{2+\alpha}\lambda_m\right]\mathscr{B}_m - [1-(p-nb)\lambda_m]\mathscr{C}_m = o(1), \\ \frac{n-2}{2}\left[1-\frac{2(4+\alpha-n)}{2+\alpha}\lambda_m\right]\mathscr{A}_m + \frac{n+\alpha}{4}\left[1-\frac{2(4+\alpha-n)}{2+\alpha}\lambda_m\right]\mathscr{B}_m - [1-(p-nb)\lambda_m]\frac{n}{p}\mathscr{C}_m = o(1), \end{cases} \tag{19}$$

where $\mathscr{A}_m = \int_{\mathbb{R}^n} |\nabla u_m|^2 dx$, $\mathscr{B}_m = L(u_m)$, $\mathscr{C}_m = \mu\int_{\mathbb{R}^n} |u_m|^p dx$. Similar to the proof of Step 4 in Lemma 5 (replacing (12) by (19)), we can deduce $\lambda_m \to 0$ as $m \to \infty$. Combining with (18), we get

$$J'(u_m) \to 0 \text{ as } m \to \infty.$$

Thus, $\{u_m\} \subset \mathscr{M}$ is (PS) sequence of J.

By (7), there holds

$$\frac{p(2+\alpha)-(8+2\alpha)}{2(p-nb)(2+\alpha)}\int_{\mathbb{R}^n}|\nabla u_m|^2 dx + \frac{p(2+\alpha)-(8+2\alpha)}{4(p-nb)(2+\alpha)}L(u_m) \to \inf_{\mathscr{M}} J \quad (m \to \infty).$$

Since $p > \frac{8+2\alpha}{2+\alpha}$, we obtain that $\{\|u_m\|_{X^{1,\alpha}}\}$ is bounded. In addition, by the Step 2 in the proof of Lemma 5, the level value is positive. By Lemma 2, up to a subsequence, and there exist $k \in \mathbb{N} \cup \{0\}$ and a finite sequence

$$(v_0, v_1, \dots, v_k) \subset X^{1,\alpha}, v_i \not\equiv 0, \; for \; i > 0$$

of solutions of problem (4) and k sequences $\{\xi_m^1\}, \dots, \{\xi_m^k\} \subset \mathbb{R}^n$, such that as $m \to +\infty$,

$$\left\| u_m - v_0 - \sum_{i=1}^{k} v_i(\cdot - \xi_m^i) \right\|_{X^{1,\alpha}} \to 0; \tag{20}$$

$$|\xi_m^i| \to +\infty, \; |\xi_m^i - \xi_m^j| \to +\infty, \; i \neq j;$$

$$\sum_{i=0}^{k} J(v_i) = \inf_{\mathscr{M}} J. \tag{21}$$

Since v_i $(i = 0, 1, \dots, k)$ is a solution of equation (4), we have $J'(v_i) = 0$ and $\mathscr{P}(v_i) = 0$ for $i = 0, 1, \dots, k$. This implies that $v_i \in \mathscr{M}$, and thus $J(v_i) \geq \inf_{\mathscr{M}} J$ for $i = 0, 1, \dots, k$. Applying (21) and noting that Step 2 in Lemma 5, there are two possibilities: either $v_0 \neq 0$ and $k = 0$, or $v_0 = 0$

and $k = 1$. In the first case, $u_m(\cdot + \xi_m^1) \to v_0(\cdot)$ in $X^{1,\alpha}$ (by (20)) and v_0 is a solution of equation (4) (by Step 4 in Lemma 5) with $J(v_0) = \inf_{\mathcal{M}} J$ (by (21)), and so v_0 is a ground state solution of (4). In the latter, $u_m(\cdot + \xi_m^1) \to v_1(\cdot)$ in $X^{1,\alpha}$ as $m \to \infty$ (by (20)) and v_1 is a ground state solution of equation (4) with $J(v_1) = \inf_{\mathcal{M}} J$ (by (21)). The proof is complete. $\qquad \square$

2.2. Case 2. $\frac{2n}{n-2} < p < \frac{2(\alpha+4)}{2+\alpha}$ with $n > 4 + \alpha$

Similar to Lemma 4, we remark that the function $f(t) := a_1 t^{\frac{2(n-\alpha-4)}{2+\alpha}} - a_2 t^{-p-n\overline{b}}$ (with $t \geq 0, a_1, a_2 > 0$ and $\overline{b} := \frac{-2}{2+\alpha}$) has a unique critical point, corresponding to its maximum provided $\frac{2n}{n-2} < p < \frac{2(\alpha+4)}{2+\alpha}$ with $n > 4 + \alpha$.

Define

$$\widetilde{I}(u) := \frac{n-\alpha-4}{2+\alpha} \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x + \frac{n-\alpha-4}{2(2+\alpha)} L(u) - \frac{-p-n\overline{b}}{p} \mu \int_{\mathbb{R}^n} |u|^p \mathrm{d}x.$$

It is clear that $\widetilde{I}(u) \neq -I(u)$, and $\widetilde{I}(u) = -\langle J'(u), u \rangle - \overline{b} \mathscr{P}(u)$. Now,

$$\mathcal{M} = \{u \in X^{1,\alpha} \setminus \{0\} : \widetilde{I}(u) = 0\}.$$

It is similar to Lemma 5 that we also have the following lemma.

Lemma 6. *Assume that $\frac{2n}{n-2} < p < \frac{8+2\alpha}{2+\alpha}$ with $n > 4+\alpha$. Then $\mathcal{M}$ is a C^1-manifold and every critical point of J in $\mathcal{M}$ is a critical point of J.*

Proof. Similar to the Step 1 in Lemma 5, we can prove that $0 \notin \partial \mathcal{M}$.

(i). We claim $\inf_{\mathcal{M}} J > 0$.

Actually, for any $u \in \mathcal{M}$, set $\mathscr{A} = \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x$, $\mathscr{B} = L(u)$, $\mathscr{C} = \mu \int_{\mathbb{R}^n} |u|^p \mathrm{d}x$, and $\delta = -p - n\overline{b}$. Clearly, $\delta > 0$ and $\widetilde{I}(u) = \frac{n-\alpha-4}{2+\alpha} \mathscr{A} + \frac{n-\alpha-4}{2(2+\alpha)} \mathscr{B} - \frac{\delta}{p} \mathscr{C} = 0$, which implies $\mathscr{C} = \frac{p}{\delta} \frac{n-\alpha-4}{2+\alpha} \mathscr{A} + \frac{p}{\delta} \frac{n-\alpha-4}{2(2+\alpha)} \mathscr{B}$. Therefore, by $p < \frac{8+2\alpha}{2+\alpha}$, there holds

$$\begin{aligned} J(u) &= \frac{1}{2} \mathscr{A} + \frac{1}{4} \mathscr{B} - \frac{1}{p} \mathscr{C} \\ &= \frac{8+2\alpha - p(2+\alpha)}{2\delta(2+\alpha)} \mathscr{A} + \frac{8+2\alpha - p(2+\alpha)}{4\delta(2+\alpha)} \mathscr{B} > 0. \end{aligned} \tag{22}$$

(ii). We claim that $\widetilde{I}'(u) \neq 0$ for any $u \in \mathcal{M}$, then $\mathcal{M}$ is a C^1-manifold.

We prove it by argument of contradiction. Suppose that $\widetilde{I}'(u) = 0$ for some $u \in \mathcal{M}$. In a weak sense, the equation $\widetilde{I}'(u) = 0$ means that

$$-\frac{2(n-\alpha-4)}{2+\alpha} \Delta u + \frac{2(n-\alpha-4)}{2+\alpha} (|x|^{\alpha-n} * u^2) u = \delta \mu u^{p-1}. \tag{23}$$

Then, we establish the following system

$$\begin{cases} J(u) = \frac{1}{2} \mathscr{A} + \frac{1}{4} \mathscr{B} - \frac{1}{p} \mathscr{C}, \\ \widetilde{I}(u) = \frac{n-\alpha-4}{2+\alpha} \mathscr{A} + \frac{n-\alpha-4}{2(2+\alpha)} \mathscr{B} - \frac{1}{p} \mathscr{C}\delta = 0, \\ \frac{2(n-\alpha-4)}{2+\alpha} \mathscr{A} + \frac{2(n-\alpha-4)}{2+\alpha} \mathscr{B} - \mathscr{C}\delta = 0, \\ \frac{n-2}{2} \frac{2(n-\alpha-4)}{2+\alpha} \mathscr{A} + \frac{n+\alpha}{4} \frac{2(n-\alpha-4)}{2+\alpha} \mathscr{B} - \frac{n}{p} \mathscr{C}\delta = 0, \end{cases} \tag{24}$$

It follows from the second equation and the third equation in (24) that $\frac{n-\alpha-4}{2+\alpha} \mathscr{B} = \frac{p-2}{p} \mathscr{C}\delta$. Multiplying the second equation by $(n-2)$, and together with the fourth equation in (24), we get $\frac{n-\alpha-4}{2} \mathscr{B} = \frac{2}{p} \mathscr{C}\delta$. Therefore, $\frac{(p-2)(2+\alpha)-4}{p} \mathscr{C}\delta = 0$. Since $\delta, \mathscr{C} > 0$, we have $p = \frac{8+2\alpha}{2+\alpha}$. It is impossible. Thus, $\mathcal{M}$ is a C^1-manifold.

(iii). We claim every critical point of J on $\mathcal{M}$ is a critical point of J in $X^{1,\alpha}$.

Suppose that u is a critical point of J on $\mathcal{M}$, there exists a Lagrange multiplier λ such that $J'(u) = \lambda \widetilde{I}'(u)$. Consequently, u solves the equation

$$-\left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\Delta u + \left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right](|x|^{\alpha-n} * u^2)u = [1 - \delta\lambda]\mu u^{p-1}. \qquad (25)$$

In the following we prove $\lambda \equiv 0$ (since $\widetilde{I}'(u) \neq 0$ by (ii)).

Recalling the definitions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \delta$, we can establish the following equations

$$\begin{cases} J(u) = \frac{1}{2}\mathcal{A} + \frac{1}{4}\mathcal{B} - \frac{1}{p}\mathcal{C}, \\ \widetilde{I}(u) = \frac{n-\alpha-4}{2+\alpha}\mathcal{A} + \frac{n-\alpha-4}{2(2+\alpha)}\mathcal{B} - \frac{1}{p}\mathcal{C}\delta = 0, \\ \left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{A} + \left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{B} - (1-\delta\lambda)\mathcal{C} = 0, \\ \frac{n-2}{2}\left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{A} + \frac{n+\alpha}{4}\left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{B} - (1-\delta\lambda)\frac{n}{p}\mathcal{C} = 0, \end{cases} \qquad (26)$$

where the third equation follows by multiplying (25) by u and integrating, and the fourth equality is the Pohozaev identity corresponding to equation (25).

It follows from the second equation and the third equation in (26) that

$$\mathcal{A} + \left[1 - \frac{n-\alpha-4}{2+\alpha}\lambda\right]\mathcal{B} = \left[1 + \frac{2\delta}{p}\lambda - \delta\lambda\right]\mathcal{C}. \qquad (27)$$

From the fourth equation in in (26), we have

$$\left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{A} + \frac{n+\alpha}{2(n-2)}\left[1 - \frac{2(n-\alpha-4)}{2+\alpha}\lambda\right]\mathcal{B} = \frac{2n(1-\delta\lambda)}{p(n-2)}\mathcal{C}. \qquad (28)$$

Applying (28) and the third equation in (26), we get

$$\frac{2+\alpha - 2(n-\alpha-4)\lambda}{2+\alpha}\mathcal{B} = [1-\delta\lambda]\frac{2[2n-(n-2)p]}{p(4+\alpha-n)}\mathcal{C}. \qquad (29)$$

The second equation in (26) can be rewritten as $\mathcal{A} = \frac{\delta(2+\alpha)}{p(n-\alpha-4)}\mathcal{C} - \frac{1}{2}\mathcal{B}$. Inserting into (27), one obtains

$$\frac{2+\alpha - 2(n-\alpha-4)\lambda}{2+\alpha}\mathcal{B} = 2\left[1 + \frac{2\delta}{p}\lambda - \frac{\delta(2+\alpha)}{p(n-\alpha-4)} - \delta\lambda\right]\mathcal{C}. \qquad (30)$$

Noting that $\delta, \mathcal{C} > 0$, it follows from (29) and (30) that

$$\frac{p(n-2)-2n}{p(n-\alpha-4)} - \frac{p(n-\alpha-4)-\delta(2+\alpha)}{p(n-\alpha-4)} = -\delta\left[\frac{p-2}{p} - \frac{2n-p(n-2)}{p(4+\alpha-n)}\right]\lambda.$$

By computing, we have $p(n-2) - 2n - [p(n-\alpha-4) - \delta(2+\alpha)] = 0$, and $\frac{p-2}{p} - \frac{2n-p(n-2)}{p(4+\alpha-n)} > 0$. Consequently, $\lambda \equiv 0$. We conclude that $J'(u) = 0$, i.e., u is a critical point of J. The proof is complete. $\qquad \square$

Proof of Theorem 1 in Case 2. Let $\{u_m\} \subset \mathcal{M}$ be a minimizing sequence of J in $\mathcal{M}$, that is, $J(u_m) \to \inf_{\mathcal{M}} J$ as $m \to \infty$, similar to the proof of Case 1, we can deduce that $J'(u_m) \to 0$ as $m \to \infty$. By (22), there holds

$$\frac{8+2\alpha - p(2+\alpha)}{2\delta(2+\alpha)}\int_{\mathbb{R}^n} |\nabla u_m|^2 \mathrm{d}x + \frac{8+2\alpha - p(2+\alpha)}{4\delta(2+\alpha)}L(u_m) \to \inf_{\mathcal{M}} J \quad (m \to \infty).$$

Since $p < \frac{8+2\alpha}{2+\alpha}$, we obtain that $\{\|u_m\|_{X^{1,\alpha}}\}$ is bounded. Similar to the proof of Case 1, we can obtain that equation (4) has a ground state solution. The proof is complete. $\qquad \square$

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