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Elliptical Partial Differential Equations / *Équations aux dérivées partielles elliptiques*

A Liouville theorem for the fractional Ginzburg–Landau equation

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Abstract. In this paper, we are concerned with a Liouville-type result of the nonlinear integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(1-|u|^2)}{|x-y|^{n-\alpha}} \, \mathrm{d}y,$$

where $u: \mathbb{R}^n \to \mathbb{R}^k$ with $k \geq 1$ and $1 < \alpha < n/2$. We prove that $u \in L^2(\mathbb{R}^n) \Rightarrow u \equiv 0$ on $\mathbb{R}^n$, as long as u is a bounded and differentiable solution.

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If a harmonic function u is bounded on $\mathbb{R}^n$, then $u \equiv$ Const. (this is the Liouville theorem). Moreover, if u is integrable (i.e. $u \in L^s(\mathbb{R}^n)$ for some $s \geq 1$), then $u \equiv 0$ on $\mathbb{R}^n$.

In 1994, Brezis, Merle and Rivière [2] studied the quantization effects of the following equation

$$-\Delta u = u(1-|u|^2) \quad \text{on } \mathbb{R}^2. \tag{1}$$

Here $u: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector valued function. It is the Euler–Lagrange equation of the Ginzburg–Landau energy

$$E_{GL}(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4}\|1-|u|^2\|^2_{L^2(\mathbb{R}^2)}.$$

In particular, they proved the finite energy solution (i.e., u satisfies $\nabla u \in L^2(\mathbb{R}^2)$) is bounded (see also [4] and [6])

$$|u| \leq 1 \quad \text{on } \mathbb{R}^n. \tag{2}$$

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(Here $n = 2$.) Based on this result, they obtained a Liouville type theorem for finite energy solutions (cf. [2, Theorem 2]):

Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a classical solution of (1)*. If $\nabla u \in L^2(\mathbb{R}^2)$, then either $u \in L^2(\mathbb{R}^2)$ which implies $u \equiv 0$, or $1-|u|^2 \in L^1(\mathbb{R}^2)$ which implies $u \equiv C$ with $|C| = 1$.*

The boundedness and the integrability of solutions are the important conditions which ensure that the Liouville theorem holds. The Pohozaev identity plays a key role in the proof.

In this paper, we are concerned with the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(1-|u|^2)}{|x-y|^{n-\alpha}} \mathrm{d}y. \tag{3}$$

Here $u : \mathbb{R}^n \to \mathbb{R}^k$, $k \geq 1$, $n \geq 3$, and $1 < \alpha < n/2$. We also apply the integral form of the Pohozaev identity (which was used for the Lane–Emden equations in [3], [5] and [12]) to establish a Liouville theorem.

Theorem 1. *Assume that $u : \mathbb{R}^n \to \mathbb{R}^k$ is bounded and differentiable, and solves* (3) *with $\alpha \in (1, n/2)$. If $u \in L^2(\mathbb{R}^n)$, then $u(x) \equiv 0$.*

Proof. For convenience, we denote $B_R(0)$ by B_R here.

Step 1. We claim that the improper integral

$$\int_{\mathbb{R}^n} \frac{z \cdot \nabla[u(z)(1-|u(z)|^2)]}{|x-z|^{n-\alpha}} \mathrm{d}z \tag{4}$$

is convergent at each $x \in \mathbb{R}^n$.

In fact, since $u \in L^2(\mathbb{R}^n)$, we can find $R = R_j \to \infty$ such that

$$R \int_{\partial B_R} |u(z)|^2 \mathrm{d}s \to 0. \tag{5}$$

Since u is bounded, by the Hölder inequality, we obtain that for sufficiently large R, there holds

$$\begin{aligned} R \left| \int_{\partial B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} \mathrm{d}s \right| &\leq C R^{1-n+\alpha} \int_{\partial B_R} |u(z)| \mathrm{d}s \\ &\leq C R^{1-n+\alpha} \left(R \int_{\partial B_R} |u(z)|^2 \mathrm{d}s \right)^{\frac{1}{2}} R^{-\frac{1}{2}+\frac{n-1}{2}}. \end{aligned}$$

Let $R = R_j \to \infty$. Noting $\alpha < n/2$, and using (5) we get

$$R \int_{\partial B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} \mathrm{d}s \to 0 \tag{6}$$

when $R = R_j \to \infty$.

Next, we claim that the improper integral

$$I(\mathbb{R}^n) := \int_{\mathbb{R}^n} \frac{u(z)(1-|u(z)|^2)(x-z) \cdot z}{|x-z|^{n-\alpha+2}} \mathrm{d}z \tag{7}$$

absolutely converges for each $x \in \mathbb{R}^n$.

In fact, we observe that the defect points of $I(\mathbb{R}^n)$ are x and ∞. When z is near ∞, we have

$$|I(\mathbb{R}^n \setminus B_r)| \leq C \int_{\mathbb{R}^n \setminus B_r} \frac{|u(z)| \mathrm{d}z}{|x-z|^{n-\alpha}} \leq C \left(\int_{\mathbb{R}^n} |u|^2 \mathrm{d}z \right)^{\frac{1}{2}} \left(\int_r^\infty \rho^{n-2(n-\alpha)} \frac{\mathrm{d}\rho}{\rho} \right)^{\frac{1}{2}}.$$

In view of $u \in L^2(\mathbb{R}^n)$ and $\alpha < n/2$, we get

$$|I(\mathbb{R}^n \setminus B_r)| < \infty. \tag{8}$$

When z is near x, we first take

$$s \in \left(\frac{n}{\alpha - 1}, \infty \right). \tag{9}$$

Clearly, $1 < \alpha < n/2$ implies $s > 2$. In addition,

$$u \in L^s(\mathbb{R}^n) \tag{10}$$

because u is bounded and $u \in L^2(\mathbb{R}^n)$. Note that

$$|I(B_\delta(x))| \le C\int_{B_\delta(x)} \frac{|u(z)|\mathrm{d}z}{|x-z|^{n-\alpha+1}} \le C\left(\int_{\mathbb{R}^n}|u|^s\mathrm{d}z\right)^{\frac{1}{s}}\left(\int_0^r \rho^{n-\frac{s}{s-1}(n-\alpha+1)}\frac{\mathrm{d}\rho}{\rho}\right)^{1-\frac{1}{s}}.$$

By (9) and (10), we get

$$|I(B_\delta(x))| < \infty.$$

Combining this with (8), we prove that (7) is absolutely convergent.

Finally we prove that (4) is convergent. Integrating by parts yields

$$\int_{B_R} \frac{z\cdot\nabla[u(z)(1-|u(z)|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z = R\int_{\partial B_R}\frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}}\mathrm{d}s - n\int_{B_R}\frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}}\mathrm{d}z - (n-\alpha)\int_{B_R}\frac{u(z)(1-|u(z)|^2)(x-z)\cdot z}{|x-z|^{n-\alpha+2}}\mathrm{d}z. \tag{11}$$

Letting $R = R_j \to \infty$ in (11) and using (3) and (6), we can see that

$$\int_{\mathbb{R}^n} \frac{z\cdot\nabla[u(z)(1-|u(z)|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z = -nu(x) + (\alpha-n)I(\mathbb{R}^n),$$

and hence it is convergent at each $x \in \mathbb{R}^n$.

Step 2. Proof of Theorem 1. For any $\lambda > 0$, from (3) it follows

$$u(\lambda x) = \lambda^\alpha \int_{\mathbb{R}^n}\frac{u(\lambda z)(1-|u(\lambda z)|^2)}{|x-z|^{n-\alpha}}\mathrm{d}z.$$

Differentiating both sides with respect to λ yields

$$x\cdot\nabla u(\lambda x) = \alpha\lambda^{\alpha-1}\int_{\mathbb{R}^n}\frac{u(\lambda z)(1-|u(\lambda z)|^2)}{|x-z|^{n-\alpha}}\mathrm{d}z + \lambda^\alpha\int_{\mathbb{R}^n}\frac{(z\cdot\nabla u(\lambda z))(1-|u(\lambda z)|^2) + u(\lambda z)[-2u(\lambda z)(z\cdot\nabla u(\lambda z))]}{|x-z|^{n-\alpha}}\mathrm{d}z.$$

Letting $\lambda = 1$ yields

$$x\cdot\nabla u(x) = \alpha u(x) + \int_{\mathbb{R}^n}\frac{z\cdot\nabla[u(1-|u|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z. \tag{12}$$

Since u is bounded and $u \in L^2(\mathbb{R}^n)$, it follows that $u \in L^4(\mathbb{R}^n)$, and hence

$$R\int_{\partial B_R}|u|^4\mathrm{d}s \to 0 \tag{13}$$

for some $R = R_j \to \infty$. Thus, integrating by parts and using (5) and (13), we respectively obtain

$$\int_{\mathbb{R}^n} u(x)(x\cdot\nabla u(x))\mathrm{d}x = \frac{-n}{2}\int_{\mathbb{R}^n}|u(x)|^2\mathrm{d}x, \tag{14}$$

and

$$\int_{\mathbb{R}^n} u(x)|u(x)|^2(x\cdot\nabla u(x))\mathrm{d}x = \frac{-n}{4}\int_{\mathbb{R}^n}|u(x)|^4\mathrm{d}x. \tag{15}$$

These results show that

$$\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)(x\cdot\nabla u(x))\mathrm{d}x < \infty. \tag{16}$$

Multiply (12) by $u(x)(1-|u(x)|^2)$ and integrate over B_R. Letting $R = R_j \to \infty$, from $u \in L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ and (16), we get

$$\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)\int_{\mathbb{R}^n}\frac{z\cdot\nabla[u(1-|u|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z\mathrm{d}x < \infty,$$

and

$$\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)(x\cdot\nabla u(x))\mathrm{d}x - \alpha\int_{\mathbb{R}^n}|u(x)|^2(1-|u(x)|^2)\mathrm{d}x = \int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)\int_{\mathbb{R}^n}\frac{z\cdot\nabla[u(1-|u|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z\mathrm{d}x. \quad (17)$$

We use the Fubini theorem and (3) to handle the right hand side term. Thus,

$$\begin{aligned}\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)\int_{\mathbb{R}^n}\frac{z\cdot\nabla[u(1-|u|^2)]}{|x-z|^{n-\alpha}}\mathrm{d}z\mathrm{d}x &= \int_{\mathbb{R}^n} z\cdot\nabla[u(1-|u|^2)]\int_{\mathbb{R}^n}\frac{u(x)(1-|u(x)|^2)}{|x-z|^{n-\alpha}}\mathrm{d}x\mathrm{d}z\\ &= \int_{\mathbb{R}^n}(x\cdot\nabla[u(1-|u|^2)])u(x)\mathrm{d}x\\ &= \int_{\mathbb{R}^n} u(x)(1-|u|^2)(x\cdot\nabla u(x))\mathrm{d}x - \int_{\mathbb{R}^n}|u|^2(x\cdot\nabla|u(x)|^2)\mathrm{d}x.\end{aligned} \quad (18)$$

Inserting this result into (17), and using (15) we have

$$\int_{\mathbb{R}^n}[\alpha|u(x)|^2 + (\frac{n}{2}-\alpha)|u(x)|^4]\mathrm{d}x = 0. \quad (19)$$

In view of $\alpha\in(1,n/2)$, (19) leads to $|u|\equiv 0$. Theorem 1 is proved. □

Remark 2. Clearly, (19) implies

$$\int_{\mathbb{R}^n}[\alpha(|u(x)|^2-|u(x)|^4)+\frac{n}{2}|u(x)|^4]\mathrm{d}x = 0. \quad (20)$$

When u satisfies (2), (20) also implies $|u|\equiv 0$. For (1), the bound $|u|\le 1$ for solutions $u:\mathbb{R}^n\to\mathbb{R}^k$ was first proved by Brezis (cf. [1]). Ma also pointed out that (2) holds true (cf. [6]).

Remark 3. In 2016, Ma [7] proved (2) for the Ginzburg–Landau-type equation with fractional Laplacian

$$(-\Delta)^{\frac{\alpha}{2}}u = (1-|u|^2)u \quad \text{on } \mathbb{R}^n \quad (21)$$

under the assumption

$$1-|u|^2\in L^2(\mathbb{R}^n), \quad (22)$$

where $n\ge 2$ and $0<\alpha<2$. The physical background of (21) can be found in [9] and [11]. Such an equation with $\alpha=1$ was well studied in [8]. Recall the definition of fractional Laplacian on $\mathbb{R}^n$. Let $n\ge 2$ and $0<\alpha<2$. Write

$$E = L_\alpha\cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n),$$

where $L_\alpha = \left\{u\in L^1_{\mathrm{loc}}(\mathbb{R}^n); \int_{\mathbb{R}^n}\frac{|u(x)|\mathrm{d}x}{1+|x|^{n+\alpha}}<\infty\right\}$. For a vector value function $u\in E$ from $\mathbb{R}^n$ to $\mathbb{R}^k$, define

$$(-\Delta)^{\frac{\alpha}{2}}u := C_{n,\alpha}P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\mathrm{d}y = C_{n,\alpha}\lim_{\varepsilon\to 0^+}\int_{|x-y|\ge\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\mathrm{d}y. \quad (23)$$

Here $C_{n,\alpha}$ is a positive constant.

Clearly, (22) and $u\in L^2(\mathbb{R}^n)$ are incompatible.

Remark 4. Another definition of the fractional order Laplacian involves the Riesz potential (cf. [10, Chapter 5]). Assume $\alpha\in(0,n)$, and $u, u(1-|u|^2)\in\mathscr{S}'(\mathbb{R}^n)$, then (21) can be explained as (3). In fact, (3) is equivalent to

$$\widehat{u}(\xi) = (|x|^{\alpha-n}*[u(1-|u|^2)])^\wedge(\xi) = C|\xi|^{-\alpha}[u(1-|u|^2)]^\wedge(\xi), \quad (24)$$

where C is a positive constant. By the property of the Riesz potential, we have

$$[(-\Delta)^{\alpha/2}u]^\wedge(\xi) = C|\xi|^\alpha\widehat{u}(\xi), \quad (25)$$

where C is another positive constant. Therefore, the above equality (24) amounts to (21). In addition, let $u \in E$ be a solution of (21) with $0 < \alpha < 2$. From (23), it follows that (25) is still true. If the Fourier inversion formula of (24) holds, then u also solves (3) (if we omit the constants).

Remark 5. If u is a finite energy solution of (1), then [2] shows that

$$\int_{\mathbb{R}^n} |u|^2(1-|u|^2)\mathrm{d}x < \infty. \tag{26}$$

Therefore, we sometimes call u a finite energy solution of (3) if u satisfies (26). Moreover, if u is uniformly continuous, we can see that either $u \in L^2(\mathbb{R}^n)$ or $1-|u|^2 \in L^1(\mathbb{R}^n)$ by the same argument of (3.9) and (3.10) in [2]. Therefore, if a bounded, uniformly continuous, differentiable function u is a finite energy solution of (3), then either $u \equiv 0$, or $|u(x)| \to 1$ when $|x| \to \infty$.

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