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Combinatorics / *Combinatoire*

# Associated $r$ -Dowling numbers and some relatives

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**Abstract.** In this paper, we introduce a new generalization of Bell numbers, the  $s$ -associated  $r$ -Dowling numbers by combining two investigational directions. Here,  $r$  distinguished elements have to be in distinct blocks, some elements are coloured according to a colouring rule, and the cardinality of certain blocks is bounded from below by  $s$ . Along with them, we define some relatives, the  $s$ -associated  $r$ -Dowling factorials and the  $s$ -associated  $r$ -Dowling–Lah numbers, when the underlying set is decomposed into cycles or ordered blocks. The study of these numbers is based on their combinatorial meaning, and the exponential generating functions of their sequences derived from the so-called  $r$ -compositional formula.

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## 1. Introduction

Before discussing the topic of the paper, we introduce the notations

$$(x|m)^{\overline{n}} = \prod_{j=0}^{n-1} (x+jm), \quad (x|m)^{\underline{n}} = \prod_{j=0}^{n-1} (x-jm)$$

for the  $n^{\text{th}}$  rising and falling factorials of  $x$  with difference  $m$  ( $n \geq 0$ ,  $m \geq 1$ ), where we do not indicate the difference if  $m = 1$ .

Bell numbers  $B_n$  are basic objects in enumerative combinatorics, they count the number of partitions of the set  $\{1, \dots, n\}$ . Specifying certain restrictions on the partitions, we can obtain

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numerous generalizations and variants of these numbers. Here, we give a brief overview of those ones which will play a crucial role in the rest of the paper.

It is possible to forbid certain elements of the underlying set to belong to the same block. In this direction, the  $r$ -Bell numbers are widely studied. A partition of  $\{1, \dots, n+r\}$  is called an  $r$ -partition if  $1, \dots, r$  belong to distinct blocks. In this situation,  $1, \dots, r$  will be called *distinguished elements*, and a block containing a distinguished element will be referred to as a *distinguished block*. L. Carlitz [4] and I. Mező [17] defined the  $r$ -Bell number  $B_{n,r}$  as the number of  $r$ -partitions of  $\{1, \dots, n+r\}$ . Obviously, 0- and 1-partitions are simply ordinary partitions, therefore  $B_{n,0} = B_n$  and  $B_{n,1} = B_{n+1}$ . A graph theoretical treatment of these numbers can be found in [16].

A further generalization, the  $r$ -Dowling numbers were introduced by G.-S. Cheon, J.-H. Jung [5] and R. B. Corcino, C. B. Corcino, R. Aldema [7] (the latter authors called them  $(r, \beta)$ -Bell numbers). E. Gyimesi and G. Nyul [11] could describe these numbers through a purely combinatorial interpretation given in [12]. Namely, we call an  $r$ -partition a *Whitney coloured  $r$ -partition* with  $m$  colours if

- the smallest elements of the blocks are not coloured,
- elements in distinguished blocks are not coloured,
- the remaining elements are coloured with  $m$  colours.

Then, the  $r$ -Dowling number  $D_{n,m,r}$  is the total number of Whitney coloured  $r$ -partitions of the set  $\{1, \dots, n+r\}$  with  $m$  colours. As we can immediately see,  $D_{n,1,r} = B_{n,r}$ .

Another direction is to prescribe restrictions on the cardinality of all or certain blocks. From our viewpoint, associated Bell numbers are relevant. The  $s$ -associated Bell number  $B_n^{\geq s}$  counts the number of those partitions of  $\{1, \dots, n\}$ , where each block contains at least  $s$  elements. Obviously,  $B_n^{\geq 1} = B_n$ . These numbers appear with some of their properties by E. A. Enneking, J. C. Ahuja [8], F. T. Howard [13, 14], V. H. Moll, J. L. Ramírez, D. Villamizar [20], and in case of  $s = 2$  by M. Bóna, I. Mező [2], which special case is also mentioned in exercises of [22]. Surprisingly,  $B_n^{\geq 2}$  is just equal to the Bell number belonging to the  $n$ -vertex cycle graph in the sense of [16], although no bijective proof of this fact is known yet.

Until recently, the only attempt to combine the above two directions was made by F. T. Howard [15]. The  $s$ -associated  $r$ -Bell number  $B_{n,r}^{\geq s}$  is defined as the number of those  $r$ -partitions of  $\{1, \dots, n+r\}$ , where each non-distinguished block contains at least  $s$  elements. We note that F. T. Howard treated only the case  $s = 2$ , and discussed a few of the properties of 2-associated  $r$ -Bell numbers. Again,  $B_{n,r}^{\geq 1} = B_{n,r}$ . After the completion of this article, we have learnt about two very recent papers [1, 3] which deal with associated  $r$ -Bell numbers, but in an alternative way, see the closing section of the present paper about alternative definitions.

Now, we turn our attention to the “Bell-like numbers of the first kind”, the permutational variants of the above numbers. As it is well known, every permutation of  $\{1, \dots, n\}$  can be decomposed into disjoint cycles. A permutation of  $\{1, \dots, n+r\}$  is called an  $r$ -permutation if  $1, \dots, r$  belong to distinct cycles, and we use again the phrases *distinguished element* and *distinguished cycle*.

For the Dowling type generalization, the main difficulty is finding a suitable colouring interpretation. The colouring rule, which was given in [12], is less straightforward. An  $r$ -permutation is called a *Whitney coloured  $r$ -permutation* with  $m$  colours if

- the smallest elements of the cycles are not coloured,
- an element in a distinguished cycle is not coloured if there are no smaller numbers on the arc from the distinguished element to this element,
- the remaining elements are coloured with  $m$  colours.

Compared to partitions, these objects can be enumerated easier. Indeed, the total number of

permutations of  $\{1, \dots, n\}$ ,  $r$ -permutations and Whitney coloured  $r$ -permutations of  $\{1, \dots, n+r\}$  are simply  $A_n = n!$ ,  $A_{n,r} = (r+1)^{\overline{n}}$ ,  $DA_{n,m,r} = (r+1|m)^{\overline{n}}$ , respectively.

Contrary to this, associated factorial numbers are really interesting on their own. The  $s$ -associated factorial number  $A_n^{\geq s}$  gives the number of permutations of  $\{1, \dots, n\}$  for which each cycle is of length at least  $s$ . Notice that  $A_n^{\geq 2}$  is commonly known as  $n$  subfactorial or the  $n^{\text{th}}$  derangement number.

We can introduce  $s$ -associated  $r$ -factorial number  $A_{n,r}^{\geq s}$  as the number of those  $r$ -permutations of  $\{1, \dots, n+r\}$ , where each non-distinguished cycle has length at least  $s$ . An alternative variant of this kind of approach appeared recently in [19, 24], see again the closing section.

Finally, we can be interested in partitions into ordered blocks. The number of partitions of  $\{1, \dots, n\}$  and  $r$ -partitions of  $\{1, \dots, n+r\}$  into ordered blocks are counted by the summed Lah numbers  $L_n$  and summed  $r$ -Lah numbers  $L_{n,r}$ , respectively, which are studied by G. Nyul and G. Rácz [21].

According to the colouring rule given in [12], E. Gyimesi [10] introduced  $r$ -Dowling–Lah numbers  $DL_{n,m,r}$  as the total number of Whitney–Lah coloured  $r$ -partitions of  $\{1, \dots, n+r\}$  with  $m$  colours. Here, an  $r$ -partition into ordered blocks is called a *Whitney–Lah coloured  $r$ -partition* with  $m$  colours if

- the smallest elements of the ordered blocks are not coloured,
- an element in a distinguished ordered block is not coloured if there are no smaller numbers between the distinguished element and this element,
- the remaining elements are coloured with  $m$  colours.

The  $s$ -associated summed Lah number  $L_n^{\geq s}$  and  $s$ -associated summed  $r$ -Lah number  $L_{n,r}^{\geq s}$  can be defined analogously as the number of those partitions of  $\{1, \dots, n\}$ / $r$ -partitions of  $\{1, \dots, n+r\}$  into ordered blocks, where each ordered block/each non-distinguished ordered block contains at least  $s$  elements. We know about no attempts in this direction.

In this paper, we combine the most general variants of the above mentioned two directions, the  $r$ -generalized numbers with Whitney colourings and the  $s$ -associated numbers. More precisely, we introduce the  $s$ -associated  $r$ -Dowling numbers, and in addition, their factorial and Lah kind relatives. We emphasize that our results are new even in the special cases when we treat  $r$ -partitions or  $r$ -permutations without Whitney colourings, that is, for  $s$ -associated  $r$ -Bell numbers and their relatives, the  $s$ -associated  $r$ -factorials and  $s$ -associated summed  $r$ -Lah numbers. The strategy that we follow is to prove a useful tool, the  $r$ -compositional formula first. This allows us to give the exponential generating functions of sequences of the numbers in question. Based on the exponential generating functions and the combinatorial definitions, we are able to derive some identities, for instance, recurrence relations, Dobiński type formulas for these numbers. The proofs are mainly demonstrated through the  $s$ -associated  $r$ -Dowling numbers, but they can be performed similarly for the other two relatives.

## 2. The $r$ -compositional formula

In the following Theorem 1, we give the  $r$ -generalization of the well known compositional formula for exponential generating functions. One might prefer instead the usage of the language of Flajolet’s symbolic method (see [9]) to derive such exponential generating functions. Here,  $\mathbb{N}_0$  denotes the set of nonnegative integers,  $\mathbb{K}$  is a field of characteristic 0.

**Theorem 1.** *Let  $f_1, f_2, g : \mathbb{N}_0 \rightarrow \mathbb{K}$  be functions such that  $f_2(0) = 0$  and  $g(0) = 1$ . Denote their exponential generating functions by  $F_1(x), F_2(x)$  and  $G(x)$ , respectively. Define the function  $h : \mathbb{N}_0 \rightarrow \mathbb{K}$  as follows:  $h(0) = 1$ , and for  $n \geq 1$  let*

$$h(n) = \sum f_1(|Y_1|) \cdots f_1(|Y_r|) f_2(|Z_1|) \cdots f_2(|Z_k|) g(k),$$

where the sum is taken for all  $r$ -partitions  $\{Y_1 \cup \{1\}, \dots, Y_r \cup \{r\}, Z_1, \dots, Z_k\}$  of  $\{1, \dots, n+r\}$ . Then the exponential generating function of  $h$  is

$$H(x) = (F_1(x))^r G(F_2(x)).$$

**Proof.** First, we define functions  $h_k : \mathbb{N}_0 \rightarrow \mathbb{K}$  ( $k \geq 0$ ). Let

$$h_0(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

For  $k \geq 1$ , let  $h_k(0) = 0$  and

$$h_k(n) = \sum f_1(|Y_1|) \cdots f_1(|Y_r|) f_2(|Z_1|) \cdots f_2(|Z_k|) g(k)$$

for  $n \geq 1$ , where the sum is taken for all  $(k+r)$ -element  $r$ -partitions  $\{Y_1 \cup \{1\}, \dots, Y_r \cup \{r\}, Z_1, \dots, Z_k\}$  of  $\{1, \dots, n+r\}$ . Since  $f_2(0) = 0$ , we can allow here the sets  $Z_j$  ( $j = 1, \dots, k$ ) to be empty, therefore

$$h_k(n) = \frac{g(k)}{k!} \sum f_1(|Y_1|) \cdots f_1(|Y_r|) f_2(|Z_1|) \cdots f_2(|Z_k|),$$

where the sum is taken for all  $(k+r)$ -element weak ordered partitions  $(Y_1, \dots, Y_r, Z_1, \dots, Z_k)$  of  $\{r+1, \dots, n+r\}$ . Now, from the so-called product formula we have that the exponential generating function of  $h_k$  is

$$H_k(x) = \frac{g(k)}{k!} (F_1(x))^r (F_2(x))^k.$$

Finally, we can observe that  $h(n) = \sum_{k=0}^{\infty} h_k(n)$  (in fact, this sum is a finite sum for all  $n$ ), hence

$$H(x) = \sum_{k=0}^{\infty} H_k(x) = (F_1(x))^r \sum_{k=0}^{\infty} \frac{g(k)}{k!} (F_2(x))^k = (F_1(x))^r G(F_2(x)). \quad \square$$

This formula seems to be very useful in the study of  $r$ -generalization of various previously known combinatorial numbers. Indeed, for instance, it immediately implies the exponential generating functions of the sequences of  $r$ -Bell, summed  $r$ -Lah,  $r$ -Fubini,  $r$ -Dowling and  $r$ -Dowling–Lah numbers which were obtained by other means in [4, 17], [21], [18], [5–7, 11, 25], [10] respectively. The application of the  $r$ -compositional formula will be demonstrated for  $s$ -associated  $r$ -Dowling numbers and their relatives in the next Section 3.

### 3. The $s$ -associated $r$ -Dowling type numbers

Now, we define the subjects of this paper, the  $s$ -associated  $r$ -Dowling numbers and their relatives, which will be jointly referred to as  $s$ -associated  $r$ -Dowling type numbers.

**Definition.** For  $n, r \geq 0, n+r \geq 1$  and  $s, m \geq 1$ , denote by  $D_{n,m,r}^{\geq s}, DA_{n,m,r}^{\geq s}, DL_{n,m,r}^{\geq s}$  the total number of Whitney coloured  $r$ -partitions/Whitney coloured  $r$ -permutations/Whitney–Lah coloured  $r$ -partitions of  $\{1, \dots, n+r\}$  with  $m$  colours, where each non-distinguished block/cycle/ordered block contains at least  $s$  elements. In addition, let  $D_{0,m,0}^{\geq s} = DA_{0,m,0}^{\geq s} = DL_{0,m,0}^{\geq s} = 1$ . We call these numbers  $s$ -associated  $r$ -Dowling numbers,  $s$ -associated  $r$ -Dowling factorials,  $s$ -associated  $r$ -Dowling–Lah numbers, respectively.

**Remark.** In particular,  $D_{n,m,r}^{\geq 1} = D_{n,m,r}, D_{n,1,r}^{\geq s} = B_{n,r}^{\geq s}$ , and they hold similarly for the other two  $s$ -associated  $r$ -Dowling type numbers. This also means that if  $m = 1$  is substituted to the identities below, we obtain the corresponding formulas for  $s$ -associated  $r$ -Bell numbers,  $s$ -associated  $r$ -factorials and  $s$ -associated summed  $r$ -Lah numbers, as well.

As it was mentioned before, the  $r$ -compositional formula is our main tool to derive the exponential generating functions of the sequences of  $s$ -associated  $r$ -Dowling type numbers.

**Theorem 2.** *If  $r \geq 0$  and  $s, m \geq 1$ , then*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,m,r}^{\geq s}}{n!} x^n &= \exp\left(rx + \frac{\exp(mx) - 1}{m}\right) \exp\left(-\frac{1}{m} \sum_{j=1}^{s-1} \frac{1}{j!} (mx)^j\right), \\ \sum_{n=0}^{\infty} \frac{DA_{n,m,r}^{\geq s}}{n!} x^n &= (1 - mx)^{-\frac{r+1}{m}} \exp\left(-\frac{1}{m} \sum_{j=1}^{s-1} \frac{1}{j} (mx)^j\right), \\ \sum_{n=0}^{\infty} \frac{DL_{n,m,r}^{\geq s}}{n!} x^n &= (1 - mx)^{-\frac{2r}{m}} \exp\left(\frac{1}{m} \left(\frac{1}{1 - mx} - 1\right)\right) \exp\left(-\frac{1}{m} \sum_{j=1}^{s-1} (mx)^j\right). \end{aligned}$$

**Proof.** In this proof, we use the notations of Theorem 1.

First, by the definition of  $s$ -associated  $r$ -Dowling numbers, if

$$f_1(n) = 1, \quad f_2(n) = \begin{cases} 0 & \text{if } n \leq s - 1 \\ m^{n-1} & \text{if } n \geq s \end{cases}, \quad g(n) = 1,$$

then  $h(n) = D_{n,m,r}^{\geq s}$ . For these sequences, we have

$$F_1(x) = \exp(x), \quad F_2(x) = \frac{1}{m} \left( \exp(mx) - \sum_{j=0}^{s-1} \frac{1}{j!} (mx)^j \right), \quad G(x) = \exp(x),$$

and Theorem 1 gives the desired exponential generating function.

Similarly,  $h(n) = DA_{n,m,r}^{\geq s}$  for

$$f_1(n) = (1|m)^{\bar{n}}, \quad f_2(n) = \begin{cases} 0 & \text{if } n \leq s - 1 \\ (n - 1)! m^{n-1} & \text{if } n \geq s \end{cases}, \quad g(n) = 1$$

and

$$\begin{aligned} F_1(x) &= \sum_{n=0}^{\infty} \frac{(1|m)^{\bar{n}}}{n!} x^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{m}}{n} (-mx)^n = (1 - mx)^{-\frac{1}{m}}, \\ F_2(x) &= \frac{1}{m} \left( \ln\left(\frac{1}{1 - mx}\right) - \sum_{j=1}^{s-1} \frac{1}{j} (mx)^j \right), \quad G(x) = \exp(x). \end{aligned}$$

Finally,  $h(n) = DL_{n,m,r}^{\geq s}$  if

$$f_1(n) = (2|m)^{\bar{n}}, \quad f_2(n) = \begin{cases} 0 & \text{if } n \leq s - 1 \\ n! m^{n-1} & \text{if } n \geq s \end{cases}, \quad g(n) = 1$$

and

$$F_1(x) = (1 - mx)^{-\frac{2}{m}}, \quad F_2(x) = \frac{1}{m} \left( \frac{1}{1 - mx} - \sum_{j=0}^{s-1} (mx)^j \right), \quad G(x) = \exp(x). \quad \square$$

The following recurrences for  $s$ -associated  $r$ -Dowling type numbers can be obtained by differentiating the above exponential generating functions.

**Theorem 3.** *If  $r \geq 0$ ,  $s, m \geq 1$  and  $n \geq s - 1$ , then*

$$\begin{aligned} D_{n+1,m,r}^{\geq s} &= r D_{n,m,r}^{\geq s} + \sum_{j=0}^{n-s+1} \binom{n}{j} D_{j,m,r}^{\geq s} m^{n-j}, \\ DA_{n+1,m,r}^{\geq s} &= r \sum_{j=0}^n \binom{n}{j} DA_{j,m,r}^{\geq s} m^{n-j} (n-j)! + \sum_{j=0}^{n-s+1} \binom{n}{j} DA_{j,m,r}^{\geq s} m^{n-j} (n-j)!, \\ DL_{n+1,m,r}^{\geq s} &= 2r \sum_{j=0}^n \binom{n}{j} DL_{j,m,r}^{\geq s} m^{n-j} (n-j)! + \sum_{j=0}^{n-s+1} \binom{n}{j} DL_{j,m,r}^{\geq s} m^{n-j} (n-j+1)!. \end{aligned}$$

**Proof.** Denote by  $d_{m,r}^{\geq s}(x)$  the exponential generating function of the sequence  $(D_{n,m,r}^{\geq s})_{n=0}^{\infty}$  given in Theorem 2. Its formal derivative is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n+1,m,r}^{\geq s}}{n!} x^n &= d_{m,r}^{\geq s}(x) \left( r + \exp(mx) - \sum_{j=1}^{s-1} \frac{1}{(j-1)!} (mx)^{j-1} \right) \\ &= r d_{m,r}^{\geq s}(x) + d_{m,r}^{\geq s}(x) \left( \sum_{j=s-1}^{\infty} \frac{1}{j!} (mx)^j \right) \\ &= r \sum_{n=0}^{\infty} \frac{D_{n,m,r}^{\geq s}}{n!} x^n + \sum_{n=s-1}^{\infty} \left( \sum_{j=0}^{n-s+1} \frac{D_{j,m,r}^{\geq s}}{j!} \frac{1}{(n-j)!} m^{n-j} \right) x^n, \end{aligned}$$

from which the assertion follows.

In the case of  $s$ -associated  $r$ -Dowling numbers, we provide an additional combinatorial proof of this recurrence. Unfortunately, this argument fails to work for  $s$ -associated  $r$ -Dowling factorials or  $s$ -associated  $r$ -Dowling–Lah numbers.

We need to count Whitney coloured  $r$ -partitions of  $\{1, \dots, n+r+1\}$  with  $m$  colours, where each non-distinguished block contains at least  $s$  elements. We examine two cases.

If  $n+r+1$  is contained in a distinguished block, then without it we have a Whitney coloured  $r$ -partition of  $\{1, \dots, n+r\}$  with  $m$  colours, where each non-distinguished block contains at least  $s$  elements, and  $n+r+1$  can be put back to any of the  $r$  distinguished blocks.

If  $n+r+1$  is contained in a non-distinguished block, then let  $j$  be the number of non-distinguished elements outside this block ( $j = 0, \dots, n-s+1$ ). These elements can be chosen in  $\binom{n}{j}$  ways, and there are  $D_{j,m,r}^{\geq s}$  possibilities to  $r$ -partition them together with the distinguished elements in Whitney coloured sense such that each non-distinguished block contains at least  $s$  elements. Finally, all but one elements are coloured with  $m$  colours in the  $(n-j+1)$ -element block of  $n+r+1$ . Hence, for a fixed  $j$ , we have  $\binom{n}{j} D_{j,m,r}^{\geq s} m^{n-j}$  possibilities.  $\square$

After some manipulation of the above identities, recurrence relations of fixed order can be reached for  $s$ -associated  $r$ -Dowling factorials and  $s$ -associated  $r$ -Dowling–Lah numbers.

**Theorem 4.** *If  $r \geq 0, s, m \geq 1$  and  $n \geq s-1$ , then*

$$DA_{n+1,m,r}^{\geq s} = (mn+r) DA_{n,m,r}^{\geq s} + (mn|m)^{s-1} DA_{n-s+1,m,r}^{\geq s}.$$

*If  $r \geq 0, s, m \geq 1$  and  $n \geq s$ , then*

$$\begin{aligned} DL_{n+1,m,r}^{\geq s} &= (2mn+2r) DL_{n,m,r}^{\geq s} + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} \\ &\quad - mn(mn-m+2r) DL_{n-1,m,r}^{\geq s} - (s-1)(mn|m)^s DL_{n-s,m,r}^{\geq s}. \end{aligned}$$

**Proof.** By using Theorem 3 and the fundamental identities

$$(n-j+1)! = (n-j)(n-j)! + (n-j)!, \quad \binom{n}{j} = \frac{n}{n-j} \binom{n-1}{j} \quad (0 \leq j \leq n-1),$$

we have

$$\begin{aligned}
 DL_{n+1,m,r}^{\geq s} &= 2rDL_{n,m,r}^{\geq s} + 2r \sum_{j=0}^{n-1} \binom{n}{j} DL_{j,m,r}^{\geq s} m^{n-j} (n-j)! \\
 &\quad + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} + \sum_{j=0}^{n-s} \binom{n}{j} DL_{j,m,r}^{\geq s} m^{n-j} (n-j+1)! \\
 &= 2rDL_{n,m,r}^{\geq s} + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} \\
 &\quad + 2mnr \sum_{j=0}^{n-1} \binom{n-1}{j} DL_{j,m,r}^{\geq s} m^{n-j-1} (n-j-1)! \\
 &\quad + mn \sum_{j=0}^{n-s} \binom{n-1}{j} DL_{j,m,r}^{\geq s} m^{n-j-1} (n-j)! + \sum_{j=0}^{n-s} (mn|m)^{\overline{n-j}} DL_{j,m,r}^{\geq s} \\
 &= (mn+2r) DL_{n,m,r}^{\geq s} + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} + \sum_{j=0}^{n-s} (mn|m)^{\overline{n-j}} DL_{j,m,r}^{\geq s}.
 \end{aligned}$$

From this identity it follows that

$$\begin{aligned}
 DL_{n+1,m,r}^{\geq s} - mnDL_{n,m,r}^{\geq s} &= (mn+2r) DL_{n,m,r}^{\geq s} + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} \\
 &\quad + \sum_{j=0}^{n-s} (mn|m)^{\overline{n-j}} DL_{j,m,r}^{\geq s} - mn(m(n-1)+2r) DL_{n-1,m,r}^{\geq s} \\
 &\quad - s(mn|m)^s DL_{n-s,m,r}^{\geq s} - \sum_{j=0}^{n-s-1} (mn|m)^{\overline{n-j}} DL_{j,m,r}^{\geq s} \\
 &= (mn+2r) DL_{n,m,r}^{\geq s} + s(mn|m)^{s-1} DL_{n-s+1,m,r}^{\geq s} \\
 &\quad - mn(m(n-1)+2r) DL_{n-1,m,r}^{\geq s} - (s-1)(mn|m)^s DL_{n-s,m,r}^{\geq s}. \quad \square
 \end{aligned}$$

A purely combinatorial argument allows us to give the connections between  $s$ -associated  $r$ -Dowling and  $s$ -associated  $r'$ -Dowling type numbers.

**Theorem 5.** *If  $n \geq 0, r \geq r' \geq 0$  and  $s, m \geq 1$ , then*

$$\begin{aligned}
 D_{n,m,r}^{\geq s} &= \sum_{j=0}^n \binom{n}{j} D_{j,m,r'}^{\geq s} (r-r')^{n-j}, \\
 DA_{n,m,r}^{\geq s} &= \sum_{j=0}^n \binom{n}{j} DA_{j,m,r'}^{\geq s} (r-r'|m)^{\overline{n-j}}, \\
 DL_{n,m,r}^{\geq s} &= \sum_{j=0}^n \binom{n}{j} DL_{j,m,r'}^{\geq s} (2r-2r'|m)^{\overline{n-j}}.
 \end{aligned}$$

**Proof.** The Whitney coloured  $r$ -partitions of  $\{1, \dots, n+r\}$  with  $m$  colours having non-distinguished blocks containing at least  $s$  elements can be enumerated in the following way.

Let  $j$  be the number of those non-distinguished elements which belong to the distinguished blocks of  $1, \dots, r'$  or a non-distinguished block ( $j = 0, \dots, n$ ). We have  $\binom{n}{j}$  possible choices of these elements, and  $D_{j,m,r'}^{\geq s}$  ways to  $r'$ -partition them together with  $1, \dots, r'$  in Whitney coloured sense such that each non-distinguished block contains at least  $s$  elements. Finally, there exist  $(r-r')^{n-j}$  placements of the remaining  $n-j$  elements (in case of  $s$ -associated  $r$ -Dowling factorials and  $s$ -associated  $r$ -Dowling–Lah numbers, we insert them in increasing order, and they might be coloured according to the appropriate colouring rule). Summarizing, for a fixed  $j$ , we have  $\binom{n}{j} D_{j,m,r'}^{\geq s} (r-r')^{n-j}$  possibilities.  $\square$



The Dobiński type formulas for  $s$ -associated  $r$ -Dowling type numbers can be derived from their exponential generating functions. As it can be seen, they become much more complicated in the  $s$ -associated case.

**Theorem 6.** *If  $n, r \geq 0$  and  $s, m \geq 1$ , then*

$$\begin{aligned}
 D_{n,m,r}^{\geq s} &= e^{-\frac{1}{m}} \sum_{k=0}^{\infty} \frac{1}{m^k k!} \sum_{*} \frac{n!}{l!} (mk+r)^l \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j!}\right)^{i_j}, \\
 DA_{n,m,r}^{\geq s} &= \sum_{*} \frac{n!}{l!} (r+1|m)^{\bar{l}} \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j}\right)^{i_j}, \\
 DL_{n,m,r}^{\geq s} &= e^{-\frac{1}{m}} \sum_{k=0}^{\infty} \frac{1}{m^k k!} \sum_{*} \frac{n!}{l!} (mk+2r|m)^{\bar{l}} \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-m^{j-1}\right)^{i_j},
 \end{aligned}$$

where the sums indicated with a star symbol are taken over all  $s$ -tuples  $(i_1, i_2, \dots, i_{s-1}, l)$  of nonnegative integers satisfying  $i_1 + 2i_2 + \dots + (s-1)i_{s-1} + l = n$ .

**Proof.** The formula can be deduced from Theorem 2 as

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{D_{n,m,r}^{\geq s}}{n!} x^n &= \exp(rx) \exp\left(\frac{\exp(mx)}{m}\right) \exp\left(-\frac{1}{m}\right) \prod_{j=1}^{s-1} \exp\left(-\frac{m^{j-1}}{j!} x^j\right) \\
 &= \exp\left(-\frac{1}{m}\right) \sum_{k=0}^{\infty} \frac{\exp((mk+r)x)}{m^k k!} \prod_{j=1}^{s-1} \exp\left(-\frac{m^{j-1}}{j!} x^j\right) \\
 &= \exp\left(-\frac{1}{m}\right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(mk+r)^l}{m^k k! l!} x^l \prod_{j=1}^{s-1} \sum_{i_j=0}^{\infty} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j!}\right)^{i_j} x^{j i_j} \\
 &= \exp\left(-\frac{1}{m}\right) \sum_{k=0}^{\infty} \frac{1}{m^k k!} \sum_{*} \frac{1}{l!} (mk+r)^l \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j!}\right)^{i_j} x^n. \quad \square
 \end{aligned}$$

As an interesting, instant consequence of the above, we can obtain that 2-associated  $r$ -Dowling numbers coincide with  $(r-1)$ -Dowling numbers, by comparing any of our Theorems 2, 3, 6 with the corresponding results in [11]. We note that a special case appeared in [15] for 2-associated  $r$ -Bell numbers. Here, we provide a direct, combinatorial proof of this fact.

**Corollary 7.** *If  $n \geq 0$  and  $r, m \geq 1$ , then  $D_{n,m,r}^{\geq 2} = D_{n,m,r-1}$ .*

**Proof.** Consider a Whitney coloured  $r$ -partition of  $\{1, \dots, n+r\}$ , where each non-distinguished block contains at least 2 elements. If we delete  $r$ , split its block into singletons, and decrease all non-distinguished elements by 1 in the whole partition, then we arrive at a Whitney coloured  $(r-1)$ -partition of  $\{1, \dots, n+r-1\}$ . It can be easily verified that this mapping is bijective, which proves the equality.  $\square$

**Remark.** Corollary 7 remains valid for  $r = 0$  in the sense that 2-associated 0-Dowling numbers behave like they were  $(-1)$ -Dowling numbers. This observation is somehow similar to the famous theorem of R. P. Stanley [23] about chromatic polynomials.

#### 4. Alternative definitions

We say a few closing remarks about the possible alternative definitions of the numbers studied in our paper. If we require each block/cycle/ordered block (not only the non-distinguished ones) to contain at least  $s$  elements, then we call them alternative  $s$ -associated  $r$ -Dowling numbers, alternative  $s$ -associated  $r$ -Dowling factorials, alternative  $s$ -associated  $r$ -Dowling-Lah numbers (especially, alternative  $s$ -associated  $r$ -Bell numbers, alternative  $s$ -associated  $r$ -factorials, alternative

$s$ -associated summed  $r$ -Lah numbers if  $m = 1$ ), and denote them by  $D_{n,m,r}^{\geq s,*}$ ,  $DA_{n,m,r}^{\geq s,*}$ ,  $DL_{n,m,r}^{\geq s,*}$ , respectively. Alternative  $s$ -associated  $r$ -Bell numbers appear in two recent papers [1, 3] (the former one allows a more general restriction on the cardinality of the blocks), while in [19, 24], the authors investigated alternative 2-associated and alternative  $s$ -associated  $r$ -factorials under the names of  $r$ -derangement and generalized  $r$ -derangement numbers.

However, we prefer our variants since our previous experiences on  $r$ -generalized combinatorial numbers highly suggest that usually different requirements are imposed for distinguished and non-distinguished blocks/cycles/ordered blocks. Also, it fits to the usage of 2-associated  $r$ -Bell numbers by F. T. Howard [15]. On the other hand, our sequences consist of nonzero numbers already from their initial elements, while the alternative numbers are equal to 0 for  $0 \leq n \leq (s-1)r-1$ .

But, of course, the exponential generating function of the sequences

$$\left(D_{n,m,r}^{\geq s,*}\right)_{n=0}^{\infty}, \quad \left(DA_{n,m,r}^{\geq s,*}\right)_{n=0}^{\infty}, \quad \left(DL_{n,m,r}^{\geq s,*}\right)_{n=0}^{\infty}$$

can be also derived using the  $r$ -compositional formula (Theorem 1).

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