



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique


Alin Bostan and Antonio Jiménez-Pastor

On the exponential generating function of labelled trees

Volume 358, issue 9-10 (2020), p. 1005-1009.

<<https://doi.org/10.5802/crmath.108>>

© Académie des sciences, Paris and the authors, 2020.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org



Combinatorics / *Combinatoire*

On the exponential generating function of labelled trees

Sur la série génératrice des arbres étiquetés

Alin Bostan^a and Antonio Jiménez-Pastor^{*, b}

^a Inria, 1 rue Honoré d'Estienne d'Orves 91120 Palaiseau, France

^b Johannes Kepler University Linz, Doctoral Program Computational Mathematics,
DK15

E-mails: Alin.Bostan@inria.fr (A. Bostan), ajpastor@risc.uni-linz.ac.at
(A. Jiménez-Pastor)

Abstract. We show that the generating function of labelled trees is not D^∞ -finite.

Résumé. Nous montrons que la série génératrice des arbres étiquetés n'est pas D^∞ -finie.

2020 Mathematics Subject Classification. 05A15, 12H05, 34A34.

Funding. This research was partially funded by the Austrian Science Fund (FWF): W1214-N15, project DK15. It was also supported in part by the ANR DeRerumNatura project, grant ANR-19-CE40-0018 of the French *Agence Nationale de la Recherche*.

Manuscript received 29th June 2020, revised and accepted 20th August 2020.

Version française abrégée

Nous montrons que la série génératrice exponentielle des arbres étiquetés, $T(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n$, n'est pas D^∞ -finie. En particulier, cela implique que, bien que $T(x)$ vérifie des équations différentielles non-linéaires, ces dernières ne peuvent pas être « trop simples ». En particulier, $T(x)$ n'est pas un quotient de deux fonctions D -finies (vérifiant des équations différentielles à coefficients polynomiaux), et plus généralement, $T(x)$ ne vérifie aucune équation différentielle linéaire à coefficients des fonctions D -finies. La preuve repose ultimement sur un résultat de théorie de Galois différentielle. Plusieurs questions ouvertes sont proposées, dont une sur la nature de la série génératrice ordinaire des arbres étiquetés, $\sum_{n \geq 1} n^{n-1} x^n$.

* Corresponding author.

1. Context and main result

A formal power series $f(x) = \sum_{n \geq 0} a_n x^n$ in $\mathbb{C}[[x]]$ is called *differentially finite*, or simply *D-finite* [23], if it satisfies a *linear* differential equation with polynomial coefficients in $\mathbb{C}[x]$. Many generating functions in combinatorics and many special functions in mathematical physics are D-finite [2, 9].

DD-finite series and more generally D^n -finite series are larger classes of power series, recently introduced in [13]. DD-finite power series satisfy linear differential equations, whose coefficients are themselves D-finite power series. One of the simplest examples is $\tan(x)$, which is DD-finite (because it satisfies $\cos(x)f(x) - \sin(x) = 0$), but is not D-finite (because it has an infinite number of complex singularities, a property which is incompatible with D-finiteness). Another basic example is the exponential generating function of the Bell numbers B_n , which count partitions of $\{1, 2, \dots, n\}$, namely:

$$B(x) := \sum_{n \geq 0} \frac{B_n}{n!} x^n. \tag{1}$$

Indeed, it is classical [9, p. 109] that $B(x) = e^{e^x - 1}$, therefore $B(x)$ is DD-finite. On the other hand, $B(x)$ is not D-finite: this can be proved either analytically (using the too fast growth of $B(x)$ as $x \rightarrow \infty$), or purely algebraically (using [22], and the fact that the power series e^x is not algebraic).

More generally, given a differential ring R , the set of *differentially definable* functions over R , denoted by $D(R)$, is the differential ring of formal power series satisfying linear differential equations with coefficients in R . In particular, $D(\mathbb{C}[x])$ is the ring of D-finite power series, $D^2(\mathbb{C}[x]) := D(D(\mathbb{C}[x]))$ is the ring of DD-finite power series, and $D^n(\mathbb{C}[x]) := D(D^{n-1}(\mathbb{C}[x]))$ is the ring of D^n -finite power series. We say that a power series $f(x) \in \mathbb{C}[[x]]$ is D^∞ -finite if there exists an n such that $f(x)$ is D^n -finite.

It is known [14] that D^n -finite power series form a strictly increasing sequence of sets and that any D^∞ -finite power series is *differentially algebraic*, in short *D-algebraic*, that is, it satisfies a differential equation, possibly *non-linear*, with polynomial coefficients in $\mathbb{C}[x]$. This class, as well as its complement (of *D-transcendental* series), are quite well studied [11, 21].

Let now $(t_n)_{n \geq 0} = (0, 1, 2, 9, 64, 625, 7776, \dots)$ be the sequence whose general term t_n counts *labelled rooted trees* with n nodes. It is well known that $t_n = n^{n-1}$, for any $n \geq 1$. This beautiful and non-trivial result is usually attributed to Cayley [6], although an equivalent result had been proved earlier by Borchardt [4], and even earlier by Sylvester, see [3, Ch. 4]. Due to the importance of the combinatorial class of trees, and to the simplicity of the formula, Cayley’s result has attracted a lot of interest over the time, and it admits several different proofs, see e.g., [16, §4] and [1, §30]. One of the more conceptual proofs goes along the following lines (see [9, §II. 5.1] for details). Let

$$T(x) := \sum_{n \geq 0} \frac{t_n}{n!} x^n \tag{2}$$

be the exponential generating function of the sequence $(t_n)_n$. The class \mathcal{T} of all rooted labelled trees is definable by a *symbolic equation* $\mathcal{T} = \mathcal{Z} \star \text{SET}(\mathcal{T})$ reflecting their recursive definition, where \mathcal{Z} represents the atomic class consisting of a single labelled node, and \star denotes the labelled product on combinatorial classes. This symbolic equation provides, by syntactic translation, an implicit equation on the level of exponential generating functions:

$$T(x) = x e^{T(x)}, \tag{3}$$

which can be solved using *Lagrange inversion*

$$t_n = n! \cdot [x^n] T(x) = n! \cdot \left(\frac{1}{n} [z^{n-1}] (e^z)^n \right) = n^{n-1}. \tag{4}$$

From (3), it follows easily that $T(x)$ is D-algebraic and satisfies the non-linear equation

$$x(1 - T(x))T'(x) = T(x),$$

and from there, that the sequence $(t_n)_{n \geq 0}$ satisfies the non-linear recurrence relation

$$t_{n+1} = \frac{n+1}{n} \cdot \sum_{i=1}^n \binom{n}{i} t_i t_{n-i+1}, \quad \text{for all } n \geq 1.$$

This recurrence can also be proved using (4), by taking $y = n, x = w = 1$ in Abel's identity [12]

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x + wk)^{k-1} (y - wk)^{n-k},$$

and then by isolating the term $k = n$ in the resulting equality.

On the other hand, it is known that the power series $T(x)$ is not D-finite, see [10, Thm. 7], or [8, Thm. 2]. This raises the natural question whether $T(x)$ is DD-finite, or D^n -finite for some $n \geq 2$. Our main result is that this is not the case:

Theorem 1. *The power series $T(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n$ in (2) is not D^∞ -finite.*

To our knowledge, this is the first explicit example of a natural combinatorial generating function which is provably D-algebraic but not D^∞ -finite. In particular, Theorem 1 implies that $T(x)$ is not equal to the quotient of two D-finite functions, and more generally, that it does not satisfy any linear differential equation with D-finite coefficients.

2. Proof of the main result

Our proof of Theorem 1 builds upon the following recent result by Noordman, van der Put and Top.

Theorem 2 ([18]). *Assume that $u(x) \in \mathbb{C}[[x]] \setminus \mathbb{C}$ is a solution of $u' = u^3 - u^2$. Then u is not D^∞ -finite.*

The proof of Theorem 2 is based on two ingredients. The first one is a result by Rosenlicht [20] stating that any set of non-constant solutions (in any differential field) of the differential equation $u' = u^3 - u^2$ is algebraically independent over \mathbb{C} (see also [18, Prop. 7.1]); the proof is elementary. The second one [18, Prop. 7.1] is that any non-constant power series solution of an autonomous first-order differential equation with this independence property cannot be D^∞ -finite; the proof is based on differential Galois theory.

Proof of Theorem 1. We will use Theorem 2 and a few facts about the (principal branch of the) Lambert W function, satisfying $W(x) \cdot e^{W(x)} = x$ for all $x \in \mathbb{C}$.

Recall [7] that the Taylor series of W around 0 is given by

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \dots.$$

In other words, our $T(x)$ and $W(x)$ are simply related by $W(x) = -T(-x)$.

The function defined by this series can be extended to a holomorphic function defined on all complex numbers with a branch cut along the interval $(-\infty, -\frac{1}{e}]$; this holomorphic function defines the principal branch of the Lambert W function.

We can substitute $x \mapsto e^{x+1}$ in the functional equation for $W(x)$ obtaining then

$$W(e^{x+1})e^{W(e^{x+1})} = e^{x+1},$$

or, renaming $Y(x) = W(e^{x+1})$, we have a new functional equation: $Y(x)e^{Y(x)-1} = e^x$. From this equality it follows by logarithmic differentiation that $Y'(x) \cdot (1 + Y(x)) = Y(x)$.

Take now $U(x) := \frac{1}{1+Y(x)} = \frac{1}{2} - \frac{1}{8}x + \frac{1}{64}x^2 + \frac{1}{768}x^3 + \dots$. We have that

$$U'(x) = \frac{-Y'(x)}{(1+Y(x))^2} = \frac{-Y(x)}{(1+Y(x))^3} = U(x)^3 - U(x)^2.$$

By Theorem 2, $U(x)$ is not D^∞ -finite. By closure properties of D^∞ -finite functions (see [14, Thm. 4] and [13, §3]), it follows that $Y(x)$ is not D^∞ -finite either.

To conclude, note that by definition, for real x in the neighborhood of 0, we have $W(x) = Y(\log(x) - 1)$, and by Theorem 10 in [14], it follows that $W(x)$ and $T(x)$ are not D^∞ -finite either, proving Theorem 1. □

3. Open questions

The class of D -finite power series is closed under Hadamard (term-wise) product. This is false for D^∞ -finite power series; for instance, Klazar showed in [15] that the ordinary generating function $\sum_{n \geq 0} B_n x^n$ of the Bell numbers is not differentially algebraic, contrary to its exponential generating function (1), which is DD -finite.

Moreover, it was conjectured by Pak and Yeliussizov [19, Open Problem 2.4] that this is an instance of a more general phenomenon.

Conjecture 3 ([19, Open Problem 2.4]). *If for a sequence $(a_n)_{n \geq 0}$ both ordinary and exponential generating functions $\sum_{n \geq 0} a_n x^n$ and $\sum_{n \geq 0} a_n \frac{x^n}{n!}$ are D -algebraic, then both are D -finite. (Equivalently, $(a_n)_{n \geq 0}$ satisfies a linear recurrence with polynomial coefficients in n .)*

This conjecture has been recently proven for large (infinite) classes of generating functions [5]. However, the very natural example of the generating function for labelled trees escapes the method in [5].

We therefore leave the following as an open question.

Open question 4. *Is the power series $\sum_{n \geq 1} n^{n-1} x^n$ D^∞ -finite? Is it at least differentially algebraic?*

According to Conjecture 3, the answer should be “no” for both questions in Open question 4.

Another natural question concerns the generating function for partition numbers:

$$\sum_{n \geq 0} p_n x^n := \prod_{n \geq 1} \frac{1}{1 - x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \dots,$$

which is known to be differentially algebraic [17].

Open question 5. *Is it true that $\sum_{n \geq 0} p_n x^n$ is not D^∞ -finite?*

One may also ask for the nature of the exponential variant of the generating function for partition numbers.

Open question 6. *Is the power series $\sum_{n \geq 0} \frac{p_n}{n!} x^n$ D^∞ -finite, or at least differentially algebraic?*

Acknowledgements

We would like to warmly thank the referees for their careful reading and for their constructive remarks.

References

- [1] M. Aigner, G. M. Ziegler, *Proofs from The Book*, fourth ed., Springer, 2010, viii+274 pages.
- [2] G. E. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia of Mathematics and Its Applications, vol. 71, Cambridge University Press, 1999, xvi+664 pages.
- [3] N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph theory. 1736–1936*, second ed., Clarendon Press, 1986, xii+239 pages.
- [4] C. W. Borchardt, “Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante”, *J. Reine Angew. Math.* **57** (1860), p. 111-121.
- [5] A. Bostan, L. Di Vizio, K. Raschel, “Differential transcendence of Bell numbers and relatives — a Galois theoretic approach”, 2020, In preparation.
- [6] A. Cayley, “A theorem on trees”, *Q. J. Math.* **23** (1889), p. 376-378.
- [7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, “On the Lambert W function”, *Adv. Comput. Math.* **5** (1996), no. 4, p. 329-359.
- [8] P. Flajolet, S. Gerhold, B. Salvy, “On the non-holonomic character of logarithms, powers, and the n th prime function”, *Electron. J. Comb.* **11** (2005), no. 2, article ID A2 (16 pages).
- [9] P. Flajolet, R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009, xiv+810 pages.
- [10] S. Gerhold, “On some non-holonomic sequences”, *Electron. J. Comb.* **11** (2004), no. 1, article ID 87 (8 pages).
- [11] J. van der Hoeven, “Computing with D-algebraic power series”, *Appl. Algebra Eng. Commun. Comput.* **30** (2019), no. 1, p. 17-49.
- [12] J. L. W. V. Jensen, “Sur une identité d’Abel et sur d’autres formules analogues”, *Acta Math.* **26** (1902), no. 1, p. 307-318.
- [13] A. Jiménez-Pastor, V. Pillwein, “A computable extension for D-finite functions: DD-finite functions”, *J. Symb. Comput.* **94** (2019), p. 90-104.
- [14] A. Jiménez-Pastor, V. Pillwein, M. F. Singer, “Some structural results on D^n -finite functions”, *Adv. Appl. Math.* **117** (2020), article ID 102027 (29 pages).
- [15] M. Klazar, “Bell numbers, their relatives, and algebraic differential equations”, *J. Comb. Theory, Ser. A* **102** (2003), no. 1, p. 63-87.
- [16] L. Lovász, *Combinatorial problems and exercises*, second ed., North-Holland, 1993, 635 pages.
- [17] A. M. Mian, S. Chowla, “The differential equations satisfied by certain functions”, *J. Indian Math. Soc., New Ser.* **8** (1944), p. 27-28.
- [18] M. P. Noordman, M. van der Put, J. Top, “Combinatorial Autonomous first order differential equations”, 2019, preprint, <https://arxiv.org/abs/1904.08152v1>.
- [19] I. Pak, “Complexity problems in enumerative combinatorics”, in *Proceedings of the International Congress of Mathematicians — Rio de Janeiro 2018. Vol. IV. Invited lectures*, World Scientific, 2018, p. 3153-3180.
- [20] M. Rosenlicht, “The nonminimality of the differential closure”, *Pac. J. Math.* **52** (1974), p. 529-537.
- [21] L. A. Rubel, “A survey of transcendently transcendental functions”, *Am. Math. Mon.* **96** (1989), no. 9, p. 777-788.
- [22] M. F. Singer, “Algebraic relations among solutions of linear differential equations”, *Trans. Am. Math. Soc.* **295** (1986), no. 2, p. 753-763.
- [23] R. P. Stanley, “Differentiably finite power series”, *Eur. J. Comb.* **1** (1980), no. 2, p. 175-188.