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p-parts of co-degrees of irreducible characters

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Abstract. For a character χ of a finite group G, the co-degree of χ is $\chi^c(1) = \frac{[G:\ker\chi]}{\chi(1)}$. Let p be a prime and let e be a positive integer. In this paper, we first show that if G is a p-solvable group such that $p^{e+1} \nmid \chi^c(1)$, for every irreducible character χ of G, then the p-length of G is not greater than e. Next, we study the finite groups satisfying the condition that p^2 does not divide the co-degrees of their irreducible characters.

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1. Introduction and preliminaries

In this paper, G is a finite group, p is a prime number and e is a positive integer. Let $Z(G)$ be the center of G and let $O_p(G)$ and $O_{p'}(G)$ be the largest normal p-subgroup and the largest normal p'-subgroup of G, respectively. Also, $O^{p'}(G)$ is the largest normal subgroup of G whose index in G is co-prime to p. For a p-solvable group G, the p-length of G, denoted by $\ell_p(G)$, is the minimum possible number of factors that are p-groups in any normal series of G which every factor is either a p-group or a p'-group. Let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of G. For a normal subgroup N of G and a character θ of N, let $I_G(\theta)$ denote the inertia group of θ in G and let $\mathrm{Irr}(G|\theta)$ be the set of the irreducible constituents of the induced character θ^G. Also, we use e_p to show the p-part of e. For a character χ of G, the number $\chi^c(1) = \frac{[G:\ker\chi]}{\chi(1)}$ is called the co-degree of χ (see [11]). Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$. In [1–3, 11], some properties of the co-degrees of irreducible characters of finite groups have been studied.

In [1], it has been proved that the p-length of a finite p-solvable group is not greater than the number of the distinct co-degrees of its irreducible characters which are divisible by p. In this paper, we prove that:

Theorem 1. *If G is a p-solvable group and $p^{e+1} \nmid \chi^c(1)$, for every $\chi \in \mathrm{Irr}(G)$, then $\ell_p(G) \le e$.*

In [8–10], it has been shown that if $p^2 \nmid \chi(1)$, for every $\chi \in \mathrm{Irr}(G)$, then $[G : O_p(G)]_p \le p^3$. In this paper, we also prove that:

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Theorem 2. *Let G be a non-p-solvable group. If $\chi^c(1)_p \le p$, for every $\chi \in \mathrm{Irr}(G)$, then $|G|_p = p$.*

Corollary 3. *If $\chi^c(1)_p \le p$, for every $\chi \in \mathrm{Irr}(G)$, then the Sylow p-subgroups of G are elementary abelian p-groups.*

In Examples 9, 10 and 11, we show that in Theorem 2, "non-p-solvability" cannot be substituted with "non-solvability" and in Corollary 3, there is not necessarily an upper bound for $|G|_p$ or $|G/O_p(G)|_p$.

2. Proofs of the main results

We first state a lemma that will be used frequently in this paper without explicit reference.

Lemma 4 (cf. [11, Lemma 2.1]). *Let N be a normal subgroup of G. Then,* $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. *Also, if $\psi \in \mathrm{Irr}(N)$, then $\psi^c(1) \mid \chi^c(1)$, for every $\chi \in \mathrm{Irr}(G|\psi)$.*

Lemma 5. *Let S be a non-abelian simple group.*

(i) *If p is a prime divisor of the order of the Schur multiplier of S, then $|S|_p \ge p^2$.*
(ii) *If p is a prime divisor of $|\mathrm{Out}(S)|$ such that p divides $|S|$, then $|S|_p \ge p^2$.*

Proof. Since S is a non-abelian simple group, $|S|_2 \ge 4$. So, the lemma follows when $p = 2$. Next, assume that $p \ge 3$. If p is a prime divisor of the order of the Schur multiplier of S, then since $p \ge 3$, [7, Section 5.1] shows that $S \cong PSL_n(q)$, $p \mid q-1$ and $p \mid n$, $S \cong PSU_n(q)$, $p \mid q+1$ and $p \mid n$,

$$S \in \{PSL_2(9), Alt_7, PSU_4(3), G_2(3), J_3, M_{22}, Fi_{22}, Mcl, Suz, B_3(3), {}^2E_6(4), Fi'_{24}, O'N\}$$

(under isomorphism) and $p = 3$, $S \cong E_6(q)$ and $p = 3 \mid q-1$ or $S \cong {}^2E_6(q)$ and $p = 3 \mid q+1$. Thus, we can check at once that $|S|_p \ge p^2$, as desired in (i). Next, let p be a prime divisor of $|\mathrm{Out}(S)|$ and $|S|$. Then, [8, Lemma 3.1] shows that $|S|_p > |\mathrm{Out}(S)|_p \ge p$. Thus, $|S|_p \ge p^2$, as wanted. □

In order to prove the main results, we need to prove the following propositions:

Proposition 6. *Let N be a minimal normal subgroup of G.*

(i) *If N is abelian and $\chi \in \mathrm{Irr}(G)$ such that $N \not\le \ker\chi$, then $|N|$ divides $\chi^c(1)$.*
(ii) *If $p \mid |N|$ and $\chi^c(1)_p \le p$, for every $\chi \in \mathrm{Irr}(G)$, then $|N|_p = p$ and N is a simple group.*

Proof. (i). Since N is a minimal normal subgroup of G and $N \ne N \cap \ker\chi \trianglelefteq G$, $N \cap \ker\chi = \{1\}$. So, $N \cong N\ker\chi/\ker\chi$ is an abelian normal subgroup of $G/\ker\chi$. By Ito's theorem (see [6, Theorem 6.15]), $\chi(1) \mid \left[\frac{G}{\ker\chi} : \frac{N\ker\chi}{\ker\chi}\right] = [G : N\ker\chi]$. Thus, $|N| \mid \chi^c(1)$, as desired in (i).

(ii). First suppose that $N \le O_p(G)$, $\theta \in \mathrm{Irr}(N) - \{1_N\}$ and $\chi \in \mathrm{Irr}(G|\theta)$. Then, $N \not\le \ker\chi$. So, $|N| \mid \chi^c(1)$, by (i). Thus, $|N|_p \le \chi^c(1)_p \le p$. However, N is a p-group. Hence, $|N| = p$, as desired. Now, let N be non-abelian. Then, $N = S_1 \times \cdots \times S_t$, where $S_1, \ldots, S_t$ are isomorphic non-abelian simple groups. For every $i \in \{1, \ldots, t\}$, $p \mid |S_i|$ and there exists $\theta_i \in \mathrm{Irr}(S_i) - \{1_{S_i}\}$ such that $p \nmid \theta_i(1)$, by [6, Corollary 12.2]. Set $\theta = \theta_1 \times \cdots \times \theta_t$ and let $\chi \in \mathrm{Irr}(G|\theta)$. Then, $\theta \in \mathrm{Irr}(N)$, $\ker\theta = \{1\}$ and $p \nmid \theta(1)$. Thus, $|N|_p \mid \theta^c(1)$, hence $|N|_p \mid \chi^c(1)$. So, $|N|_p \le p$. Consequently, $t = 1$ and N is a non-abelian simple group. □

Proposition 7. *Let $N \le Z(G)$ and G/N be a non-abelian simple group. If p divides $|N|$ and $|G/N|$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^c(1)_p \ge p^2$.*

Proof. Since $N \le Z(G)$, N is abelian. Hence, N has a maximal normal subgroup M such that $|N/M| = p$. However, $M \le N \le Z(G)$. Thus, $M \trianglelefteq G$ and $N/M \le Z(G/M)$. If we can show that there exists $\chi \in \mathrm{Irr}(G/M)$ such that $\chi^c(1)_p \ge p^2$, then Lemma 4 completes the proof. So, without loss of generality, assume that $|N| = p$. Then, $G' \cap N = \{1\}$ or N, where G' denotes the derived subgroup of G. Since G/N is a non-abelian simple group, $G'N = G$. Thus, either $G' \times N = G$ and $G' \cong G/N$ or $N \le G' = G$. In the former case, there exists $\theta \in \mathrm{Irr}(G') - \{1_{G'}\}$ such that $p \nmid \theta(1)$, by [6, Corollary 12.2]. Set $\chi = \theta \times \varphi$, for some $\varphi \in \mathrm{Irr}(N) - \{1_N\}$. Then, $\chi \in \mathrm{Irr}(G)$, $p \nmid \chi(1)$ and $\ker\chi = \{1\}$. Thus, $|G|_p$ divides $\chi^c(1)_p$. Hence, $\chi^c(1)_p \ge p^2$, as desired. In the latter case, G is a quasi-simple group with $Z(G) = N$. Consequently, $|N|$ divides $|M(G/N)|$, the order of the Schur multiplier of G/N. Therefore, $p \mid |M(G/N)|$. It follows from Lemma 5 (i) that $|G/N|_p \ge p^2$. Since G/N is a non-abelian simple group, there exists $\psi \in \mathrm{Irr}(G/N)$ such that $p \nmid \psi(1)$ and $\ker\psi = \{1\}$, by [6, Corollary 12.2]. So, $|G/N|_p$ divides $\psi^c(1)$. Consequently, $\psi^c(1)_p \ge p^2$. Hence, the proposition follows because $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. □

Proof of Theorem 1. Let G be a minimal counterexample. Then, since the hypothesis is inherited by quotients and normal subgroups and $\ell_p(G/O_{p'}(G)) = \ell_p(G) = \ell_p(O^{p'}(G))$, we can assume that $O_{p'}(G) = \{1\}$ and $O^{p'}(G) = G$. Thus, every minimal normal subgroup M of G is a p-group and $\ell_p(G/M) \le e$. Suppose that M and W are two distinct minimal normal subgroups of G. Then, since $M \cap W = \{1\}$, $\ell_p(G/W) \le e$ and $\ell_p(G/M) \le e$, $\ell_p(G) = \ell_p(G/(M\cap W)) \le \max\{\ell_p(G/M), \ell_p(G/W)\} \le e$, by [5, VI. 6.4]. This is a contradiction. Now let M be the unique minimal normal subgroup of G. Let $l = \ell_p(G/M)$ and define a normal series $\{1\} = P_0(G/M) \trianglelefteq M_0(G/M) \trianglelefteq P_1(G/M) \trianglelefteq M_1(G/M) \trianglelefteq \cdots \trianglelefteq P_l(G/M) \trianglelefteq M_l(G/M) = G/M$ of G/M such that $\frac{M_i(G/M)}{P_i(G/M)} = O_{p'}\left(\frac{G/M}{P_i(G/M)}\right)$ and $\frac{P_i(G/M)}{M_{i-1}(G/M)} = O_p\left(\frac{G/M}{M_{i-1}(G/M)}\right)$. Set $P_i/M = P_i(G/M)$ and $M_i/M = M_i(G/M)$. We claim that $O_{p'}(G/M) \ne \{1\}$. If not, $M_0 = P_0$. Thus, $P_1 = O_p(G)$ and $\{1\} \trianglelefteq P_1 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq P_l \trianglelefteq M_l = G$ is a normal series of G such that $\frac{M_i}{P_i} = O_{p'}\left(\frac{G}{P_i}\right)$ and $\frac{P_i}{M_{i-1}} = O_p\left(\frac{G}{M_{i-1}}\right)$, for every $1 \le i \le l$. Therefore, $\ell_p(G) = l = \ell_p(G/M) \le e$. This is a contradiction. Thus, $O_{p'}(G/M) \ne \{1\}$. Set $N/M = O_{p'}(G/M)$. By Schur–Zassenhaus theorem, N has a p-complement L. Then, $G = NN_G(L) = MN_G(L)$. Since M is abelian, $M \cap N_G(L) \trianglelefteq G$. However, M is a minimal normal subgroup of G and $O_{p'}(G) = \{1\}$. Thus, we can check that $M \cap N_G(L) = \{1\}$. So, every $\lambda \in \mathrm{Irr}(M)$ extends to $I_G(\lambda)$, by [6, Exercise 6.18]. Let $1_M = \lambda_1, \dots, \lambda_t$ be the representatives of the action of G on $\mathrm{Irr}(M)$. If O_i is the G-orbit of λ_i, then $1 + \Sigma_{i=2}^t |O_i| \lambda_i(1)^2 = \Sigma_{\lambda\in\mathrm{Irr}(M)} \lambda(1)^2 = |M| \equiv_p 0$. Hence, there exists $i > 1$ such that $p \nmid |O_i| = [G : I_G(\lambda_i)]$. So, $I_G(\lambda_i)$ contains a Sylow p-subgroup P of G. Since λ_i extends to $I_G(\lambda_i)$, there exists $\widehat{\lambda}_i \in \mathrm{Irr}(I_G(\lambda_i))$ such that $\widehat{\lambda}_{iM} = \lambda_i$. Set $\chi = \widehat{\lambda}_i^G$. By Clifford theory (see [6, Theorem 6.4]), $\chi \in \mathrm{Irr}(G)$ and $\chi(1) = [G : I_G(\lambda_i)]$. Also, $\ker\chi \cap M$ is a normal subgroup of G and M is a minimal normal subgroup of G. Thus, either $\ker\chi \cap M = M$ or $\ker\chi \cap M = \{1\}$. In the former case, $M \le \ker\chi$, so χ_M is trivial and $\lambda_i = \lambda_1$, which is a contradiction. Therefore, $\ker\chi \cap M = \{1\}$. Consequently, $\ker\chi = \{1\}$, because M is the unique minimal normal subgroup of G. Hence, $\chi^c(1) = |I_G(\lambda_i)|$, which is divisible by $|P|$. So, $|G|_p \le p^e$. Therefore, $\ell_p(G) \le e$, which is a contradiction. Now, the proof is complete. □

Note that in some parts of the proof of Theorem 1, we follow the ideas in the proof of [4, Theorem 2.3].

Proof of Theorem 2. First, let G be a non-p-solvable group of the minimal order such that $|G|_p \ge p^2$. Since the hypothesis is inherited by quotients and normal subgroups, we may assume that $O_{p'}(G) = \{1\}$ and $O^{p'}(G) = G$. We continue the proof in the following cases:

Case a. Assume that $O_p(G) \ne \{1\}$. Let $N \le O_p(G)$ be a minimal normal subgroup of G. Then, $|N| = p$, by Proposition 6 (ii). However, G/N is not p-solvable and $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. So, $|G/N|_p = p$, by minimality of G. Hence, $|G|_p = p^2$. Since $|N| = p$ and $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, we have $O^{p'}(G) \le C_G(N)$. However, $O^{p'}(G) = G$. So, $C_G(N) = G$. Consequently, $N \le Z(G)$. Set $\overline{G} = G/N$ and let $M/N = \overline{M}$ be a minimal normal subgroup of $\overline{G}$. If $\overline{M} \le O_{p'}(\overline{G})$, then $|N|$ and $|M/N|$ are co-prime. By Schur–Zassenhaus theorem, M has a p-complement H. Since $N \le Z(G)$, $M = H \times N$. Thus, $\overline{M} = HN/N \cong H/(H\cap N) = H = O_{p'}(M) \le O_{p'}(G) = \{1\}$, which is a contradiction. Now let $O_{p'}(\overline{G}) = \{1\}$. Then, since $\overline{G}$ is not p-solvable and $|\overline{G}|_p = p$, we get that $\overline{M}$

is the unique minimal normal subgroup of $\overline{G}$ and $|\overline{M}|_p = p$. So, $\overline{M}$ is a non-abelian simple group. By Proposition 7, there exists $\theta \in \mathrm{Irr}(M)$ such that $\theta^c(1)_p \geq p^2$. Therefore, $\chi^c(1)_p \geq p^2$, for every $\chi \in \mathrm{Irr}(G|\theta)$. This is a contradiction.

Case b. Let $O_p(G) = \{1\}$. Since $O_{p'}(G) = \{1\}$, every minimal normal subgroup of G is a non-abelian simple group of order divisible by p, by Proposition 6 (ii). If G has two distinct minimal normal subgroups M_1 and M_2, then $p \mid |M_1|, |M_2|$. However, $|G|_p = p^2$ and $O^{p'}(G) = G$. Thus, $G = M_1 \times M_2$. So, there exists $\theta = \theta_1 \times \theta_2 \in \mathrm{Irr}(M_1) \times \mathrm{Irr}(M_2) = \mathrm{Irr}(G)$ such that $p \nmid \theta(1)$ and $\ker\theta = \{1\}$. Therefore, $p^2 = |G|_p \mid \theta^c(1)$, which is a contradiction. Next let G have the unique minimal normal subgroup, say M. Then, $C_G(M) = \{1\}$. Consequently, $G \lesssim \mathrm{Aut}(M)$. By Proposition 6 (ii), $|M|_p = p$. Thus, $p \mid |G/M|$, hence $p \mid |\mathrm{Out}(M)|$. Lemma 5 (ii) shows that $|M|_p \geq p^2$. This is a contradiction.

So, $|G|_p = p$, as desired. □

Remark 8. If $\chi^c(1)_p \leq p$, for every irreducible character χ of G, then by Theorems 1 and 2, either $|G|_p = p$ or G is a p-solvable group of p-length one.

Proof of Corollary 3. If G is non-p-solvable, then $|G|_p = p$, by Theorem 2. Thus, the corollary follows. Now, let G be p-solvable. By Theorem 1, G has p-length one. Let $L = O_{p'}(G)$ and $K/L = O_p(G/L)$. Then, K/L is isomorphic to a Sylow p-subgroup of G. By Lemma 4, $\mathrm{Codeg}(K/L) = \{1, p\}$. Thus, [3, Lemma 2.4] forces K/L to be elementary abelian, as desired. □

Example 9. Assume that $G = L_1 \times \cdots \times L_t$, where $L_1, \ldots, L_t$ are Symmetric groups of degrees 3. Let $\chi \in \mathrm{Irr}(G) - \{1_G\}$. Then, there exist $\theta_1 \in \mathrm{Irr}(L_1), \ldots, \theta_t \in \mathrm{Irr}(L_t)$ such that $\chi = \theta_1 \times \cdots \times \theta_t$. Set $\Omega_1 = \{1 \leq i \leq t : \theta_i(1) = 2\}$ and $\Omega_2 = \{1 \leq i \leq t : i \not\in \Omega_1\}$. Let $\chi_1 = \Pi_{i \in \Omega_1}\theta_i$ and $\chi_2 = \Pi_{i \in \Omega_2}\theta_i$. If $\chi = \chi_1$, then $\chi(1) = |G|_2$. Hence, $\chi^c(1)_2 = 1$. Otherwise, fix $H = \Pi_{i \in \Omega_2} L_i$. Then, H/H' is an elementary abelian 2-group of order $|H|_2$ and $\chi_2 \in \mathrm{Irr}(H/H')$. Therefore, $|\ker\chi_2|_2 = |H|_2/2$, by [3, Lemma 2.4]. Since, $\chi = \chi_1 \times \chi_2$, $(\Pi_{i \in \Omega_1} 1_{L_i}) \times \ker\chi_2 \leq \ker\chi$. Thus, $2^{|\Omega_2|-1} \leq |\ker\chi|$. Also, $\chi(1) = 2^{|\Omega_1|}$. Therefore, $\chi^c(1)_2 \leq 2$. This example shows that in Corollary 3, $|G/O_p(G)|_p$ is not necessarily bounded.

Example 10. Let K be an elementary abelian 3-group of order 3^n. Then, the cyclic group $P = \langle z \rangle$ of order 2 acts on K by $x^z = x^2$, for every $x \in K$. Let G be a semi-direct product $K \rtimes P$ and let $\chi \in \mathrm{Irr}(G) - \{1_G\}$. If $K \leq \ker\chi$, then $\chi^c(1) = 2$. Otherwise, there exists $\theta \in \mathrm{Irr}(K) - \{1_K\}$ such that $\langle \chi_K, \theta \rangle \neq 0$. By [3, Lemma 2.4], $|\ker\theta| = 3^{n-1}$. It is easy to check that $\ker\theta \trianglelefteq G$. Therefore, $\ker\theta \leq \ker\chi$. Thus, $\chi^c(1)_3 \mid |G/\ker\theta|_3 = 3$. Consequently, $\chi^c(1)_3 \leq 3$. This example shows that in Corollary 3, $|O_p(G)|$ and $|G/Z(G)|_p$ are not necessarily bounded.

Example 11. Let $p \neq 2$ and S be a non-abelian simple group such that $p \nmid |S|$. Suppose that P is an elementary abelian p-group of order p^n and $G = P \times S$. For every $\chi \in \mathrm{Irr}(G)$, we can see that $p^{n-1} \mid |\ker\chi|$, so $\chi^c(1)_p \mid p^n/p^{n-1} = p$. This example shows that in Theorem 2, "non-p-solvability" cannot be substituted with "non-solvability".

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