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Contact unimodal map germs from the plane to the plane

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Abstract. In this article, we correct the classification of unimodal map germs from the plane to the plane of Boardman symbol $(2,2)$ given by Dimca and Gibson. Also, we characterize this classification of unimodal map germs in terms of certain invariants. Moreover, on the basis of this characterization we present an algorithm to compute the type of unimodal map germs of the Boardman symbol $(2,2)$ without computing the normal form and give its implementation in the computer algebra system Singular [8].

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1. Introduction

After the classical work of Whitney [14] the singularities of maps from plane to the plane became a subject of great interest due to its nice visual interpretations. Different authors concentrated to study the degenerate singularities of low codimension contained in families of maps from the plane to the plane [2, 6, 7]. Rieger gave the classification of all $\mathscr{A}-$simple and $\mathscr{A}-$unimodal map germs from the plane to the plane of corank at most 1 in [12, 13]. These classifications are characterized in [3, 5, 11] in terms of certain invariants. In [9], Dimca and Gibson gave the classification of all map germs from the plane to the plane of Boardman symbol $(2,1)$ and $(2,2)$ with respect to $\mathscr{K}$-equivalence. A classifier for the classification of Dimca and Gibson, in the case of Boardman symbol $(2,1)$ is given in [4].

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Let $\mathbb{C}$ be the field of complex numbers and $A(2,2) = \langle x, y\rangle\mathbb{C}[[x, y]]^2$. Let $\mathscr{K} = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]]) \times \mathrm{Gl}_2(\mathbb{C}[[x, y]])$ acting on $A(2,2)$ by

$$((\varphi, M), f) \mapsto \varphi^{-1} \circ M f.$$

This is equivalent to consider $A(2,2)$ as the set of map germs $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ under the action of the group $\mathscr{K} = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^2,0) \times \mathrm{Gl}_2(\mathbb{C}^2,0)$.

The map germs $f, g \in A(2,2)$ are called $\mathscr{K}$-equivalent ($f \sim_{\mathscr{K}} g$) if they are in the same orbit under the action of $\mathscr{K}$. In the classification of map germs with respect to the action of the group $\mathscr{K}$ the tangent spaces to the orbit under the action of this group and their codimension play an important role (cf. [1]). Given $f \in A(2,2)$ the orbit map $\theta_f : \mathscr{K} \to A(2,2)$ is defined by $\theta_f(\varphi, M) = \varphi^{-1} \circ M f$. The corresponding tangent map has as image the tangent space to the orbit at $f = (f_1, f_2)$:

$$T_{\theta_f,\mathrm{id}} = \langle x, y\rangle \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathbb{C}[[x,y]]} + \langle f_1, f_2\rangle\mathbb{C}[[x, y]]^2.$$

In this article we characterize the classification of maps from the plane to the plane of Boardman symbol $(2,2)$ given by Dimca and Gibson, in terms of invariants and on the basis of this characterization we give the implementation of this classification in the computer algebra system Singular [8, 10].

We use the following invariants for our characterization:

Let $f \in A(2,2)$ be a map germ then the codimension of the tangent space, $c = \dim_{\mathbb{C}} \frac{\langle x,y\rangle\mathbb{C}[[x,y]]}{T_{\theta_f,\mathrm{id}}}$, a number closely connected with the order of k-determinacy, $\sigma = \min\{s : \langle x, y\rangle^s\mathbb{C}[[x, y]]^2 \subset T_{\theta_f,\mathrm{id}}\}$, the Milnor number, $\mu = \dim \frac{\mathbb{C}[[x,y]]}{f} - 1$ and the double fold number d, which is the number of times 2 occurs in the sequence $\dim \frac{\langle x,y\rangle^k}{I_f \cap \langle x,y\rangle^k + \langle x,y\rangle^{k+1}}$, are the invariants used in the classification. An other important invariant is Boardman symbol $\sum f$, which is define as follows:

Definition 1. *Let $I = \langle f_1, \dots, f_p\rangle$ be an ideal in $\mathbb{C}[[x_1, \dots, x_n]]$. Let the s-th Jacobian extension of I to be the ideal $\Delta^s(I) = I + I_1$, where I_1 is the ideal in $\mathbb{C}[[x_1, \dots, x_n]]$ generated by all $n-s+1 \times n-s+1$ minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{j=1\dots n}^{i=1\dots p}$. Then one has a sequence of inclusions*

$$I = \Delta^0(I) \subset \Delta^1(I) \subset \cdots \subset \Delta^n(I).$$

If I is a proper ideal then the critical Jacobian extension of I is the last ideal $\Delta^{i_1}(I)$ in the sequence which is proper. This ideal $\Delta^{i_1}(I)$ has in turn its critical Jacobian extension $\Delta^{i_2}(\Delta^{i_1}(I))$ and so on. The sequence of integers $(i_1, i_2, \dots)$ obtained in this way is called Boardman symbol of the ideal I.

Definition 2. *The Boardman symbol of a map germ $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is the Boardman symbol of the ideal I_f generated by the components $f_1, f_2, \dots, f_p$ of f.*

2. Classification of unimodal map germs from the plane to the plane

Table 1 contains all map germs from the plane to the plane of Boardman symbol $(2,2)$ given by Dimca and Gibson [9].

In the following proposition, we correct the classification of Dimca and Gibson by showing that Y_1 is contact equivalent to Y_2, Z_0 is contact equivalent to Z_2 and Q_4 is contact equivalent to X_1.

Proposition 3. *The following hold:*

(1) $(x^3 + y^5 + xy^4, x^2y) \sim_{\mathscr{K}} (x^3 + y^5, x^2y + y^5)$.
(2) $(x^3 + 12xy^3, x^2y + y^4) \sim_{\mathscr{K}} (x^3 + 12xy^3 + y^6, x^2y + y^4)$.
(3) $(x^3 + xy^3 + y^4, x^2y + y^4) \sim_{\mathscr{K}} (x^3 + y^4, x^2y + y^4)$.

Table 1.

Type	Normal form	Conditions
$\widetilde{H}$	$(x^3+3x^2y, 3xy^2+\lambda y^3)$	$\lambda \neq 0,1,9$
I	(x^3, xy^2+y^3)	–
$\mathfrak{I}$	(x^3, y^3)	–
K_p	$(x^3+x^2y+y^p, xy^2)$	$p \geq 4$
$L_{p,q}$	(x^2y+y^p, xy^2+x^q)	$p \geq q \geq 4$
M_p	(x^3+y^p, xy^2)	$p \geq 4$
$\widetilde{N}$	$(x^3+\lambda xy^3, x^2y+y^4)$	$\lambda \neq 1,12$
P_p	$(x^3+xy^3+xy^p, x^2y+y^4)$	$p \geq 4$
Q_p	$(x^3+xy^3+y^p, x^2y+y^4)$	$p \geq 4$
R_p	(x^3+xy^3, x^2y+y^p)	$p \geq 5$
X_0	(x^3+y^4, x^2y)	–
X_1	(x^3+y^4, x^2y+y^4)	–
Y_0	(x^3+y^5, x^2y)	–
Y_1	$(x^3+y^5+xy^4, x^2y)$	–
Y_2	(x^3+y^5, x^2y+y^5)	–
Z_0	(x^3+12xy^3, x^2y+y^4)	–
Z_1	$(x^3+12xy^3+y^5, x^2y+y^4)$	–
Z_2	$(x^3+12xy^3+y^6, x^2y+y^4)$	–

Proof. **(1).** Let $f=(x^3+y^5+xy^4, x^2y)$ and $g=(x^3+y^5, x^2y+y^5)$. Note that $I_g=\langle x^3+y^5, x^2y-x^3\rangle$. Then the transformation

$$x \to x, \quad y \to x+y,$$

transform I_g into $\langle x^3+y^5+5xy^4+x^5, x^2y\rangle$. Since

$$x^3+y^5+5xy^4+x^5-x^2(x^3+y^5+5xy^4+x^5)=x^3+y^5+5xy^4-x^7,$$

therefore $I_g \sim_{\mathscr{K}} \langle x^3+y^5+5xy^4-x^7, x^2y\rangle$. Iterating in a similar way, we get

$$I_g \sim_{\mathscr{K}} \langle x^3+y^5+5xy^4, x^2y\rangle.$$

Applying $\mathbb{C}^*$-action on $\langle x^3+y^5+5xy^4, x^2y\rangle$, we get

$$I_g \sim_{\mathscr{K}} \langle x^3+y^5+xy^4, x^2y\rangle = I_f.$$

This gives $f \sim_{\mathscr{K}} g$.

(2). Let $f=(x^3+12xy^3, x^2y+y^4)$ and $g=(x^3+12xy^3+y^6, x^2y+y^4)$. We find g in the deformation $f+t(y^6,0)$ of f. Computing the module of first order infinitesimal deformations and the tangent space of f of the orbit of f under the action of the contact group.

$$T^1=\frac{\mathbb{C}[[x,y]]^2}{\left\langle (x^3+12xy^3,0),(0,x^3+12xy^3),(x^2y+y^4,0),(0,x^2y+y^4),\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle}.$$

We find that $(y^6,0)$ is zero in T^1, mainly it is easy to check that

$$\begin{aligned}(y^6,0) &= \frac{1}{9}y^3\frac{\partial f}{\partial x} - \frac{2}{99}y(0,x^3+12xy^3) - \frac{1}{3}y^2(x^2y+y^4,0) + \frac{2}{99}x(0,x^2y+y^4);\\ &= \frac{1}{9}y^3\frac{\partial f}{\partial x} + f\begin{pmatrix} 0 & -\frac{2}{99}y\\ -\frac{1}{3}y^2 & \frac{2}{99}x\end{pmatrix}.\end{aligned}$$

Define $\varphi:\mathbb{C}[[x,y]]\to\mathbb{C}[[x,y]]$ by $\varphi(x)=x+\frac{1}{9}y^3$, $\varphi(y)=y$ and $M=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)-\begin{pmatrix}0 & -\frac{2}{99}y\\ -\frac{1}{3}y^2 & \frac{2}{99}x\end{pmatrix}$ then

$$\varphi(f) = (f+(y^6,0))M + \left(\frac{1}{729}y^9+\frac{1}{27}xy^6, -\frac{7}{891}y^7\right).$$

Note that f is homogeneous of degree 9 with respect to $\deg((x)=3$, $\deg((y)=2$, $\deg((1,0)=0$ and $\deg((0,1)=1$. Using this, we obtain $\varphi(f)=gM+(\frac{1}{27}xy^6,-\frac{7}{891}y^7)+$ terms of higher degree. Since all homogeneous elements of degree $\geq 12=\deg(y^6,0)$ are zero in T^1 and $\deg((\frac{1}{27}xy^6,-\frac{7}{891}y^7)=15$, we can repeat this procedure. Assume we have an automorphism $\varphi_k:\mathbb{C}[[x,y]]\to\mathbb{C}[[x,y]]$, a matrix $M_k\in \mathrm{GL}_2(\mathbb{C}[[x,y]])$ such that $\varphi_k(f)=gM_k+u_k+$ terms of higher degree, where u_k is homogeneous and $\deg((u_k)=k\geq 15$, then we can write

$$\begin{aligned}u_k &= a_k\frac{\partial f}{\partial x}+b_k\frac{\partial f}{\partial y}+c_k(x^3+12xy^3,0)+d_k(0,x^3+12xy^3)+e_k(x^2y+y^4,0)+f_k(0,x^2y+y^4)\\ &= a_k\frac{\partial f}{\partial x}+b_k\frac{\partial f}{\partial y}+f\begin{pmatrix}c_k & d_k\\ e_k & f_k\end{pmatrix},\end{aligned}$$

where a_k,b_k,c_k,d_k,e_k,f_k are homogeneous and $\deg((a_k)=k-6$, $\deg((b_k)=k-7$, $\deg((c_k)=\deg((d_k)=\deg((e_k)=\deg((f_k)=k-9$. We define $\varphi_{k+1}(x)=\varphi_k(x)-a_k$, $\varphi_{k+1}(y)=\varphi_k(y)-b_k$, $M_{k+1}=M_k-\left(\begin{smallmatrix}c_k & d_k\\ e_k & f_k\end{smallmatrix}\right)$ and obtain $\varphi_{k+1}(f)=gM_{k+1}+$ terms of degree $\geq k+1$. Induction implies that $f\sim_{\mathscr{K}} g$.

(3). The following Singular computation shows that Q_4 is contact equivalent to X_1.

```
ring R=0,(x,y),ds;

poly p=x3+xy3+y4;
poly q=x2y+y4;
poly p1=x3+y4;
poly q1=x2y+y4;
matrix G[2][1]=p1,q1;

matrix T[2][2]=16777216/244140625,301989888/6103515625*y,
-4194304/244140625,4194304/48828125;

map phi=R,256/390625*x^2-2048/234375*x*y+2048/15625*y^2+256/625*x,
-128/390625*x^2+1664/234375*x*y-512/46875*y^2-64/625*x+64/125*y;

matrix N=T*G;

poly p2=jet(phi(p)-N[1,1],4);
poly q2=jet(phi(q)-N[2,1],4);
p2;q2;
0
0
```

The computation shows that (x^3+y^4, x^2y+y^4), denoted by X_1 is contact equivalent to $\overline{g} = (x^3+xy^3+y^4, x^2y+y^4)+$ terms of order ≥ 5. $\overline{g}$ has 4-jet $(x^3+xy^3+y^4, x^2y+y^4)$, denoted by Q_4. In the paper of Dimca and Gibson [9] it is proved (Lemma 3.6) that $\overline{g}$ is contact equivalent to its 4-jet $(x^3+xy^3+y^4, x^2y+y^4)$. This implies that X_1 is contact equivalent to Q_4. □

Table 2 contains the correct classification of unimodal map germs from the plane to the plane of Boardman symbol $(2,2)$ given by Dimca and Gibson.

Table 2.

Type	Normal form	Conditions
$\widetilde{H}$	$(x^3+3x^2y, 3xy^2+\lambda y^3)$	$\lambda \neq 0,1,9$
I	(x^3, xy^2+y^3)	–
$\Im$	(x^3, y^3)	–
K_p	$(x^3+x^2y+y^p, xy^2)$	$p \geq 4$
$L_{p,q}$	(x^2y+y^p, xy^2+x^q)	$p \geq q \geq 4$
M_p	(x^3+y^p, xy^2)	$p \geq 4$
$\widetilde{N}$	$(x^3+\lambda xy^3, x^2y+y^4)$	$\lambda \neq 1,12$
P_p	$(x^3+xy^3+xy^p, x^2y+y^4)$	$p \geq 4$
Q_p	$(x^3+xy^3+y^p, x^2y+y^4)$	$p \geq 4$
R_p	(x^3+xy^3, x^2y+y^p)	$p \geq 5$
X_0	(x^3+y^4, x^2y)	–
Y_0	(x^3+y^5, x^2y)	–
Y_1	$(x^3+y^5+xy^4, x^2y)$	–
Z_0	(x^3+12xy^3, x^2y+y^4)	–
Z_1	$(x^3+12xy^3+y^5, x^2y+y^4)$	–

3. Characterization of Unimodal Map Germs by Invariants

In this section we characterize all unimodal map germs from the plane to the plane of Boardman symbol $(2,2)$ in terms of certain invariants.

Proposition 4. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $\sum f = (2,2)$ and H_p denote the Hilbert polynomial of $j^3(f)$. Then*

(1) *if $H_p = 0$ then f is unimodal of type $\widetilde{H}$ or I or $\Im$;*
(2) *if $H_p = 1$ then f is unimodal of type K_p or M_p;*
(3) *if $H_p = 2$ then f is unimodal of type $L_{p,q}$ or f is one of the type with $j^3(f) = (x^3, x^2y)$;*
(4) *if $H_p \neq 0,1,2$ then contact modality of f is ≥ 2.*

Proof. From the classification of the pencils of binary cubic forms and the classification of Dimca and Gibson [9] it follows that f is unimodal if and only if $H_p \in \{0,1,2\}$.

(1). If $H_p = 0$ then we obtain that $\widetilde{H}$, I and $\Im$ have these properties. Now if $c = 11$ and $\sqrt{\det(J(j^3(f)))}$ is the intersection of a smooth curve and D_4-singularity then f is of type $\widetilde{H}$ and if $c = 11$ and $\sqrt{\det(J(j^3(f)))}$ is the intersection of a smooth curve and A_1-singularity then f is of type I and if $c = 12$ and $\sqrt{\det(J(j^3(f)))}$ is the intersection of two smooth curves then f is of type $\Im$.

Table 3. The Invariants used for the Classification.

Type	c	μ	d	σ
$\widetilde{H}$	11	8	1	4
I	11	8	1	4
$\Im$	12	8	1	4
K_p	$p+7$	$p+5$	1	p
$L_{p,q}$	$p+q+4$	$p+q+2$	$q-2$	p
M_p	$p+8$	$p+5$	1	p
$\widetilde{N}$	15	11	2	5
P_p	$2p+8$	$2p+5$	$p-1$	$p+2$
Q_p	$2p+5$	$2p+2$	$p-2$	p
R_p	$p+10$	$p+7$	2	p
X_0	14	10	2	5
Y_0	17	12	3	6
Y_1	16	12	3	5
Z_0	16	11	2	6
Z_1	15	11	2	5

(2). If $H_p = 1$ then we obtain that K_p and M_p have these properties. Now if $c = \sigma + d + 6$ and $\sqrt{\det(J(j^3(f)))}$ is the intersection of three smooth curves then f is of type K_p and if $c = \sigma + d + 7$ and $\sqrt{\det(J(j^3(f)))}$ is the intersection of two smooth curves then f is of type M_p.

(3). If $H_p = 2$ then if $j^3(f)$ has 3 primary ideals then f is of type $L_{p,q}$ with $\sigma = p$, $d = q-2$ and $c = \sigma + d + 6$ and if $j^3(f)$ has 2 primary ideals then f is one of the type with $j^3(f) = (x^3, x^2y)$.

(4). $\sum f = (2,2)$ implies that the modality is at least 1. So the statement follows from the classification of Dimca and Gibson. □

Proposition 5. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $j^3(f) = (x^3, x^2y)$. Then*

(1) *if $c \le 18$ then f is unimodal.*
(2) *if $c \ge 19$ then f is unimodal if and only if $f \sim_{\mathscr{K}} (x^3 + xy^3a(y) + y^q b(y), x^2y + y^r c(y))$ with $a(0) \neq 0$, $b(0) \neq 0$, $c(0) \neq 0$ and $q, r \ge 4$.*

Proof. We obtain (1) because of [9, Lemma 3.5 (ii)] and (2) is a consequence of the classification. From the list it follows the codimension is ≥ 19 only in this case. □

Proposition 6. *Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $j^3(f) = (x^3, x^2y)$. Then*

(1) *if $c = 13$ then f is unimodal of type Q_4;*
(2) *if $c = 14$ then f is unimodal of type X_0;*
(3) *if $c = 15$ then f is unimodal of type $\widetilde{N}$ or Z_1 or Q_5 or R_5;*
(4) *if $c = 16$ then f is unimodal of type Y_1 or Z_0 or P_4 or R_6;*
(5) *if $c = 17$ then f is unimodal of type Y_0 or Q_6 or R_7;*
(6) *if $c > 17$ and $f \sim_{\mathscr{K}} (x^3 + xy^3a(y) + y^q b(y), x^2y + y^r c(y))$ with $a(0) \neq 0$, $b(0) \neq 0$, $c(0) \neq 0$ and $q, r \ge 4$ then f is unimodal of type P_p, Q_p or R_p.*

Proof. (1) and (2) are clear.

(3). If $c = 15$ then we obtain that $\widetilde{N}$, Z_1, Q_5 and R_5 have these properties. If $\mu = 11$, $d = 2$ and $\sigma = 5$ then f is of type $\widetilde{N}$ or Z_1. To differentiate these two types, compute normal form up to order 4 to obtain λ, if $\lambda = 12$ then f is of type Z_1 and if $\lambda \neq 1, 12$ then f is of type $\widetilde{N}$. Moreover if $\mu = 12$, $d = 2$ and $\sigma = 5$ then f is of type R_5 and if $\mu = 12$, $d = 3$ and $\sigma = 5$ then f is of type Q_5.

(4). If $c = 16$ then we observe that Y_1, Z_0, P_4, R_6 have these properties. If $\mu = 11$, $d = 2$ and $\sigma = 6$ then f is of type Z_0 and if $\mu = 12$, $d = 3$ and $\sigma = 5$ then f is of type Y_1. Moreover if $\mu = 13$, $d = 2$ and $\sigma = 6$ then f is of type R_6 and if $\mu = 13$, $d = 3$ and $\sigma = 6$ then f is of type P_4.

(5). If $c = 17$ then we observe that Y_0, Q_6 and R_7 have these properties. If $\mu = 12$, $d = 3$ and $\sigma = 6$ then f is of type Y_0 and if $\mu = 14$, $d = 4$ and $\sigma = 6$ then f is of type Q_6 and if $\mu = 14$, $d = 2$ and $\sigma = 7$ then f is of type R_7.

(6). If $c > 17$ then we observe that P_p, Q_p and R_p have these properties. If c is even then f is of type P_p or R_p, if $d = 2$ then f is of type R_p with $\sigma = p$ and if $d > 2$ then f is of type P_p with $\sigma = p + 2$. Similarly if c is odd then f is of type Q_p or R_p, if $d = 2$ then f is of type R_p with $\sigma = p$ and if $d > 2$ then f is of type Q_p with $\sigma = p$. □

We use the following algorithm to differentiate the type of $\widetilde{N}$ and Z_1.

Algorithm 1 Normal form of $\widetilde{N}$ or Z_1

Require: Ideal $I = f, g$ and an integer $k > 0$.
Ensure: Ideal $I = h, l$ required normal form.

1: Define a map φ, $\varphi(x) = \sum_{1\leq i+j\leq k-2} a_{ij}x^i y^j, \varphi(y) = \sum_{1\leq i+j\leq k-2} b_{ij}x^i y^j$ with parameters up to the degree $k-2$.
2: Define a 2×2-matrix T with polynomial entries of degree $k-3$ and parameters as coefficients, i.e., $T = (t_{ij})$, $t_{ij} = \sum_{0\leq l+m\leq k-3} a_{ij,l,m}x^l y^m$, $a_{ij,l,m}$ are parameters.
3: Define the expected normal form (h_1, h_2) depending on parameters.
4: Define $J := \varphi(I) - T\binom{h_1}{h_2}$.
5: Let K be the ideal generated by all the coefficients of J with respect to x, y.
6: Compute the minimal associated primes of K.
7: Let $h := \det\big(a_{ij,0,0}\big)\cdot\big(a_{10}b_{01} - a_{01}b_{10}\big)$.
8: For the prime ideal $P \supseteq K$, compute $n = NF\big(h/P\big)$.
9: If $h \neq 0$, elminate in P the a_{ij}, b_{ij} and $a_{ij,l,m}$. The elimination just contains the conditions on the parameters of (h_1, h_2).
10: Use the conditions to produce the required normal form.

4. Singular Examples

We have implemented the Algorithm in the computer algebra system Singular [8]. Code can be downloaded from mathcity.org/junaid.

We give some examples.

```
ring R=0,(x,y),ds;

> ideal f=2x3+4x2y+2xy2+x10-10x9y+45x8y2-120x7y3+210x6y4-252x5y5+210x4y6
-120x3y7+45x2y8-10xy9+y10,x3-x2y-xy2+y3;
> contactMapgerms(f);
f is unimodal of type K_10

> ideal f=x3+3x2y+3xy2+y3+77x4-154x3y+154xy3-77y4,
x3+x2y-xy2-y3+x4-4x3y+6x2y2-4xy3+y4;
```

```
> contactMapgerms(f);
f is unimodal of type tild(N)

> ideal f=x3+3x2y+3xy2+y3+xy3+y4,x2y+2xy2+y3+y7;
> contactMapgerms(f);
f is unimodal of type R_7
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