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A note on bias reduction

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Abstract. Let $\widehat{w}$ be an unbiased estimate of an unknown $w \in R$. Given a function $t(w)$, we show how to choose a function $f_n(w)$ such that for $w^* = \widehat{w} + f_n(w)$, $E\ t\left(w^*\right) = t(w)$. We illustrate this with $t(w) = w^a$ for a given constant a. For $a = 2$ and $\widehat{w}$ normal, this leads to the convolution equation $c_r = c_r \otimes c_r$.

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1. Introduction

Let $\widehat{w}$ be an unbiased estimate of an unknown $w \in R$. Given a function $t(w)$, we show how to choose a function $f_n(w)$ such that $E\ t(w^*) = t(w)$ for $w^* = \widehat{w} + f_n(w)$. We illustrate this in Section 2 with $\widehat{w}$ normal and $t(w) = w^a$ for some constant a. For $a = 2$ this gives the convolution equation $c_r = c_r \otimes c_r$ to solve.

w^* is not an estimate since it depends on the unknown w. The method extends to $\widehat{\mathbf{w}}$ any *standard estimate* of an unknown $\mathbf{w} \in R^p$ with respect to a given parameter n. That is, $E\ \widehat{\mathbf{w}} \to \mathbf{w}$ as $n \to \infty$ and, for $r \geq 1$, its rth order cumulants have magnitude n^{1-r} and can be expanded as power series in n^{-1}:

$$\kappa\left(\widehat{w}_{i_1}, \ldots, \widehat{w}_{i_r}\right) = \sum_{e=r-1}^{\infty} n^{-e} k_e^{i_1,\ldots,i_r} \tag{1}$$

for $1 \leq i_1, \ldots, i_r \leq p$ and $k_0^{i_1} = w_{i_1}$, where w_i is the ith component of $\mathbf{w}$, and the *cumulant coefficients* $k_e^{i_1,\ldots,i_r}$ are bounded as $n \to \infty$, but may depend on $\mathbf{w}$. For $p = 1$, (1) can be written

$$\kappa\left(\widehat{w}\right) = \sum_{e=r-1}^{\infty} n^{-e} k_{r,e}$$

for $r \geq 1$, where $k_{1,0} = w$. Cumulant coefficients are the building blocks of analytic methods for statistical inference. For example, methods for constructing estimates of low bias for any smooth function $t\left(\mathbf{w}\right) : R^p \to R$ were given in Mynbaev et al. [3] and Withers and Nadarajah [4–14].

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Given a sequence $a_1, a_2, \ldots$, the exponential partial Bell polynomial $B_{i,k}(\mathbf{a})$ is defined by

$$\left(\sum_{j=1}^{\infty} a_j t^j / j!\right)^k / k! \equiv \sum_{j=k}^{\infty} B_{j,k}(\mathbf{a}) t^j / j!$$

for $t \in R$ and $k = 0, 1, \ldots$. It is tabled on p. 307–308 of Comtet [2] for $1 \le r \le 12$. Given two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, their discrete convolution is defined by

$$a_r \otimes b_r = \sum_{i=1}^{r} a_i b_{r-i}.$$

Set $\delta_{i,1} = I(i=1)$ and $(a)_j = a!/(a-j)!$.

2. Adding bias to $\widehat{w}$ to reduce the bias of $t(\widehat{w})$

Suppose that $\widehat{w} \sim \mathcal{N}(w, v(w)/n)$, a normal estimate. Set

$$v = v(w), \quad f_n(w) = \sum_{i=1}^{\infty} k_i n^{-i}, \tag{2}$$

where $k_i = b_i / i!$ may depend on w. Theorems 1 and 3 show how to choose k_i or b_i so that for any given a,

$$E\left(w^*\right)^a = w^a, \ w^* = \widehat{w} + f_n(w). \tag{3}$$

Theorem 2.1 considers the case $a = 2$. Theorem 3 considers the case $a = 3$. Throughout, we set

$$\delta = \widehat{w} - w, \quad \nu = v/w^2, \quad m_n = E\, w^* = w + f_n(w). \tag{4}$$

Theorem 1. *Take $w^* = \widehat{w} + f_n(w)$ with*

$$k_r = -c_r v^r / D^{2r-1},$$

where $D = 2w$ and, for $r \ge 2$,

$$c_1 = 1, \quad c_r = c_r \otimes c_r. \tag{5}$$

Then $E(w^*)^2 = w^2$.

Proof. For w^* of (3),

$$E\left(w^*\right)^2 = m_n^2 + v/n = w^2 + 2w f_n(w) + f_n(w)^2 + v/n = w^2 + \sum_{i=1}^{\infty} T_i n^{-i},$$

where $T_i = D k_i + s_i + v\delta_{i,1}$, $s_1 = 0$, and for $i \ge 2$,

$$s_i = k_i \otimes k_i = \sum_{j=1}^{i-1} k_j k_{i-j}.$$

So, $T_i = 0$ if we take $k_i = -\left(s_i + v\delta_{i,1}\right)/D$ for $i \ge 1$. This gives the result. □

Corollary 2 gives an explicit form for c_r in (5).

Corollary 2. *For $r \ge 2$,*

$$c_r = -\binom{1/2}{r}(-4c_1)^r/2 = 2^{r-1} 1\, 3 \cdots (2r-3)/r!.$$

Proof. Set

$$C(t) = \sum_{r=1}^{\infty} c_r t^r,$$

where c_1 is now arbitrary. By (5), $C(t) = c_1 t + C(t)^2$. Since $C(0) = 0$, this gives

$$C(t) = \left[1 - (1 - 4c_1 t)^{1/2}\right]/2 = -\sum_{r=1}^{\infty} \binom{1/2}{r}(-4c_1 t)^r/2$$

which implies the result. □

Theorem 3. *Take $w^* = \widehat{w} + f_n(w)$ with*

$$f_n(w) = w\sum_{j=0}^{\infty} A_j \left(c/n^3\right)^j/j! + wu\sum_{j=0}^{\infty} B_j\left(c/n^3\right)^j/j! - w,$$

where $c = 4\nu^3$, $a_j = (1/2)_j/2$, $u = -\nu n^{-1}$, $B_k = a_{k+1}/a_1(k+1)$ and

$$A_j = \sum_{k=0}^{j} (1/3)_k\, B_{j,k}(\mathbf{a}).$$

Then $E(w^*)^3 = w^3$.

Proof. For w^* of (3),

$$E\left(w^*\right)^3 = m_n^3 + 3m_n v/n = w^3$$

if $m_n^3 + 3m_n v/n - w^3 = 0$. Set $\gamma = (v/n)^3 + w^6/4$. Since $\gamma > 0$, this cubic has one real root given by Equation (3.8.2) of Abramowitz and Stegun [1]:

$$m_n = S_1^{1/3} + S_2^{1/3},$$

where $S_j = w^3/2 \pm \gamma^{1/2}$. Suppose that $w > 0$. (If not, replace w by $|w|$.) Then for ν of (4),

$$\gamma^{1/2} = w^3(1+d)^{1/2}/2,$$

where $d = cn^{-3}$. Furthermore,

$$(1+d)^{1/2} = \sum_{j=0}^{\infty}\binom{1/2}{j} d^j, \quad S_1 = 1 + D$$

for

$$D = \sum_{j=1}^{\infty}\binom{1/2}{j} d^j/2 = \sum_{j=1}^{\infty} a_j d^j/j!.$$

Furthermore,

$$D^k/k! = \sum_{j=k}^{\infty} B_{j,k}(\mathbf{a}) d^j/j!$$

implies

$$S_1^{1/3} = \sum_{k=0}^{\infty}\binom{1/3}{k} D^k = \sum_{j=0}^{\infty} A_j d^j/j!.$$

Also

$$S_2 = 1/2 - (1+d)^{1/2}/2 = -\sum_{j=1}^{\infty} a_j d^j/j! = -a_1 d(1+U)$$

for

$$U = \sum_{k=1}^{\infty} B_k d^k/k!$$

and

$$U^j/j! = \sum_{k=j}^{\infty} B_{k,j}(\mathbf{B}) d^k/k!.$$

Then

$$S_2^{1/3} = u(1+U)^{1/3} = u\sum_{j=0}^{\infty}\binom{1/3}{j} U^j = u\sum_{j=0}^{\infty}(1/3)_j U^j/j! = \sum_{k=0}^{\infty} C_k d^k/k!,$$

where

$$C_k = \sum_{j=0}^{k}(1/3)_j\, B_{k,j}(\mathbf{B}).$$

Hence, for the choice of $f_n(w)$, $E(w^*)^3 = w^3$. □

The method of Theorems 1 and 3 will not work for $t(w) = w^5$ since there is no explicit solution to a quintic. However, we now show how to obtain an unbiased or bias-reduced estimate of w^a for *any* $a > 0$. Set $\Delta = \widehat{w} - w = w^* - m_n$. Then

$$E\left(w^*\right)^a = E\left(m_n + \Delta\right)^a = \sum_{j=0}^{\infty} \binom{a}{j} m_n^{a-j} \mu_j\left(\widehat{w}\right) = \sum_{j=0}^{\infty} \binom{a}{2j} m_n^{a-2j} \mu_{2j}\left(\widehat{w}\right),$$

where

$$\mu_{2j}\left(\widehat{w}\right) = N_j v^j, \quad N_0 = 1, \quad N_j = 1\,3\cdots(2j-1)$$

for $j \geq 1$. By (2),

$$f_n(w)^k / k! = \sum_{i=k}^{\infty} B_{i,k}(\mathbf{b}) n^{-i} / i!.$$

So,

$$\left(m_n / w\right)^a = \left[1 + f_n(w)/w\right]^a = \sum_{k=0}^{\infty} (a)_k w^{-k} f_n(w)^k / k! = \sum_{i=0}^{\infty} D_{a,i} n^{-i} / i!$$

for

$$D_{a,i} = \sum_{k=0}^{i} (a)_k w^{-k} B_{i,k}(\mathbf{b}).$$

This implies

$$E\left(w^*\right)^a / w^a = \sum_{k=0}^{\infty} n^{-k} E_k$$

for

$$E_k = \sum_{i+j=k} \binom{a}{2j} N_j v^j D_{a-2j,i} / i!.$$

So,

$$b_1 = -(a-1)w/2, \qquad E_1 = 0, \qquad E\left(w^*\right)^a = w^a + O\left(n^{-2}\right),$$
$$b_2 = w(a-1)\left[-(a-1)^2 + (a-1)_2 wv - (a-2)_2 v^2\right], \qquad E_2 = 0, \qquad E\left(w^*\right)^a = w^a + O\left(n^{-3}\right).$$

In this way, we can construct $f_n(w)$ so that for any given $a > 0$ and $k \geq 1$, $E\left[\widehat{w} + f_n(w)\right]^a = w^a + O\left(n^{-k}\right)$.

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