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Complex Analysis / *Analyse complexe*

Picard-Hayman behavior of derivatives of meromorphic functions

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Abstract. Let f be a transcendental meromorphic function on $\mathbb{C}$, and $P(z), Q(z)$ be two polynomials with $\deg P(z) > \deg Q(z)$. In this paper, we prove that: if $f(z) = 0 \Rightarrow f'(z) = a$(a nonzero constant), except possibly finitely many, then $f'(z) - P(z)/Q(z)$ has infinitely many zeros. Our result extends or improves some previous related results due to Bergweiler–Pang, Pang–Nevo–Zalcman, Wang–Fang, and the author, et. al.

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1. Introduction and Main Results

In 1959, Hayman [2, 3] proved the following result, which has come to be known as Hayman's alternative.

Theorem 1. *Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$, and let $k \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{C}\backslash\{0\}$. Then either $f - a$ or $f^{(k)} - b$ has infinitely many zeros.*

Considering $g = f - a$, it suffices to take $a = 0$ in Theorem 1.

In the past years, a number of improvements and extensions of Theorem 1 have been obtained. Wang and Fang [8] proved the following result.

Theorem 2. *Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$ and $k \in \mathbb{N}$. If*

(i) *all zeros of f have multiplicity at least $k+1$ and all poles of f are multiple, or*

(ii) *all zeros of f have multiplicity at least 3,*

then, for each $b \in \mathbb{C}\backslash\{0\}$, $f^{(k)} - b$ has infinitely many zeros.

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For $k = 1$, Nevo, Pang and Zalcman [5] proved the following result, in which they use an argument involving quasinormal families.

Theorem 3. *Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. If all zeros of f are multiple, then, for each $b \in \mathbb{C}\backslash\{0\}$, $f' - b$ has infinitely many zeros.*

It is a natural to ask: *Does the above results still hold if we replace the nonzero constant b by a small function function $R(z)(\not\equiv 0)$ of f?*

By using the theory of normal families, Bergweiler and Pang [1] (for $k = 1$), and the author [9] (for $k \geq 2$) obtained

Theorem 4. *Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$, and $R(\not\equiv 0)$ be a rational function. If*

(i) *all zeros of f have multiplicity at least $k+1$ and all poles of f are multiple, or*
(ii) *all zeros of f have multiplicity at least $k+2$, except possibly finitely many,*

then $f^{(k)} - R$ has infinitely many zeros.

For the case $k = 1$, using the theory of quasinormal families, Pang, Nevo and Zalcman [6] proved the following stronger result.

Theorem 5. *Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. If all zeros of f are multiple except possibly finitely many, then, for each rational function $R \not\equiv 0$, $f' - R$ has infinitely many zeros.*

Clearly, the condition "all zeros of f are multiple" in Theorem 5 is equivalent to the condition "$f = 0 \Rightarrow f'(z) = 0$". We note that the function $f(z) = e^z - a$ ($a \in \mathbb{C}$) satisfying "$f = 0 \Rightarrow f'(z) = a$", and $f'(z) - R(z)$ has infinitely many zeros for rational function $R(z) \not\equiv -a$.

Inspired by this observation, we prove the following result by using the theory of normal family.

Theorem 6. *Let f be a transcendental meromorphic function on $\mathbb{C}$, and $a \in \mathbb{C}$. If $f(z) = 0 \Rightarrow f'(z) = a$, except possibly finitely many, then $f'(z) - R(z)$ has infinitely many zeros, where $R(z) = P(z)/Q(z)(\not\equiv 0)$, and $P(z), Q(z)$ are two polynomials with $\deg P(z) > \deg Q(z)$.*

Remark. At present, we are not clear whether the condition $\deg P(z) > \deg Q(z)$ in Theorem 6 can be omitted.

Corollary 7. *Let f be a transcendental meromorphic function on $\mathbb{C}$, and $a \in \mathbb{C}$. If $f(z) = 0 \Rightarrow f'(z) = a$, except possibly finitely many, then for nonconstant polynomial $P(z)$, $f'(z) - P(z)$ has infinitely many zeros.*

2. Some Lemmas

First we recall some definitions. If there exists a curve $\Gamma \subset \mathbb{C}$ tending to ∞ such that $f(z) \to a$ as $z \to \infty$ and $z \in \Gamma$, we call that a is an asymptotic value of f.

A meromorphic function f is called a Julia exceptional function if $f^{\#}(z) = O(1/|z|)$ as $z \to \infty$. Here, as usual, $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of f. It follows easily from the Ahlfors-Shimizu characteristic function that if f is a Julia exceptional function, then $T(r, f) = O((\log r)^2)$ as $r \to \infty$.

The following result is due to Lehto and Virtanen [4].

Lemma 8. *A transcendental Julia exceptional function does not have an asymptotic value.*

Lemma 9 (see [1]). *Let f be a transcendental meromorphic function, and let R be a rational function satisfying $R(z) \sim cz^d$ as $z \to \infty$, with $c \in \mathbb{C}\backslash\{0\}$ and $d \in \mathbb{Z}$. Suppose that $f' - R$ has only finitely many zeros and $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Set $g := f(z)/z^{d+1}$. Then g has an asymptotic value.*

Lemma 10 (see [2, 3]). *Let f be a meromorphic function in the complex plane, and k a positive integer. If $f \neq 0$ and $f^{(k)}(z) \neq 1$, then $f(z)$ is a constant.*

The next is a local version of Zalcaman's lemma due to Pang and Zalcman [7].

Lemma 11. *Let k be a positive integer and let $\mathscr{F}$ be a family of functions meromorphic in a domain D, such that each function $f \in \mathscr{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathscr{F}$ is not normal at $z_0 \in D$, then, for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathscr{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. Moreover, g has order at most 2.

3. Proof of Theorem 6

By Theorem 5, we only need to consider the case $a \neq 0$.

Since $R(z) = P(z)/Q(z) (\not\equiv 0)$, where $P(z)$ and $Q(z)$ are two polynomials with $\deg P(z) > \deg Q(z)$. We may assume that $R(z) \sim cz^d$ as $z \to \infty$, where $c \in \mathbb{C}\backslash\{0\}$ and d is a positive integer. Define

$$g(z) := \frac{f(z)}{z^{d+1}}.$$

Suppose that $f' - R$ has only finitely many zeros. If g is a Julia exceptional function, then $T(r, g) = O((\log r)^2)$ and hence $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Lemma 9 implies that g has an asymptotic value. But, by Lemma 8, g has no asymptotic value, a contradiction.

Thus g is not a Julia exceptional function. Hence, by the definition of Julia exceptional function, there exists $\{a_n\}$ such that $a_n \to \infty$ and $a_n g^{\#}(a_n) \to \infty$ as $n \to \infty$.

Let $D = \{z \in \mathbb{C} : |z - 1| < 1/2\}$, and set

$$\mathscr{G} = \{g_n(z) := g(a_n z) z^{d+1} = \frac{f(a_n z)}{a_n^{d+1}}, z \in D\}.$$

The family $\mathscr{G}$ is not normal at $z = 1$. Indeed, by computation, we have

$$g_n^{\#}(1) = \frac{|a_n g'(a_n) + (d+1) g(a_n)|}{1 + |g(a_n)|^2} \geq |a_n| g^{\#}(a_n) - \frac{|d+1|}{2} \to \infty$$

as $n \to \infty$. So, by Marty's criterion, $\mathscr{F}$ is not normal at $z = 1$.

If $g_n(z) = 0$, that is, $f(a_n z) = 0$, then from the hypotheses of theorem, we have $f'(a_n z) = a$. Thus there exists $M \geq 1$ such that (for large n)

$$|g_n'(z)| = \left| \frac{f'(a_n z)}{a_n^d} \right| = \left| \frac{a}{a_n^d} \right| \leq M$$

whenever $g_n(z) = 0$. Then, applying Lemma 11, we can find $z_n \in D$, $z_n \to 1$, $\rho_n \to 0^+$, and $g_n \in \mathscr{G}$ such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n} = \frac{f(a_n(z_n + \rho_n \zeta))}{\rho_n a_n^{d+1}} \to G(\zeta) \tag{1}$$

locally uniformly with respect to the spherical metric, where $G(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$.

Claim. $G(\zeta) \neq 0$ *on* $\mathbb{C}$.

Suppose that there exists a point ζ_0 such that $G(\zeta_0)=0$. Then, there exist ζ_n, $\zeta_n \to \zeta_0$, such that $G_n(\zeta_n)=0$ (for n sufficiently large) since G is not constant, and thus $f(a_n(z_n+\rho_n\zeta_n))=0$. By the assumption of theorem, we have $f'(a_n(z_n+\rho_n\zeta_n))=a$.

From (1), we have

$$G_n'(\zeta)=g_n'(z_n+\rho_n\zeta)=\frac{f'(a_n(z_n+\rho_n\zeta))}{a_n^d}\to G'(\zeta), \tag{2}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G. It follows that $G'(\zeta_0)=\lim_{n\to\infty}G_n'(\zeta_n)=\lim_{n\to\infty}a/a_n^d=0$. Thus, all zeros of G are multiple.

Let ζ_0 be a zero of G with multiplicity $m(\geq 2)$, then $G^{(m)}(\zeta_0)\neq 0$. By (1) and Rouché's theorem, there exist $\delta>0$, and m sequences $\{\zeta_n^{(i)}\}$ $(i=1,2,\dots,m)$ on $D_\delta(\zeta_0)=\{\zeta:|\zeta-\zeta_0|<\delta\}$, tending to ζ_0, such that

$$G_n(\zeta_n^{(i)})=0,\quad (i=1,2,\dots,m).$$

From (1), we have $f(a_n(z_n+\rho_n\zeta_n^{(i)}))=0$ $(i=1,2,\dots,m)$. It follows that $f'(a_n(z_n+\rho_n\zeta_n^{(i)}))=a$, and thus $G_n'(\zeta_n^{(i)})=a/a_n^d\neq 0 (i=1,2,\dots,m)$. This means that each $\zeta_n^{(i)}$ is a simple zero of G_n, which rules out the possibility that each two of $\{\zeta_n^{(i)}\}$ $(i=1,2,\dots,m)$ might coincide. So, $\zeta_n^{(i)}$ $(i=1,2,\dots,m)$ are m distinct zeros of $G_n'(\zeta)-a/a_n^d$ on $D_\delta(\zeta_0)$. Noting that $\zeta_n^{(i)}\to\zeta_0$ and

$$G_n'(\zeta)-\frac{a}{a_n^d}\to G'(\zeta),$$

Rouché's theorem implies that ζ_0 is the zero of $G'(\zeta)$ with multiplicity at least m. We get $G^{(m)}(\zeta_0)=0$, a contradiction. We thus proved our claim.

Since $R(z)\sim cz^d$ as $z\to\infty$, by (2), we have

$$G_n'(\zeta)-\frac{R(a_n(z_n+\rho_n\zeta))}{a_n^d}\to G'(\zeta)-c \tag{3}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G. On the other hand, for n sufficiently large

$$G_n'(\zeta)-\frac{R(a_n(z_n+\rho_n\zeta))}{a_n^d}=\frac{f'(a_n(z_n+\rho_n\zeta))-R(a_n(z_n+\rho_n\zeta))}{a_n^d}\neq 0.$$

By (3) and Hurwitz's theorem, either $G'(\zeta)\neq c$ or $G'(\zeta)\equiv c$ on $\mathbb{C}\setminus G^{-1}(\infty)$. Clearly, these also hold on $\mathbb{C}$. If $G'\equiv c$, then G is a polynomial with degree 1. This is impossible since $G\neq 0$. Hence $G'\neq c$.

So, by Lemma 10, $G(\zeta)$ must be a constant, a contradiction. This completes the proof of Theorem 6. □

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