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Probabilities / *Probabilités*

# A backward Itô–Ventzell formula with an application to stochastic interpolation

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**Abstract.** This Note and its extended version [7] present a novel backward Itô–Ventzell formula and an extension of the Aleksev–Gröbner interpolating formula to stochastic flows. We also present some natural spectral conditions that yield direct and simple proofs of time uniform estimates of the difference between the two stochastic flows when their drift and diffusion functions are not the same.

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## Version française abrégée

Cette Note et sa version étendue [7] présentent une nouvelle formule à rebours de type Itô–Ventzell (Theorème 1). Cette formule permet notamment de déduire une extension de la formule d’interpolation d’Aleksev–Gröbner aux flots stochastiques (Corollaire 2). Ces formules d’interpolation peuvent aussi s’obtenir par extension naturelle du calcul intégral stochastique bilatéral développé par Pardoux and Protter dans l’article [10] aux interpolations de flots stochastiques. Ces extensions sont développées en détails dans la version étendue [7]. Les approximations associées à cette approche offrent une variante de la construction originale de l’intégrale de Itô. Ces dernières sont décrites sur la base d’un échantillonnage temporel dans la formule (9).

Supposons donnés deux flots stochastiques  $X_{s,t}(x)$  et  $\bar{X}_{s,t}(x)$  associés à des équations (3) pour des champs de vitesse et diffusion  $(b_t, \sigma_t)$  et  $(\bar{b}_t, \bar{\sigma}_t)$  distincts. On notera par la suite  $a_t(x) := \sigma_t(x) \sigma_t(x)'$  et  $\bar{a}_t(x) := \bar{\sigma}_t(x) \bar{\sigma}_t(x)'$ .

Un flot interpolant naturel dans l’analyse de la différence entre ces deux flots stochastiques est donné le flot composé  $u \in [s, t] \mapsto X_{u,t} \circ \bar{X}_{s,u}$ . Comme il est précisé dans le Corollaire 3, la formule à rebours de type Itô–Ventzell décrite dans le Theorème 1, ainsi que les formules d’interpolations

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associées développées dans le Corollaire 2, nous permettent d’appliquer la règle différentielle de type Itô décrite dans la formule (12).

La formule d’interpolation (6) ainsi obtenue s’exprime en fonction du gradient et de la matrice Hessienne du flot de référence  $X_{s,t}$ . Cette formulation variationnelle permet d’analyser finement la distance entre les deux flots  $X_{s,t}$  et  $\bar{X}_{s,t}$  en fonctions des propriétés spectrales de ces flots. Le Théorème 4 présente des estimations uniformes par rapport au temps (16) entre ces deux flots stochastiques. La version étendue de cette Note [7] illustre ces formules d’interpolation en théorie des perturbations, ainsi que dans le cadre des diffusions en interaction et dans l’étude des approximations temporelles de processus stochastiques.

### 1. A generalized backward Itô–Ventzell formula

We represent the gradient  $\nabla f$  of a real valued function  $f$  of several variables as a column vector while the gradient  $\nabla F$  and the Hessian  $\nabla^2 F$  of a (column) vector valued function  $F$  as tensors of type (1, 1) and (2, 1). The transpose  $A'$  of a  $(p, q)$ -tensor  $A$  is the  $(q, p)$ -tensor with entries  $A'_{i,j} = A_{j,i}$  for any multiple indices  $i = (i_k)_{1 \leq k \leq p}$  and  $j = (j_l)_{1 \leq l \leq q}$ .

We denote by  $\lambda_{max}(S)$  the maximal eigenvalue of a symmetric matrix  $S$ , and by  $\rho(A) = \lambda_{max}((A + A')/2)$  the logarithmic norm of some matrix  $A$ .

We denote by  $\Omega := C(\mathbb{R}_+, \mathbb{R}^r)$  the set of continuous paths from  $\mathbb{R}_+$  into  $\mathbb{R}^r$ , for some integer parameter  $r \geq 1$ .

Let  $W_t$  be an  $r$ -dimensional Brownian motion and denote by  $\mathcal{W}_{s,t}$  the  $\sigma$ -field over  $\Omega$  generated by the increments  $(W_u - W_v)$ , with  $u, v \in [s, t]$ . Without further mention, all the anticipating stochastic integrals discussed in the present article are understood as Skorohod integrals.

We denote by  $D_t$  the Malliavin derivative from some dense domain  $\mathbb{D}_{2,1} \subset \mathbb{L}_2(\Omega)$  into the space  $\mathbb{L}_2(\Omega \times \mathbb{R}_+; \mathbb{R}^r)$ . For multivariate  $d$ -column vector random variables  $F$  with entries  $F^j$ , we use the same rules as for the gradient and  $D_t F$  is the  $(r, p)$ -matrix with entries  $(D_t F)_{i,j} := D_t^i F^j$ . For  $(p \times q)$ -matrices  $F$  with entries  $F^j_k$  we let  $D_t F$  be the tensor with entries  $(D_t F)_{i,j,k} = D_t^i F^j_k$ .

Let  $F$  be some function from  $\mathbb{R}^p$  into  $\mathbb{R}^q$ , and let  $y \in \mathbb{R}^p$  be some given state, for some  $p, q \geq 1$ . Suppose we are given a forward  $p$ -dimensional continuous semi-martingale  $Y_{s,t}$  and a backward random field  $F_{s,t}$  from  $\mathbb{R}^p$  into  $\mathbb{R}^q$  with a column-vector type canonical representation of the following form :

$$\begin{cases} Y_{s,t} &= y + \int_s^t B_{s,u} du + \int_s^t \Sigma_{s,u} dW_u \\ F_{s,t}(x) &= F(x) + \int_s^t G_{u,t}(x) du + \int_s^t H_{u,t}(x) dW_u \end{cases} \tag{1}$$

for some  $\mathcal{W}_{s,t}$ -adapted functions  $B_{s,t}, G_{s,t}, H_{s,t}, \Sigma_{s,t}$  with appropriate dimensions and satisfying the following conditions :

- ( $\mathcal{H}$ )<sub>1</sub> : The functions  $G_{u,t}, \nabla H_{u,t}, \nabla^2 F_{u,t}$  and the derivatives  $D_v \nabla F_{u,t}(x)$  and  $D_v H_{u,t}(x)$  are continuous w.r.t.  $x$  for any given  $u, v \in [s, t]$  and  $\omega \in \Omega$ .
- ( $\mathcal{H}$ )<sub>2</sub> The function  $G_{u,t}, \nabla H_{u,t}, \nabla^2 F_{u,t}$ , and the derivatives  $D_v H_{u,t}, D_v \nabla F_{u,t}$  have at most polynomial growth w.r.t. the state variable, uniformly with respect to  $\omega \in \Omega$ .
- ( $\mathcal{H}$ )<sub>2</sub> The processes  $B_{s,u}, \Sigma_{s,u}$  as well as the derivatives  $D_v \Sigma_{s,u}$  have moments of any order.

Next Theorem 1 is the first main result of this Note.

**Theorem 1.** *Assume conditions  $(\mathcal{H}_i)_{i=1,2,3}$  are satisfied. In this situation, for any  $s \leq u \leq v \leq t$  we have the generalized backward Itô–Ventzell formula*

$$F_{v,t}(Y_{s,v}) - F_{u,t}(Y_{s,u}) = \int_u^v (\nabla F_{r,t}(Y_{s,r})' B_{s,r} + \frac{1}{2} \nabla^2 F_{r,t}(Y_{s,r})' \Sigma_{s,r} \Sigma'_{s,r} - G_{r,t}(Y_{s,r})) dr + \int_u^v (\nabla F_{r,t}(Y_{s,r})' \Sigma_{s,r} - H_{r,t}(Y_{s,r})) dW_r \quad (2)$$

The above Theorem 1 can be seen as the backward version of the generalized Itô–Ventzell formula presented in [9] (see also [8, Theorem 3.2.11]).

### 2. Stochastic flows interpolation

Let  $b_t(x)$  be a vector-valued function from  $\mathbb{R}^d$  into  $\mathbb{R}^d$  and  $\sigma_t(x) = [\sigma_{t,1}(x), \dots, \sigma_{t,r}(x)]$  be a matrix-valued function from  $\mathbb{R}^d$  into  $\mathbb{R}^{d \times r}$ , for some parameters  $d, r \geq 1$ .

For any time horizon  $s \geq 0$  we denote by  $X_{s,t}(x)$  be the stochastic flow defined for any  $t \in [s, \infty[$  and any starting point  $X_{s,s}(x) = x \in \mathbb{R}^d$  by the stochastic integral equation

$$X_{s,t}(x) = x + \int_s^t b_u(X_{s,u}(x)) du + \int_s^t \sigma_u(X_{s,u}(x)) dW_u \quad (3)$$

We assume that the drift  $b_t(x)$  and the diffusion matrix  $\sigma_t(x)$  have continuous and uniformly bounded derivatives up to the third order. This condition is met for linear Gaussian models as well as for the geometric Brownian motion. It ensures that the stochastic flow  $x \mapsto X_{s,t}(x)$  is a twice differentiable function of the initialisation  $x$ . In addition, all absolute moments of the flow and the ones of its first and second order derivatives exists for any time horizon.

For any  $p \geq 1$  and any twice differentiable function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^p$  with at most polynomial growth the function

$$F_{s,t}(x) = \mathbb{P}_{s,t}(f)(x) := f(X_{s,t}(x)) \quad (4)$$

satisfies the backward formula (1) with  $F(x) = f(x)$  and the random fields

$$G_{u,t}(x) := \nabla F_{u,t}(x)' b_u(x) + \frac{1}{2} \nabla^2 F_{u,t}(x)' a_u(x) \quad \text{and} \quad H_{u,t}(x) := \nabla F_{u,t}(x)' \sigma_u(x)$$

In addition, the regularity conditions on the drift and the diffusion function ensure that conditions  $(\mathcal{H}_i)$  with  $i = 1, 2, 3$  are satisfied.

Let  $\bar{X}_{s,t}(x)$  be the stochastic flow associated with a stochastic integral equation defined as (3) by replacing  $(b_t, \sigma_t)$  by some drift and diffusion functions  $(\bar{b}_t, \bar{\sigma}_t)$  with the same regularity properties. Also let  $\bar{\mathbb{P}}_{s,t}$  be the operator defined as in (4) by replacing  $X_{s,t}$  by  $\bar{X}_{s,t}$ .

Theorem 1 allows to describe the differences of operators  $\mathbb{P}_{s,t} - \bar{\mathbb{P}}_{s,t}$  in terms of the difference of their corresponding drifts and diffusion functions,

$$\Delta a_t := a_t - \bar{a}_t \quad \Delta b_t := b_t - \bar{b}_t \quad \text{and} \quad \Delta \sigma_t := \sigma_t - \bar{\sigma}_t \quad (5)$$

where  $a_t(x) := \sigma_t(x) \sigma_t(x)'$  and  $\bar{a}_t(x) := \bar{\sigma}_t(x) \bar{\sigma}_t(x)'$ . More precisely, rewritten in terms of the stochastic semigroups  $\mathbb{P}_{s,t}$  and  $\bar{\mathbb{P}}_{s,t}$  the generalized backward Itô–Ventzell formula (2) yields the following corollary.

**Corollary 2.** *For any twice differentiable function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^p$  with at most polynomial growth we have the forward-backward multivariate interpolation formula*

$$\mathbb{P}_{s,t}(f)(x) - \bar{\mathbb{P}}_{s,t}(f)(x) = \mathbb{T}_{s,t}(f, \Delta a, \Delta b)(x) + \mathbb{S}_{s,t}(f, \Delta \sigma)(x) \quad (6)$$

with the stochastic integro-differential operator

$$\mathbb{T}_{s,t}(f, \Delta a, \Delta b)(x) := \int_s^t \left[ \nabla \mathbb{P}_{u,t}(f) \left( \bar{X}_{s,u}(x) \right)' \Delta b_u \left( \bar{X}_{s,u}(x) \right) + \frac{1}{2} \nabla^2 \mathbb{P}_{u,t}(f) \left( \bar{X}_{s,u}(x) \right)' \Delta a_u \left( \bar{X}_{s,u}(x) \right) \right] du \quad (7)$$

and the Skorohod stochastic integral term given by

$$\mathbb{S}_{s,t}(f, \Delta \sigma)(x) := \int_s^t \nabla \mathbb{P}_{u,t}(f) \left( \bar{X}_{s,u}(x) \right)' \Delta \sigma_u \left( \bar{X}_{s,u}(x) \right) dW_u \quad (8)$$

The interpolation formula (6) with a fluctuation term given by the Skorohod stochastic integral (8) can be seen as a Aleksev–Gröbner formula of Skorohod type [1]. For a more thorough discussion on these stochastic interpolation formulae and their applications we refer to [7] and the references therein.

Consider the discrete time interval  $[s, t]_h := \{u_0, \dots, u_{n-1}\}$  associated with some refining time mesh  $u_{i+1} = u_i + h$  from  $u_0 = s$  to  $u_n = t$ , for some time step  $h > 0$ . In this notation, the Skorohod stochastic integral (8) can alternatively be defined by the  $\mathbb{L}_2$ -approximation of two-sided stochastic integrals

$$\mathbb{S}_{s,t}(f, \Delta \sigma_u)(x) = \lim_{h \rightarrow 0} \sum_{u \in [s,t]_h} \nabla \mathbb{P}_{u+h,t}(f) \left( \bar{X}_{s,u}(x) \right)' \Delta \sigma_u \left( \bar{X}_{s,u}(x) \right) (W_{u+h} - W_u) \quad (9)$$

This alternative approach can be seen as a variation of Itô original construction of the stochastic integral and it relies on an extended version [7] of the two-sided stochastic integration calculus introduced by Pardoux and Protter in [10].

Using elementary differential calculus, for twice differentiable (column vector-valued) function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^p$  we readily check the gradient and the Hessian formulae

$$\begin{aligned} \nabla \mathbb{P}_{s,t}(f)(x) &= \nabla X_{s,t}(x) \mathbb{P}_{s,t}(\nabla f)(x) \\ \nabla^2 \mathbb{P}_{s,t}(f)(x) &= [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \mathbb{P}_{s,t}(\nabla^2 f)(x) + \nabla^2 X_{s,t}(x) \mathbb{P}_{s,t}(\nabla f)(x) \end{aligned} \quad (10)$$

Choosing  $p = d$  and the identity function  $f(x) = e(x) := x$  the above formula reduces to

$$\mathbb{P}_{s,t}(e)(x) = \nabla X_{s,t}(x) \quad \text{and} \quad \nabla^2 \mathbb{P}_{s,t}(e)(x) = \nabla^2 X_{s,t}(x)$$

In this context, we have backward stochastic flow differential equation

$$d_s X_{s,t}(x) = - \left[ \left( \nabla X_{s,t}(x)' b_s(x) + \frac{1}{2} \nabla^2 X_{s,t}(x)' a_s(x) \right) ds + \nabla X_{s,t}(x)' \sigma_s(x) dW_s \right] \quad (11)$$

In the above display,  $d_s X_{s,t}^i(x)$  represents the change in  $X_{s,t}^i(x)$  w.r.t. the variable  $s$ . A proof of the above formula based on Taylor expansions is presented in [6], see also the appendix of [3].

In differential form, the forward-backward multivariate interpolation formula (6) applied to the identity function yields the following Corollary 3.

**Corollary 3.** *For any  $u \in [s, t]$  and any  $x \in \mathbb{R}^d$  we have the forward-backward stochastic interpolation differential equation*

$$\begin{aligned} d_u \left( X_{u,t} \circ \bar{X}_{s,u} \right)(x) &= \left( d_u X_{u,t} \right) \left( \bar{X}_{s,u}(x) \right) + \left( \nabla X_{u,t} \right) \left( \bar{X}_{s,u}(x) \right)' d_u \bar{X}_{s,u}(x) \\ &\quad + \frac{1}{2} \left( \nabla^2 X_{u,t} \right) \left( \bar{X}_{s,u}(x) \right)' \bar{a}_u \left( \bar{X}_{s,u}(x) \right) du \end{aligned} \quad (12)$$

Forward-backward interpolation formulae of the same form as (12) without the Skorohod fluctuation term for stochastic matrix Riccati diffusion flows are also discussed in [5]. The articles [2, 4] also discuss similar interpolation formulae for mean field particle systems and deterministic nonlinear measure valued semigroups.

### 3. Uniform perturbation estimates

The second order perturbation methodology developed in the present article takes advantage of the stability properties of the tangent and the Hessian flow in the estimation of Skorohod fluctuation term and this sharpen analysis of the difference between stochastic flows.

For some multivariate function  $f_t(x)$ , for  $(t, x) \in [0, \infty) \times \mathbb{R}^d$ , let  $\|f(x)\| := \sup_t \|f_t(x)\|$  and the uniform norm be  $\|f\| := \sup_{t,x} \|f_t(x)\|$ . For any  $n \geq 1$  we also set

$$\| \|f(x)\| \|_n := \sup_{s \geq 0} \sup_{t \geq s} \mathbb{E} \left( \left\| f_t(\bar{X}_{s,t}(x)) \right\|^n \right)^{1/n} \quad \text{and} \quad \rho(\nabla\sigma) := \sup_{1 \leq k \leq r} \sup_{t,x} \rho(\nabla\sigma_{t,k}(x)) \quad (13)$$

We denote by  $\kappa_n$  and  $\kappa_{\delta,n}$  some constants that depends on some parameters  $n$  and  $(\delta, n)$  but do not depend on the time horizon, nor on the space variable. We also consider the following collection of regularity conditions indexed by  $\alpha \geq 1$  :

$(\mathcal{M})_\alpha$  For any  $x \in \mathbb{R}^d$  we have the uniform moment inequality w.r.t. the time horizon

$$\sup_{s \leq t} \mathbb{E} \left( \|X_{s,t}(x)\|^\alpha \right)^{1/\alpha} \vee \sup_{s \leq t} \mathbb{E} \left( \|\bar{X}_{s,t}(x)\|^\alpha \right)^{1/\alpha} \leq \kappa_\alpha (1 \vee \|x\|)$$

$(\mathcal{F})_\alpha$  There exists some parameters  $\lambda > 0$  such that

$$\nabla b_t + (\nabla b_t)' + \sum_{1 \leq k \leq r} \nabla\sigma_{t,k}(\nabla\sigma_{t,k})' \leq -2\lambda I \quad \text{and} \quad \lambda > \frac{d(\alpha-2)}{2} \rho(\nabla\sigma)^2$$

In addition, the drift and diffusion matrix  $(\bar{b}_t, \bar{\sigma}_t)$  satisfy the same condition for some  $\bar{\lambda} > 0$ .

For constant diffusion functions

$$\sigma_t(x) = \Sigma_t \quad \text{and} \quad \bar{\sigma}_t(x) = \bar{\Sigma}_t \quad \text{for some matrices } \Sigma_t \text{ and } \bar{\Sigma}_t. \quad (14)$$

the condition  $(\mathcal{F})_n$  is met for any  $n \geq 2$  as soon as the following log-norm inequalities are met

$$\nabla b_t + (\nabla b_t)' \leq -2\lambda I \quad \text{and} \quad \nabla \bar{b}_t + (\nabla \bar{b}_t)' \leq -2\bar{\lambda} I \quad \text{for some } \lambda \wedge \bar{\lambda} > 0, \quad (15)$$

We end this Note with a uniform interpolation theorem w.r.t. the time parameter.

**Theorem 4.** Assume conditions  $(\mathcal{M})_{2n/\delta}$  and  $(\mathcal{F})_{2n/(1-\delta)}$  are satisfied for some parameters  $n \geq 2$  and  $\delta \in ]0, 1[$ . In this situation, for any  $x \in \mathbb{R}^d$  and  $s \leq t$  we have the time-uniform estimates

$$\mathbb{E} \left[ \left\| X_{s,t}(x) - \bar{X}_{s,t}(x) \right\|^n \right]^{1/n} \leq \kappa_{\delta,n} \left( \|\Delta a(x)\|_{2n/(1+\delta)} + \|\Delta b(x)\|_{2n/(1+\delta)} + \|\Delta\sigma(x)\|_{2n/\delta} (1 \vee \|x\|) \right) \quad (16)$$

Whenever (14) and (15) are satisfied, for any  $n \geq 2$  we have the time-uniform estimates

$$\mathbb{E} \left[ \left\| X_{s,t}(x) - \bar{X}_{s,t}(x) \right\|^n \right]^{1/n} \leq \kappa_n \left( \|\Delta b(x)\|_n + \|\Sigma - \bar{\Sigma}\| \right) \quad (17)$$

Illustrations of the forward-backward interpolation formulae presented in this Note in the context of diffusion perturbation theory, interacting diffusions and discrete time approximations are discussed in the extended version [7].

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