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Control Theory / *Théorie du contrôle*

Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay

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Abstract. This work exploits structural properties of a class of functional Vandermonde matrices to emphasize some qualitative properties of a class of linear autonomous n^{th} order differential equation with forcing term consisting in the delayed dependent-variable. More precisely, it deals with the stabilizing effect of delay parameter coupled with the coexistence of the maximal number of real spectral values. The derived conditions are *necessary and sufficient* and, to the best of the authors' knowledge, represent a novelty in the literature. Under appropriate conditions, such a configuration characterizes the spectral abscissa corresponding to the studied equation. A new stability criterion is proposed. This criterion extends recent results in factorizing quasipolynomial functions. The applicative potential of the proposed method is illustrated through the stabilization of coupled oscillators.

Résumé. Ce travail exploite les propriétés structurelles d'une classe de matrices de Vandermonde fonctionnelles, pour mettre en évidence certaines propriétés qualitatives d'une classe d'équation différentielle d'ordre n, autonome linéaire avec un terme source dépendant de la variable retardée. Plus précisément, il traite de l'effet stabilisateur du paramètre de retard couplé à la coexistence du nombre maximal de valeurs spectrales réelles. Les conditions dérivées sont *nécessaires et suffisantes* et, à la connaissance des auteurs, représentent une nouveauté dans la littérature. Sous des conditions appropriées, une telle configuration caractérise l'abscisse spectrale correspondant à l'équation étudiée. Un nouveau critère de stabilité est proposé. Ce critère étend les résultats récents sur la factorisation de fonctions quasi-polynomiales. Le potentiel applicatif du procédé proposé est illustré par la stabilisation d'oscillateurs couplés.

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1. Introduction

Matrices arising from a wide range of problems in mathematics and engineering typically display characteristic structures. In particular, exploiting such a structure in problems from dynamical systems is known to be challenging for understanding of complex qualitative behaviors and for characterizing system's properties, see, for instance, [6] and references therein. This study is a crossroad between the investigation of the invertibility of a class of such structured matrices which is related to *Multivariate Interpolation Problems* (namely, the well-known *Lagrange Interpolation Problem*) and the localisation of *spectral values* of linear *time-delay systems*. The study of conditions on the time-delay systems parameters guaranteeing the exponential stability of solutions is a question of ongoing interest and, to the best of the authors' knowledge, it remains an *open problem*. In particular, in frequency-domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic function, which is an entire function called *characteristic quasipolynomial*), see for instance [4, 14, 16, 24, 36, 37, 39].

The starting point of the present work is a property, discussed in recent studies, called *Multiplicity-Induced-Dominancy*, see, for instance, [3, 9]. As a matter of fact, it is shown that multiple spectral values for time-delay systems can be appropriately characterized by using the so-called *Birkhoff/Vandermonde-based approach*; see for instance [5–7, 13]. More precisely, in previous works, it is emphasized that the admissible multiplicity of the real spectral values is bounded by the generic *Polya and Szegö bound* (denoted PS_B), which is nothing else but the *degree* of the corresponding quasipolynomial (i.e the number of the involved polynomials plus their degree minus one), see for instance [32, Problem 206.2 p. 144 and p. 347]. It is worth mentioning that such a bound was recovered by using structured matrices in [6] rather than the standard principle argument as it was proved by Polya and Szegö in [32]. It is important to point out that the multiplicity of a root itself is not essential as such but its connection with the dominancy of this root is a meaningful tool for control synthesis.

To the best of the authors' knowledge, the first analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented and discussed in the 50s, see [17]. The dominancy property is further explored and analytically shown in the scalar delay system case in [13], and in second-order systems controlled by a delayed proportional controller is proposed in [10, 12] where its applicability in damping active vibrations for a piezo-actuated beam is proved. An extension to the delayed proportional-derivative controller case is proposed in [8, 11] where the dominancy property is parametrically characterized and proven using the argument principle. Recently, in [2] it is shown that, under appropriate conditions, the coexistence of exactly PS_B distinct negative zeros of quasipolynomial of reduced degree guarantees the exponential stability of the zero solution of the corresponding time-delay system. The dominancy of such real spectral values is proved by using an extended factorization technique which generalizes the one provided in [2]. Finally, to the best of the authors' knowledge, the *necessary and sufficient conditions* derived in the present paper as well as corresponding control strategy represent a novelty.

The present work investigates the effect of structural properties of a class of functional Vandermonde matrices and its effect on qualitative properties of a corresponding *linear autonomous time-delay system of retarded type*. More precisely, the aim of this work is three-fold: first, it emphasizes the link between the invertibility of a class of structured functional Vandermonde matrices and the coexistence of distinct real spectral values of linear time-delay systems, which allows to recover the maximal number of distinct real spectral values that may coexist for a given time-delay system. Second, if the number of coexistent real spectral values reaches the PS_B, then a *necessary and sufficient condition* for the asymptotic stability is provided (which is equivalent to the exponential stability [19, p. 79]), see also [26] for an estimate of the exponential decay rate for stable linear delay systems. Notice also that the constructive approach we propose, which consists in providing an appropriate factorization of a given quasipolynomial function and then to focus on the location of zeros of one of its factors, gives further insights on such a qualitative property. Namely, it furnishes the exact exponential decay rate rather than just counting the number of the quasipolynomial roots on the left-half plane as may be done by using the principle argument, see, for instance, [37]. Finally, we present the main ingredients some control-oriented algorithmic procedure that can be useful for developing a systematic toolbox for testing all the properties mentioned above.

The class of dynamical systems considered in this work is represented by an n^{th} order linear autonomous system of ordinary differential equations with a forcing delay term. This class of systems has an applicative interest particularly in control design problems. As a matter of fact, the forcing term may be seen as a delayed-input able to stabilize the system's solutions. The idea of exploiting the delay effect in controllers design was first introduced in [38] where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus can replace the proportional-derivative controller, see also [34]. Furthermore, it was stressed in [28] that time-delay has a stabilizing effect in the control design. Indeed, the closed-loop stability is guaranteed precisely by the existence of the delay in the control loop. Also in [27] it is shown that a chain of n integrators can be stabilized using n distinct delay blocks, where a delay block is described by two parameters: a "gain" and a "delay". The interest of considering control laws of the form $\sum_{k=1}^{m} \gamma_k \, y(t-\tau_k)$ lies in the simplicity of the controller as well as in its easy practical implementation.

From a control theory point of view, the problem we consider and the approach we propose give rise to an *exponential decay assignment* method using two "control" parameters a "gain" and a "delay". Notice that the idea of using roots assignment for controller-design for time-delay system is not new. For instance, in [22] a feedback law yields a finite spectrum of the closed-loop system, located at an arbitrarily preassigned set of points in the complex plane. In the case of systems with delays in control only, a necessary and sufficient condition for finite spectrum assignment is obtained. Notice that the resulting feedback law involves integrals over the past control. In case of delays in state variables it is shown that a technique based on the finite Laplace transform leads to a constructive design procedure. The resulting feedback consists of proportional and (finite interval) integral terms over present and past values of state variables. In [21], a similar finite pole placement for time-delay systems with commensurate delays is proposed. Feedback laws defined in terms of Volterra equations are obtained due to the properties of the Bezout ring of operators including derivatives, localized and distributed delays. Other analytical/numerical placement methods for retarded time-delay systems are proposed in [23, 25], see also [40] for further insights on pole-placement methods for retarded time-delays systems with proportional-integral-derivative controller-design.

The remaining paper is organized as follows. In Section 2, the problem formulation is presented and some technical lemmas are derived. Section 3 is devoted to the main results of the paper. Section 4 gives an illustrative example showing the potential of the method to address some practical applications. Some concluding remarks end the paper. Finally, the reader finds proofs of the technical lemmas in the Appendix A.

2. Problem settings and prerequisites

In this paper, we are interested in studying the stabilizing effect of the coexistence of the maximal number of real spectral values for the generic n-order ordinary differential equation perturbed by a forcing delay term:

$$y^{(n)}(t)+\sum_{k=0}^{n-1} a_k y^{(k)}(t)+\alpha y(t-\tau)=0, \quad t\in\mathbb{R}_+, \tag{1}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathscr{C}([-\tau,0],\mathbb{R})$ which is an infinite-dimensional differential equation with a single constant delay $\tau>0$.

From a control theory point of view, the aim is to establish a delayed-output-feedback controller $u(t)=-\alpha\, y(t-\tau)$ able to stabilize solutions of the following control system:

$$y^{(n)}(t)+\sum_{k=0}^{n-1} a_k y^{(k)}(t)=u(t). \tag{2}$$

The particular cases of first and second order equations are considered in [2], where a stabilizing effect of the coexistence of respectively 2 and 3 negative real roots is shown. By this paper, one generalizes such a result for arbitrary order n.

In the Laplace domain, the corresponding quasipolynomial characteristic function defined by $\Delta_n:\mathbb{C}\times\mathbb{R}_+^*\longrightarrow\mathbb{C}$ writes

$$\Delta_n(s,\tau):=s^n+\sum_{k=0}^{n-1} a_k s^k+\alpha e^{-\tau s}. \tag{3}$$

One can prove that the quasipolynomial function (3) admits an infinite number of zeros, see for instance the references [1,4,20,33]. The study of zeros of an entire function [20] of the form (3) plays a crucial role in the analysis of asymptotic stability of the zero solution of Equation (1). Indeed, the zero solution is asymptotically stable if, and only if, all the zeros of (3) are in the open left-half complex plane [24].

2.1. *Counting quasipolynomial roots in horizontal strips*

The following result was first introduced and claimed in the problems collection published in 1925 by G. Pólya and G. Szegö. In the fourth edition of their book [32, Problem 206.2, p. 144 and p. Z347], G. Pólya and G. Szegö emphasize that the proof was obtained by N. Obreschkoff in 1928 using the principle argument, see [29]. Such a result gives a bound for the number of quasipolynomial's roots in any horizontal strip. As a consequence, a bound for the number of quasipolynomial's real roots can be easily deduced.

Theorem 1 ([32]). *Let $\tau_1,\ldots,\tau_N$ denote real numbers such that $\tau_1<\tau_2<\ldots<\tau_N$ and $d_1,\ldots,d_N$ positive integers such that $d_1+d_2+\ldots+d_N=D$. Let $f_{i,j}$ stand for the function $f_{i,j}(s)=s^{i-1}e^{\tau_j s}$, for $1\le i\le d_j$ and $1\le j\le N$. Let $\sharp_{PS}$ be the number of zeros of the function*

$$f(s)=\sum_{\substack{1\le j\le N\\ 1\le i\le d_j}} c_{i,j} f_{i,j}(s) \tag{4}$$

that are contained in the horizontal strip $\alpha\le\operatorname{Im}(z)\le\beta$. Assuming that

$$\sum_{1\le k\le d_1}|c_{k,1}|>0 \quad \textit{and} \quad \sum_{1\le k\le d_N}|c_{k,N}|>0$$

then

$$\frac{(\tau_N-\tau_1)(\beta-\alpha)}{2\pi}-D+1\leq \sharp_{PS}\leq \frac{(\tau_N-\tau_1)(\beta-\alpha)}{2\pi}+D+N-1. \tag{5}$$

Setting $\alpha=\beta=0$, the above theorem yields $\sharp_{PS}\leq D+N-1$ where D stands for the sum of the degrees of the polynomials involved in the quasipolynomial function f and N designates the associated number of polynomials. This gives a sharp bound for the number of f's real roots. Notice that $D+N-1$ corresponds to the degree of the quasipolynomial f. [1]

Let's investigate the coexistence of $n+1$ real (negative) roots for the quasipolynomial $\Delta_n(.,\tau)$. Due to the linearity of Δ_n with respect to its coefficients $(a_k)_{0\leq k\leq n-1}$ and α, one reduces the system $\Delta_n(s_1,\tau)=\ldots=\Delta_n(s_{n+1},\tau)=0$ to the linear system $V_n(X^{n+1},\tau).\mathcal{V}=b$ where $\mathcal{V}=(a_{n-1},\ldots,a_0,\alpha)^T$, $b=-(s_1^n,\ldots,s_{n+1}^n)^T$ and $X^{n+1}\triangleq(s_1,s_2,\cdots,s_{n+1})$:

$$V_n\left(X^{n+1},\tau\right)=\begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & 1 & e^{-\tau s_1}\\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & 1 & e^{-\tau s_2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & 1 & e^{-\tau s_n}\\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}}\end{bmatrix}. \tag{6}$$

In the sequel, such a matrix is called *structured functional Vandermonde type matrix* due to its form and its structural properties.

2.2. *The determinant of a structured functional Vandermonde type matrix*

As reported in [18, p. 121], the Vandermonde matrix appears in a control problems when studying the controllability of a finite dimensional dynamical system. More precisely, the controllability property is guaranteed by the invertibility of such a matrix, see also [15,35]. Next, in the context of time-delay systems, the use of the standard Vandermonde matrix properties was proposed by [24,27] when controlling some chain of integrators by delay blocks. Analogously to the Birkhoff interpolation problem, in [6] the non degeneracy of some functional Birkhoff matrices represents a fundamental assumption for investigating the codimension of the zero spectral values for time-delay systems. Here, we further exploit the algebraic properties of such structured matrices into a different context.

The following auxiliary result explicitly gives the determinant of the structured functional Vandermonde type matrix (6). Its proof is presented in the Appendix. In the sequel, we adopt the notation $[x,y]_t$ to designate the $t-$convex combination of the real (or complex) numbers x and y, that is: $[x,y]_t=tx+(1-t)y$ for $t\in[0,1]$.

Theorem 2. *For any distinct real numbers $s_{n+1}<\cdots<s_2<s_1$, and $\tau>0$, the structured functional Vandermonde type matrix $V_n(X^{n+1},\tau)$ is invertible. Moreover, its determinant is*

$$Q_n\left(X^{n+1},\tau\right)=\det\left(V_n\left(X^{n+1},\tau\right)\right)=\tau^n\prod_{\substack{i<j\\ i,j=1}}^{n+1}\left(s_i-s_j\right)F_{\tau,n}\left(X^{n+1}\right), \tag{7}$$

which is always positive and where $F_{\tau,n}:\mathbb{R}^{n+1}\to\mathbb{R}_+^*$ is defined as follows:

$$F_{\tau,n}\left(X^{n+1}\right)=\underbrace{\int_0^1\cdots\int_0^1}_{n\ \ times}\prod_{k=1}^{n-1}(1-t_k)^{n-k}.e^{-\tau\left[s_1,\left[s_2,\cdots\left[s_n,s_{n+1}\right]_{t_n}\cdots\right]_{t_2}\right]_{t_1}}dt_n\cdots dt_1$$

[1] The quasipolynomial degree is defined as the sum of degrees of the involved polynomials plus the corresponding number of delays

Remark 3. It is worth mentioning that the product in the expression of Q_n given by (7) corresponds to the determinant of the standard Vandermonde matrix, see for instance [30].

2.3. *Symmetry property*

The multivariate function $F_{\tau,n}$ admits an interesting *invariance property* that will be emphasized in the following Lemma 4 which will be used in the proof of the main results. Its proof is presented in the Appendix A.

Lemma 4. *For any positive delay τ the functional $F_{\tau,n}$ is invariant for any permutation of the finite sequence $(s_1, s_2, \cdots, s_{n+1})$, namely, for any permutation σ of X^{n+1}, we have*

$$F_{\tau,n}\left(X^{n+1}\right) = F_{\tau,n}\left(\sigma\left(X^{n+1}\right)\right).$$

For instance, for $n = 2$, Lemma 4 allows saying that for all $(x, y, z) \in \mathbb{R}^3$,

$$F_{\tau,2}\left(x, y, z\right) = \int_0^1 \int_0^1 (1-t_1)\, e^{-\tau\left(t_1 x + (1-t_1)\left(t_2 y + (1-t_2) z\right)\right)} dt_1\, dt_2$$

and

$$F_{\tau,2}\left(x, y, z\right) = F_{\tau,2}\left(x, z, y\right) = F_{\tau,2}\left(y, x, z\right) = F_{\tau,2}\left(y, z, x\right) = F_{\tau,2}\left(z, x, y\right) = F_{\tau,2}\left(z, y, x\right).$$

Remark 5. The symmetry property emphasized in the above Lemma 4 is justified by the convexity property on the argument of the exponential kernel. Its proof (see the Appendix A) relies on some simple change of coordinates.

2.4. *Shifting properties*

The following Lemmas 6 and 7 exhibit some shifting properties that will be used in proving the main results. Their proofs are presented in the Appendix A.

Lemma 6. *Let $(s_i)_{i=1}^{n+1}$ be a sequence of distinct real numbers. For $1 \leq m \leq n$, let $(j_k)_{1\leq k\leq m+1}$ be any subsequence from $\{1, \ldots, n+1\}$. Let*

$$(S_{j_k})_{1\leq k\leq m+1} \subset (S_i)_{1\leq i\leq n+1}$$

For $1 \leq M \leq n-1$, let the corresponding set of $m-$tuple partitions be

$$\mathrm{I}_{m,M} = \left\{(i_1, i_2, \cdots, i_m) \in \mathbb{N}^m,\ \sum_{j=1}^{m} i_j = M\right\}.$$

Then

$$\sum_{(i_1,i_2,\cdots,i_m)\in \mathrm{I}_{m,M}} \prod_{k=1}^{m} s_{j_k}^{i_k} - \sum_{(i_1,i_2,\cdots,i_m)\in \mathrm{I}_{m,M}} \prod_{k=1}^{m} s_{j_{k+1}}^{i_k} = \left(s_{j_1} - s_{j_{m+1}}\right) \sum_{(i_1,i_2,\cdots,i_{m+1})\in \mathrm{I}_{m+1,M-1}} \prod_{k=1}^{m+1} s_{j_k}^{i_k}$$

Lemma 7. *Let $\tau > 0$ and $n \geq 1$. Let $(s_i)_{i=1}^{n+1}$ be a sequence of distinct real numbers. For any subsequence $(s_{i_k})_{k=1}^{k=m+1}$ from $(s_i)_{i=1}^{n+1}$, the function $F_{\tau,m}$ satisfies*

$$F_{\tau,m-1}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right) - F_{\tau,m-1}\left(s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right) = -\tau\left(s_{i_1} - s_{i_{m+1}}\right) F_{\tau,m}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right).$$

Remark 8.

- Lemma 6 and Lemma 7 remain valid even if the elements of the sequence $(s_i)_{1\leq i\leq n+1}$ are distinct and complex.
- Under the conditions of Lemma 7 it is obvious that $F_{\tau,n} > 0$ for any $\tau > 0$.

2.5. *Factorization property*

The following Lemma 9 provides an explicit way to factorize a given quasipolynomial function (3) having at least n distinct real roots. This will be used in the proof of the main results.

Lemma 9. *Assume that the quasipolynomial* (3) *admits n distinct real roots $s_n < \ldots < s_1$ then it can be written under the following factorized form:*

$$\Delta_n(s,\tau) = \prod_{i=1}^{n}(s-s_i)\left[1+(-\tau)^n\,\alpha F_{\tau,n}(s,s_1,\cdots,s_n)\right]. \tag{8}$$

3. Main results

In this section, we provide mainly two Theorems 10 and 12 exploiting the structural properties of the considered class of functional Vandermonde matrices to give some qualitative properties of the solutions of (1). More precisely, the first Theorem 10 gives conditions on the coexistence of real roots of the quasipolynomial Δ_n. The second Theorem 12 emphasizes the effect of the coexistence of such real roots on the remaining roots of Δ_n. Finally, the combination of those results allows to give some important insights on the exponential stability of the solutions of (1).

3.1. *Coexistence of $n+1$ real roots of Δ_n*

The following Theorem 10 allows recovering PS_B as a bound of the admissible number of coexisting real roots for the quasipolynomial (3), see for instance [32]. This provides an alternative constructive analytical proof based on factorization techniques. Furthermore, explicit conditions on the parameters guaranteeing the coexistence of such a number of real roots is provided allowing to Vieta's-like formulas for quasipolynomials.

Theorem 10.

(i) *The maximal number of coexisting real roots of the quasipolynomial* (3) *is $n+1$.*

(ii) *For a fixed $\tau > 0$, Equation* (3) *admits $n+1$ distinct real spectral values $s_{n+1}, s_n, \cdots, s_2$ and s_1 with $s_{n+1} < \cdots < s_2 < s_1$ if, and only if, the coefficients $(a_k)_{0\le k\le n-1}$ and α are respectively given by the following functions in τ and $X^{n+1} = (s_1,\ldots,s_{n+1})$*

$$a_0\left(X^{n+1},\tau\right) = \frac{1}{Q_n\left(X^{n+1},\tau\right)}\det\begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & -s_1^n & e^{-\tau s_1}\\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & -s_2^n & e^{-\tau s_2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & -s_n^n & e^{-\tau s_n}\\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & -s_{n+1}^n & e^{-\tau s_{n+1}}\end{bmatrix}, \tag{9}$$

and for $1\le k\le n-1$ one has:

$$a_k\left(X^{n+1},\tau\right) = \frac{1}{Q_n\left(X^{n+1},\tau\right)}\det\begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1^{k+1} & -s_1^n & s_1^{k-1} & \cdots & s_1 & 1 & e^{-\tau s_1}\\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2^{k+1} & -s_2^n & s_2^{k-1} & \cdots & s_2 & 1 & e^{-\tau s_2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n^{k+1} & -s_n^n & s_n^{k-1} & \cdots & s_n & 1 & e^{-\tau s_n}\\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1}^{k+1} & -s_{n+1}^n & s_{n+1}^{k+1} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}}\end{bmatrix}, \tag{10}$$

and

$$\alpha\left(X^{n+1},\tau\right)=\frac{1}{Q_n\left(X^{n+1},\tau\right)}\det\begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & 1 & -s_1^n \\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & 1 & -s_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & 1 & -s_n^n \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & -s_{n+1}^n \end{bmatrix}. \tag{11}$$

Remark 11.

- From a control theory point of view, recall the design problem presented in (2), which consists in tuning the controller gain α and the delay parameter τ such that the closed-loop system's solution is asymptotically stable. In such a problem, the sign of the controller gain is important with respect to the system's structure. More precisely, it is important to emphasize that in our design procedure, the coefficient α is of alternate sign with respect to the parity of the derivative order n.
- One can observe that the asymptotic expansion of the coefficients a_k allows to recover the well-know Vieta's formulas. This comes from the fact that when $\tau\to\infty$ the quasipolynomial Δ_n reduces to a polynomial of degree n. So here the important fact to emphasize is the disappearance of the $(n+1)^{\text{th}}$ real root of the quasipolynomial Δ_n.

Proof of Theorem 10. Let us start by the proof of (ii) and we conclude by (i). (ii) Assume that (3) admits $n+1$ real spectral values $s_1>s_2>\cdots>s_{n+1}$. This means that the coefficients $(a_k)_{0\le k\le n-1}$ and α satisfy the linear system

$$\Delta_n(s_i,\tau)=s_i^n+\sum_{k=0}^{n-1}a_k s_i^k+\alpha e^{-\tau s_i}=0,\quad \text{for all } i=1,\cdots,n+1. \tag{12}$$

Thanks to the invertibility of structured functional Vandermonde type matrix $V_n(X^{n+1},\tau)$ as asserted in Theorem 2, one deals with a Cramer system with respect to the coefficients $(a_k)_{0\le k\le n-1}$ and α. So that, one easily computes these coefficients with the standard formulas allowing to get (9), (10) and (11) respectively. In particular, the expression of $\alpha(X^{n+1},\tau)$ is reduced to

$$\alpha\left(X^{n+1},\tau\right)=\frac{(-1)^{n+1}\prod\limits_{\substack{i<j\\ i,j=1}}^{n+1}\left(s_i-s_j\right)}{\det V_n\left(X^{n+1},\tau\right)}=(-1)^{n+1}\left[\tau^n F_{\tau,n}\left(X^{n+1}\right)\right]^{-1} \tag{13}$$

showing the alternating sign of α. i) Let proceed by contradiction in assuming the coexistence of $n+2$ real roots of (3). We shall use the factorization of (3) derived in Lemma 9, that is:

$$\Delta_n(s,\tau)=\prod_{i=1}^n(s-s_i)\left[1+(-\tau)^n\alpha\left(X^{n+1},\tau\right)F_{\tau,n}(s,s_1,\cdots,s_n)\right].$$

Since we assumed that s_{n+1} and s_{n+2} are two distinct real roots of Δ_n then one has

$$\begin{cases}\Delta_n(s_{n+1},\tau) &= \prod\limits_{i=1}^n(s_{n+1}-s_i)\left[1+(-\tau)^n\alpha\left(X^{n+1},\tau\right)F_{\tau,n}(s_{n+1},s_1,\cdots,s_n)\right]=0,\\ \Delta_n(s_{n+2},\tau) &= \prod\limits_{i=1}^n(s_{n+2}-s_i)\left[1+(-\tau)^n\alpha\left(X^{n+1},\tau\right)F_{\tau,n}(s_{n+2},s_1,\cdots,s_n)\right]=0.\end{cases} \tag{14}$$

Since $\prod\limits_{i=1}^n(s_{n+1}-s_i)\neq 0$ and $\prod\limits_{i=1}^n(s_{n+2}-s_i)\neq 0$, one gets:

$$\begin{cases}\left[1+(-\tau)^n\alpha\left(X^{n+1},\tau\right)F_{\tau,n}(s_{n+1},s_1,\cdots,s_n)\right] &= 0,\\ \left[1+(-\tau)^n\alpha\left(X^{n+1},\tau\right)F_{\tau,n}(s_{n+2},s_1,\cdots,s_n)\right] &= 0.\end{cases} \tag{15}$$

Hence, by subtracting equations in (15), and because $\alpha(X^{n+1},\tau) \neq 0$ (see (13)), we obtain the following equation

$$F_{\tau,n}(s_{n+1}, s_1, \cdots, s_n) - F_{\tau,n}(s_{n+2}, s_1, \cdots, s_n) = 0.$$

Furthermore, using the shifting property from Lemma 7, one gets:

$$-\tau\,(s_{n+1} - s_{n+2})\,F_{\tau,n+1}(s_1, \cdots, s_{n+2}) = 0.$$

Finally, from the definition of $F_{\tau,n+1}$, it follows the inconsistency in assuming the coexistence of $n+2$ distinct real roots. □

3.2. *On qualitative properties of s_1 as a root of Δ_n*

To study the stability of solutions of Equation (3), one needs to study the negativity as well as the dominancy of the root s_1 by using an adequate factorization of the quasipolynomial $\Delta_n(s,\tau)$ defined in (3).

Theorem 12. *For a fixed $\tau > 0$, assume that Equation* (3) *admits $n+1$ distinct real spectral values $s_{n+1} < \cdots < s_2 < s_1$.*

The following assertions hold:

(i) (*Negativity) The spectral value s_1 is negative if, and only if, there exists $\tau^* > 0$ such that*

$$a_{n-1}\left(X^{n+1},\tau^*\right) + \sum_{k=2}^{n} s_k = 0. \tag{16}$$

(ii) (*Dominancy) The spectral value s_1 is the spectral abscissa of Equation* (1).

Proof of Theorem 12.

(i) Assume that $s_1 < 0$. Since the parameter a_{n-1} given by (10) is a continuous function with respect to the delay τ and thanks to the l'Hospital's rule one asserts that its asymptotic behavior is described by:

$$\lim_{\tau\to 0} a_{n-1}(X^{n+1},\tau) = -\infty \text{ and } \lim_{\tau\to\infty} a_{n-1}(X^{n+1},\tau) = -\sum_{k=1}^{n} s_k > 0,$$

which proves the existence of

$$\tau^* > 0 \text{ such that } a_{n-1}(X^{n+1},\tau^*) + \sum_{k=2}^{n} s_k = 0.$$

Conversely, to show the negativity of s_1, one exploits the determinant expressions provided in Theorem 2, allowing to write for any $\tau > 0$ one has:

$$a_{n-1}\left(X^{n+1},\tau\right) = -\sum_{k=1}^{n} s_k - \frac{1}{\tau}\,\frac{F_{\tau,n-1}(s_1,\ldots,s_n)}{F_{\tau,n}(s_1,\ldots,s_{n+1})}.$$

In particular

$$a_{n-1}\left(X^{n+1},\tau^*\right) + \sum_{k=2}^{n} s_k = -s_1 - \frac{1}{\tau^*}\,\frac{F_{\tau^*,n-1}(s_1,\ldots,s_n)}{F_{\tau^*,n}(s_1,\ldots,s_{n+1})}.$$

Using (16) and the positivity of τ^* as well as the positivity of both $F_{\tau^*,n}$ and $F_{\tau^*,n-1}$ one concludes

$$s_1 = -\frac{1}{\tau^*}\,\frac{F_{\tau^*,n-1}(s_1,\ldots,s_n)}{F_{\tau^*,n}(s_1,\ldots,s_{n+1})} < 0.$$

(ii) The proof of dominancy property for s_1 is based on the quasipolynomial factorization established in the proof of Theorem 10, especially, the formula (8). Let us assume that there exists some $s_0 = \zeta + j\eta$ a root of $\Delta_n(s,\tau) = 0$ such that $\zeta > s_1$. This means that $P(s_0,\tau) = 0$. Hence

$$\begin{aligned} 1 &= (-1)^{n+1}\tau^n \alpha(X^{n+1},\tau) F_{\tau,n}(s_0, s_1, \cdots, s_n) \\ &= (-1)^{n+1}\tau^n \alpha(X^{n+1},\tau)\operatorname{Re}\left(F_{\tau,n}(s_0, s_1, \cdots, s_n)\right) \\ &= \tau^n \left|\alpha(X^{n+1},\tau)\right| \operatorname{Re}\left(F_{\tau,n}(s_0, s_1, \cdots, s_n)\right) \\ &\le \tau^n \left|\alpha(X^{n+1},\tau)\right| F_{\tau,n}(\zeta, s_1, \cdots, s_n). \end{aligned} \tag{17}$$

Denote by $x_{2,n}$ the quantity $[s_2, \cdots [s_{n-1}, s_n]_{t_n} \cdots]_{t_3}$. Rewriting the term $[\zeta, [x_{2,n}, s_1]_{t_2}]_{t_1}$ as follows

$$\begin{aligned} \left[\zeta, \left[x_{2,n}, s_1\right]_{t_2}\right]_{t_1} &= t_1(\zeta - s_1) + s_1 + t_2(1-t_1)\left(x_{2,n} - s_1\right) \\ &= t_1(\zeta - s_1) + \left[x_{2,n}, s_1\right]_{t_2(1-t_1)} \\ &= t_1(\zeta - s_1) + \left[s_1, \left[x_{2,n}, s_1\right]_{t_2}\right]_{t_1}. \end{aligned}$$

Then, using the following estimates

$$\left[s_1, \left[x_{2,n}, s_1\right]_{t_2}\right]_{t_1} > \left[s_1, \left[x_{2,n}, s_{n+1}\right]_{t_2}\right]_{t_1} \quad \text{and} \quad e^{-\tau t_1(\zeta - s_1)} < 1, \quad \forall\, t_1 \in]0,1[$$

we get from (17) and Lemma 4

$$\begin{aligned} 1 &\le \tau^n \left|\alpha(X^{n+1},\tau)\right| \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k} e^{-\tau t_1(\zeta - s_1)} e^{-\tau\left[\zeta, [x_{2,n}, s_1]_{t_2}\right]_{t_1}} dt_n \cdots dt_1 \\ &< \tau^n \left|\alpha(X^{n+1},\tau)\right| \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k} e^{-\tau\left[s_1, [x_{2,n}, s_{n+1}]_{t_2}\right]_{t_1}} dt_n \cdots dt_1 \\ &= \tau^n \left|\alpha(X^{n+1},\tau)\right| F_{\tau,n}(s_1, s_2, \cdots, s_{n+1}) = 1 \quad \text{(thanks to (13))}, \end{aligned}$$

which is inconsistent. Thus, the dominancy of s_1 is proved. The proof of Theorem 12 is achieved.

□

Remark 13. Note that the factorization (8) of $\Delta_n(.,\tau)$ allows to retrieve the explicit expression of the coefficient α defined in (11), since s_{n+1} is a root of quasipolynomial $\Delta_n(.,\tau)$. Just replace s by s_{n+1} in (8).

3.3. *Exponential stability*

Note that for linear retarded functional differential equations, the exponential stability is equivalent to the uniform asymptotic stability, [19, p. 79]. Further, for the linear autonomous retarded functional differential equations, asymptotic stability implies uniform asymptotic stability and, hence, exponential stability. Recall that Theorem 10 gives necessary and sufficient conditions for the coexistence of $n+1$ real roots of (3). Theorem 12 gives a necessary and sufficient conditions for the negativity of all such real roots and asserts that the roots of (3) have necessarily $\operatorname{Re}(s) < s_1$. So, the following result which is a direct consequence of Theorems 10-12 allows an appropriate characterization of the exponential stability.

Corollary 14. *If equation* (3) *admits* $(n+1)$ *distinct real spectral values* $s_{n+1} < \ldots < s_1$ *and* (16) *is satisfied then the trivial solution of* (1) *is exponentially stable with* s_1 *as a decay rate.*

3.4. *Control design perspectives: Strategy steps on a comprehensive example*

Corollary 14 gives rise a new *partial pole placement methodology* which consists in assigning $n+1$ distinct roots $s_{n+1} < \ldots < s_1 < 0$ for Δ_n. This assignment guarantees the exponential stability of the closed-loop system with s_1 as exponential decay rate.

Let us exhibit the *steps* of the proposed design on some comprehensive example of second-order dynamics subject to some delay in the input. It is well known that second-order linear systems capture the dynamic behavior of many natural phenomena, and have found wide applications in a variety of fields, such as vibration and structural analysis.

Consider the following control problem:

$$\ddot{x}(t) + 2\xi\,\omega\,\dot{x}(t) + \omega^2\,x(t) = u(t) \tag{18}$$

where $\omega > 0$ and $0 < \xi < 1$ stand respectively for the oscillator natural frequency and the damping factor and we consider a controller u having a *proportional-minus-delay* structure as suggested in [38]; that is

$$u(t) = -\alpha_0\,x(t) - \alpha_1\,x(t-\tau).$$

Thus, the corresponding closed-loop characteristic function is given by:

$$\Delta_2(s,\tau) = s^2 + 2\xi\,\omega\,s + \omega^2 + \alpha_0 + \alpha_1 \mathrm{e}^{-\tau s}.$$

(**Step 1**) Since the degree of the quasipolynomial $\deg(\Delta_2) = 3$ then the first step of our approach consists in assigning three negative roots $s_3 < s_2 < s_1$. For simplicity, let consider the case of equidistributed roots, which corresponds to $s_3 = s_1 - 2\,d$ and $s_2 = s_1 - d$ with $d > 0$.

(**Step 2**) One solves the system of the three transcendental equations for the control parameters $(\alpha_0, \alpha_1, \tau)$ in terms of the system physical parameters (ξ, ω) as well as the assigned root s_1 and the distance between two successive roots "d". One obtains the following solution:

$$\begin{cases} \tau &= \frac{\sigma}{d}, \\ \alpha_0 &= 3/2\,(-2\xi\,\omega + d - 2\,s_1)\,(-2/3\,\xi\,\omega + d - 2/3\,s_1)\,\mathrm{e}^{-\sigma} - 2\xi\,\omega\,s_1 - \omega^2 - {s_1}^2, \\ \alpha_1 &= -1/2\,(-2\xi\,\omega + d - 2\,s_1)\,(-2\xi\,\omega + 3\,d - 2\,s_1)\,\mathrm{e}^{-\frac{\sigma\,(-s_1+d)}{d}} \end{cases} \tag{19}$$

with

$$\sigma = \ln\left(\frac{-2\xi\,\omega + 3\,d - 2\,s_1}{-2\xi\,\omega + d - 2\,s_1}\right).$$

At this stage, the distance d is not yet fixed.

(**Step 3**) The distance d has to be chosen such that the positivity of the delay τ is guaranteed. To do so, one has to chose d such that:

$$\frac{-2\xi\,\omega + 3\,d - 2\,s_1}{-2\xi\,\omega + d - 2\,s_1} = 1 + \frac{2\,d}{-2\xi\,\omega + d - 2\,s_1} > 1$$

which is equivalent to chose d such that $d > 2\,(s_1 + \xi\,\omega)$. In particular, if s_1 is set such that $s_1 < -\xi\,\omega$ then d can be arbitrarily chosen.

4. Stabilizing coupled oscillators using delayed output feedback

To show the potential of the obtained results for applications, consider as an illustrative example a more involved system consisting in two coupled oscillators. Coupled oscillations occur when two or more oscillating systems are connected in such a way the motion energy is transferred between them. The dynamics of coupled oscillators plays an important role in a variety of systems in nature and technology, see for instance [31] and references therein. Their ability to display complex self-organized dynamical phenomena makes them an important tool to explain fundamental mechanism of emergent dynamics in coupled systems. It is known that when the coupling is small then each oscillator operates at its natural frequency and the system is then said to be incoherent. However, when the coupling exceeds a certain threshold then the system spontaneously synchronizes. Here we consider the mechanical system of two coupled oscillators as depicted in Figure 1 and we aim to design a stabilizing delayed controller, which corresponds to *oscillation quenching*. Using the fundamental principle of dynamics and the standard assump-

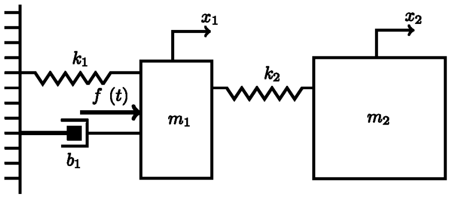

Figure 1. Coupled damped oscillators.

tion about the linearity of the damping lead to the following differential equations governing the motion of the system:

$$\begin{cases} m_1\,\ddot{x}_1(t) & = -b_1\,\dot{x}_1(t) - k_1 x_1(t) + k_2\,(x_2(t) - x_1(t)) + f(t), \\ m_2\,\ddot{x}_2(t) & = -k_2\,(x_2(t) - x_1(t)). \end{cases} \tag{20}$$

where the parameters values are chosen accordingly to some experimental setting: $b_1 = 2$, $k_1 = 1$, $k_2 = 2/3$, $m_1 = 1/2$, $m_2 = 3$. If the forcing term f acts on the system as an input and takes a *proportional-minus-delay* structure as suggested in [38], that is:

$$f(t) = -\alpha_1\,x_1(t) - \alpha_2\,x_2(t) - \alpha_0\,x_2(t-\tau), \tag{21}$$

and by setting $\zeta(t) = (x_1(t), \dot{x}_1(t), x_2(t), \dot{x}_2(t))$, then the closed-loop system writes as:

$$\dot{\zeta}(t) = A_0\zeta(t) + A_1\zeta(t-\tau), \tag{22}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_2+k_1+\alpha_1}{m_1} & -\frac{b_1}{m_1} & \frac{k_2-\alpha_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_0}{m_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding characteristic quasipolynomial function has the form (3) and writes explicitly as follows:

$$\Delta_4(s,\tau) = s^4 + \frac{b_1 s^3}{m_1} + \frac{(\alpha_1 m_2 + k_1 m_2 + k_2 m_1 + k_2 m_2)\, s^2}{m_1 m_2} + \frac{b_1 k_2 s}{m_1 m_2} + \frac{\alpha_1 k_2 + \alpha_2 k_2 + k_1 k_2}{m_1 m_2} + \frac{\mathrm{e}^{-s\tau}\alpha_0 k_2}{m_1 m_2}. \tag{23}$$

The aim is to establish values for controller's gains α_0, α_1, α_2 as well as the value of the delay parameter $\tau > 0$ enabling us to assign 5 negative roots of the quasipolynomial (23) guaranteeing the exponential stability of the trivial solution of the closed-loop system as asserted in Theorem 12 and Corollary 14. To simplify the design task, we consider the case of equidistributed negative spectral values where the distance between two consecutive roots is d which, for the moment, is left as a "free" parameter. By setting a targeted decay rate or equivalently the rightmost root, for instance at $s_1 = -1$; that is $s_{k+1} = -(1+kd)$ for $k = 1,\dots,4$ one then applies Theorem 12. Solving, the obtained set of equations gives the appropriate distance and the controller's parameters the gains values

$$\begin{cases} d &= \frac{\sqrt[3]{26900+300\sqrt{7329}}}{90} + \frac{40}{9\sqrt[3]{26900+300\sqrt{7329}}} + 2/9 \approx 0.7571217245, \\ \tau &= -\frac{45\,d(3\,d-2)(-\ln(5)+\ln(3))}{7} \approx 0.6746941850, \\ \alpha_0 &= -\frac{4860\,d^2+567\,d+378}{40}\, 3^{\frac{135\,d^2}{7}-\frac{90\,d}{7}}\, 5^{-\frac{135\,d^2}{7}+\frac{90\,d}{7}} \approx -45.75153460, \\ \alpha_1 &= 10\,d^2 + \frac{11}{9} \approx 6.954555281, \\ \alpha_2 &= \frac{4860\,d^2+567\,d+378}{40}\, 3^{\frac{135\,d^2}{7}-\frac{90\,d}{7}}\, 5^{-\frac{135\,d^2}{7}+\frac{90\,d}{7}}\, \mathrm{e}^{-\frac{45\,d(3\,d-2)(-\ln(5)+\ln(3))}{7}} - 55\,d^2 - \frac{251}{36} \\ & \approx 51.32999300. \end{cases} \tag{24}$$

The obtained positive value of d guarantees the positivity of the delay τ and allows the spectrum distribution illustrated in Figure 2 in closed-loop.

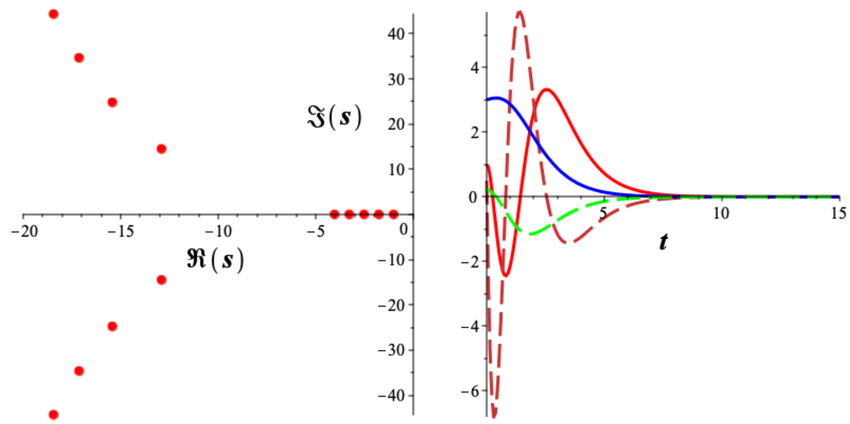

Figure 2. (Left) Spectrum distribution of the closed-loop system (22) using a proportional-minus-delay controller. The parameters values are given in (24). (Right) The closed-loop time-domain simulation of the state variable $\zeta(t)$ in the time-window of 15 seconds. $x_1(t)$ in solid red plot, $\dot{x}_1(t)$ in dashed orange, $x_2(t)$ in solid blue and $\dot{x}_2(t)$ in dashed green. Initial conditions are taken $\zeta(t) = (1, 1/2, 3, 1/4)$ for all $t \in [-\tau, 0]$.

5. Concluding remarks

In this paper, we investigated conditions on the coefficients of the n^{th} order linear ordinary differential equations with delayed-state forcing term guaranteeing the coexistence of the maximal number of real spectral values. Such a number corresponds to the well-known Polya and Szegö bound for quasipolynomial's real roots n and it was recovered by using an analytical constructive approach. Furthermore, an easy to check criterion was provided, allowing the characterization of the stabilizing effect of the coexistence of such spectral values. It is worth noting that such a configuration guarantees the exponential stability and explicitly describes the corresponding exponential decay rate. The potential of the derived results to applications is illustrated through the problem of constructing appropriate stabilizing controllers for some system of coupled oscillators.

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Appendix A. Proof of the technical lemmas

Proof of Lemma 4. Without any loss of generality, it suffices to consider the following permutation

$$\sigma_i\left(s_1, s_2, \cdots, s_i, s_{i+1}, \cdots, s_{n+1}\right) = \left(s_1, s_2, \cdots, s_{i+1}, s_i, \cdots, s_{n+1}\right),$$

where $1 \le i \le n$. Any other permutation of sets of indices is none other than the composition of such permutations. For example, if $\sigma_{i,j}$, with $j-i>1$, is such that

$$\sigma_{i,j}\left(s_1, s_2, \cdots, s_i, \cdots, s_j, \cdots, s_{n+1}\right) = \left(s_1, s_2, \cdots, s_j, \cdots, s_i, \cdots, s_{n+1}\right),$$

then

$$\sigma_{i,j} = \sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \cdots \circ \sigma_{i+1} \circ \sigma_i.$$

Write

$\left[s_1, \left[s_2, \cdots \left[s_n, s_{n+1}\right]_{t_n} \cdots\right]_{t_2}\right]_{t_1}$ as

$$t_1 s_1 + (1-t_1)\, t_2 s_2 + \cdots + \prod_{k=1}^{i-1}(1-t_k)\, t_i s_i + \prod_{k=1}^{i}(1-t_k)\, t_{i+1} s_{i+1} + \cdots$$
$$+ \prod_{k=1}^{n-1}(1-t_k)\, t_n s_n + \prod_{k=1}^{n}(1-t_k)\, s_{n+1}.$$

It is then necessary to introduce a suitable change of variable, that switches the coefficient of s_i with the coefficient of s_{i+1}, without affecting the other coefficients. Let

$$\begin{cases} \begin{cases} u_k &= t_k, \quad k \neq i \wedge i+1, \\ u_i &= (1-t_i)\, t_{i+1} \\ u_{i+1} &= \frac{t_i}{1-t_{i+1}+t_i\, t_{i+1}} \end{cases} & \text{if} \quad 1 \le i \le n-1 \\ \text{and} & \\ \begin{cases} u_k &= t_k,\ 1 \le k \le n-1 \\ u_n &= 1-t_n \end{cases} & \text{if} \quad i = n \end{cases} \tag{25}$$

Clearly, $u_i \in]0,1[$ for all $1 \le i \le n-1$. Moreover, from $(1-t_i)(1-t_{i+1}) > 0$, we have $1 - t_{i+1} + t_i\, t_{i+1} > t_i > 0$, hence $u_{i+1} \in]0,1[$. The Jacobian matrix

$$J = \frac{D(u_1, u_2, \cdots, u_n)}{D(t_1, t_2, \cdots, t_n)}$$

is such that

$$\det J = \frac{t_i - 1}{t_i\, t_{i+1} - t_{i+1} + 1} \neq 0, \ \text{ for all } 1 \le i \le n-1.$$

So, (25) defines a C^1- diffeomorphism from $]0,1[^n$ into $]0,1[^n$, for all $1 \le i \le n-1$, and the following properties

$$\begin{aligned}
t_i \prod_{k=1}^{i-1} (1-t_k) &= u_{i+1} \prod_{k=1}^{i} (1-u_k),\\
t_{i+1} \prod_{k=1}^{i} (1-t_k) &= u_i \prod_{k=1}^{i-1} (1-u_k),\\
t_m \prod_{k=1}^{m-1} (1-t_k) &= u_m \prod_{k=1}^{m-1} (1-u_k),\ \ \forall\, m \in \{2, \cdots, n\},\ m \neq i \wedge i+1
\end{aligned}$$

are satisfied.

On the other hand, from

$$du_1 du_2 \cdots du_n = \left|\det \frac{D(u_1, u_2, \cdots, u_n)}{D(t_1, t_2, \cdots, t_n)}\right| dt_1\, dt_2 \cdots dt_n = \frac{1-t_i}{1-u_i}\, dt_1\, dt_2 \cdots dt_n,$$

one gets

$$\prod_{k=1}^{n-1} (1-t_k)^{n-k}\, dt_1 dt_2 \cdots dt_n = \prod_{k=1}^{n-1} (1-u_k)^{n-k}\, du_1\, du_2 \cdots du_n.$$

The case $i = n$ is simpler so omitted. The symmetry property is well proven. □

Proof of Lemma 6. Let us first observe that the preceding sums (or the homogeneous forms of degree M and $M-1$ respectively) are invariant under any permutation between the s_{i_k}, for $k \in \{1, \cdots, m+1\}$. Thus, by using the well-known factorization

$$s_{j_1}^{i_{m-1}} - s_{j_{m+1}}^{i_{m-1}} = \left(s_{j_1} - s_{j_{m+1}}\right) \sum_{i_m=0}^{i_{m-1}-1} s_{j_1}^{i_m} s_{j_{m+1}}^{i_{m-1}-i_m-1},$$

we have

$$\begin{aligned}
&\sum_{(i_1, i_2, \cdots, i_m) \in \mathrm{I}_{m,M}} \prod_{k=1}^{m} s_{j_k}^{i_k} - \sum_{(i_1, i_2, \cdots, i_m) \in \mathrm{I}_{m,M}} \prod_{k=1}^{m} s_{j_{k+1}}^{i_k}\\
&= \sum_{i_1=0}^{M} \cdots \sum_{i_k=0}^{i_{k-1}} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} s_{j_1}^{i_{m-1}} s_{j_2}^{i_{m-2}-i_{m-1}} \cdots s_{j_k}^{i_{m-k}-i_{m-k+1}} \cdots s_{j_{m-1}}^{i_1-i_2} s_{j_m}^{M-i_1}\\
&\quad - \sum_{i_1=0}^{M} \cdots \sum_{i_k=0}^{i_{k-1}} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} s_{j_{m+1}}^{i_{m-1}} s_{j_2}^{i_{m-2}-i_{m-1}} \cdots s_{j_k}^{i_{m-k}-i_{m-k+1}} \cdots s_{j_m}^{M-i_1}\\
&= \sum_{i_1=0}^{M} \cdots \sum_{i_k=0}^{i_{k-1}} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} \left(s_{j_1}^{i_{m-1}} - s_{j_{m+1}}^{i_{m-1}}\right)\left(s_{j_2}^{i_{m-2}-i_{m-1}} \cdots s_{j_k}^{i_{m-k}-i_{m-k+1}} \cdots s_{j_m}^{M-i_1}\right)\\
&= \left(s_{j_1} - s_{j_{m+1}}\right) \sum_{i_1=0}^{M} \cdots \sum_{i_k=0}^{i_{k-1}} \cdots \sum_{i_{m-1}=0}^{i_{m-2}} \sum_{i_m=0}^{i_{m-1}-1} \left(s_{j_1}^{i_m} s_{j_{m+1}}^{i_{m-1}-i_m-1}\right)\left(s_{j_2}^{i_{m-2}-i_{m-1}} \cdots s_{j_k}^{i_{m-k}-i_{m-k+1}} \cdots s_{j_m}^{M-i_1}\right)\\
&= \left(s_{j_1} - s_{j_{m+1}}\right) \sum_{(i_1, i_2, \cdots, i_{m+1}) \in \mathrm{I}_{m+1,M-1}} \prod_{k=1}^{m+1} s_{j_k}^{i_k}.
\end{aligned}$$

□

Proof of Lemma 7. In view of Lemma 4, we have

$$F_{\tau, m-1}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right) = F_{\tau, m-1}\left(s_{i_2}, \cdots, s_{i_m}, s_{i_1}\right),$$

hence

$$F_{\tau,m-1}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right) - F_{\tau,m-1}\left(s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right)$$
$$= \underbrace{\int_0^1 \cdots \int_0^1}_{(m-1)\text{ times}} \prod_{k=1}^{m-2} (1-t_k)^{m-1-k} . \left(e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m}, s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}} - e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m}, s_{i_{m+1}}\right]_{t_{m-1}}\cdots\right]_{t_1}} \right) dt_{m-1}\cdots dt_1$$

Using the judicious form of the exponential in brackets,

$$e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m}, s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}} = e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2}, t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}} e^{-\tau\prod_{k=1}^{m-1}(1-t_k)s_{i_1}}$$

which allows isolating the last term of the convex combination, we obtain by virtue of the mean value theorem applied to

$$x \longmapsto e^{-\tau\prod_{k=1}^{m-1}(1-t_k)x}:$$

$$e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m}, s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}} - e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m}, s_{i_{m+1}}\right]_{t_{m-1}}\cdots\right]_{t_1}}$$
$$= e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2},\, t_{m-1}\, s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}} \left(-\tau \prod_{k=1}^{m-1}(1-t_k)\left(s_{i_1} - s_{i_{m+1}}\right) \int_0^1 e^{-\tau\left(t_m s_{i_1} + (1-t_m)s_{i_{m+1}}\right)} dt_m \right)$$
$$= -\tau \prod_{k=1}^{m-1}(1-t_k)\left(s_{i_1} - s_{i_{m+1}}\right) e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2}, t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}} \int_0^1 e^{-\tau\prod_{k=1}^{m-1}(1-t_k)\left(t_m s_{i_1} + (1-t_m)s_{i_{m+1}}\right)} dt_m$$

Hence, as a consequence of the two properties

$$\prod_{k=1}^{m-1}(1-t_k) . \prod_{k=1}^{m-2}(1-t_k)^{m-1-k} = \prod_{k=1}^{m-1}(1-t_k)^{m-k}$$

and

$$\left[s_{i_2}, \cdots \left[s_{m-2,}\, t_{m-1}\, s_{i_m}\right]_{t_{m-2}} \cdots\right]_{t_1} + \prod_{k=1}^{m-1}(1-t_k)\left(t_m s_{i_1} + (1-t_m)\, s_{i_{m+1}}\right)$$
$$= \left[s_{i_2}, \cdots \left[s_{m-2,} \left[s_{i_m,} \left[s_{i_{m+1}}, s_{i_1}\right]_{t_m}\right]_{t_{m-1}}\right]_{t_{m-2}} \cdots\right]_{t_1}$$

and Lemma 4, one obtains

$$F_{\tau,m-1}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right) - F_{\tau,m-1}\left(s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right)$$
$$= -\tau\left(s_{i_1} - s_{i_{m+1}}\right) \underbrace{\int_0^1 \cdots \int_0^1}_{m\text{ times}} \prod_{k=1}^{m-1}(1-t_k)^{m-k} . e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2,}\left[s_{i_m,}\left[s_{i_{m+1}}, s_{i_1}\right]_{t_m}\right]_{t_{m-1}}\right]_{t_{m-2}}\cdots\right]_{t_1}}$$
$$dt_m dt_{m-1}\cdots dt_1 = -\tau\left(s_{i_1} - s_{i_{m+1}}\right) F_{\tau,m}.$$

This achieves the proof of the Lemma 7. □

Proof of Theorem 2. The calculation is done in n steps. The idea is to have at each step k in the next-to-last column only "1". Then, a linear combination of the lines makes it possible to reduce the size of the determinant of a unit, as well as to recover the factors $(s_i - s_j)$, with $i - j = k$, using

Lemmas 7 and 6. To do so, denoting by L_i the i^{th} line of $V_n(\tau) := V_n(s_1, s_2, \cdots, s_{n+1}, \tau)$. Replacing L_i by $L_i - L_{i+1}$, for $i = 1, \cdots, n$, in $\det V_n(\tau)$, we get

$$\det V_n(\tau) = \det \begin{bmatrix} s_1^{n-1} - s_2^{n-1} & s_1^{n-2} - s_2^{n-2} & \cdots & s_1 - s_2 & 0 & e^{-\tau s_1} - e^{-\tau s_2} \\ s_2^{n-1} - s_3^{n-1} & s_2^{n-2} - s_3^{n-2} & \cdots & s_2 - s_3 & 0 & e^{-\tau s_2} - e^{-\tau s_3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} - s_{n+1}^{n-1} & s_n^{n-2} - s_{n+1}^{n-2} & \cdots & s_n - s_{n+1} & 0 & e^{-\tau s_n} - e^{-\tau s_{n+1}} \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}} \end{bmatrix}.$$

Using the mean value theorem as follows,

$$e^{-\tau s_i} - e^{-\tau s_{i+1}} = -\tau (s_i - s_{i+1}) \int_0^1 e^{-\tau(t s_i + (1-t)s_{i+1})} dt, \quad i = 1, \cdots, n,$$

we obtain

$$\det V_n(\tau) = \tau \prod_{k=1}^{n} (s_k - s_{k+1}) \times \det \begin{bmatrix} \sum_{i=0}^{n-2} s_1^i s_2^{n-2-i} & \sum_{i=0}^{n-3} s_1^i s_2^{n-3-i} & \cdots & s_1 + s_2 & 1 & \int_0^1 e^{-\tau[s_1, s_2]_{t_1}} dt_1 \\ \sum_{i=0}^{n-2} s_2^i s_3^{n-2-i} & \sum_{i=0}^{n-3} s_2^i s_3^{n-3-i} & \cdots & s_2 + s_3 & 1 & \int_0^1 e^{-\tau[s_2, s_3]_{t_1}} dt_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{n-2} s_n^i s_{n+1}^{n-2-i} & \sum_{i=0}^{n-3} s_n^i s_{n+1}^{n-3-i} & \cdots & s_n + s_{n+1} & 1 & \int_0^1 e^{-\tau[s_n, s_{n+1}]_{t_1}} dt_1 \end{bmatrix}$$

we get using the same linear combination as above

$$\det V_n(\tau) = \tau^2 \prod_{k=1}^{n} \left(s_k - s_{k+1}\right) (s_1 - s_3)(s_2 - s_4) \cdots (s_{n-1} - s_{n+1}) \times$$
$$\det \begin{bmatrix} \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k \geq 0}} s_1^i s_2^j s_3^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k \geq 0}} s_1^i s_2^j s_3^k & \cdots & \sum_{l=1}^{3} s_l & 1 & \int_0^1 \int_0^1 (1-t) e^{-\tau[s_1, [s_2, s_3]_\theta]_t} d\theta dt \\ \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k \geq 0}} s_2^i s_3^j s_4^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k \geq 0}} s_2^i s_3^j s_4^k & \cdots & \sum_{l=2}^{4} s_l & 1 & \int_0^1 \int_0^1 (1-t) e^{-\tau[s_2, [s_3, s_4]_\theta]_t} d\theta dt \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k \geq 0}} s_{n-1}^i s_n^j s_{n+1}^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k \geq 0}} s_{n-1}^i s_n^j s_{n+1}^k & \cdots & \sum_{l=n-1}^{n+1} s_l & 1 & \int_0^1 \int_0^1 (1-t) e^{-\tau[s_{n-1}, [s_n, s_{n+1}]_\theta]_t} d\theta dt \end{bmatrix}.$$

Repeating the same process as above, in the last step, only the term $(s_1 - s_{n+1})$ remains to be recovered. Thus, the determinant is reduced to the following expression:

$$\det V_n(\tau) = \tau^{n-1} \left(\prod_{0 < i-j \leq n} \left(s_i - s_j\right) \right) \det \begin{bmatrix} 1 & F_{\tau,n}(s_1, s_2, \cdots, s_n) \\ 1 & F_{\tau,n}(s_2, \cdots, s_n, s_{n+1}) \end{bmatrix}.$$

By using Lemma 7, we get

$$\det V_n(\tau) = \tau^n \left(\prod_{\substack{i<j \\ i,j=1}}^{n+1} \left(s_i - s_j\right) \right) F_{\tau,n}(s_1, s_2, \cdots, s_{n+1}),$$

which is always positive since $F_{\tau,n}$ is positive and $s_i > s_j$. $\square$

Proof of Lemma 9. We start first by carrying out the following factorization of Δ_n by writing

$$\Delta_n(s,\tau) = \prod_{i=1}^{n}(s-s_i).P_n(s,\tau)$$

with

$$P_n(s,\tau) = \left[\prod_{i=1}^{n}(s-s_i)\right]^{-1}\left(s^n + \sum_{k=1}^{n} a_{n-k}s^{n-k} + \alpha\exp(-\tau s)\right) \tag{26}$$

where the coefficients a_n, $n = 0,\ldots,n-1$ and α satisfy (9), (10), and (11).

From the standard Vieta's formulas, introduce now the following coefficients:

$$\widetilde{a}_{n-k} = (-1)^k\left(\sum_{i_1<\cdot<i_k}^{n}\prod_{j=1}^{k} s_{i_j}\right), \quad \text{for } k = 1,\cdots,n.$$

Then by performing an Euclidean division in (26) one gets:

$$P_n(s,\tau) = 1 + \frac{\sum_{k=1}^{n}(a_{n-k}-\widetilde{a}_{n-k})\,s^{n-k}}{\prod_{i=1}^{n}(s-s_i)} + \frac{\alpha\exp(-\tau s)}{\prod_{i=1}^{n}(s-s_i)}.$$

Let B_n be defined as follows: $B_n(s) := \sum_{k=1}^{n}(a_{n-k}-\widetilde{a}_{n-k})\,s^{n-k}$, which satisfies

$$B_n(s_k) = s_k^n + a_{n-1}\,s_k^{n-1} + \cdots + a_1 s_k + a_0, \quad \forall\, k = 1,\cdots,n.$$

Then one performs the partial fractions corresponding to

$$\frac{B_n(s)}{\prod_{i=1}^{n}(s-s_i)} \quad\text{and}\quad \frac{1}{\prod_{i=1}^{n}(s-s_i)}:$$

$$\frac{B_n(s)}{\prod_{i=1}^{n}(s-s_i)} = \sum_{k=1}^{n}\frac{c_k}{s-s_k} \quad\text{and}\quad \frac{1}{\prod_{i=1}^{n}(s-s_i)} = \sum_{k=1}^{n}\frac{d_k}{s-s_k} \tag{27}$$

where

$$c_k = \frac{B_n(s_k)}{\prod_{i=1,\,i\neq k}^{n}(s_k-s_i)} = \frac{s_k^n + \sum_{i=1}^{n} a_{n-i}\,s_k^{n-i}}{\prod_{i=1,\,i\neq k}^{n}(s_k-s_i)}, \quad k = 1,\cdots,n.$$

$$d_k = \left[\prod_{i=1,\,i\neq k}^{n}(s_k-s_i)\right]^{-1}, \quad k = 1,\cdots,n.$$

Thus

$$P_n(s,\tau) = 1 + \sum_{k=1}^{n}\left[\frac{B_n(s_k)}{(s-s_k)\prod_{i=1,\,i\neq k}^{n}(s_k-s_i)} + \frac{\alpha e^{-\tau s}}{(s-s_k)\prod_{i=1,i\neq k}^{n}(s_k-s_i)}\right].$$

Next, one substitutes $B_n(s_k)$ as $-\alpha\, e^{-\tau s_k}$ for $1 \le k \le n$ which allows to:

$$P_n(s,\tau) = 1 + \alpha\sum_{k=1}^{n}\left(\frac{e^{-\tau s} - e^{-\tau s_k}}{(s-s_k)\prod_{i=1,\,i\neq k}^{n}(s_k-s_i)}\right).$$

At this step, we use the following integral representation

$$e^{-\tau s} - e^{-\tau s_k} = -\tau\,(s - s_k)\int_0^1 e^{-\tau\,[s,s_k]_{t_1}}\,dt_1 \tag{28}$$

to get

$$P_n(s,\tau) = 1 - \tau\alpha\sum_{k=1}^{n}\left(\left[\prod_{i=1,\,i\neq k}^{n}(s_k - s_i)\right]^{-1}\int_0^1 e^{-\tau\,[s,s_k]_{t_1}}\,dt_1\right). \tag{29}$$

From the property

$$\sum_{k=1}^{n}\left[\prod_{i=1,\,i\neq k}^{n}(s_k - s_i)\right]^{-1} = 0$$

which can be easily shown using the decomposition in partial fractions as illustrated in the second equality in (27). One can extract the first term (corresponding to $k = 1$) of (29) in terms of the remaining $n-1$-terms, and namely one gets:

$$\left[\prod_{i=2}^{n}(s_1 - s_i)\right]^{-1}\int_0^1 e^{-\tau\,[s,s_1]_{t_1}}\,dt_1 = -\sum_{k=2}^{n}\left[\prod_{i=1,\,i\neq k}^{n}(s_k - s_i)\right]^{-1}\int_0^1 e^{-\tau\,[s,s_1]_{t_1}}\,dt_1.$$

The expression of P_n given by (29) becomes:

$$P_n(s,\tau) = 1 - \tau\alpha\sum_{k=2}^{n}\left(\left[\prod_{i=1,\,i\neq k}^{n}(s_k - s_i)\right]^{-1}\int_0^1\left(e^{-\tau\,[s,s_k]_{t_1}} - e^{-\tau\,[s,s_1]_{t_1}}\right)dt_1\right).$$

From Lemma 7

$$\int_0^1\left(e^{-\tau\,[s,s_k]_{t_1}} - e^{-\tau\,[s,s_1]_{t_1}}\right)dt_1 = -\tau\,(s_k - s_1)\int_0^1\int_0^1(1-t_1)\,e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}}\,dt_2\,dt_1,\quad \text{for } k = 2,\cdots,n$$

we get

$$P_n(s,\tau) = 1 + (-\tau)^2\,\alpha\sum_{k=2}^{n}\left(\left[\prod_{i=2,i\neq k}^{n}(s_k - s_i)\right]^{-1}\int_0^1\int_0^1(1-t_1)\,e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}}\,dt_2\,dt_1\right).$$

Observe that the coefficients $\left(\left[\prod_{i=2,\,i\neq k}^{n}(s_k - s_i)\right]^{-1}\right)_{2\leq k\leq n}$ satisfy

$$\sum_{k=2}^{n}\left[\prod_{i=2,\,i\neq k}^{n}(s_k - s_i)\right]^{-1} = 0,$$

(and are independent of s_1), hence repeating the previous step, by rewriting the first term in the last sum as follows:

$$\left[\prod_{i=3}^{n}(s_2 - s_i)\right]^{-1} = -\sum_{k=3}^{n}\left[\prod_{i=2,\,i\neq k}^{n}(s_k - s_i)\right]^{-1}$$

one obtains

$$
\begin{aligned}
P_n(s,\tau) &= 1+(-\tau)^2\,\alpha \sum_{k=3}^{n} \left(\frac{\int_0^1\int_0^1 (1-t_1)\left(e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}} - e^{-\tau\left[s,[s_1,s_2]_{t_2}\right]_{t_1}}\right) dt_2\, dt_1}{\prod_{i=2,\, i\neq k}^{n} (s_k - s_i)} \right) \\
&= 1+(-\tau)^3\,\alpha \sum_{k=3}^{n} \left(\frac{\int_0^1\int_0^1\int_0^1 (1-t_1)^2 (1-t_2)\, e^{-\tau\left[s,\left[s_1,[s_2,s_k]_{t_3}\right]_{t_2}\right]_{t_1}} dt_3\, dt_2\, dt_1}{\prod_{i=3,\, i\neq k}^{n} (s_k - s_i)} \right).
\end{aligned}
$$

In reiterating the same process, one observes that the order of denominator decreases by one at each step, and one gets the general formula for the intermediate step l:

$$
P_n(s,\tau) = 1+(-\tau)^l\,\alpha \sum_{k=l}^{n} \frac{F_{\tau,l}(s, s_1, \dots, s_{l-1}, s_k)}{\prod_{i=l,\, i\neq k}^{n} (s_k - s_i)}
$$

which, by induction, allows to obtain at the step $n-1$

$$
P_n(s,\tau) = 1+(-\tau)^{n-1}\,\alpha \frac{F_{\tau,n-1}(s, s_1, \dots, s_{n-2}, s_{n-1}) - F_{\tau,n-1}(s, s_1, \dots, s_{n-2}, s_n)}{s_{n-1} - s_n}
$$

Finally, using the shifting formula given in Lemma 7, one gets

$$
\begin{aligned}
P_n(s,\tau) &= 1+(-\tau)^n\,\alpha \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k}\, e^{-\tau\left[s,[s_1,\cdots[s_{n-1},s_n]_{t_n}\cdots]_{t_2}\right]_{t_1}} dt_n \cdots dt_1 \\
&= 1+(-\tau)^n\,\alpha F_{\tau,n}(s, s_1, \cdots, s_n)
\end{aligned}
\tag{30}
$$

which allows deriving the following factorization of the quasipolynomial $\Delta_n(.,\tau)$

$$
\Delta_n(s,\tau) = \prod_{i=1}^{n} (s - s_i)\left[1+(-\tau)^n\,\alpha(X^{n+1},\tau) F_{\tau,n}(s, s_1, \cdots, s_n)\right].
$$

□

References

[1] L. V. Ahlfors, *Complex Analysis. An introduction to the theory of analytic functions of one complex variable*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, 1979.

[2] S. Amrane, F. Bedouhene, I. Boussaada, S.-I. Niculescu, "On qualitative properties of low-degree quasipolynomials: further remarks on the spectral abscissa and rightmost-roots assignment", *Bull. Math. Soc. Sci. Math. Roum.* **61(109)** (2018), no. 4, p. 361-381.

[3] F. Bedouhene, I. Boussaada, S.-I. Niculescu, "On Pole Placement and Spectral Abscissa Characterization for Timedelay Systems", *IFAC-PapersOnLine* **52** (2019), no. 18, p. 55-60, Proceeding of the 15th IFAC Workshop on Time Delay Systems TDS 2019.

[4] R. Bellman, K. L. Cooke, *Differential-difference equations*, Mathematics in Science and Engineering, vol. 6, Academic Press Inc., 1963.

[5] I. Boussaada, S.-I. Niculescu, "Computing the Codimension of the Singularity at the Origin for Delay Systems: The Missing Link with Birkhoff IncidenceMatrices", in *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems, MTNS 2014*, University of Groningen, 2014, p. 1-8.

[6] ———, "Characterizing the Codimension of Zero Singularities for Time-Delay Systems", *Acta Appl. Math.* **145** (2016), no. 1, p. 47-88.

[7] ———, "Tracking the Algebraic Multiplicity of Crossing Imaginary Roots for Generic Quasipolynomials: A Vandermonde-Based Approach", *IEEE Trans. Autom. Control* **61** (2016), no. 6, p. 1601-1606.

[8] ———, "On the Dominancy of Multiple Spectral Values for Time-delay Systems with Applications", *IFAC-PapersOnLine* **51** (2018), no. 14, p. 55-60, 14th IFAC Workshop on Time Delay Systems TDS.

[9] I. Boussaada, S.-I. Niculescu, E.-A. Ati, R. Perez-Ramos, K. Trabelsi, "Multiplicity-Induced-Dominancy in parametric second-order delay differential equations: Analysis and application in control design", submitted, 34 pages, 2019.

[10] I. Boussaada, S.-I. Niculescu, S. Tliba, T. Vyhlídal, "On the Coalescence of Spectral Values and its Effect on the Stability of Time-delay Systems: Application to Active Vibration Control", *Procedia IUTAM* **22** (2017), p. 75-82.

[11] I. Boussaada, S.-I. Niculescu, K. Trabelsi, "Toward a Decay Rate Assignment Based Design for Time-Delay Systems with Multiple Spectral Values", in *Proceeding of the 23rd International Symposium on Mathematical Theory of Networks and Systems*, MTNS, 2018, p. 864-871.

[12] I. Boussaada, S. Tliba, S.-I. Niculescu, H. U. Ünal, T. Vyhlídal, "Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. Application to the control of a mechanical system", *Linear Algebra Appl.* **542** (2018), p. 589-604, Proceedings of the 20th ILAS Conference, Leuven, Belgium 2016.

[13] I. Boussaada, H. Unal, S.-I. Niculescu, "Multiplicity and Stable Varieties of Time-delay Systems: A Missing Link", in *Proceeding of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, MTNS, 2016, p. 1-6.

[14] K. L. Cooke, P. van den Driessche, "On zeroes of some transcendental equations", *Funkc. Ekvacioj, Ser. Int.* **29** (1986), no. 1, p. 77-90.

[15] T. T. Ha, J. A. Gibson, "A note on the determinant of a functional confluent Vandermonde matrix and controllability", *Linear Algebra Appl.* **30** (1980), p. 69-75.

[16] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences, vol. 99, Springer, 1993.

[17] N. D. Hayes, "Roots of the Transcendental Equation Associated with a Certain Difference-Differential Equation", *J. Lond. Math. Soc.* **25** (1950), no. 3, p. 226-232.

[18] T. Kailath, *Linear Systems*, Prentice-Hall information and system sciences series, Prentice Hall, 1998.

[19] V. Kolmanovskii, V. Nosov, *Stability of Functional Differential Equations*, Mathematics in Science and Engineering, vol. 180, Academic Press Inc., 1986.

[20] B. Y. Levin, *Distribution of zeros of entire functions*, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, 1964.

[21] J.-J. Loiseau, "Invariant factors assignment for a class of time-delay systems", *Kybernetika* **37** (2001), no. 3, p. 265-275.

[22] A. Z. Manitius, A. W. Olbrot, "Finite spectrum assignment problem for systems with delays", *IEEE Trans. Autom. Control* **24** (1979), no. 4, p. 541-553.

[23] W. Michiels, K. Engelborghs, P. Vansevenant, D. Roose, "Continuous pole placement for delay equations", *Automatica* **38** (2002), no. 5, p. 747-761.

[24] W. Michiels, S.-I. Niculescu, *Stability and stabilization of time-delay systems*, 2nd revised ed., Advances in Design and Control, vol. 12, Society for Industrial and Applied Mathematics, 2007.

[25] S. Mondié, J.-J. Loiseau, "Finite spectrum assignment for input delay systems", *IFAC Proceedings Volumes* **34** (2001), no. 23, p. 201-206.

[26] T. Mori, N. Fukuma, M. Kuwahara, "On an estimate of the decay rate for stable linear delay systems", *Int. J. Control* **36** (1982), p. 95-97.

[27] S.-I. Niculescu, W. Michiels, "Stabilizing a chain of Integrators Using Multiple Delays", *IEEE Trans. Autom. Control* **49** (2004), no. 5, p. 802-807.

[28] S.-I. Niculescu, W. Michiels, K. Gu, C. T. Abdallah, "Delay Effects on Output Feedback Control of Dynamical Systems", in *Complex Time-Delay Systems. Theory and Applications*, Understanding Complex Systems, Springer, 2010, p. 63-84.

[29] N. D. Obreschkoff, "Nullstellen linearer Kombinationen von Exponential funktionen", *Jber. der Deutsch. Math. Verein.* **37** (1928), p. 81-84.

[30] P. J. Olver, "On multivariate interpolation", *Stud. Appl. Math.* **116** (2006), no. 2, p. 201-240.

[31] A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge Nonlinear Science Series, vol. 12, Cambridge University Press, 2003.

[32] G. Pólya, G. Szegö, *Problems and Theorems in Analysis. Vol. I: Series, Integral Calculus, Theory of Functions*, Grundlehren der Mathematischen Wissenschaften, vol. 193, Springer, 1972.

[33] L. S. Pontrjagin, "On zeros of some transcendental functions", *Izv. Akad. Nauk SSSR, Ser. Mat.* **6** (1942), no. 3, p. 115-134.

[34] K. Pyragas, "Continuous control of chaos by self-controlling feedback", *Phys. Lett.* **170** (1992), no. 6, p. 421-428.

[35] J. S. Respondek, "On the confluent Vandermonde matrix calculation algorithm", *Appl. Math. Lett.* **24** (2011), no. 2, p. 103-106.

[36] R. Sipahi, S.-I. Niculescu, C. T. Abdallah, W. Michiels, K. Gu, "Stability and stabilization of systems with time delay: limitations and opportunities", *IEEE Control Sys.* **31** (2011), no. 1, p. 38-65.

[37] G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Pitman Research Notes in Mathematics Series, vol. 210, Longman Scientific & Technical; John Wiley & Sons, 1989.

[38] I. Suh, Z. Bien, "Proportional minus delay controller", *IEEE Trans. Autom. Control* **24** (1979), p. 370-372.
[39] K. Walton, J. E. Marshall, "Direct method for TDS stability analysis", *IEE Proc., Part D* **134** (1987), no. 2, p. 101-107.
[40] P. Zítek, J. Fišer, T. Vyhlídal, "Dimensional analysis approach to dominant three-pole placement in delayed PID control loops", *J. Process Control* **23** (2013), no. 8, p. 1063-1074.