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# A $q$-deformation of true-polyanalytic Bargmann transforms when $q^{-1}>1$

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**Abstract.** We combine continuous $q^{-1}$-Hermite Askey polynomials with new $2D$ orthogonal polynomials introduced by Ismail and Zhang as $q$-analogs for complex Hermite polynomials to construct a new set of coherent states depending on a nonnegative integer parameter $m$. Our construction leads to a new $q$-deformation of the $m$-true-polyanalytic Bargmann transform on the complex plane. In the analytic case $m=0$, the obtained coherent states transform can be associated with the Arïk-Coon oscillator for $q'=q^{-1}>1$. These result may be used to introduce a $q$-deformed Ginibre-type point process.

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## 1. Introduction and statement of the results

In [13], Bargmann introduced a transform which maps isometrically the space $L^2(\mathbb{R})$ onto the Fock space $\mathfrak{F}(\mathbb{C})$ of entire functions belonging to $\mathfrak{H}:=L^2\left(\mathbb{C},e^{-z\bar{z}}\mathrm{d}\lambda(z)/\pi\right)$ where $\mathrm{d}\lambda(z)$ is the Lebesgue measure on $\mathbb{C}$. Since this transform is strongly linked to the Heisenberg group, it can be seen as a windowed Fourier transform [18]. Hence, the important role it plays in signal processing and harmonic analysis on the phase space [16]. It is also possible to interpret the kernel of this transform in terms of coherent states [5] of the quantum harmonic oscillator whose eigenstates are given by Hermite functions

$$\varphi_j(\xi)=\left(\sqrt{\pi}2^j j!\right)^{-1/2}H_j(\xi)e^{-\frac{1}{2}\xi^2}, \tag{1}$$

$H_j(\cdot)$ being the $j$th Hermite polynomial ( [22, p. 50]). A coherent state can be defined by a normalized vector $\Psi_z$ in $L^2(\mathbb{R})$, as a special superposition with the form

$$\Psi_z:=\left(e^{z\bar{z}}\right)^{-1/2}\sum_{j\geq 0}\frac{\bar{z}^j}{\sqrt{j!}}\varphi_j,\quad z\in\mathbb{C}. \tag{2}$$

It turns out that the coefficients

$$h_j(z) := \frac{z^j}{\sqrt{j!}},\ j = 0,1,2,\dots, \tag{3}$$

form an orthonormal basis of $\mathfrak{F}(\mathbb{C})$. If we denote by $\mathscr{B}_0$ the Bargmann transform, the image of an arbitrary function $f \in L^2(\mathbb{R})$ can be written as

$$\mathscr{B}_0[f](z) := \pi^{-\frac{1}{4}} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}\xi^2 + \sqrt{2}\xi z} f(\xi)\mathrm{d}\xi, \quad z \in \mathbb{C}. \tag{4}$$

Otherwise, it was proven [9] that $\mathfrak{F}(\mathbb{C})$ coïncides with the null space

$$\mathscr{A}_0(\mathbb{C}) := \{F \in \mathfrak{H},\ \widetilde{\Delta}F = 0\} \tag{5}$$

of the second-order differential operator

$$\widetilde{\Delta} := -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z}\frac{\partial}{\partial \bar{z}}. \tag{6}$$

The latter one, which acts on the Hilbert space, can be unitarily intertwined to appear as the Schrödinger operator for the motion of a charged particle evolving in a constant and uniform magnetic field normal to the plane. The spectrum of $\widetilde{\Delta}$ in $\mathfrak{H}$ is the set of eigenvalues $m \in \mathbb{Z}_+$, each of which has an infinite multiplicity, usually called Euclidean Landau levels. For $m \in \mathbb{Z}_+$, the associated eigenspace [9] :

$$\mathscr{A}_m(\mathbb{C}) := \left\{F \in \mathfrak{H},\ \widetilde{\Delta}F = mF\right\} \tag{7}$$

is also the $m$th-true-polyanalytic space [4, 27] or the generalized Bargmann space [9]. An orthonormal basis for this space is given by the functions

$$h_j^m(z) := (-1)^{m\wedge j}\left(m!j!\right)^{-1/2}(m\wedge j)!|z|^{|m-j|}e^{-i(m-j)arg(z)}L_{m\wedge j}^{(|m-j|)}(z\bar{z}),\ j = 0,1,\dots, \tag{8}$$

$L_n^{(\alpha)}(\cdot)$ being the Laguerre polynomial ( [22, p. 47]), $m \wedge j = \min(m, j)$ and $i^2 = -1$. Note that when $m = 0$, $h_j^0(z)$ reduces to $h_j(z)$ in (3). Therefore, we may replace the coefficients $h_j(z)$ by $h_j^m(z)$ to construct a family of coherent states depending on the parameter $m$. This leads to the coherent states transform $\mathscr{B}_m : L^2(\mathbb{R}) \to \mathscr{A}_m(\mathbb{C})$, defined for any $f \in L^2(\mathbb{R})$ by [24]:

$$\mathscr{B}_m[f](z) = (-1)^m(2^m m!\sqrt{\pi})^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}\xi^2 + \sqrt{2}\xi z} H_m\left(\xi - \frac{z+\bar{z}}{\sqrt{2}}\right) f(\xi)\mathrm{d}\xi, \tag{9}$$

where $H_m(\cdot)$ denotes the Hermite polynomial. This transform, also called $m$-true-polyanalytic Bargmann transform, has found applications in time-frequency analysis [1], discrete quantum dynamics [2] and determinantal point processes [3]. For more details on (9), see [4] and reference therein.

We also observe that the coefficients (8) can be rewritten in terms of $2D$ complex Hermite polynomials introduced by Itô [20], as $h_j^m(z) = \left(m!j!\right)^{-1/2} H_{m,j}(z,\bar{z})$ where

$$H_{r,s}(z,w) = \sum_{k=0}^{r\wedge s}(-1)^k k!\binom{r}{k}\binom{s}{k} z^{r-k}w^{s-k}, \quad r,s = 0,1,2,\dots. \tag{10}$$

For the latter ones, Ismail and Zhang have introduced the following $q$-analogs ( [19, p. 9]) :

$$H_{r,s}(z,w|q) := \sum_{k=0}^{r\wedge s} \begin{bmatrix} r \\ k \end{bmatrix}_q \begin{bmatrix} s \\ k \end{bmatrix}_q q^{(r-k)(s-k)}(-1)^k q^{\binom{k}{2}}(q;q)_k z^{r-k}w^{s-k},\ z,w \in \mathbb{C} \tag{11}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k},\ k \in \mathbb{Z}_+, \quad (a;q)_n = \prod_{l=0}^{n-1}\left(1 - aq^l\right) \quad \text{and} \quad (a;q)_\infty = \prod_{l=0}^{\infty}\left(1 - aq^l\right). \tag{12}$$

The polynomials (11) can also be rewritten in a form similar to (8) as

$$H_{r,s}(z,w|q) = (-1)^{r\wedge s}(q;q)_{r\wedge s}|z|^{|r-s|}e^{-i(r-s)arg(z)}L_{r\wedge s}^{(|r-s|)}\left(zw;q\right) \tag{13}$$

in terms of $q$-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ ( [22, p. 108]).

Here, we introduce a new $q$-deformation of the transform (9) with the parameter range $q^{-1}>1$. The kernel of such a transform may be obtained, up to a normalization factor depending on $z$, as the closed form of a generalized coherent state (a special superposition) that we now construct by replacing the coefficients $h_j^m(z)$ by a slight modification of the polynomials $H_{m,j}(z,\bar{z}|q)$. More precisely, our superposition combines the new coefficients with continuous $q^{-1}$-Hermite Askey functions [8], which are chosen as $q$-analogs of eigenstates of the harmonic oscillator and may also be associated with the Arïk-Coon oscillator for $q'=q^{-1}>1$ [14]. Precisely, by setting $w=\bar{z}$ in (13), we will be concerned with the following new coefficients

$$\mathfrak{h}_j^{m,q}(z):=\frac{(-1)^{m\wedge j}(q;q)_{m\wedge j}\sqrt{q^{-1}(1-q)}^{|m-j|}|z|^{|m-j|}e^{-i(m-j)arg(z)}}{q^{\frac{-1}{4}((m-j)^2+m+j)}\sqrt{(q;q)_m(q;q)_j}}L_{m\wedge j}^{(|m-j|)}\left(q^{-1}\alpha;q\right) \tag{14}$$

where $\alpha=(1-q)z\bar{z}$. Since $\lim_{q\to1}L_n^{(\alpha)}\left((1-q)x;q\right)=L_n^{(\alpha)}(x)$ it follows, after straightforward calculations, that $\lim_{q\to1}\mathfrak{h}_j^{m,q}(z)=h_j^m(z)$ which justifies our choice for the functions (14). Next, as $q$-analogs of eigenstates of the Hamiltonian of the harmonic oscillator, we will be dealing with the functions

$$\varphi_j^q(\xi):=\sqrt{\omega_q(\xi)}\left(\frac{q^{\frac{j(j+1)}{2}}}{(q;q)_j}\right)^{\frac{1}{2}}h_j\left(\sqrt{\frac{1-q}{2}}\xi|q\right),\ \xi\in\mathbb{R},\quad j=0,1,2,\ldots, \tag{15}$$

where $h_j(x|q)$ are the continuous $q^{-1}$-Hermite Askey polynomials [8] defined by

$$h_j(x|q)=i^{-j}H_j(ix|q^{-1}), \tag{16}$$

$H_j(x|p)$ being the continuous $p$-Hermite polynomial with $p>1$ ( [22, p. 115]) and

$$\omega_q(\xi)=\pi^{-\frac{1}{2}}q^{\frac{1}{8}}\cosh(\sqrt{\frac{1-q}{2}}\xi)e^{-\xi^2}. \tag{17}$$

Furthermore, in ( [10, p. 5]) Atakishiyev showed that the functions (15) satisfy a Ramanujan-type orthogonality relation on the full real line, which translates to

$$\int_{\mathbb{R}}\varphi_j^q(\xi)\varphi_k^q(\xi)\mathrm{d}\xi=\delta_{jk} \tag{18}$$

in terms of the functions $\{\varphi_j^q\}$. The latter ones also satisfy $\lim_{q\to1}\varphi_j^q(\xi)=\varphi_j(\xi)$ where $\varphi_j(\xi)$ are the Hermite functions (1). This justifies our choice in (15).

Now, with the above material, we are able to define *"à la Iwata"* [15, 21] a new family of generalized coherent states belonging to $L^2(\mathbb{R})$ by setting

$$\Psi_{z,m,q}:=(\mathcal{N}_{m,q}(z\bar{z}))^{-\frac{1}{2}}\sum_{j\geq0}\overline{\mathfrak{h}_j^{m,q}(z)}\varphi_j^q, \tag{19}$$

where the normalization factor

$$\mathcal{N}_{m,q}(z\bar{z})=\frac{q^{2m}((q-1)z\bar{z};q)_\infty(q^{-1}(q-1)z\bar{z};q)_m}{((q-1)z\bar{z};q)_m}, \tag{20}$$

is defined for every $z\in\mathbb{C}$. These states satisfy the resolution of the identity operator on $L^2(\mathbb{R})$ as

$$\int_{\mathbb{C}}|\Psi_{z,m,q}\rangle\langle\Psi_{z,m,q}|\mathrm{d}\nu_{m,q}(z)=\mathbf{1}_{L^2(\mathbb{R})}. \tag{21}$$

Here, the Dirac's bra-ket notation $|\Psi_{z,m,q}\rangle\langle\Psi_{z,m,q}|$ means the rank-one operator $\phi\longmapsto\langle\Psi_{z,m,q},\phi\rangle\cdot\Psi_{z,m,q}$, $\phi\in L^2(\mathbb{R})$ and $\mathrm{d}\nu_{m,q}(z):=\mathcal{N}_{m,q}(z\bar{z})\mathrm{d}\mu_q(z)$ where $\mathrm{d}\mu_q(z)$ is one of many orthogonal measures for the polynomials $\mathfrak{h}_j^{m,q}(z)$ and it is given by ( [19, p. 11]) :

$$\mathrm{d}\mu_q(z):=\frac{q-1}{q\,\mathrm{Log}\,q}(E_q(q^{-1}z\bar{z}))^{-1}\mathrm{d}\lambda(z)/\pi, \tag{22}$$

where $E_q(x) = ((q-1)x;q)_\infty$ defines a $q$-exponential function ( [17, p. 11]). Moreover, in the limit $q \to 1$ the measure $\mathrm{d}\mu_q$ reduces to the Gaussian measure $e^{-z\bar{z}}\mathrm{d}\lambda(z)/\pi$. Eq. (21) may also be understood in the weak sense as

$$\int_{\mathbb{C}} \langle f, \Psi_{z,m,q}\rangle\langle\Psi_{z,m,q}, g\rangle \mathrm{d}\nu_{m,q}(z) = \langle f, g\rangle, \qquad f,g \in L^2(\mathbb{R}). \tag{23}$$

Furthermore, straightforward calculations give the overlapping function of two coherent states (19). See Subsection 2.1 below for the proof.

**Proposition 1.** *For $m \in \mathbb{Z}_+$ and $q^{-1} > 1$, the following assertion holds true*

$$\langle\Psi_{z,m,q}, \Psi_{w,m,q}\rangle_{L^2(\mathbb{R})} = \frac{q^{2m}((q-1)z\bar{w};q)_\infty}{(\mathcal{N}_{m,q}(z\bar{z})\mathcal{N}_{m,q}(w\bar{w}))^{\frac{1}{2}}}\, {}_3\phi_2\left(\begin{matrix} q^{-m}, q^{\frac{\bar{w}}{\bar{z}}}, q^{\frac{z}{w}} \\ q, (q-1)z\bar{w}\end{matrix}\,\middle|\, q; q^{m-1}(q-1)w\bar{z}\right) \tag{24}$$

*for every $z, w \in \mathbb{C}$.*

Here, the ${}_3\phi_2$ $q$-series is defined by ( [17, p. 4]) :

$${}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, d\end{matrix}\,\middle|\, q; x\right) = \sum_{k\geq 0} \frac{(q^{-n};q)_k(a;q)_k(b;q)_k}{(c;q)_k(d;q)_k}\frac{x^k}{(q;q)_k}. \tag{25}$$

In particular, for $z = w$ in (24), the condition $\langle\Psi_{z,m,q}, \Psi_{z,m,q}\rangle_{L^2(\mathbb{R})} = 1$ may provide us with the normalization factor (20). Furthermore, (24) gives an explicit expression for the function

$$K_{m,q}(z,w) := (\mathcal{N}_{m,q}(z\bar{z})\mathcal{N}_{m,q}(w\bar{w}))^{\frac{1}{2}}\langle\Psi_{z,m,q}, \Psi_{w,m,q}\rangle_{L^2(\mathbb{R})} \tag{26}$$

which satisfies the limit

$$\lim_{q\to 1} K_{m,q}(z,w) = e^{z\bar{w}} L_m^{(0)}\left(|z-w|^2\right). \tag{27}$$

The proof of (27) is given in Subsection 2.2 below. Hence, one can say that the closure in $\mathfrak{H}_q := L^2(\mathbb{C}, \mathrm{d}\mu_q)$ of the linear span of $\{\mathfrak{h}_j^{m,q}\}_{j\geq 0}$ is a Hilbert space whose reproducing kernel is given in (26) and it will be denoted $\mathscr{A}_m^q(\mathbb{C})$. This space can also be viewed as a $q$-analog of the $m$th-true-polyanalytic space $\mathscr{A}_m(\mathbb{C})$ in (7) whose reproducing kernel was given by $e^{z\bar{w}}L_m^{(0)}\left(|z-w|^2\right)$, see [9].

Eq. (23) also means that the *coherent states transform* $\mathscr{B}_m^q : L^2(\mathbb{R}) \longrightarrow \mathscr{A}_m^q(\mathbb{C})$ defined as usual (see [5, p. 27 for the general theory]) by

$$\mathscr{B}_m^q[f](z) = (\mathcal{N}_{m,q}(z\bar{z}))^{\frac{1}{2}}\langle f, \Psi_{z,m,q}\rangle_{L^2(\mathbb{R})},\ z \in \mathbb{C}, \tag{28}$$

is an isometric map for which we establish the following precise result, see Subsection 2.3 below for the proof.

**Theorem 2.** *For $m \in \mathbb{Z}_+$ and $q^{-1} > 1$, the transform* (28) *is explicitly given by*

$$\begin{aligned}\mathscr{B}_m^q[f](z) = \gamma_{q,m}\int_{\mathbb{R}} &\left(-q^{\frac{1+m}{2}}\sqrt{1-q}\,z e^{\operatorname{argsinh}(\sqrt{\frac{1-q}{2}}\xi)}, q^{\frac{1+m}{2}}\sqrt{1-q}\,z e^{-\operatorname{argsinh}(\sqrt{\frac{1-q}{2}}\xi)}; q\right)_\infty \\ &\times \widetilde{Q}_m\left(\sqrt{\frac{1-q}{2}}\xi; i q^{\frac{m-1}{2}}\sqrt{1-q}\,z, i q^{\frac{m-3}{2}}\sqrt{1-q}\,\bar{z}; q\right)\sqrt{\omega_q(\xi)}f(\xi)\mathrm{d}\xi,\end{aligned} \tag{29}$$

*where $\gamma_{q,m} = \frac{(-1)^m q^{\frac{1}{2}\binom{m}{2}}}{\sqrt{(q;q)_m}}$ and $\widetilde{Q}_m$ denotes the $q^{-1}$-AL-Salam-Chihara polynomials.*

Here, the polynomial $\widetilde{Q}_m$ is defined by ( [10, p. 6]) :

$$\widetilde{Q}_n(\sinh\kappa; t, \tau; q) = q^{-\binom{n}{2}}(it)^n(it^{-1}e^{\kappa}, -it^{-1}e^{-\kappa}; q)_n\, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{1-n}t\tau, 0 \\ iq^{1-n}te^{\kappa}, -iq^{1-n}te^{\kappa}\end{matrix}\,\middle|\, q; q\right) \tag{30}$$

where $\kappa \in \mathbb{R}$ and $t, \tau \in \mathbb{C}$. The isometry $\mathscr{B}_m^q$ will be called a $q$-deformation of the true-polyanalytic Bargmann transform $\mathscr{B}_m$ when $q^{-1} > 1$. Indeed, when $q \to 1$ (29) reduces to (9), see Subsection 2.4 below for the proof.

**Corollary 3.** *For $m = 0$, the transform* (29) *reduces to* $\mathscr{B}_0^q : L^2(\mathbb{R}) \longrightarrow \mathscr{A}_0^q(\mathbb{C})$, *defined by*

$$\mathscr{B}_0^q[f](z) = \int_{\mathbb{R}} \left(-\sqrt{q(1-q)}ze^{\operatorname{argsinh}(\sqrt{\frac{1-q}{2}}\xi)}, \sqrt{q(1-q)}ze^{-\operatorname{argsinh}(\sqrt{\frac{1-q}{2}}\xi)}; q\right)_\infty \sqrt{\omega_q(\xi)}f(\xi)\mathrm{d}\xi$$

*for every $z \in \mathbb{C}$. In particular, when $q \to 1$, $\mathscr{B}_0^q$ goes to the Bargmann transform* (4).

Here, $\mathscr{A}_0^q(\mathbb{C})$ is the completed space of entire functions in $\mathfrak{H}_q$, for which the elements

$$\mathfrak{h}_j^{0,q}(z) = ([j]_q!)^{-1/2} q^{\frac{1}{2}\binom{j}{2}} z^j, \tag{31}$$

where $[j]_q! = \frac{(q;q)_j}{(1-q)^j}$, constitute an orthonormal basis. Note that by replacing in (31) the parameter $q$ by its inverse $q' = q^{-1}$, we recover the well known orthonormal basis $([j]_{q'}!)^{-1/2}z^j$ of the classical Arïk-Coon type space with $q' = q^{-1} > 1$ [25].

**Remark 4.** For $m = 0$, we recover in $L^2\left(\mathbb{R}, \sqrt{\omega_q(\xi)}\,\mathrm{d}\xi\right)$ the state $\langle \xi | z, 0, q\rangle \equiv \left(\omega_q(\xi)\right)^{-\frac{1}{2}} \Psi_{z,0,q}(\xi)$ as a coherent state for the Arïk-Coon oscillator with the deformation parameter $q' = q^{-1} > 1$, which was constructed by Burban ( [14, p. 5]).

**Remark 5.** In [19, p. 4] Ismail and Zhang have also introduced another class of $2D$ orthogonal $q$-polynomials, here denoted by $\widetilde{H}_{m,j}(z, w|q)$, which also generalize the complex Hermite polynomials [20] and are connected to ones in (11) by

$$\widetilde{H}_{m,j}(z, w|q) = q^{mj} i^{m+j} H_{m,j}(z/i, w/i|q^{-1}). \tag{32}$$

In our previous joint work with Arjika [7], we have combined the polynomials $\widetilde{H}_{m,j}(z, \bar{z}|q)$ with the continuous $q$-Hermite polynomials $H_j(\xi|q)$ and we have obtained a $q$-deformed $m$-true-polyanalytic Bargmann transform on $L^2\left(\left]\frac{-\sqrt{2}}{\sqrt{1-q}}, \frac{\sqrt{2}}{\sqrt{1-q}}\right[, \mathrm{d}\xi\right)$ with $q^{-1} > 1$.

**Remark 6.** The expression (26) may also constitute a starting point to construct a $q$-deformation for the determinantal point process associated with an $m$th Euclidean Landau level or Ginibre-type point process in $\mathbb{C}$ as discussed by Shirai [26].

## 2. Proofs

### 2.1. *Proof of Proposition 1*

By (18)-(19), the overlapping function of two coherent states is given by

$$\langle \Psi_{z,m,q}, \Psi_{w,m,q}\rangle_{L^2(\mathbb{R})} = \left(\mathscr{N}_{m,q}(z\bar{z})\mathscr{N}_{m,q}(w\bar{w})\right)^{-\frac{1}{2}} \sum_{j=0}^{\infty} \overline{\mathfrak{h}_j^{m,q}(z)}\mathfrak{h}_j^{m,q}(w) \tag{33}$$

$$= \left(\mathscr{N}_{m,q}(z\bar{z})\mathscr{N}_{m,q}(w\bar{w})\right)^{-\frac{1}{2}} S^{(m)}. \tag{34}$$

Replacing $\mathfrak{h}_j^{m,q}(z)$ by their expressions in (14), we can write $S^{(m)} = S_{<\infty}^{(m)} + S_{\infty}^{(m)}$, where

$$S_{<\infty}^{(m)} = \sum_{j=0}^{m-1} \frac{(q,q;q)_j q^{\frac{(m-j)^2+m+j}{2}} (q^{-1}-1)^{m-j} (\bar{z}w)^{m-j}}{(q;q)_m (q;q)_j} L_j^{(m-j)}\left(q^{-1}\alpha; q\right) L_j^{(m-j)}\left(q^{-1}\beta; q\right)$$
$$- \sum_{j=0}^{m-1} \frac{(q,q;q)_m q^{\frac{(m-j)^2+m+j}{2}} (q^{-1}-1)^{j-m} (z\bar{w})^{j-m}}{(q;q)_m (q;q)_j} L_m^{(j-m)}\left(q^{-1}\alpha; q\right) L_m^{(j-m)}\left(q^{-1}\beta; q\right),$$

and

$$S_{\infty}^{(m)} = \sum_{j\geq 0} \frac{(q,q;q)_m q^{\frac{(m-j)^2+m+j}{2}} (q^{-1}-1)^{j-m} (z\bar{w})^{j-m}}{(q;q)_m (q;q)_j} L_m^{(j-m)}\left(q^{-1}\alpha;q\right) L_m^{(j-m)}\left(q^{-1}\beta;q\right)$$
$$= \frac{q^{\frac{m^2+3m}{2}}(q;q)_m}{\lambda^m} \sum_{j\geq 0} \frac{q^{\binom{j}{2}}\left(\lambda q^{-m}\right)^j}{(q;q)_j} L_m^{(j-m)}\left(q^{-1}\alpha;q\right) L_m^{(j-m)}\left(q^{-1}\beta;q\right),$$

where $\lambda = (1-q)z\bar{w}$, $\alpha = (1-q)z\bar{z}$ and $\beta = (1-q)w\bar{w}$. Now, we apply the relation ( [23, p. 3]):

$$L_n^{(-N)}(x;q) = (-1)^{-N} x^N \frac{(q;q)_{n-N}}{(q;q)_n} L_{n-N}^{(N)}(x;q) \tag{35}$$

for $N = j-m,\ n = j,\ x = \alpha$ in a first time and next for $x = \beta$. To obtain that $S_{<\infty}^{(m)}(z,w;q) = 0$. For the infinite sum, we rewrite the $q$-Laguerre polynomial as ( [22, p. 110]):

$$L_n^{(\gamma)}(x;q) = \frac{1}{(q;q)_n} {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, -x \\ 0 \end{matrix}\right| q; q^{n+\gamma+1}\right) \tag{36}$$

with $n = m, \gamma = j-m, x = q^{-1}\alpha$ for $L_m^{(j-m)}\left(q^{-1}\alpha;q\right)$ and $x = q^{-1}\beta$ for $L_m^{(j-m)}\left(q^{-1}\beta;q\right)$. This gives

$$S_{\infty}^{(m)} = \frac{q^{\frac{m^2+3m}{2}}}{\lambda^m (q;q)_m} \mathsf{S}_q^{(m)}(\alpha;\beta) \tag{37}$$

where

$$\mathsf{S}_q^{(m)}(\alpha;\beta) := \sum_{j\geq 0} \frac{q^{\binom{j}{2}}\left(\lambda q^{-m}\right)^j}{(q;q)_j} {}_2\phi_1\left(\left.\begin{matrix} q^{-m}, -q^{-1}\alpha \\ 0 \end{matrix}\right| q; q^{j+1}\right) {}_2\phi_1\left(\left.\begin{matrix} q^{-m}, -q^{-1}\beta \\ 0 \end{matrix}\right| q; q^{j+1}\right). \tag{38}$$

Recalling ( [17, p. 3]):

$${}_2\phi_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| q; x\right) = \sum_{k\geq 0} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{x^k}{(q;q)_k}, \tag{39}$$

the r.h.s of (38) becomes

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = \sum_{j\geq 0} \frac{q^{\binom{j}{2}}\left(\lambda q^{-m}\right)^j}{(q;q)_j} \sum_{k\geq 0} \frac{(q^{-m}, -q^{-1}\alpha;q)_k}{(q;q)_k} (q^{j+1})^k \sum_{l\geq 0} \frac{(q^{-m}, -q^{-1}\beta;q)_l}{(q;q)_l} (q^{j+1})^l \tag{40}$$
$$= \sum_{k,l\geq 0} \frac{(q^{-m}, -q^{-1}\alpha;q)_k\, q^k}{(q;q)_k} \frac{(q^{-m}, -q^{-1}\beta;q)_l\, q^l}{(q;q)_l} \sum_{j\geq 0} \frac{q^{\binom{j}{2}}\left(q^{-m+k+l}\lambda\right)^j}{(q;q)_j}. \tag{41}$$

Now, by applying the $q$-binomial theorem ( [17, p. 11]):

$$\sum_{n\geq 0} \frac{q^{\binom{n}{2}}}{(q;q)_n} a^n = (-a;q)_\infty \tag{42}$$

for $a = q^{-m+k+l}\lambda$, the r.h.s of (40) takes the form

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = \sum_{k,l\geq 0} \frac{(q^{-m}, -q^{-1}\alpha;q)_k\, q^k}{(q;q)_k} \frac{(q^{-m}, -q^{-1}\beta;q)_l\, q^l}{(q;q)_l} (-q^{-m+k+l}\lambda;q)_\infty. \tag{43}$$

By making use of the identity ( [22, p. 9]):

$$(a;q)_\gamma = \frac{(a;q)_\infty}{(aq^\gamma;q)_\infty} \tag{44}$$

for the factor $(-q^{-m+k+l}\lambda;q)_\infty$, (43) transforms to

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = (-q^{-m}\lambda;q)_\infty \sum_{k\geq 0} \frac{(q^{-m}, -q^{-1}\alpha;q)_k}{(q;q)_k} q^k \sum_{l\geq 0} \frac{(q^{-m}, -q^{-1}\beta;q)_l}{(-q^{-m}\lambda;q)_{k+l}(q;q)_l} q^l. \tag{45}$$

Next, by the fact that $(q^{-m}\lambda;q)_{l+k} = (q^{-m}\lambda;q)_k(q^{k-m}\lambda;q)_l$, it follows that

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = (-q^{-m}\lambda;q)_\infty \sum_{k\geq 0} \frac{(q^{-m},-q^{-1}\alpha;q)_k}{(-q^{-m}\lambda,q;q)_k}\, q^k\, {}_2\phi_1\left(\begin{matrix} q^{-m},-q^{-1}\beta \\ -q^{k-m}\lambda \end{matrix}\middle| q;q\right).$$

Using the identity ( [17, p. 10]):

$$ {}_2\phi_1\left(\begin{matrix} q^{-n},b \\ c \end{matrix}\middle| q;q\right) = \frac{(b^{-1}c;q)_n}{(c;q)_n} b^n \tag{46}$$

for $n = m$, $b = -q^{-1}\beta$ and $c = -q^{k-m}\lambda$, leads to

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = (-q^{-m}\lambda;q)_\infty \sum_{k\geq 0} \frac{(q^{-m},-q^{-1}\alpha;q)_k}{(-q^{-m}\lambda,q;q)_k}\, q^k\, \frac{(q^{k+1-m}\frac{z}{w};q)_m}{(-q^{k-m}\lambda;q)_m}(-q^{-1}\beta)^m.$$

Applying the identity ( [22, p. 9]):

$$(aq^n;q)_r = \frac{(a;q)_r(aq^r;q)_n}{(a;q)_n} \tag{47}$$

for $r = m$, $n = k$, $a = q^{1-m}z/w$ and $a = -q^{-m}\lambda$ in a second time, we arrive at

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = \frac{(-q^{-1}\beta)^m(q^{1-m}\frac{z}{w};q)_m(-q^{-m}\lambda;q)_\infty}{(-q^{-m}\lambda;q)_m}\, {}_3\phi_2\left(\begin{matrix} q^{-m},-q^{-1}\alpha,q\frac{z}{w} \\ q^{1-m}\frac{z}{w},-\lambda \end{matrix}\middle| q;q\right). \tag{48}$$

Finally, by the finite Heine transformation ( [6, p. 2]):

$$ {}_3\phi_2\left(\begin{matrix} q^{-n},\xi,\sigma \\ \gamma,q^{1-n}/\tau \end{matrix}\middle| q;q\right) = \frac{(\xi\tau;q)_n}{(\tau;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},\gamma/\sigma,\xi \\ \gamma,\xi\tau \end{matrix}\middle| q;\sigma\tau q^n\right) \tag{49}$$

for parameters $\xi = q\frac{z}{w}$, $\sigma = -q^{-1}\alpha$, $\gamma = -\lambda$ and $\tau = \frac{w}{z}$, (48) reads

$$\mathsf{S}_q^{(m)}(\alpha;\beta) = \frac{(-q^{-1}\beta)^m(q^{1-m}\frac{z}{w},q;q)_m(-q^{-m}\lambda;q)_\infty}{(-q^{-m}\lambda,\frac{w}{z};q)_m}\, {}_3\phi_2\left(\begin{matrix} q^{-n},q\frac{\bar{w}}{\bar{z}},q\frac{z}{w} \\ q,-\lambda \end{matrix}\middle| q;-q^{m-1}(1-q)w\bar{z}\right). \tag{50}$$

Summarizing the above calculations and taking into account the previous prefactors, we arrive at the announced result (24). ☐

## 2.2. *Proof of the limit (27)*

Recalling that $E_q(x) = ((q-1)x;q)_\infty$, then we get that

$$\lim_{q\to 1} q^{2m}((q-1)z\bar{w};q)_\infty = e^{z\bar{w}}. \tag{51}$$

By another side, using (25) together with the fact that $(q^{-n};q)_k = 0, \forall\, k > n$, the series ${}_3\phi_2$ in (24) terminates as

$$\sigma_{m,q}(z,w) := \sum_{k=0}^{m} \frac{(q^{-m},q\frac{\bar{w}}{\bar{z}},q\frac{z}{w};q)_k}{((q-1)z\bar{w},q;q)_k}\, \frac{\left(q^{m-1}(q-1)w\bar{z}\right)^k}{(q;q)_k}. \tag{52}$$

Thus, from the identity ( [22, p. 10]):

$$\begin{bmatrix}\gamma \\ k\end{bmatrix}_q = (-1)^k q^{k\gamma-\binom{k}{2}}\frac{(q^{-\gamma};q)_k}{(q;q)_k},$$

we, successively, have

$$\lim_{q\to 1}\sigma_{m,q}(z,w)=\sum_{k=0}^{m}\lim_{q\to 1}\left(\frac{(q^{-m};q)_k}{(q;q)_k}\frac{(q\frac{\bar{w}}{\bar{z}},q\frac{z}{w};q)_k}{((q-1)z\bar{w};q)_k}\frac{(1-q)^k}{(q;q)_k}(-1)^k\left(q^{m-1}w\bar{z}\right)^k\right) \tag{53}$$

$$=\sum_{k=0}^{m}\lim_{q\to 1}\left(\begin{bmatrix} m\\ k\end{bmatrix}_q q^{\binom{k}{2}-mk}\frac{(q\frac{\bar{w}}{\bar{z}},q\frac{z}{w};q)_k}{((q-1)z\bar{w};q)_k}\frac{\left(q^{m-1}w\bar{z}\right)^k}{[k]_q!}\right) \tag{54}$$

$$=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{|z-w|^{2k}}{k!}. \tag{55}$$

By noticing that the sum in (55) is the evaluation of the Laguerre polynomial $L_m^{(0)}$ at $|z-w|^2$, the proof of the limit (27) is completed. □

### 2.3. *Proof of Theorem 2*

To apply (28), we seek for a closed form for the following series

$$\left(\mathcal{N}_{m,q}(z\bar{z})\right)^{\frac{1}{2}}\Psi_{z,m,q}(\xi)=\sum_{j\geq 0}\frac{(-1)^{m\wedge j}q^{\frac{(m-j)^2+(m+j)}{4}}(q;q)_{m\wedge j}\sqrt{q^{-1}(1-q)}^{|m-j|}}{\sqrt{(q;q)_m(q;q)_j}}$$
$$\times|z|^{|m-j|}e^{-i(m-j)arg(z)}L_{m\wedge j}^{(|m-j|)}\left(q^{-1}(1-q)z\bar{z};q\right)\varphi_j^q(\xi) \tag{56}$$

which may also be written as

$$\frac{(-1)^m q^{\frac{m^2+3m}{4}}\sqrt{\omega_q(\xi)(q;q)_m}}{(z\sqrt{1-q})^m}\eta^{m,q}(\xi,z), \tag{57}$$

with

$$\eta^{m,q}(\xi,z)=\sum_{j\geq 0}\frac{q^{\frac{-2mj+2j^2}{4}}(\sqrt{1-q}z)^j}{(q;q)_j}L_m^{(j-m)}\left(q^{-1}\alpha;q\right)h_j\left(\sqrt{\frac{1-q}{2}}\xi\middle|q\right) \tag{58}$$

where $\alpha=(1-q)z\bar{z}$. Next, replacing the $q$-Laguerre polynomial by its expression (36), (58) becomes

$$\eta^{m,q}(\xi,z)=\sum_{j\geq 0}\frac{q^{\frac{-2mj+2j^2}{4}}(\sqrt{1-q}z)^j}{(q;q)_j}h_j\left(\sqrt{\frac{1-q}{2}}\xi\middle|q\right)\frac{1}{(q;q)_m}{}_2\phi_1\left(\begin{matrix}q^{-m},-q^{-1}\alpha\\ 0\end{matrix}\middle|q;q^{j+1}\right)$$

$$=\frac{1}{(q;q)_m}\sum_{j\geq 0}\frac{q^{\frac{-2mj+2j^2}{4}}(\sqrt{1-q}z)^j}{(q;q)_j}h_j\left(\sqrt{\frac{1-q}{2}}\xi\middle|q\right)\sum_{k\geq 0}\frac{(q^{-m},-q^{-1}\alpha;q)_k}{(q;q)_k}q^{k(j+1)}$$

$$=\frac{1}{(q;q)_m}\sum_{k\geq 0}\frac{(q^{-m},-q^{-1}\alpha;q)_k}{(q;q)_k}q^k\sum_{j\geq 0}\frac{q^{\binom{j}{2}}(q^{\frac{1-m}{2}+k}\sqrt{1-q}z)^j}{(q;q)_j}h_j\left(\sqrt{\frac{1-q}{2}}\xi\middle|q\right). \tag{59}$$

By using the generating function of the $q^{-1}$-Hermite polynomials ( [10, p. 6]) :

$$\sum_{n\geq 0}\frac{t^n q^{\binom{n}{2}}}{(q;q)_n}h_n(x|q)=(-te^{\theta},te^{-\theta};q)_{\infty},\qquad \sinh\theta=x \tag{60}$$

for the parameters $t=q^{\frac{1-m}{2}+k}\sqrt{1-q}z$ and

$$\sinh\theta=\sqrt{\frac{1-q}{2}}\xi, \tag{61}$$

the r.h.s of (59) takes the form

$$\eta^{m,q}(\xi,z)=\frac{1}{(q;q)_m}\sum_{k\geq 0}\frac{(q^{-m},-q^{-1}\alpha;q)_k}{(q;q)_k}q^k(-ye^{\theta}q^k,ye^{-\theta}q^k;q)_{\infty}, \tag{62}$$

where $y = q^{\frac{1-m}{2}}\sqrt{1-q}z$. By applying (44), it follows that

$$\eta^{m,q}(\xi,z) = \frac{(-ye^{\theta}, ye^{-\theta};q)_\infty}{(q;q)_m}\sum_{k\geq 0}\frac{(q^{-m},-q^{-1}\alpha;q)_k}{(-ye^{\theta},ye^{-\theta};q)_k}\frac{q^k}{(q;q)_k} \tag{63}$$

which can also be expressed as

$$\eta^{m,q}(\xi,z) = \frac{(-ye^{\theta}, ye^{-\theta};q)_\infty}{(q;q)_m}{}_3\phi_2\left(\begin{matrix}q^{-m},-q^{-1}\alpha,0\\ -ye^{\theta},ye^{-\theta}\end{matrix}\middle| q;q\right). \tag{64}$$

Next, recalling the definition of the $q^{-1}$-Al-Salam-Chihara polynomials in (30) for $\kappa=\theta$, $t = iq^{m-1}y$ and $\tau = iq^{\frac{m-3}{2}}\sqrt{1-q}\bar{z}$, (64) reads

$$\eta^{m,q}(\xi,z) = \frac{(-1)^m q^{\binom{m}{2}}(-ye^{\theta},ye^{-\theta};q)_\infty\,\widetilde{Q}_m(\sinh\theta; iq^{\frac{m-1}{2}}y, iq^{\frac{m-3}{2}}\sqrt{1-q}\bar{z};q)}{(yq^{m-1})^m(q^{1-m}y^{-1}e^{\theta},-q^{1-m}y^{-1}e^{-\theta};q)_m(q;q)_m}. \tag{65}$$

After some simplifications, we arrive at the following form for the series (56)

$$\begin{aligned}\sqrt{\omega_q(\xi)}&(-q^{\frac{1+m}{2}}\sqrt{1-q}ze^{\theta}, q^{\frac{1+m}{2}}\sqrt{1-q}ze^{-\theta};q)_\infty\\ &\times\frac{(-1)^m q^{\frac{1}{2}\binom{m}{2}}}{\sqrt{(q;q)_m}}\widetilde{Q}_m\left(\sqrt{\frac{1-q}{2}}\xi; iq^{\frac{m-1}{2}}\sqrt{1-q}z, iq^{\frac{m-3}{2}}\sqrt{1-q}\bar{z};q\right).\end{aligned} \tag{66}$$

This ends the proof. □

### 2.4. *Proof of the limit in (29)*

To compute the limit of the quantity in (66) as $q\to 1$, we first observe that

$$\lim_{q\to 1}\sqrt{\omega_q(\xi)} = \lim_{q\to 1}\left(\pi^{-\frac{1}{2}}q^{\frac{1}{8}}\cosh(\sqrt{\frac{1-q}{2}}\xi)e^{-\xi^2}\right)^{1/2} = \pi^{-\frac{1}{4}}e^{-\xi^2/2}. \tag{67}$$

Next, we denote

$$G_q(z;\xi) := (-q^{\frac{1+m}{2}}\sqrt{1-q}ze^{\theta}, q^{\frac{1+m}{2}}\sqrt{1-q}ze^{-\theta};q)_\infty. \tag{68}$$

Then by (12), we successively obtain

$$\begin{aligned}\operatorname{Log}G_q(z;\xi) &= \sum_{k\geq 0}\operatorname{Log}\left(1-q^{\frac{1+m}{2}+k}\sqrt{1-q}ze^{-\theta}+q^{\frac{1+m}{2}+k}\sqrt{1-q}ze^{\theta}-q^{m+1+2k}(1-q)z^2\right)\\ &= q^{\frac{1+m}{2}}\sqrt{1-q}z(e^{\theta}-e^{-\theta})\sum_{k\geq 0}q^k - q^{m+1}(1-q)z^2\sum_{k\geq 0}q^{2k}+o(1-q)\\ &= q^{\frac{1+m}{2}}z(e^{\theta}-e^{-\theta})\frac{1}{\sqrt{1-q}} - q^{m+1}z^2\frac{1}{1+q}+o(1-q).\end{aligned}$$

Thus, form (61) the last equality also reads

$$\operatorname{Log}G_q(z;\xi) = q^{\frac{1+m}{2}}\sqrt{2}z\xi - q^{m+1}z^2\frac{1}{1+q}+o(1-q). \tag{69}$$

Therefore, when $q\to 1$, we have $\lim_{q\to 1}G_q(z;\xi) = e^{\sqrt{2}z\xi-\frac{1}{2}z^2}$. To obtain the limit of the polynomial quantity in (66) as $q\to 1$, we recall that the $q^{-1}$-Al-Salam-Chihara polynomials can be expressed as ( [11, p. 6]) :

$$\widetilde{Q}_n(s;a,b|q) = q^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}(ia)^{n-k}h_k(s;b|q) \tag{70}$$

in terms of the continuous big $q^{-1}$-Hermite polynomials. The latter ones satisfy the limit ( [12, p. 4]) :

$$\lim_{q\to 1}\kappa^{-n}h_n(\kappa s;2\kappa b|q) = H_n(s+ib),$$

and from (70) we conclude that

$$\lim_{q\to 1} \kappa^{-n} \widetilde{Q}_n(\kappa s; 2i\kappa a, 2i\kappa b; q) = H_n(s-a-b). \tag{71}$$

By applying (71) for $n = m,\ s = \xi,\ a = q^{\frac{m-1}{2}} z/\sqrt{2}, b = q^{\frac{m-3}{2}} \bar{z}/\sqrt{2}$ and $\kappa = \sqrt{\frac{1-q}{2}}$, we establish the following

$$\lim_{q\to 1} \frac{(-1)^m q^{\frac{1}{2}\binom{m}{2}}}{\sqrt{(q;q)_m}} \widetilde{Q}_m\left(\sqrt{\frac{1-q}{2}}\xi; i q^{\frac{m-1}{2}}\sqrt{1-q}z, i q^{\frac{m-3}{2}}\sqrt{1-q}\bar{z}; q\right) = (-1)^m (2^m m!)^{-\frac{1}{2}} H_m\left(\xi - \frac{z+\bar{z}}{\sqrt{2}}\right).$$

Finally, by grouping the obtained three limits, we arrive at the assertion in (29). □

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## References

[1] L. D. Abreu, "Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions", *Appl. Comput. Harmon. Anal.* **29** (2010), no. 3, p. 287-302.

[2] L. D. Abreu, P. Balázs, M. de Gosson, Z. Mouayn, "Discrete coherent states for higher Landau levels", *Ann. Phys.* **363** (2015), p. 337-353.

[3] L. D. Abreu, J. M. Pereira, J. L. Romero, S. Torquato, "The Weyl–Heisenberg ensemble: hyperuniformity and higher Landau levels", *J. Stat. Mech. Theory Exp.* **2017** (2017), no. 4, article no. 043103 (16 pages).

[4] L. D. Abreu, H. G. Feichtinger, "Function spaces of polyanalytic functions", in *Harmonic and complex analysis and its applications*, Trends in Mathematics, Birkhäuser, 2014, p. 1-38.

[5] S. T. Ali, J.-P. Antoine, J.-P. Gazeau, *Coherent states, Wavelets and their Generalizations*, 2nd ed., Theoretical and Mathematical Physics, Springer, 2014.

[6] G. E. Andrews, "The finite Heine transformation", in *Combinatorial number theory*, Walter de Gruyter, 2009, p. 1-6.

[7] S. Arjika, O. El Moize, Z. Mouayn, "Une $q$-déformation de la transformation de Bargmann vraie-polyanalytique", *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 8, p. 903-910.

[8] R. Askey, "Continuous $q$-Hermite polynomials when $q > 1$. $q$-Series and Partitions", in *$q$-Series and partitions (IMA, Minneapolis, 1988)*, The IMA Volumes in Mathematics and its Applications, vol. 18, Springer, 1989, p. 151-158.

[9] N. Askour, A. Intissar, Z. Mouayn, "Espaces de Bargmann généralisés et formules explicites pour leurs noyaux reproduisants", *C. R. Math. Acad. Sci. Paris* **325** (1997), no. 7, p. 707-712.

[10] N. M. Atakishiev, "Orthogonality of Askey–Wilson polynomials with respect to a measure of Ramanujan type", *Theor. Math. Phys.* **102** (1995), no. 1, p. 23-28, translated from *Teoret. Mat. Fiz* **104** (1995), no. 1, p. 32-39.

[11] M. K. Atakishiyeva, N. M. Atakishiev, "Fourier–Gauss transforms of the Al-Salam-Chihara polynomials", *J. Phys. A, Math. Gen.* **30** (1997), no. 19, p. 655-661.

[12] ———, "Fourier–Gauss transforms of the continuous big $q$-Hermite polynomials", *J. Phys. A, Math. Gen.* **30** (1997), no. 16, p. 559-565.

[13] V. Bargmann, "On a Hilbert space of analytic functions and an associated integral transform", *Commun. Pure Appl. Math.* **14** (1961), p. 174-187.

[14] I. M. Burban, "Arik–Coon oscillator with $q > 1$ in the framework of unified $(q;\alpha,\beta,\gamma;\nu)$-deformation", *J. Phys. A, Math. Theor.* **43** (2010), no. 62, article no. 305204 (9 pages).

[15] V. V. Dodonov, "Nonclassical states in quantum optics: a 'squeezed' review of the first 75 years", *J. Opt. B: Quantum Semiclassical Opt.* **4** (2002), p. 1-33.

[16] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, 1989, x+277 pages.

[17] G. Gasper, M. Rahman, *Basic hypergeometric series*, 2nd ed., Encyclopedia of Mathematics and Its Applications, vol. 96, Cambridge University Press, 2004.

[18] B. C. Hall, "Bounds on the Segal-Bargmann transform of $L_p$ functions", *J. Fourier Anal. Appl.* **7** (2001), no. 6, p. 553-569.

[19] M. E. H. Ismail, R. Zhang, "On some $2D$ Orthogonal $q$-polynomials", *Trans. Am. Math. Soc.* **369** (2017), no. 10, p. 6779-6821.

[20] K. Itô, "Complex multiple Wiener integral", *Jap. J. Math.* **22** (1952), p. 63-86.

[21] G. Iwata, "Transformation functions in the complex domain", *Prog. Theor. Phys.* **6** (1951), p. 524-528.

[22] R. Koekoek, R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogues", 1998, Delft University of Technology.

[23] S. G. Moreno, E. M. García-Caballero, "$q$-Sobolev orthogonality of the $q$-Laguerre polynomials $\{L_n^{(-N)}(\cdot,q)\}_{n=0}^{\infty}$ for positive integers $N$", *J. Korean Math. Soc.* **48** (2011), no. 5, p. 913-926.

[24] Z. Mouayn, "Coherent state transforms attached to generalized Bargmann spaces on the complex plane", *Math. Nachr.* **284** (2011), no. 14-15, p. 1948-1954.

[25] C. Quesne, K. A. Penson, V. M. Tkachuk, "Maths-type $q$-deformed coherent states for $q > 1$", *Phys. Lett., A* **33** (2003), no. 1-2, p. 29-36.

[26] T. Shirai, "Ginibre-type point processes and their asymptotic behavior", *J. Math. Soc. Japan* **67** (2015), no. 2, p. 763-787.

[27] N. L. Vasilevski, "Poly-Fock spaces", in *Differential operators and related topics*, Operator Theory: Advances and Applications, vol. 117, Birkhäuser, 2000, p. 371-386.