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Abstract. Intersection norms are integer norms on the first homology group of a surface. In this article, we give examples of polytopes which are not the dual unit balls of intersection norms, answering a question asked in [2]. On the way, we investigate the set of collections of curves on Σ_2 whose complement is a disk.

1. Introduction

Intersection norms on surfaces were first quickly introduced by Turaev [8, p. 143], and studied by M. Cossarini and P. Dehornoy [2]. They use intersection norms to classify, up to isotopy, all surfaces transverse to the geodesic flow on the complement of special links in the unit tangent bundle of a closed oriented surface. Their result makes explicit Thurston's fibered faced theory for Thurston norms on compact oriented 3-manifolds. It tells us that an intersection norm on a surface (respectively the Thurston norm on a 3-manifold) encodes the open book decompositions of the unit tangent bundle of that surface (respectively the topology of foliated 3-manifold).

Our purpose in this article is to study intersection norms for their own sake. Let Σ_g be a closed oriented surface of genus $g \geq 1$, and Γ a collection of closed curves on Σ_g. Throughout this article, we consider collections in generic position: collections with transverse double intersection points. Let α be a closed curve on Σ_g; we define the number $i_\Gamma(\alpha)$ as follows:

$$i_\Gamma(\alpha) = \inf\{\#\{\alpha' \cap \Gamma\}; \alpha' \sim \alpha; \alpha' \pitchfork \Gamma\};$$

where the symbol $\sim$ (respectively $\pitchfork$) denotes the free homotopy relation (respectively transversality).

The function

$$N_\Gamma : H_1(\Sigma_g, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$a \longmapsto \inf\{i_\Gamma(\alpha); [\alpha] = a\}.$$

defines a semi-norm on $H_1(\Sigma_g, \mathbb{R})$ and it takes integer values on the lattice $H_1(\Sigma_g, \mathbb{Z})$ (see [2]). Using a standard basis for the homology, we shall identify $H_1(\Sigma_g, \mathbb{R})$ and $H^1(\Sigma_g, \mathbb{R})$ with $\mathbb{R}^{2g}$. By a theorem of Thurston [7], the dual unit ball of N_Γ is a lattice polytope, ie, the convex hull of

finitely many integer vectors (integer vector here means a vector in the lattice $H^1(\Sigma_g, \mathbb{Z}) \approx \mathbb{Z}^{2g}$). Moreover, if Γ fills Σ_g, that is $\Sigma_g - \Gamma$ is a union of topological disks, then N_Γ defines a norm and as a consequence, its dual unit ball has non-empty interior in $H^1(\Sigma_g, \mathbb{R})$.

One constraint on the dual unit balls of intersection norms is that their vertices are congruent modulo 2. This comes from the fact that geometric and algebraic intersections have the same parity. In genus 1, this constraint happens to be the only one. So, every symmetric convex lattice polygon with mod 2 congruent vertices is the dual unit ball of an intersection norm on the torus. The proof of this fact follows from an implicit argument in Thurston's paper [7]. We will explain it in Section 2 for completeness.

Now we raise the following problem:

Question 1 (Pierre Dehornoy).   *Fix $g \geq 2$, and let $P \subset H^1(\Sigma_g, \mathbb{R})$ be a symmetric lattice polytope all of whose vertices are congruent mod 2. Is P the dual unit ball of an intersection norm on Σ_g?*

This question is natural when we deal with integer norms coming from topology (for instance, we have an analogue of this question for the Thurston norm).

Definition 2.   *Let P be a polytope obtained by taking the convex hull of finitely many vectors $\{v_i\}$. A* sub-polytope P' *of P is a polytope obtained by taking the convex hull of a subset of $\{v_i\}$.*

In this article, we give examples of lattice polytopes on $\mathbb{R}^4$ with mod 2 congruent vertices, which are not dual unit balls of intersection norms. More precisely, we show that sub-polytopes (with eight vertices and non-empty interior) of the cube $[-1, 1]^4$ are not the dual unit balls of intersection norms. This means that in higher dimension, dual unit balls of intersection norms come with other constraints.

Let $\mathscr{P}_8$ be the set of all symmetric sub-polytopes of $[-1, 1]^4$ having eight vertices and non-empty interior. The set $\mathscr{P}_8$ is not empty; it contains the polytope generated by the following vectors (and their opposites):

$$v_1 = (1, 1, 1, 1), \quad v_2 = (1, -1, 1, 1), \quad v_3 = (-1, 1, 1, 1), \quad v_4 = (1, 1, -1, 1).$$

Now, we state the main result of this article:

Theorem 3.   *Elements of $\mathscr{P}_8$ are not dual unit balls of intersection norms.*

If Γ is a filling collection of curves on a surface, whose complement is a disk, we say that Γ is a *one-faced collection*. The proof of Theorem 3 relies on:

Theorem 4.   *There are four orbits (under the action of orientation-preserving homeomorphisms) of one-faced collections on Σ_2, whose dual unit balls are in the cube $[-1, 1]^4$.*

Thurston showed that every symmetric polygon with mod 2 congruent vertices is the dual unit ball of a Thurston norm on a 3-manifold (see [7]). His construction is closely related to intersection norms on the torus. Roughly speaking, he showed that from a filling collection Γ on the torus, one can construct a 3-manifold M_Γ such that the dual unit ball of the Thurston norm on M_Γ and the dual unit ball of the intersection norm N_Γ are the same. Thurston's construction extends to higher genus surfaces (see [6]), and Theorem 3 suggests that there are probably polytopes in dimension greater than two which are not the dual unit balls of Thurston norms on 3-manifolds.

For the proof of Theorem 1, we define a partial order on the set of filling collections of closed curves and we use it to show that if an element of $\mathscr{P}_8$ is the dual unit ball of an intersection norm N_Γ, then Γ is a one-faced collection. Finally, we check that none of the four collections of Theorem 4 realizes an element of $\mathscr{P}_8$.

For $g \geq 2$, sub-polytopes of $[-1, 1]^{2g}$ with non-empty interiors have at least $4g$ vertices. Let $\mathscr{P}_{4g}$ denotes the set of sub-polytopes of $[-1, 1]^{2g}$ with non-empty interiors and with $4g$ vertices.

Proposition 5. *Let $P \in \mathscr{P}_{4g}$. If P is the dual unit ball of an intersection norm N_Γ, then Γ is a one-faced collection.*

Organization of this article

In Section 2, we recall some facts on intersection norms and we show that for the question of realizability, we can restrict our attention to minimal collections.

In Section 3, we show that any intersection norm is bounded from below by a norm defined by a one-faced collection.

In Section 4, we count orbits (under the action of orientation-preserving homeomorphisms) of one-faced collections (whose dual unit balls are sub-polytopes of the cube $[-1,1]^4$) on Σ_2 and we prove Theorem 3. We finish this section by establishing a link between the realization problem and the determination of one-faced collections by proving Proposition 5.

2. Preliminaries on intersection norms

In this section, we first recall some facts about integer (semi)-norms. Then, we define the intersection (semi)-norm associated to a collection of curves and we recall some basic notions about them (For more details on intersection norms, see [2]). We end this section by proving that, concerning the realizability of polytopes, we can restrict our attention to minimal collections.

Let $\sim$ denote the free homotopy relation between curves; let $\pitchfork$ be the transversality relation and $[\cdot]$ the homology class.

Integer norms

Let E be a vector space of dimension n and

$$L = L(u_1, \ldots, u_n) := \{a_1 u_1 + \cdots + a_n u_n, a_i \in \mathbb{Z}\}$$

the lattice generate by the vectors $(u_i)_{i=1,\ldots,n}$.

Definition 6 (Integer norm). *A norm $N : E \longrightarrow \mathbb{R}_+$ is an* integer semi-norm *relatively to the lattice L if the restriction of N to L takes positive integer values.*

The dual unit ball *of N is the unit ball of the dual norm $N^* : E^* \longrightarrow \mathbb{R}_+$; where E^* is the dual space of E.*

The following theorem states that the dual unit ball of an integer semi-norm has a combinatorial description.

Theorem 7 (W. Thurston). *If N is an integer semi-norm relatively to a lattice L, then its dual unit ball is a convex hull of finitely many vectors in the lattice;*

$$B_{N^*} = \mathrm{ConvHull}\{v_1, \ldots\ldots, v_n; v_i \in L^*\}.$$

One can find a sketch of the proof of Theorem 7 in [7, p. 107–112]. For a more complete proof, see [3] (Exposé Fourteen by David Fried). More recently, M. de la Salle gives a new proof of Theorem 7 (see [5]).

Definition of intersection norms

We consider a genus g closed oriented surface Σ_g and a collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of closed curves on Σ_g. The collection Γ is allowed to change only in its isotopy class. Let $a \in H_1(\Sigma_g, \mathbb{Z})$ be a homology class and α an oriented multi-curve representing a. Then we define:

$$i_\Gamma(\alpha) := \inf\{\#\{\alpha' \cap \Gamma\}; \alpha' \sim \alpha; \alpha' \pitchfork \Gamma\}$$

and

$$N_\Gamma(a) := \inf\{i_\Gamma(\alpha); [\alpha] = a\}.$$

If a multi-curve α representing a homology class a is such that

$$N_\Gamma(a) = i_\Gamma(\alpha),$$

then α is Γ-*minimizing*.

One important thing is that Γ-minimizing multi-curves can be chosen to be simple. In fact, if α is a non-simple Γ-minimizing multi-curve then by smoothing all the self-intersection points of α with respect to their orientation, we get a new oriented multi-curve α' in the same homology class as α (its a common fact on homology that smoothing singularities preserves homology classes) and $i_\Gamma(\alpha') = i_\Gamma(\alpha)$. This implies that α' is a simple Γ-minimizing multi-curve as we claim.

Proposition 8 (Cossarini–Dehornoy).   *The function $N_\Gamma : H_1(\Sigma_g, \mathbb{Z}) \longrightarrow \mathbb{N}$ satisfies:*

- *linearity on rays: $N_\Gamma(na) = |n| N_\Gamma(a)$ for all $n \in \mathbb{Z}$ and $a \in H_1(\Sigma_g, \mathbb{Z})$*
- *convexity: $N_\Gamma(a + b) \leq N_\Gamma(a) + N_\Gamma(b)$ for all $a, b \in H_1(\Sigma_g, \mathbb{Z})$.*

The proof of Proposition 8 is not trivial and one can see [2] for the proof.

Linearity on rays implies that N_Γ can be extended to homology with rational coefficients since for all $a \in H_1(\Sigma_g, \mathbb{Z})$ and $q \in \mathbb{N}$, we have:

$$N_\Gamma(a) = N_\Gamma\left(\frac{q}{q}.a\right) = q N_\Gamma\left(\frac{1}{q}a\right).$$

It follows by convexity that N_Γ extends uniquely to a positive function on $H_1(\Sigma_g, \mathbb{R})$. Moreover, the extended function $N_\Gamma : H_1(\Sigma_g, \mathbb{R}) \longrightarrow \mathbb{R}_+$ is still linear on rays and convex. Therefore, N_Γ defines a semi-norm on $H_1(\Sigma_g, \mathbb{R})$ and it takes integer values on the lattice $H_1(\Sigma_g, \mathbb{Z})$. So, N_Γ is an *integer semi-norm*. Theorem 7 implies that the dual unit ball $B_{N_\Gamma^*}$ is a convex hull of finitely many integer vectors.

If the collection is filling, then N_Γ defines an *integer norm*.

Relation between the vectors of the dual unit ball

If α and β are two transverse oriented closed curves, then the algebraic intersection number between α and β is given by

$$\widehat{i}(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta);$$

where $\varepsilon(p, \alpha, \beta)$ is the algebraic sign of the intersection at p, relatively to the orientation of Σ_g. We recall that $\widehat{i}$ depends only on the homology classes of α and β, and defines a non degenerate skew-symmetric 2-form on $H_1(\Sigma_g, \mathbb{R})$.

Then, if α and α' are two homologous curves, by taking an orientation of Γ, we have

$$i_\Gamma(\alpha) = \widehat{i}(\alpha, \Gamma) \mod 2; \quad i_\Gamma(\alpha') = \widehat{i}(\alpha', \Gamma) \mod 2; \quad \widehat{i}(\alpha, \Gamma) = \widehat{i}(\alpha', \Gamma).$$

Thus, $i_\Gamma(\alpha) = i_\Gamma(\alpha') \mod 2$ for every orientation of Γ. Therefore, if v_1 and v_2 are two integer vertices in the dual unit sphere of N_Γ,

$$v_1 = v_2 \mod 2.$$

The relation above is a necessary condition for a symmetric lattice polytope to be the dual unit ball of an intersection norm. The following statement shows that it is sufficient in the genus one case. The idea of the proof is from Thurston [7].

Proposition 9. *If P is a symmetric lattice polygon in $\mathbb{R}^2$ with congruent mod 2 vertices, then P is the dual unit ball of an intersection norm.*

Proof. First, if P is a symmetric lattice segment in $\mathbb{R}^2$, then there is a matrix $M \in SL(2, \mathbb{Z})$ such that $P' := M(P)$ is a vertical segment with extremities in $\mathbb{Z}^2$. Moreover, M has a geometric realization since the group of isotopy classes of preserving orientation homeomorphisms of the torus is isomorphic to $SL(2, \mathbb{Z})$. That is there is a homeomorphism ϕ of $\mathbb{T}^2$ such that

$$\phi_* : H_1(\mathbb{T}^2, \mathbb{R}) \approx \mathbb{R}^2 \longrightarrow H_1(\mathbb{T}^2, \mathbb{R}) \approx \mathbb{R}^2$$

is equal to M.

Now, let $l := \frac{1}{2} \text{length}(P')$; $l \in \mathbb{N}$. If $\{\alpha, \beta\}$ is the canonical basis of $H_1(\mathbb{T}^2, \mathbb{R})$, by taking l parallel curves to β, we get a collection Γ' in $\mathbb{T}^2$ such that $B_{N^*_{\Gamma'}} = P'$. So, $\Gamma := \phi^{-1}(\Gamma')$ is such that $B_{N^*_\Gamma} = P$.

Secondly, if $\Gamma := \{\gamma_1, \dots, \gamma_n\}$ is a collection of closed geodesics on $\mathbb{T}^2$ (with the flat metric), and if a is a homology class represented by a collection α of oriented simple closed curves which are pairwise disjoint then

$$N_\Gamma(a) = i_\Gamma(\alpha) = \sum_{j=1}^n i_{\gamma_j}(\alpha) = \sum_{j=1}^n N_{\gamma_j}(a).$$

It follows that the dual unit ball of N_Γ is equal to the Minkowski sum of the dual unit balls of N_{γ_j}, which are symmetric lattice segments:

$$B_{N^*_\Gamma} = \bigoplus_j B_{N^*_{\gamma_j}}.$$

Finally, every symmetric lattice polygon of $\mathbb{R}^2$ is the Minkowski sum of finitely many symmetric lattice segments.

Combining the three arguments above we construct, for any symmetric lattice polygon P with mod 2 congruent vertices, a geodesic collection Γ such that

$$B_{N^*_\Gamma} = P. \qquad \qquad \square$$

Minimality of collections

Now, we show that we can restrict to collections in minimal position.

Definition 10. *Let γ_1 and γ_2 be two transverse closed curves on Σ_g. They are in* minimal position *if they realize the geometric intersection in their free homotopy classes:*

$$i(\gamma_1, \gamma_2) = \text{card}\{\gamma_1 \cap \gamma_2\}.$$

A collection Γ is minimal *if all the curves in Γ are pairwise in minimal position.*

If a collection Γ is not minimal, it admits a minimal representative Γ' in its homotopy class. Moreover, the isotopy classes of Γ and Γ' differ by Reidemester's moves (see Figure 1).

Lemma 11. *Let Γ be a collection of closed curves in Σ_g, then there is a minimal collection $\Gamma_{\min}$ such that $N_\Gamma = N_{\Gamma_{\min}}$.*

Proof. One can apply a generic homotopy so that we get a collection in minimal position. Such a generic homotopy consists of doing a finite number of Reidemester moves (1, 2 and 3 as depicted in Figure 1). By Hass and Scott [4], one can choose a decreasing homotopy with respect to the intersection number of the collection. Moves 1 and 3 do not change the norm, while Move 2 (deleting a bigon) changes the norm.

Then, we replace Move 2 by a crossing move (see Figure 1). This new move changes the homotopy class of Γ but it does not change the norm. The intersection decreases by one with the crossing move (Move 2bis). As we can choose a decreasing homotopy, we get a collection $\Gamma_{\min}$ in minimal position after applying finitely many Reidemester's moves 1 and 3 and crossing moves. Doing so, the norm does not change; hence we built a new collection $\Gamma_{\min}$ in minimal position such that $N_\Gamma = N_{\Gamma_{\min}}$. $\hspace{2cm}\square$
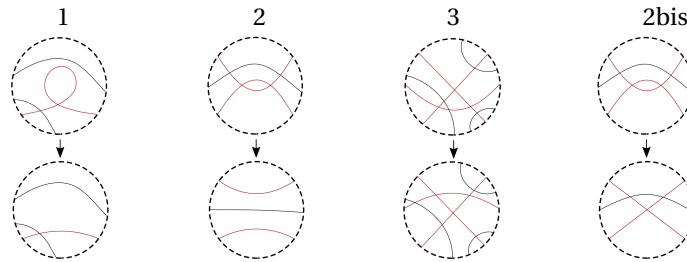

Figure 1. From left to right, we have the three Reidemeister's moves and the last one is the crossing move (2bis) on which a bigon is replaced by a transverse self-intersection. The crossing move changes the homotopy class of the collection. The curves in red color represent the local configuration of Γ, and curves in black are sub-arcs of curves in Σ_g. One can see that Reidemeister's move 1 and 3 and crossing move do not change the intersection. Otherwise, the bigon deleting does change the intersection.

Remark 12. One-faced collections are minimal.

Eulerian Co-orientation

Let Γ be a collection of closed curves on Σ_g such that $\Sigma_g - \Gamma$ is a union of topological disks. The collection Γ defines a filling graph on Σ_g. We denote by $V(\Gamma)$ the set of its vertices, defined as self-intersection points of Γ. Let $E(\Gamma)$ be the set of edges and $F(\Gamma)$ the set of faces.

The Euler characteristic of Σ_g is given by:

$$\chi(\Sigma_g) = 2 - 2g = |V| - |E| + |F|.$$

Definition 13. *A* co-orientation ν *of* Γ *is a choice of a positive direction to cross (transversally) every edge of* Γ. *We denote by* Γ^ν *the collection* Γ *together with the co-orientation* ν.

A co-orientation is Eulerian *if a small oriented circle centered at a vertex crosses positively two edges and negatively the other two, relatively to the co-orientation.*

Figure 2. Non alternating and alternating co-orientation.

Remark 14.

- Up to rotation, we distinguish two types of Eulerian co-orientations around a vertex (see Figure 2). A vertex is *non-alternating* if the arcs emanating from it, and belonging to the same curve are co-oriented in the same direction; otherwise it is *alternating*.
- A co-orientation of an arc corresponds to an orientation of it.
- A collection Γ with c curves has at least 2^c Eulerian co-orientations given by all the different ways to co-orient Γ only by non-alternating vertices (this is equal to the number of possibilities to orient Γ).

Let α be an oriented closed curve on Σ_g transverse to Γ, and let ν be a co-orientation of Γ. We define

$$\nu(\alpha) := \sum_{p \in \alpha \pitchfork \Gamma} \varepsilon(p, \alpha, \Gamma^\nu),$$

where $\varepsilon(p, \alpha, \Gamma^\nu) = \pm 1$ depending on whether α crosses Γ at p in the direction of the co-orientation ν or not. Moreover, if ν is an Eulerian co-orientation,

$$\nu(\alpha) = 0 \quad \text{if} \quad [\alpha] = 0.$$

An Eulerian co-orientation ν defines a map

$$[\nu] : H_1(\Sigma_g, \mathbb{R}) \longrightarrow \mathbb{R}_+$$
$$H_1(\Sigma_g, \mathbb{Z}) \longrightarrow \mathbb{N}.$$

Hence, an Eulerian co-orientation defines an integer cohomology class. We denote by Eulco(Γ) the set of all Eulerian co-orientations of Γ and by [Eulco(Γ)] the set of their cohomology classes (different co-orientations can give the same cohomology class).

Theorem 15 (M. Cossarini & P. Dehornoy). *The set* [Eulco(Γ)] *is a subset of the unit dual ball* $B_{N_\Gamma^*}$. *Moreover, every integer vector in* $B_{N_\Gamma^*}$, mod 2 *congruent to the vertices of* $B_{N_\Gamma^*}$ *belongs to* [Eulco(Γ)].

The proof of Theorem 15 is well explained in [2].

3. Partial order on the set of intersection norms of Σ_g

In this section, we define a topological operation on filling collections of closed curves. That operation induces a partial order on the set of filling collections.

We finish by proving that every (non-even) intersection norm is bounded from below by an intersection norm induced by a one-faced collection.

Let p be a self-intersection point of a filling collection Γ in minimal position, such that two different and opposite faces F_1 and F_2 have p as a common point.

By smoothing the point p (see Figure 3) in such a way that F_1 and F_2 merge to a unique face, we obtain a new filling collection Γ'. We call this operation *merging of faces*.

Definition 16. *Let Γ and Γ' two filling collections. We say that $\Gamma' \leq \Gamma$ if there is a sequence $\Gamma_0 = \Gamma \longrightarrow \Gamma_1 \longrightarrow \cdots \longrightarrow \Gamma_n = \Gamma'$ where each step of the sequence is a Move 1, Move 2bis, Move 3 or a merging of faces.*

Lemma 17. *If $\Gamma' \leq \Gamma$, then $N_{\Gamma'} \leq N_\Gamma$.*

Proof. Let Γ be a filling collection in minimal position and Γ' a collection obtained by merging two faces at a point p. The collections Γ and Γ' are different only in a small neighborhood of p. Then, for any closed curves α on Σ_g

$$i_{\Gamma'}(\alpha) \leq i_\Gamma(\alpha);$$

and it follows that $N_{\Gamma'} \leq N_\Gamma$. Since Move 1, Move 2bis and Move 3 do not change the intersection norm, the inequality holds for Γ' in minimal position. $\qquad\square$

Figure 3. Smoothing a self-intersection point and the merging of two faces.

Definition 18. *A filling collection Γ is* bi-colorable *if one can color the faces of Γ in black and white such that two faces with a commun edge do not have the same color.*

A filling collection Γ is one-faced *if $\Sigma_g - \Gamma$ is a disk.*

If Γ is a bi-colorable collection, then N_Γ restricted to the lattice $H_1(\Sigma_g, \mathbb{Z})$ takes only even values. In fact, a closed curve α alternates between black and white faces. That is $i_\Gamma(\alpha)$ is even; hence N_Γ is even. In this case, the coordinates of the vectors defining the dual unit ball of N_Γ are all even. If Γ' is obtained by merging faces on a bi-colorable collection Γ, then Γ' is also a bi-colorable.

For a graph defined by a collection of curves with only double points, we have $|E| = 2|V|$. Then, the Euler characteristic of the surface is given by:

$$\chi(\Sigma_g) = 2 - 2g = |F| - |V|.$$

It follows that for a filling collection, we have $|V| = |F| + 2g - 2 \geq 2g - 1$; the minimum is obtained for one-faced collections. In particular, in genus two, a one-faced collection has self-intersection number equal to 3.

The following lemma is one of the cornerstones of this article.

Lemma 19 (Lower bound for intersection norms). *If Γ is not bi-colorable, then there is a finite sequence $\Gamma_0 = \Gamma \longrightarrow \cdots \longrightarrow \Gamma_n$ of merging of faces such that Γ_n is one-faced.*

Proof. Let Γ be a filling collection on Σ_g and $\Gamma_0 = \Gamma \to \cdots \to \Gamma_n$ be a sequence of merging of faces such that no merging is possible on Γ_n.

Let p be a double point of Γ_n and $e := (v_1 = p, v_2, \ldots, v_n = p)$ an Eulerian cycle of Γ. Since all the vertices of Γ have degree 4, such cycles exist.

As Γ_n is not reducible by a merging of faces, there are at most two different faces around p; let F_a and F_b be the faces (eventually equal) around p. When we turn around p, we read $F_a - F_b - F_a - F_b$.

The faces F_a and F_b are also faces around v_2 since v_1 and v_2 share an edge. The configuration around v_2 is also $F_a - F_b - F_a - F_b$, since a merging of faces is not possible around v_2.

Step by step, following the Eulerian cycle, we show that all the vertices of Γ_n have the same configuration of faces: $F_a - F_b - F_a - F_b$.

Therefore, Γ_n has at most two faces in its complement depending on whether $F_a = F_b$ or not. If Γ_n has two faces in its complement, then two faces with a common edge have different color. So, Γ_n is bi-colorable and so is Γ.

By taking the contrapositive, we obtain the proof. $\qquad\square$

Corollary 20. *Every intersection norm with dual unit ball in the cube $[-1, 1]^{2g}$ is bounded from below by a norm defined by a one-faced collection.*

Proof. If $B_{N_\Gamma^*}$ is a sub-polytope of $[-1, 1]^{2g}$, then N_Γ is non-even and then not bi-colorable. Applying Lemma 19, we obtain the result. $\qquad\square$

4. One-faced collections with dual unit ball in the cube $[-1,1]^4$

Now, we show that there are four one-faced collections (up to orientation-preserving homeomorphisms) with dual unit ball in the cube $[-1,1]^4$ (Theorem 4).

Partial configuration

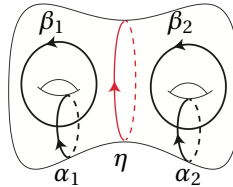We consider a one-faced collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ in Σ_2.

Figure 4. Canonical symplectic basis.

In what follows $\alpha_1, \beta_1, \alpha_2$ and β_2 are the oriented simple closed curves, that canonically represent the generators of the first homology group (see Figure 4). Let $\eta := \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}$ be the curve depicted in red and let A_η be a tubular neighborhood of η.

The following lemma gives a canonical partial configuration for one-faced collections.

Lemma 21. *If Γ is a one-faced collection on Σ_2 with dual unit ball in the cube $[-1,1]^4$, then there exists a diffeomorphism ψ of Σ_2 such that*

$$i(\alpha_i, \psi(\Gamma)) = i(\beta_i, \psi(\Gamma)) = 1; \quad i = 1,2.$$

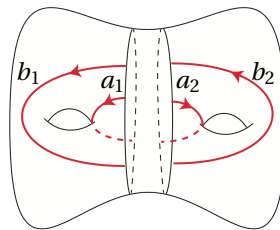*Hence, up to diffeomorphism and outside A_η, Γ looks like Figure 5.*

Figure 5. Partial configuration of the collection $\psi(\Gamma)$; with labelled arcs.

Proof. It is equivalent to show that, up to diffeomorphism, $\{\alpha_1, \beta_1, \alpha_1, \beta_1\}$ is Γ-minimizing. Since Γ is one-faced with dual unit ball in $[-1,1]^4$ then

$$N_\Gamma(a_i) = N_\Gamma(b_i) = 1$$

where $\{a_i, b_i; i = 1,2\}$ is the symplectic basis of $H_1(\Sigma_g, \mathbb{R})$. Now, as $N_\Gamma(a_i) = N_\Gamma(b_i) = 1$, there is an oriented simple closed curve α such that

$$i(\alpha, \Gamma) = 1, \quad [\alpha] = a_1.$$

Up to diffeomorphism, we can take $\alpha = \alpha_1$.

Now, let β be the Γ-minimizing simple curve in the homology class of b_1. Then,

$$\widehat{i}(\alpha_1, \beta) = 1, \quad i(\Gamma, \beta) = 1.$$

Therefore, one can perform a surgery on β along α (See Figure 6) to get a new curve β' such that β' is a simple Γ-minimizing curve in the same homology class of β and such that

$$i(\beta', \alpha) = i(\beta', \Gamma) = 1.$$

Up to diffeomorphism, we can take $\beta' = \beta_1$.

If α and β are Γ-minimizing in the homology classes of α_2 and β_2 respectively, we have

$$\widehat{i}(\alpha, \alpha_1 \cup \beta_1) = \widehat{i}(\beta, \alpha_1 \cup \beta_1) = 0.$$

Again, by performing surgery on α and β, we get α' and β' such that

$$i(\alpha', \alpha_1 \cup \beta_1) = i(\beta', \alpha_1 \cup \beta_1) = 0, \quad i(\alpha', \beta') = 1.$$

Then, up to diffeomorphism $\alpha' = \alpha_2$ and $\beta' = \beta_2$. This prove that up to diffeomorphism, $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ are Γ-minimizing. □
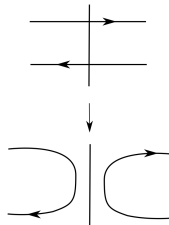

Figure 6. Surgery along the vertical curve.

Remark 22. Lemma 21 remains true for $g \geq 2$ and the proof is the same.

Lemma 21 implies that, up to diffeomorphism, a one-faced collection with dual unit ball in the cube $[-1, 1]^4$ is obtained by connecting the extremities of the partial configuration by arcs in the annulus A_η. Moreover, the self-intersection number of Γ is determined by the intersection between those arcs we used to complete the partial configuration.

Let a_1, b_1, a_2 and b_2 be the four oriented arcs in the partial configuration (see Figure 5). A closed curve from the partial configuration will be labelled by the arcs being used and the number of twists we make around η when we walk along that curve. For instance, $a_1 \eta^2 b_1^{-1} b_2$ is the closed curve depicted on Figure 7.
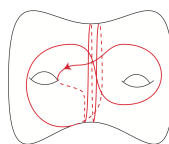

Figure 7. The curve $a_1 \eta^2 b_1^{-1} b_2$

As we are dealing with non oriented collections, the labeling of curves is defined up to cyclic permutation and reversing. For example, $a_1 \eta^2 b_1^{-1} b_2$ and $a_1^{-1} b_2^{-1} b_1 \eta^{-2}$ are labels of the same curve.

Intersection of arcs in an annulus

As we said above, the geometric intersection of a one-faced collection in Σ_2 is completely determined by the intersection of arcs in an annulus. Here, the intersection number is computed over the homotopy class of arcs with fixed end points. Now, let λ be a simple oriented arc joining the two boundaries of A_η. Cutting along λ, we obtain a rectangle with two opposite sides identified. Let X and Y be two points in the boundary components of A_η. An oriented arc from X to Y will be denoted by $\overrightarrow{XY}_p$ where $p \in \mathbb{Z}$ is the algebraic intersection between $\overrightarrow{XY}_p$ and λ.
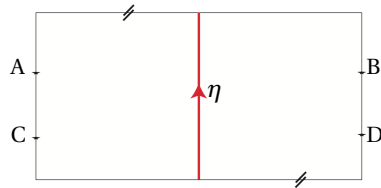

Figure 8. End-points in annulus.

Let A, B, C and D be four points in the boundaries of A_η as in Figure 8.

Lemma 23. *The following formulas give the intersection between two oriented arcs in A_η:*

- $i(\overrightarrow{AB}_p, \overrightarrow{CD}_q) = i(\overrightarrow{BA}_p, \overrightarrow{DC}_q) = |p - q|$
- $i(\overrightarrow{AB}_p, \overrightarrow{DC}_q) = i(\overrightarrow{BA}_p, \overrightarrow{CD}_q) = |p + q|$
- $i(\overrightarrow{AD}_p, \overrightarrow{CB}_q) = i(\overrightarrow{DA}_p, \overrightarrow{BC}_q) = |p - q - 1|$
- $i(\overrightarrow{AD}_p, \overrightarrow{BC}_q) = |p + q - 1|$
- $i(\overrightarrow{DA}_p, \overrightarrow{CB}_q) = |q + p + 1|$

Proof. Up to the Dehn twist τ_η^{-q} on the configuration of the arcs, one can assume that q is equal to 0 in all cases, meaning that one of the arc is untwisted.

Therefore, we have:

$$i(\overrightarrow{AB}_p, \overrightarrow{CD}_q) = i(\overrightarrow{AB}_{p'}, \overrightarrow{CD}) = |p'|$$

with

$$\overrightarrow{AB}_{p'} = \tau_\eta^{-q}(\overrightarrow{AB}_p).$$

Moreover, $p' = p - q$. Hence, we obtain the result.

Again, for the second formula, we have:

$$i(\overrightarrow{AB}_p, \overrightarrow{DC}_q) = i(\overrightarrow{AB}_{p'}, \overrightarrow{CD}) = |p'|$$

and $p' = p + q$. The difference between the first two cases shows how crucial the orientation is for the computing of intersection.

We next address the third case, and the remaining are handled in a similar way. We still have that

$$i(\overrightarrow{AD}_p, \overrightarrow{CB}_q) = i(\overrightarrow{AD}_{p'}, \overrightarrow{CB}) = |p' - 1|, \quad p' = p - q.$$

The appearance of -1 in this case comes from the cross configuration of the extremities. $\qquad \square$

List of one-faced collections with dual unit ball in the cube $[-1, 1]^4$

Now, we are able to count all one-faced collections whose dual unit balls are sub-polytopes of the cube $[-1, 1]^4$. Before that, we define some diffeomorphisms that will be useful for the counting.

If γ is an oriented simple closed curve on Σ_2, we recall that τ_γ is the right-handed Dehn twist along γ.
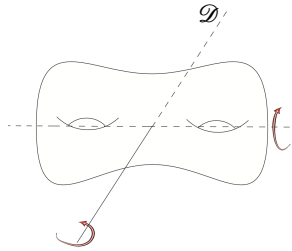
Figure 9. Rotations R_1 and R_2

Let R_1 (respectively R_2) be the rotation of angle π along the axis $\mathscr{D}$ (respectively the horizontal axis) as depicted in Figure 9. The diffeomorphism R_1 (respectively R_2) is an involution and it maps α_1 to α_2, β_1 to β_2 and η to η^{-1} (respectively α_i to α_i^{-1}, β_i to β_i^{-1} and η to η).

We recall that α_i and β_i can be interchanged by a diffeomorphism. More precisely, there is a diffeomorphism sending α_i to β_i and β_i to α_i^{-1}. This fact implies that in the writing of the label of the curves, a_i can be replaced by b_i and b_i by a_i^{-1}; we call this operation *interchanging*.

Definition 24. *Let Γ be a collection of closed curves on Σ_2. A cycle γ in Γ (Γ seen as a graph on Σ_2) is separating if $\Sigma_2 - \gamma$ has more than one component.*

The following lemma gives a necessary condition for a collection to be one-faced.

Lemma 25. *If Γ is a one-faced collection, then Γ does not contain a separating cycle.*

Proof. Assume that Γ contains a separating cycle γ, then $\Sigma_2 - \gamma$ has at least two connected components. In this case, Γ has more than one disc in its complement. So if Γ is one-faced, it does not contain a separating cycle. $\square$

Now, we can state the main result of this section which is an elaborate form of Theorem 4.

Theorem 26 (Orbits of one-faced collections). *If Γ is a one-faced collection on Σ_2 with dual unit ball in the cube $[-1,1]^4$, then Γ has at most three closed curves. Moreover, up to diffeomorphism,*

- *if Γ is made of three closed curves, then $\Gamma = \{a_1, a_2, b_1 b_2^{-1}\}$*
- *if Γ is made of two closed curves, then*

$$\Gamma = \{a_1 a_2^{-1}, b_1 b_2 \eta\} \quad or \quad \Gamma = \{a_1, b_1 b_2 \eta a_2\}$$

- *if Γ is made of one closed curve, then $\Gamma = \{a_1 a_2^{-1} b_1^{-1} b_2 \eta\}$*

Proof. If Γ is one-faced, then $i(\Gamma, \Gamma) = 3$ (this comes from an Euler characteristic argument; cf. Section 3).

Now, if Γ has at least four closed curves, then the arcs $a_i, b_i (i = 1, 2)$ belong to four different closed curves $\alpha_i \eta^{p_i}, \beta_i \eta^{q_i}$; otherwise Γ would contain a separating cycle. Therefore, $i(\Gamma, \eta) = 0$ which is absurd as Γ is filling. So, if Γ is one-faced $|\Gamma| \leq 3$.

Case 1. If $|\Gamma| = 3$, then two arcs of the partial configuration belong to the same closed curve and the other two belong to two different closed curves. Moreover, as Γ is filling, the two arcs contained in the same closed curve are in different handles. As one can interchange a_i and b_i, we can assume that the curve containing two arcs is $\gamma := b_1 \eta^p b_2^{-1} \eta^q$; the other curves being $\alpha_1 \eta^r$ and $\alpha_2 \eta^s$. Since Γ is one-faced, it does not contain a separating cycle meaning that $r = s = 0$. Up to a Dehn twist along η, one can take p=0; therefore, $\gamma = b_1 b_2^{-1} \eta^q$. The fact that $i(\Gamma, \Gamma) = 3$ implies that

$$i(\gamma, \gamma) = 1.$$

By Lemma 23 $i(\gamma, \gamma) = |q + 1| = 1$. We obtain $q = 0$ or $q = -2$ and one can check that $\Gamma_1 = \{a_1, a_2, a_1 a_2^{-1}\}$ and $\Gamma_2 = \{a_1, a_2, b_1 b_2^{-1} \eta^{-2}\}$ are in the same orbit under the mapping class group action.
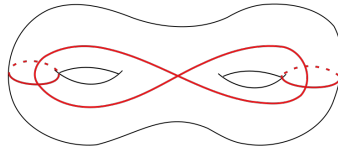

Figure 10. One-faced collection with three curves

Case 2. If $|\Gamma| = 2$, then one of the curves of Γ is simple. Otherwise if the two curves are not simple, one can smooth intersection points of one of the curves in Γ (let g_1 be that curve) such that each smoothing separates g_1 in to two components. We obtain at least two simple curves λ_j, $j = 1, \ldots, n$. The curves λ_j, as they are all parallels to g_1, intersect g_2. Then

$$i(\Gamma, \Gamma) \geq i(g_1, g_1) + i(g_2, g_2) + \sum_j i(\lambda_j, g_2) > 3$$

which is absurd since Γ is one-faced. Therefore, one of the two curves is simple, say g_1. Moreover, Σ_2 does not admit a filling pair [1] (i.e a one faced-collection made of two simple closed curves). So, g_2 is non simple.

Up to diffeomorphism (interchanging and rotations), one can assume that a_1 is contained in g_1.

Case 2.1. If g_1 does not contain another arc, then

$$g_2 = b_1 b_2 \eta^p a^\varepsilon$$

with $\varepsilon = \pm 1$. In this case,

$$i(\Gamma, \Gamma) = i(g_1, g_2) + i(g_2, g_2), \quad i(g_1, g_2) = 1.$$

This implies that $i(g_2, g_2) = 2$. The solution of this equation is $p = 1$ and $\varepsilon = 1$.

So,

$$\Gamma = \{a_1, b_1 b_2 \eta a_2\}$$

which indeed is a one-faced collection (see Figure 11).

Case 2.2. If g_1 contains an arc other than a_1, that arc cannot be in the same handle as a_1 (otherwise, Γ is not filling). Up to interchanging, we can assume that

$$g_1 = a_1 \eta^p a_2^{-1} \eta^{-p}$$

and again by applying a Dehn twist around η, one can take $g_1 = a_1 a_2^{-1}$ and $g_2 = b_1 \eta^p b_2^\varepsilon \eta^q$ with $\varepsilon = \pm 1$. Moreover,

$$i(\Gamma, \Gamma) = i(g_1, g_2) + i(g_2, g_2).$$

We have $i(\alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2) \equiv i(g_1, g_2) \mod 2$ since $\alpha_1 \cup \alpha_2$ (respectively $\beta_1 \cup \beta_2$) is homologous to g_1 (respectively g_2). It follows that

$$i(g_1, g_2) = 2, \quad i(g_1, g_2) = 1.$$

Case 2.2.1. If $\varepsilon = -1$, by applying the formulas of Lemma 23, we have:

$$i(g_2, g_2) = |p + q + 1|, \quad i(g_1, g_2) = |p| + |q| + |q + 1| + |p + 1|.$$

The solution of the equations $i(g_2, g_2) = 1$ and $i(g_1, g_2) = 2$ are $\{p = 0, q = 0\}$ and $\{p = -1, q = -1\}$. The two collections obtained are not filling since $i(b_1 b_2, \Gamma) = 0$.

Case 2.2.2. If $\varepsilon = 1$, then $i(g_2, g_2) = |p - q|$ and $i(g_1, g_2) = 2(|p| + |q|)$. The solution of the equations $i(g_2, g_2) = 1$ and $i(g_1, g_2) = 2$ are $\{p = 0, q = \pm 1\}$ and $\{p = \pm 1, q = 0\}$.

We check that $\Gamma_1 = \{a_1 a_2^{-1}, b_1 \eta^{\pm 1} b_2\}$ and $\Gamma_2 = \{a_1 a_2^{-1}, b_1 b_2 \eta^{\pm 1}\}$ are one-faced (here, Γ_1 and Γ_2 are well-defined up to the power of η). The rotation R_1 maps elements Γ_1 to elements of Γ_2. The collection $\{a_1 a_2^{-1}, b_1 b_2 \eta\}$ is the mirror image of $\{a_1 a_2, b_1 b_2 \eta^{-1}\}$.

Hence, up to diffeomorphism, we have two one-faced collections with two curves (see Figure 11); namely

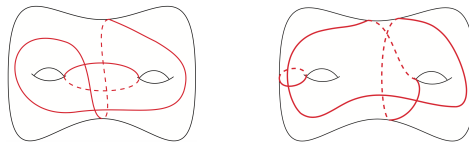$$\Gamma_1 = \{a_1 a_2^{-1}, b_1 b_2 \eta\}, \quad \Gamma_2 = \{a_1, b_1 b_2 \eta a_2\}.$$

Figure 11. One-faced collections with two curves

Case 3. If Γ has only one curve g, then up to diffeomorphism (interchangeability and rotations)

$$g = a_1 a_2^{-1} \eta^p b_1^{\varepsilon_1} \eta^q b_2^{\varepsilon_2} \eta^r$$

or

$$g = a_1 \eta^p b_1^{\varepsilon_1} a_2^{-1} \eta^q b_2^{\varepsilon_2} \eta^r,$$

where $\varepsilon_i = \pm 1$.

If $g = a_1 \eta^p b_1^{\varepsilon_1} a_2^{-1} \eta^q b_2^{\varepsilon_2} \eta^r$, we check that Γ is either not filling, or filling with more than one disk in its complement.

For $g(\varepsilon_1, \varepsilon_2) = a_1 a_2^{-1} \eta^p b_1^{\varepsilon_1} \eta^q b_2^{\varepsilon_2} \eta^r$, R_1 sends $g(\varepsilon_1, \varepsilon_2)$ to $g(-\varepsilon_1, -\varepsilon_2)$. If we start with $g = a_1 a_2^{-1} \eta^p b_1^{-1} \eta^q b_2 \eta^r$ and we change a_1 to b_1 by a diffeomorphism (that diffeomorphism will map b_1 to a_1^{-1}), g gets mapped to

$$g' = b_1 a_2^{-1} \eta^p a_1 \eta^q b_2 \eta^r.$$

Now, if we reverse the orientation of g' starting at a_1, we have

$$g' = a_1^{-1} \eta^p a_2 b_1^{-1} \eta^r b_2^{-1} \eta^q, \quad R_2(g') = a_1 \eta^p a_2^{-1} b_1 \eta^r b_2 \eta^q.$$

Finally, $\tau_{\eta^{-p}} \circ R_2(g') = a_1 a_2^{-1} \eta^p b_1 \eta^q b_2 \eta^r$. Hence, up to diffeomorphism, one can look at the case where

$$\varepsilon_1 = 1; \quad \varepsilon_2 = -1$$

In this case we have

$$i(\Gamma, \Gamma) = |p| + |q| + |r| + |p + q + 1| + |p - r| + |q + r + 1|.$$

The equation $i(\Gamma, \Gamma) = 3$ has two solutions

$$\{p = 0, q = 0, r = -1\}, \quad \{p = -1, q = 0, r = 0\}.$$

The collections $\Gamma_1 = \{a_1 a_2^{-1} b_1 b_2^{-1} \eta^{-1}\}$ and $\Gamma_2 = \{a_1 a_2^{-1} \eta^{-1} b_1 b_2\}$ are one-faced. Moreover $R_1(\Gamma_1) = \Gamma_2$. Therefore, up to diffeomorphism, we have one one-faced collection with one curve (see Figure 12), namely

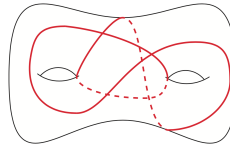$$\Gamma = \{a_1 a_2^{-1} b_1 b_2^{-1} \eta\}. \qquad \square$$

Figure 12. One-faced collection made of one curve

Proof of Theorem 3. By Lemma 11, we can restrict our attention to minimal collections. By Corollary 20, if $P \in \mathscr{P}_8$ is the dual unit ball associated to a collection Γ, then N_Γ is bounded from below by $N_{\Gamma'}$ where Γ' is one-faced. This implies that $B_{N_{\Gamma'}^*}$ has also eight vectors and has non-empty interior.

The collection Γ' is one of the collections in Theorem 26. We check that dual unit balls of those collections are not in $\mathscr{P}_8$ (see bellow for their dual unit balls); which finally proves that elements of $\mathscr{P}_8$ are not realizable.

Computation of dual unit balls. We compute the dual unit ball of an intersection norm by evaluating all Eulerian co-orientations on the canonical homology basis [2]. Doing so, we obtain the vertices (of the dual unit balls) below:

$$\{a_1, a_2, b_1 b_2^{-1}\} \mapsto [-1, 1]^4$$
$$\{a_1, b_1 b_2 \eta a_2\} \mapsto \{\pm(1, 1, -1, 1); \pm(1, 1, -1, -1); \pm(1, -1, -1, 1); \pm(1, -1, -1, -1);$$
$$\pm(1, -1, 1, 1); \pm(1, -1, 1, -1)\}$$
$$\{a_1 a_2^{-1}, b_1 b_2 \eta\} \mapsto \{\pm(1, 1, -1, 1); \pm(1, -1, -1, -1); \pm(1, -1, 1, -1); \pm(1, 1, 1, 1); \pm(1, -1, -1, 1)\}$$
$$\{a_1 a_2^{-1} b_1^{-1} b_2 \eta\} \mapsto \{\pm(1, 1, -1, -1); \pm(1, -1, -1, 1); \pm(1, -1, 1, 1); \pm(1, 1, 1, 1); \pm(-1, 1, 1, 1)\}.$$

The first collection has the whole unit cube as dual unit ball; the others three have dual unit balls with at least ten vectors. $\qquad \square$

We finish this article by giving a perspective in dimension greater than 4. Let $\mathscr{P}_{4g}$ be the set of symmetric non degenerate sub-polytopes of $[-1, 1]^{2g}$ with $4g$ vertices. Since we are in dimension $2g$, elements of $\mathscr{P}_{4g}$ are minimal (in term of number of vertices) among symmetric non degenerate sub-polytopes of $[-1, 1]^{2g}$.

Lemma 27. *If Γ' is a one-faced collection obtained by merging faces of a filling collection Γ, then $N_{\Gamma'} < N_\Gamma$.*

Proof. For the proof, it is enough to show that if Γ is two-faced (F_1 and F_2 in $\Sigma - \Gamma$) and reduced to Γ', then $N_{\Gamma'} < N_\Gamma$.

Since Γ reduced to a one-faced collection, there is a vertex v of Γ which is adjacent to F_1 and F_2 (see Figure 13).

Since Σ_g is a (non singular) surface, there is an edge x in F_1 whose identified side $\bar{x}$ is in F_2. Let α be the oriented closed curve on Σ_g from x to $\bar{x}$ passing by v. The curve α is non trivial in homology since $i(\Gamma, \alpha) = 3$. Moreover, α is Γ-minimizing in its homology class because α is not
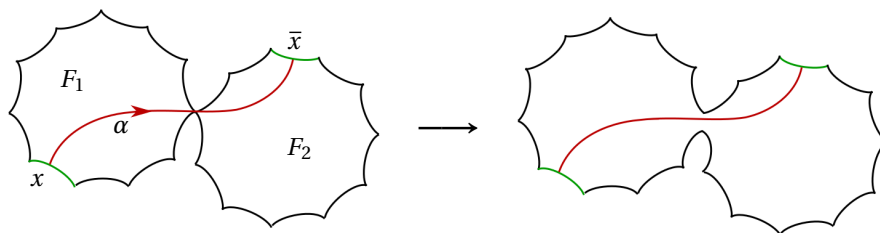
Figure 13.

homologus to any curve intersecting Γ once (which is equivalent to say that the curve is defined by an arc in F_1 or in F_2). Hence, $N_\Gamma([\alpha]) = 3$

After reducing F_1 and F_2 to a single face, we obtain Γ' and $i(\Gamma', \alpha) = 1$; that is $N_{\Gamma'}([\alpha]) = 1$.

So, we obtain the inequality $N_{\Gamma'} < N_\Gamma$.   □

Proof of Proposition 5. Let $P \in \mathscr{P}_{4g}$ and let Γ be a filling collection on Σ_g such that the dual unit ball of N_Γ is equal to P. If Γ has more than two faces in its complement, then by Lemma 27 $N_\Gamma > N_{\Gamma'}$; where Γ' is a one-faced collection. It follows that the dual unit ball of $N_{\Gamma'}$ has less than $4g$ vertices, which is absurd. So, Γ is a one-faced collection.   □

Theorem 3 makes the general problem of realization for intersection norms a *decision problem*: given a symmetric non degenerate polytope P with congruent mod 2 vertices, state whether or not P is the dual unit ball of an intersection norm. On can restricted to minimal polytopes in dimension $2g$: those non degenerate sub-polytopes of $[-1,1]^{2g}$ with $2g$ vertices, and compare their numbers (up to the action of the linear symplectomorphisms which fix $[-1,1]^{2g}$) with the number of one-faced collections in Σ_g.

Acknowledgments

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