INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

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Volume 358, issue 9-10 (2020), p. 971-980.

<https://doi.org/10.5802/crmath.73>

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Comptes Rendus
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2020, 358, nº 9-10, p. 971-980
https://doi.org/10.5802/crmath.73

Theory of Functions, Combinatorics / *Théorie des fonctions, Combinatoire*

New asymptotic expansions on hyperfactorial functions

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Abstract. In this paper, by using the Bernoulli numbers and the exponential complete Bell polynomials, we establish four general asymptotic expansions for the hyperfactorial functions $\prod_{k=1}^{n} k^{k^q}$, which have only odd power terms or even power terms. We derive the recurrences for the parameter sequences in these four general expansions and give some special asymptotic expansions by these recurrences.

2020 Mathematics Subject Classification. 41A60, 05A15.

Manuscript received 19th April 2020, revised 11th May 2020, accepted 12th May 2020.

1. Introduction

In 1933, Bendersky [2] considered the product $\prod_{k=1}^{n} k^{k^q}$ for $q = 0, 1, 2, \dots$ which degenerates into the classical factorial function $n!$ and the classical hyperfactorial function $H(n) = \prod_{k=1}^{n} k^k$ when q takes 0 and 1 respectively. By taking the logarithm of the product, he obtained the result as follows:

$$\ln(A_q) = \lim_{n\to\infty} \ln(A_q(n)) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^q \ln k - P_q(n) \right\}, \tag{1}$$

where

$$P_q(n) = \frac{n^q}{2}\ln n + \frac{n^{q+1}}{q+1}\left(\ln n - \frac{1}{q+1}\right) + q! \sum_{j=1}^{q} \frac{n^{q-j} B_{j+1}}{(j+1)!(q-j)!} \left\{ \ln n + (1-\delta_{q,j}) \sum_{l=1}^{j} \frac{1}{q-l+1} \right\},$$

B_n are the Bernoulli numbers, and $\delta_{q,j}$ is the Kronecker delta function.

In 2012, using the Euler–Maclaurin summation formula, Chen [3] obtained the asymptotic expansion of $\ln A_1(n)$ as follows:

$$\ln A_1(n) \sim \ln A_1 - \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)} \frac{1}{n^{2k}}, \quad n \to \infty,$$

where A_1 is the Glaisher–Kinkelin constant, which can be defined by

$$A_1 = \lim_{n\to\infty} \frac{(2\pi)^{\frac{n}{2}} n^{\frac{n^2}{2} - \frac{1}{12}} e^{-\frac{3n^2}{4} + \frac{1}{12}}}{G(n+1)},$$

and $G(z)$ is the Barnes-G function. Then, substituting the values of B_n into the above expression of $\ln A_1(n)$ gives

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \exp\left(\frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \frac{691}{3603600n^{10}} - \frac{1}{1872n^{12}} + \cdots\right), \quad n \to \infty. \tag{2}$$

Based on this expansion, Chen and Lin [5] established a general asymptotic expansion for the hyperfactorial function:

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\breve{\alpha}_k}{n^k}\right)^{\frac{1}{r}}, \quad n \to \infty, \tag{3}$$

and presented the expression of $(\breve{\alpha}_k)$. Wang and Liu [14] obtained two general expansions:

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{(n+h)^k}\right)^{\frac{1}{r}}, \tag{4}$$

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty} \frac{\varphi_k}{(n+h)^k}\right)^{\frac{n}{r}+q}, \tag{5}$$

as $n \to \infty$, and determined the explicit expressions of (α_k) and (φ_k).

Furthermore, Choi [7] presented the expression

$$\ln A_q(n) = \sum_{k=1}^{n} k^q \ln k - U_{q+1}(n) \ln n + V_{q+1}(n), \tag{6}$$

where $U_{q+1}(n)$ and $V_{q+1}(n)$ are defined by

$$U_{q+1}(n) = \left\{\frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \sum_{r=1}^{[\frac{q+1}{2}]} \frac{B_{2r}}{(2r)!} \left(\prod_{j=1}^{2r-1} (q-j+1)\right) n^{q+1-2r}\right\}, \tag{7}$$

$$V_{q+1}(n) = \frac{n^{q+1}}{(q+1)^2} - \sum_{r=1}^{[\frac{q+1}{2}]+\frac{(-1)^q-1}{2}} \frac{B_{2r}}{(2r)!} \left\{\prod_{j=1}^{2r-1} (q-j+1) \sum_{j=1}^{2r-1} \frac{1}{q-j+1}\right\} n^{q+1-2r}. \tag{8}$$

He also gave the following general asymptotic expansion:

$$\ln A_q(n) \sim \ln A_q + (-1)^q q! \sum_{r=[\frac{q+1}{2}]+1}^{\infty} \frac{B_{2r}}{(2r)!} \cdot \frac{(2r-q-2)!}{n^{2r-q-1}}, \quad n \to \infty, \tag{9}$$

where the constants A_q, which are called the generalized Glaisher–Kinkelin constants, can be expressed by the derivatives of the Riemann zeta function $\zeta(s)$ and the harmonic numbers H_n [1, 8]:

$$A_q = \exp\left\{\frac{B_{q+1} H_q}{q+1} - \zeta'(-q)\right\}.$$

Besides, Choi's result degenerates into Stirling's formula of $n!$ when $q = 0$.

Most recently, Wang [13] further studied this problem and presented the next two general asymptotic expansions on the hyperfactorial functions:

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\alpha_k(q;h,r)}{(n+h)^k}\right)^{\frac{1}{r}}, \tag{10}$$

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k(q;h,r,s)}{(n+h)^k}\right)^{\frac{n}{r}+s}, \tag{11}$$

as $n \to \infty$,

For more results on the asymptotic expansions of hyperfactorial functions and related functions, the readers may refer to, for example, Chen [4], Cheng and Chen [6], Lin [9], Lu and Mortici [10], Mortici [11], Xu and Wang [15], Yang and Tian [16] and Wang [12].

Inspired by these works, in this paper, we establish the following four general asymptotic expansions on hyperfactorial functions:

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{(n+u_k)^{2k}} \right)^{\frac{1}{r}}, \tag{12}$$

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\Phi + \sum_{k=0}^{\infty} \frac{\bar{\alpha}_k}{(n+\bar{u}_k)^{2k+1}} \right)^{\frac{1}{r}}, \tag{13}$$

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k}{(n+v_k)^{2k}} \right)^{\frac{n}{r}+s}, \tag{14}$$

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\Psi + \sum_{k=0}^{\infty} \frac{\bar{\varphi}_k}{(n+\bar{v}_k)^{2k+1}} \right)^{\frac{n}{r}+s}, \tag{15}$$

as $n \to \infty$, where the polynomials $U_{q+1}(n)$ and $V_{q+1}(n)$ are the same as (7) and (8). By using the exponential complete Bell polynomials, we obtain the recurrences and explicit expressions for the parameter sequences in the series, and discuss further some special cases of these four general expansions.

This paper is organized as follows. In Section 2, we give the first two general asymptotic expansions. It can be found that the general asymptotic expansion (12) holds for odd integer q, and in this case $u_k = 0$ and $\alpha_k = Y_{2k}/(2k)!$, and (13) holds for even integer q. In Section 3, we give the last two general asymptotic expansions. Similarly, (14) holds when q is even and (15) holds when q is odd. We establish the recurrences for these parameter sequences, and present many special asymptotic expansions by specifying the values of r, q and s.

2. The first two general asymptotic expansions

The exponential complete Bell polynomials Y_n can be defined by

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right) = \sum_{n=0}^{\infty} Y_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}, \tag{16}$$

then we obtain $Y_0 = 1$ and

$$Y_n(x_1, x_2, \ldots, x_n) = \sum_{c_1+2c_2+\cdots+nc_n=n} \frac{n!}{c_1! c_2! \cdots c_n!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_n}{n!}\right)^{c_n}. \tag{17}$$

Moreover, the polynomials Y_n satisfy the recurrence

$$Y_n(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{n-1} \binom{n-1}{j} x_{n-j} Y_j(x_1, x_2, \ldots, x_j), \quad n \geq 1. \tag{18}$$

From (17) or (18), the sequence $(Y_n)_{n\geq 1}$ can be determined immediately. Then, using the definition and recurrence of the Bell polynomials, we can obtain the following general asymptotic expansions which have only odd power terms or even power terms.

Theorem 1. *Let $r \neq 0$ be a real number and q be an odd integer. Define the sequence $(\beta_k)_{k\geq 1}$ by*

$$\beta_k = (-1)^q r (k-1)! \binom{k+q}{q}^{-1} \frac{B_{k+q+1}}{k+q+1}. \tag{19}$$

Then the following asymptotic expansion on the hyperfactorial functions holds:

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{n^{2k}} \right)^{\frac{1}{r}}, \tag{20}$$

as $n \to \infty$, *where* $(\alpha_k)_{k\ge 0}$ *is determined by* $\alpha_k = y_{2k}/(2k)!$, $k \ge 0$, *and the sequence* $(y_k)_{k\ge 0}$ *can be determined by*

$$y_0 = 1, \quad y_{2k} = \sum_{j=0}^{k-1} \binom{2k-1}{2j} \beta_{2k-2j} y_{2j}, \quad y_{2k+1} = 0, \quad k \ge 0. \tag{21}$$

Proof. Define the falling factorials $x^{\underline{n}}$ by $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n = 1,2,\ldots$, and the sequence $y_k = Y_k(\beta_1, \beta_2, \ldots, \beta_k)$. It is known that $B_{2k+1} = 0$ for $k = 1,2,\ldots$, then we can rewrite the asymptotic expansion (9) as follows:

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \exp\left\{ \sum_{k=1}^{\infty} \frac{(-1)^q q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}} \frac{1}{n^k} \right\}, \quad n \to \infty. \tag{22}$$

Which, togather with (16) and (19), gives

$$\begin{aligned} \left(\frac{\prod_{k=1}^{n} k^{k^q}}{A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)}} \right)^r &\sim \exp\left\{ \sum_{k=1}^{\infty} \frac{(-1)^q r k! q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}} \frac{(\frac{1}{n})^k}{k!} \right\} \\ &= \exp\left\{ \sum_{k=1}^{\infty} \beta_k \frac{(\frac{1}{n})^k}{k!} \right\} = \sum_{k=0}^{\infty} \frac{Y_k(\beta_1, \beta_2, \ldots, \beta_k)}{k!} \frac{1}{n^k}, \quad n \to \infty. \end{aligned} \tag{23}$$

On the other hand, expanding the next sum in powers of $1/n$ yields

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\alpha_j}{(n+u_j)^{2j}} &= \sum_{j=0}^{\infty} \frac{\alpha_j}{n^{2j}} (1 + \frac{u_j}{n})^{-2j} = \sum_{j=0}^{\infty} \frac{\alpha_j}{n^{2j}} \sum_{i=0}^{\infty} \binom{-2j}{i} \frac{{u_j}^i}{n^i} \\ &= \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{[k/2]} (-1)^k \alpha_j \binom{k-1}{k-2j} {u_j}^{k-2j} \right\} \frac{1}{n^k}. \end{aligned} \tag{24}$$

To establish the general asymptotic expansion (12), it suffices to show that we can determine the two sequences $(\alpha_j)_{j\ge 0}$ and $(u_j)_{j\ge 0}$ uniquely from (23) and (24). we equate the coefficients of $1/n^k$ in (23) and (24). When $k = 0$, the coefficient of $1/n^0$ in (23) is $y_0 = 1$; then we have $\alpha_0 = 1$ and $u_0 = 0$. When $k = 1$, we get $\alpha_1 = y_2/2!$ and $u_1 = 0$ by identifying the coefficients.

For $k \ge 2$, we consider the following system

$$\frac{y_k}{k!} = \sum_{j=0}^{[k/2]} (-1)^k \alpha_j \binom{k-1}{k-2j} {u_j}^{k-2j}.$$

Setting $k = 2n$, $n = 1,2,\ldots$, we have

$$\alpha_n = \frac{y_{2n}}{(2n)!} - \sum_{j=0}^{n-1} \alpha_j \binom{2n-1}{2n-2j} {u_j}^{2n-2j}. \tag{25}$$

Setting $k = 2n+1$, $n = 1,2,\ldots$, we have

$$\alpha_n u_n = -\frac{1}{2n} \left\{ \frac{y_{2n+1}}{(2n+1)!} - \sum_{j=0}^{n-1} \alpha_j \binom{2n}{2n-2j+1} {u_j}^{2n-2j+1} \right\}. \tag{26}$$

According to the values of the Bernoulli numbers and the definition of q, it can be found that $\beta_{2k+1} = 0$ for $k = 1,2,\ldots$. Then by the recurrence (18), we have $y_{2k+1} = 0$, and we can show that $u_n = 0$ by induction. Thus, $\alpha_n = y_{2n}/(2n)!$, and the asymptotic expansion (20) can be established.

Finally, using (18) as well as the fact y_{2k+1} for $k = 1,2,\ldots$, we have

$$y_{2k} = \sum_{j=0}^{2k-1} \binom{2k-1}{j} \beta_{2k-j} y_j = \sum_{j=0}^{k-1} \binom{2k-1}{2j} \beta_{2k-2j} y_{2j}, \ k \ge 1,$$

which, combined with the expression of α_k and the definition of β_k in (19), gives the recurrence of $(\alpha_k)_{k\geq 0}$. □

Theorem 2. *Let $r \neq 0$ be a real number and q be an even integer. Define the sequence $(\beta_k)_{k\geq 1}$ by* (19). *Then the following asymptotic expansion on the hyperfactorial functions holds:*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} e^{-V_{q+1}(n)} \left(1 + \sum_{k=0}^{\infty} \frac{\bar{\alpha}_k}{(n+\bar{u}_k)^{2k+1}}\right)^{\frac{1}{r}}, \tag{27}$$

as $n \to \infty$, where the sequence $(\bar{\alpha}_k)_{k\geq 0}$ and $(\bar{u}_k)_{k\geq 0}$ can be determined by

$$\bar{\alpha}_k = \frac{y_{2k+1}}{(2k+1)!} - \sum_{j=0}^{k-1} \bar{\alpha}_j \binom{2k}{2k-2j} \bar{u}_j{}^{2k-2j}, \tag{28}$$

$$\bar{\alpha}_k \bar{u}_k = -\frac{1}{2k+1} \left\{ \frac{y_{2k+2}}{(2k+2)!} + \sum_{j=0}^{k-1} \bar{\alpha}_j \binom{2k+1}{2k-2j+1} \bar{u}_j{}^{2k-2j+1} \right\}, \tag{29}$$

and the sequence $(y_k)_{k\geq 0}$ can be determined by

$$y_0 = 1, \quad y_k = \sum_{j=0}^{k-1} \binom{2k-1}{j} \beta_{2k-j} y_j, \quad k \geq 1. \tag{30}$$

Proof. The asymptotic expansion (9) can be written as the form (22). Expanding the next sum in powers of $1/n$ yields

$$\begin{aligned} \Phi + \sum_{j=0}^{\infty} \frac{\bar{\alpha}_j}{(n+\bar{u}_j)^{2j+1}} &= \Phi + \sum_{j=0}^{\infty} \frac{\bar{\alpha}_j}{n^{2j+1}} \left(1 + \frac{\bar{u}_j}{n}\right)^{-2j-1} = \Phi + \sum_{j=0}^{\infty} \frac{\bar{\alpha}_j}{n^{2j+1}} \sum_{i=0}^{\infty} \binom{-2j-1}{i} \frac{\bar{u}_j{}^i}{n^i} \\ &= \Phi + \sum_{k=1}^{\infty} \left\{ \sum_{j=0}^{[\frac{k-1}{2}]} (-1)^{k-1} \bar{\alpha}_j \binom{k-1}{k-2j-1} \bar{u}_j{}^{k-2j-1} \right\} \frac{1}{n^k}. \end{aligned} \tag{31}$$

To establish the desired general asymptotic expansion, we equate the coefficients of (23) and (31). Firstly, we have $\Phi = y_0 = 1$. Next, we show that the parameter sequences can be determined from the system

$$\sum_{j=0}^{[\frac{k-1}{2}]} (-1)^{k-1} \bar{\alpha}_j \binom{k-1}{k-2j-1} \bar{u}_j{}^{k-2j-1} = \frac{y_k}{k!}. \tag{32}$$

Setting $k = 2n$, $n = 1, 2, \ldots$, we have

$$-\sum_{j=0}^{n-1} (-1)^{k-1} \bar{\alpha}_j \binom{2n-1}{2n-2j-1} \bar{u}_j{}^{2n-2j-1} = \frac{y_{2n}}{(2n)!}. \tag{33}$$

Setting $k = 2n+1$, $n = 1, 2, \ldots$, we have

$$\sum_{j=0}^{n} \bar{\alpha}_j \binom{2n}{2n-2j} \bar{u}_j{}^{2n-2j} = \frac{y_{2n+1}}{(2n+1)!}. \tag{34}$$

Thus, the corresponding recurrences can be established and the asymptotic expansion holds. □

Corollary 3. *The coefficient sequence $(\alpha_k)_{k\geq 0}$ in the asymptotic expansion* (20) *satisfies the explicit expression*

$$\alpha_k = \sum_{d_1+2d_2+\cdots+kd_k=k} \frac{(-rq!)^{d_1+d_1+\cdots+d_k}}{d_1! d_2! \cdots d_k!} \left(\frac{B_{q+3}}{(q+3)\underline{q+2}}\right)^{d_1} \left(\frac{B_{q+5}}{(q+5)\underline{q+2}}\right)^{d_2} \times \cdots \times \left(\frac{B_{q+2k+1}}{(q+2k+1)\underline{q+2}}\right)^{d_k}.$$

Proof. Since $\alpha_k = y_{2k}/(2k)!$, we have

$$\alpha_k = \frac{y_{2k}}{(2k)!} = \sum_{\substack{c_1+2c_2+\cdots\\+2kc_{2k}=2k}} \frac{(-rq!)^{c_1+c_2+\cdots+c_{2k}}}{c_1!c_2!\cdots c_{2k}!}\left(\frac{B_{q+2}}{(q+2)^{\underline{q+2}}}\right)^{c_1}\left(\frac{B_{q+3}}{(q+3)^{\underline{q+2}}}\right)^{c_2}\times\cdots\times\left(\frac{B_{q+2k+1}}{(q+2k+1)^{\underline{q+2}}}\right)^{c_{2k}}.$$

Using further the fact $B_{2k+1} = 0$ for $k = 1,2,\ldots$, we obtain the result. □

Example 4. In Theorem 1, setting $q = 1$ and $r = 1,2$ gives

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(1 + \frac{1}{720n^2} - \frac{1433}{725600n^4} + \frac{1550887}{15676416000n^6} - \frac{365236274341}{3476402012160000n^8} + \frac{31170363588856607}{1626956141690880000000n^{10}} - \cdots\right),$$

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(1 + \frac{1}{360n^2} - \frac{713}{1814400n^4} + \frac{386647}{1959552000n^6} - \frac{45586354501}{217275125760000n^8} + \frac{1946465323055807}{50842379427840000000n^{10}} - \cdots\right)^{\frac{1}{2}},$$

as $n \to \infty$.

Example 5. In Theorem 1, setting $q = 3$ and $r = 1,2$ yields

$$\prod_{k=1}^{n} k^{k^3} \sim A_3 \cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}} \mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}} \left(1 - \frac{1}{5040n^2} + \frac{1513}{50803200n^4} - \frac{127057907}{84495588224000n^6} + \frac{7078687551763}{442893616349184000n^8} - \cdots\right),$$

$$\prod_{k=1}^{n} k^{k^3} \sim A_3 \cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}} \mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}} \left(1 - \frac{1}{2520n^2} + \frac{757}{12700800n^4} - \frac{31776959}{1056198528000n^6} + \frac{885025670587}{27680851021824000n^8} - \cdots\right)^{\frac{1}{2}},$$

as $n \to \infty$.

Example 6. In Theorem 2, let $q = 2$ and $r = 1,2$, we have

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{360(n+\frac{1}{720})} - \frac{1}{360(n-\frac{1036793}{7838208000})^3} + \frac{11923193}{282154880000(n-\frac{9137074752049}{973504862064000})^5} + \cdots\right),$$

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{180(n+\frac{1}{360})} - \frac{1}{180(n-\frac{259193}{979776000})^3} + \frac{2980793}{176359680000(n-\frac{571060368049}{60843946716000})^5} + \cdots\right)^{\frac{1}{2}},$$

as $n \to \infty$. Similarly, we can obtain other special cases of Theorem 1 and 2. In Example 5, the cases of $r = 1$ and $q = 1,3$ are presented in Wang's paper [13].

3. The last two general asymptotic expansions

In this section, we present two other general asymptotic expansions on the hyperfactorial functions.

Theorem 7. *Let r, s, q be real numbers such that $r \neq 0$ and q is odd. Define the sequence $(\psi_m)_{m\geq 1}$ by*

$$\psi_1 = 0, \quad \psi_m = m! \sum_{k=1}^{m-1} \frac{(-1)^{q+m-k-1} r^{m-k} s^{m-k-1} q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}}, \quad m \geq 2. \tag{35}$$

Then the following asymptotic expansion on hyperfactorial functions holds:

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(1 + \sum_{k=0}^{\infty} \frac{\overline{\varphi}_k}{(n+\overline{v}_k)^{2k+1}}\right)^{\frac{n}{r}+s}, \tag{36}$$

as $n \to \infty$, where $(\overline{\varphi}_k)_{k\geq 0}$ and $(\overline{v}_k)_{k\geq 1}$ are determined by

$$\overline{\varphi}_k = \frac{Y_{2k+1}}{(2k+1)!} - \sum_{j=0}^{k-1} \overline{\varphi}_j \binom{2k}{2k-2j} v_j{}^{2k-2j}, \tag{37}$$

$$\overline{\varphi}_k \overline{v}_k = -\frac{1}{2k+1} \left\{ \frac{Y_{2k+2}}{(2k+2)!} - \sum_{j=0}^{k-1} \overline{\varphi}_j \binom{2k+1}{2k-2j+1} v_j{}^{2k-2j+1} \right\}, \tag{38}$$

and the sequence $(Y_k)_{k\geq 0}$ can be determined by

$$Y_0 = 1, \quad Y_1 = Y_2 = 0, \quad Y_k = \sum_{j=0}^{k-3} \binom{k-1}{j} \psi_{k-j} Y_j, \quad k \geq 3. \tag{39}$$

Proof. Using the expansion (22) and the definition of (ψ_m), we have

$$\begin{aligned}
\left(\frac{\prod_{k=1}^{n} k^{k^q}}{A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)}}\right)^{\frac{r}{n+rs}} &\sim \exp\left\{ \frac{r}{n+rs} \sum_{k=1}^{\infty} \frac{(-1)^q q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}} \frac{1}{n^k} \right\} \\
&= \exp\left\{ \frac{r}{n} \sum_{j=0}^{\infty} \left(-\frac{rs}{n}\right)^j \sum_{k=1}^{\infty} \frac{(-1)^q q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}} \frac{1}{n^k} \right\} \\
&= \exp\left\{ \sum_{m=2}^{\infty} \sum_{k=1}^{m-1} \frac{(-1)^{q+m-k-1} r^{m-k} s^{m-k-1} q! B_{k+q+1}}{(k+q+1)^{\underline{q+2}}} \frac{1}{n^m} \right\} \\
&= \exp\left\{ \sum_{m=1}^{\infty} \psi_m \frac{\left(\frac{1}{n}\right)^m}{m!} \right\} = \sum_{k=0}^{\infty} \frac{Y_k(\psi_1, \psi_2, \ldots, \psi_k)}{k!} \frac{1}{n^k}, \quad n \to \infty.
\end{aligned} \tag{40}$$

Moreover, we have

$$\Psi + \sum_{j=0}^{\infty} \frac{\overline{\varphi}_j}{(n+\overline{v}_j)^{2j+1}} = \Psi + \sum_{k=1}^{\infty} \left\{ \sum_{j=0}^{[\frac{k-1}{2}]} (-1)^{k-1} \overline{\varphi}_j \binom{k-1}{k-2j-1} \overline{v}_j{}^{k-2j-1} \right\} \frac{1}{n^k}.$$

Define $Y_k = Y_k(\psi_1, \psi_2, \ldots, \psi_k)$. Similarly to Theorem 2.2, we have the system

$$\frac{Y_k}{k!} = \sum_{j=0}^{[\frac{k-1}{2}]} (-1)^{k-1} \overline{\varphi}_j \binom{k-1}{k-2j-1} \overline{v}_j{}^{k-2j-1} \text{ and } \Psi = 1.$$

Then by setting $k = 2n$ and $k = 2n+1$, we get the unique solution $(\overline{\varphi}_k)$ and $(\overline{v}_k)$, which can be computed by the recurrences (37), (38) and (39). □

Theorem 8. *Let r, s, q be real numbers such that $r \neq 0$ and q is even. Define the sequence $(\psi_m)_{m\geq 1}$ by* (35). *Then we have*

$$\prod_{k=1}^{n} k^{k^q} \sim A_q \cdot n^{U_{q+1}(n)} \mathrm{e}^{-V_{q+1}(n)} \left(\sum_{k=0}^{\infty} \frac{\varphi_k}{(n+v_k)^{2k}} \right)^{\frac{n}{r}+s}, \tag{41}$$

as $n \to \infty$, where $(\varphi_k)_{k\geq 0}$ and $(v_k)_{k\geq 1}$ are determined by

$$\varphi_k = \frac{Y_{2k}}{(2k)!} - \sum_{j=0}^{k-1} \varphi_j \binom{2k-1}{2k-2j} {v_j}^{2k-2j}, \tag{42}$$

$$\varphi_k v_k = \frac{1}{2k} \left\{ \frac{Y_{2k+1}}{(2k+1)!} + \sum_{j=0}^{k-1} \varphi_j \binom{2k}{2k-2j+1} {v_j}^{2k-2j+1} \right\}, \tag{43}$$

and the sequence $(Y_k)_{k\geq 0}$ can be determined by

$$Y_0 = 1,\ Y_1 = 0, \quad Y_k = \sum_{j=0}^{k-2} \binom{k-1}{j} \psi_{k-j} Y_j, \quad k \geq 2. \tag{44}$$

Proof. Using the expansion (22) and the definition of (ψ_m), we obtain (40). Moreover we have

$$\sum_{j=0}^{\infty} \frac{\varphi_j}{(n+v_j)^{2j+1}} = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)\varphi_j \binom{k-1}{k-2j} {v_j}^{k-2j} \right\} \frac{1}{n^k}.$$

To prove the theorem, it suffices to show that the parameter sequences can be determined from the system:

$$\frac{Y_k}{k!} = \sum_{j=0}^{[\frac{k}{2}]} \varphi_j \binom{k-1}{k-2j} {v_j}^{k-2j}.$$

Similarly to Theorem 7, setting $k = 2n$ and $k = 2n+1$, we obtain the recurrences (42), (43) and (44). Then the theorem holds. □

Example 9. In Theorem 7, let $q = 1, 3$, $r = 1$ and $s = 1/(2r)$, we get

$$\prod_{k=1}^{n} k^k \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(1 + \frac{1}{720(n+\frac{1}{6})^3} - \frac{1}{12096(n-\frac{199}{9000})^5} + \frac{7836601}{65318400000(n+\frac{16117869401}{211588227000})^7} - \cdots \right)^{n+\frac{1}{2}},$$

$$\prod_{k=1}^{n} k^{k^3} \sim A_3 \cdot n^{\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}} \mathrm{e}^{-\frac{n^4}{16}+\frac{n^2}{12}} \left(1 - \frac{1}{5040(n+\frac{1}{6})^3} + \frac{1}{75600(n-\frac{23}{2016})^5} - \frac{19986823}{1126611763200(n+\frac{46081010939}{60440157520})^7} + \cdots \right)^{n+\frac{1}{2}},$$

as $n \to \infty$.

Example 10. In Theorem 8, setting $q = 2,4$, $r = 1$ and $s = 1/(2r)$ gives

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{360(n-\frac{1}{4})^2} - \frac{17}{453600(n-\frac{409}{136})^4} + \frac{31086371}{9517824000(n+\frac{3868297709}{3699278149})^6} + \cdots\right)^{n+\frac{1}{2}},$$

$$\prod_{k=1}^{n} k^{k^4} \sim A_4 \cdot n^{\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}} \mathrm{e}^{-\frac{n^5}{25}+\frac{n^3}{12}+\frac{13n}{360}} \left(1 + \frac{1}{1260(n-\frac{1}{4})^2} + \frac{13}{1270080(n-\frac{821}{260})^4} - \frac{13513363343}{13730580864000(n+\frac{770640238939}{702694893836})^6} - \cdots\right)^{n+\frac{1}{2}},$$

as $n \to \infty$.

Example 11. In Theorems 7 and 8, setting $q = 1,2$, $r = 1$ and $s = 0$, we can obtain

$$\prod_{k=1}^{n} k^{k} \sim A_1 \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \mathrm{e}^{-\frac{n^2}{4}} \left(1 + \frac{1}{720n^3} - \frac{1}{5040(n+\frac{7}{7200})^5} + \frac{1728049}{17418240000(n+\frac{103682401}{261281008800})^7} - \cdots\right)^{n},$$

$$\prod_{k=1}^{n} k^{k^2} \sim A_2 \cdot n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} \mathrm{e}^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{360n^2} + \frac{247}{1814400n^4} - \frac{78487}{1959552000n^6} + \frac{6557802299}{217275125760000n^6} + \cdots\right)^{n},$$

as $n \to \infty$. Other special cases can be obtained by assigning different values to q, r and s. In Example 9, the case of $q = 2$, $r = 1$ appears in Wang's paper [13].

Acknowledgments

The author would like to thank Wang and the anonymous referee for their valuable suggestions and comments.

References

[1] V. S. Adamchik, "Polygamma functions of negative order", *J. Comput. Appl. Math.* **100** (1998), no. 2, p. 191-199.

[2] L. Bendersky, "Sur la fonction gamma généralisée", *Acta Math.* **61** (1933), p. 263-322.

[3] C.-P. Chen, "Glaisher–Kinkelin constant", *Integral Transforms Spec. Funct.* **23** (2012), no. 11, p. 785-792.

[4] ———, "Asymptotic expansions for Barnes *G*-function", *J. Number Theory* **135** (2014), p. 36-42.

[5] C.-P. Chen, L. Lin, "Asymptotic expansions related to Glaisher–Kinkelin constant based on the Bell polynomials", *J. Number Theory* **133** (2013), no. 8, p. 2699-2705.

[6] J.-X. Cheng, C.-P. Chen, "Asymptotic expansions of the Glaisher–Kinkelin and Choi–Srivastava constants", *J. Number Theory* **144** (2014), p. 105-110.

[7] J. Choi, "A set of mathematical constants arising naturally in the theory of the multiple gamma functions", *Abstr. Appl. Anal.* **2012** (2012), article ID 121795 (11 pages).

[8] B. K. Choudhury, "The Riemann zeta-function and its derivatives", *Proc. R. Soc. Lond., Ser. A* **450** (1995), no. 1940, p. 477-499.

[9] L. Lin, "Inequalities and asymptotic expansions related to Glaisher–Kinkelin constant", *Math. Inequal. Appl.* **17** (2014), no. 4, p. 1343-1352.

[10] D. Lu, C. Mortici, "Some new quicker approximations of Glaisher–Kinkelin's and Bendersky–Adamchik's constants", *J. Number Theory* **144** (2014), p. 340-352.

[11] C. Mortici, "Approximating the constants of Glaisher–Kinkelin type", *J. Number Theory* **133** (2013), no. 8, p. 2465-2469.

[12] W. Wang, "Unified approaches to the approximations of the gamma function", *J. Number Theory* **163** (2016), p. 570-595.

[13] ———, "Some asymptotic expansions on hyperfactorial functions and generalized Glaisher–Kinkelin constants", *Ramanujan J.* **43** (2017), no. 3, p. 513-533.

[14] W. Wang, H. Liu, "Asymptotic expansions related to hyperfactorial function and Glaisher–Kinkelin constant", *Appl. Math. Comput.* **283** (2016), p. 153-162.

[15] Z. Xu, W. Wang, "More asymptotic expansions for the Barnes *G*-function", *J. Number Theory* **174** (2017), p. 505-517.

[16] Z. Yang, J.-F. Tian, "Asymptotic expansions for the gamma function in terms of hyperbolic functions", *J. Math. Anal. Appl.* **478** (2019), no. 1, p. 133-155.