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Levent Kargin and Mümün Can

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Number Theory / *Théorie des nombres*

Harmonic number identities via polynomials with r -Lah coefficients

*Identités sur les nombres harmonique via des polynômes
à coefficients r -Lah*

Levent Kargin^{*, a} and Mümün Can^a

^a Department of Mathematics, Akdeniz University, Antalya, Turkey.

E-mails: leventkargin48@gmail.com, mcan@akdeniz.edu.tr.

Abstract. In this paper, polynomials whose coefficients involve r -Lah numbers are used to evaluate several summation formulae involving binomial coefficients, Stirling numbers, harmonic or hyperharmonic numbers. Moreover, skew-hyperharmonic number is introduced and its basic properties are investigated.

Résumé. Dans cet article, des polynômes à coefficients faisant intervenir les nombres r -Lah sont utilisés pour établir plusieurs formules de sommation en fonction des coefficients binomiaux, des nombres de Stirling et des nombres harmoniques ou hyper-harmoniques. De plus, nous introduisons le nombre asymétrique-hyper-harmonique et nous étudions ses propriétés de base.

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1. Introduction

The n^{th} harmonic number H_n is defined by

$$H_n = \sum_{k=1}^n \frac{1}{k},$$

with the assumption $H_0 = 0$. These numbers have a long mathematical history and are seen in various branches of mathematics, especially in number theory. Therefore, there is an enormous literature about the identities involving harmonic numbers with binomial coefficients,

* Corresponding author.

Stirling numbers and Bernoulli numbers: Benjamin and Quinn [5, Identity 4] used combinatorial technique to obtain

$$\sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} k = n! H_n = \begin{bmatrix} n+1 \\ 2 \end{bmatrix},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the Stirling number of the first kind. Kellner [35] capitalized derivative operator to achieve

$$\sum_{k=0}^n (-1)^{k+1} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{k!}{k+1} H_k = \frac{n}{2} B_{n-1},$$

where $\begin{Bmatrix} n \\ k \end{Bmatrix}$ is the Stirling number of the second kind and B_n is the n^{th} Bernoulli number. By using finite differences, Spivey [48] exhibited many combinatorial sums, for instance, the *binomial harmonic identity*

$$H_n = \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k+1}}{k}.$$

This identity was also recorded by Chu [18] and Boyadzhiev [11, 12] with different methods. Utilizing the backward difference Boyadzhiev [11, 12] reproved the *symmetric formula*

$$\frac{H_n}{n+1} = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{H_k}{k+1}.$$

Moreover, numerous evaluation formulas for binomial-harmonic sums (sums involving binomial coefficients and harmonic numbers) are produced by using generating function [47], algorithmic methods [42], hypergeometric summation theorems [19], derivative operator [17, 18, 51], Hadamard multiplication Theorem [9].

As a generalization of the harmonic numbers, hyperharmonic numbers $h_n^{(r)}$ are defined, for $r \geq 1$, by [20]

$$h_n^{(r)} = \sum_{k=1}^n h_k^{(r-1)}, \quad \text{with } h_n^{(0)} = \frac{1}{n}, n \geq 1, \quad \text{and } h_0^{(r)} = 0. \quad (1)$$

It is obvious that $h_n^{(1)} = H_n$. These numbers have the generating function [4]

$$\sum_{n=0}^{\infty} h_n^{(r)} t^n = -\frac{\ln(1-t)}{(1-t)^r} \quad (2)$$

and the explicit representation (cf. [4, 20])

$$h_n^{(r)} = \binom{n+r-1}{n} (H_{n+r-1} - H_{r-1}). \quad (3)$$

There exist many elegant identities involving hyperharmonic numbers. Some of these identities are exhibited by using combinatorial technique [4], Euler–Siedel matrix [24, 39], derivative and difference operators [22, 23, 26, 38], Pascal type matrix [15]. Whether the properties of harmonic numbers are provided by hyperharmonic numbers are actively studied. For instance, the harmonic number H_n is never an integer except for H_1 ; this is a classical result of Theisinger [49]. In [37], Mezö proved that if $r = 2$ or $r = 3$, the numbers $h_n^{(r)}$ are never integers except the trivial case when $n = 1$. He conjectured that the hyperharmonic numbers $h_n^{(r)}$ are never integers except when $n = 1$. This conjecture was handled by Ait–Amrane and Belbachir [1, 2], Cereceda [16] and recently, Göral and Sertbaş [28]. Moreover Euler showed that (see, e.g., [27, 44])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \left(1 + \frac{m}{2}\right) \zeta(m+1) - \frac{1}{2} \sum_{k=1}^{m-2} \zeta(m-k) \zeta(k+1), \quad m \in \mathbb{N} \setminus \{1\},$$

where $\zeta(s)$ is the usual Riemann zeta function. The summation on the left-hand side is known as Euler sum. Euler sum is generalized in different means and evaluated in terms of miscellaneous zeta functions. One of the generalizations is Euler sum of hyperharmonic numbers which has also been evaluated in terms of zeta functions [6, 21, 25, 40, 50].

The aim of this paper is to contribute to the theory of harmonic and hyperharmonic numbers by means of producing identities involving binomial coefficients, harmonic numbers, Stirling numbers and hyperharmonic numbers. For this purpose we capitalize some families of polynomials whose coefficients involve r -Lah numbers (see Section 2 for r -Lah numbers). In fact, these polynomials appear by applying the following Mellin type derivative to appropriate functions:

$$(xD + 2r)(xD + 2r + 1) \cdots (xD + 2r + n - 1).$$

Properties of the arising polynomials give rise to *binomial hyperharmonic identity* and several summation formulas involving (hyper)harmonic numbers. Moreover, a closed-form evaluation formula for an Euler-type sum is deduced (Theorem 8). It should be noted that the studies on the Euler-type sums containing hyperharmonic numbers depend on the lower index [6, 21, 25, 31, 40, 50], however, the sum in question is over the upper index. Furthermore, we come across a generalization of the skew-harmonic numbers H_n^- defined by

$$H_n^- = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}, \quad \text{with } H_0^- = 0,$$

i.e., partial sums of the expansion of $\log 2$. We then examine basic properties of these numbers. Additionally, several new formulas for the r -Lah numbers are presented.

2. Preliminaries

Let $x^{(n)} = x(x+1) \cdots (x+n-1)$, $x^{(0)} = 1$, and $(x)_n = (-1)^n (-x)^{(n)}$ denote the rising and falling factorial functions. The r -Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r$ and the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ can be defined by [14]

$$(x+r)^{(n)} = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r x^k, \quad (x+r)^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r (x)_k$$

Note that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_0$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_0$ are the Stirling numbers of the first and second kind.

The r -Lah numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r$ are defined by [3, 41]

$$(x+2r)^{(n)} = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r (x)_k, \quad (4)$$

and have the explicit formula

$$\left[\begin{smallmatrix} n \\ l \end{smallmatrix} \right]_r = \frac{n!}{k!} \binom{n+2r-1}{k+2r-1}, \quad (5)$$

and the generating function

$$\frac{1}{k!} \left(\frac{t}{1-t} \right)^k \left(\frac{1}{1-t} \right)^{2r} = \sum_{n=k}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r \frac{t^n}{n!}. \quad (6)$$

In particular $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_0 = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ is Lah numbers or rarely called Stirling numbers of the third kind [45].

The Mellin derivative $(xD) = x \frac{d}{dx}$ has been used for many different purposes, such as evaluating some power series, integrals [7, 13, 23, 33, 36] and also introducing some new families of polynomials [7, 22, 23, 33, 34]. When it is applied to a n -times differentiable function f we have [7]

$$(xD)^n f(x) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^k \frac{d^k}{dx^k} f(x). \quad (7)$$

According to the generalizations of the Stirling numbers of the second kind, numerous generalizations of the Mellin derivative have also been studied [7, 22, 33, 34].

Here, as a generalization of the Mellin derivative, we deal with the operator

$$(xD + 2r)^{(n)} = (xD + 2r)(xD + 2r + 1) \cdots (xD + 2r + n - 1).$$

Then, we have

$$(xD + 2r)^{(n)} f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_r x^k \frac{d^k}{dx^k} f(x), \quad (8)$$

which is a companion of (7) and follows from (4) and

$$(xD)_k f(x) = x^k \frac{d^k}{dx^k} f(x). \quad (9)$$

To see (9) we replace x by (xD) in

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

and then utilize the identities (7) and

$$\sum_{j=k}^n (-1)^{j-k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}.$$

We finally want to recall the r -Stirling transform which will be useful in the next sections:

$$a_n = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix}_r b_k \quad (n \geq 0) \quad \text{if and only if} \quad b_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_r a_k \quad (n \geq 0).$$

3. Identities via geometric r -Lah polynomials

In this section, we shall present several identities involving hyperharmonic numbers. These identities follow from the connection between hyperharmonic numbers and polynomials that appear in (8) for $f(x) = (1-x)^{-1}$. Thus,

$$(xD + 2r)^{(n)} \left(\frac{1}{1-x} \right) = \frac{1}{1-x} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_r k! \left(\frac{x}{1-x} \right)^k.$$

We denote

$$\mathfrak{L}_{n,r}(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_r k! x^k, \quad (10)$$

and call these polynomials *geometric r -Lah polynomials*. The notation $L_n(x)$ for the polynomial whose coefficient is the Lah numbers $\begin{bmatrix} n \\ i \end{bmatrix}_0 = \begin{bmatrix} n \\ i \end{bmatrix}$ was firstly used by Guo and Qi in [30] and their related papers.

Using the fact $(xD + 2r)^{(n)} x^m = (m + 2r)^{(n)} x^m$, we see that

$$(xD + 2r)^{(n)} \left(\frac{1}{1-x} \right) = \sum_{k=0}^{\infty} (k + 2r)^{(n)} x^k = \frac{1}{1-x} \mathfrak{L}_{n,r} \left(\frac{x}{1-x} \right).$$

Setting $r = 0$ and utilizing (5) and (10) give the generating function for rising factorial [43]

$$\sum_{k=1}^{\infty} k^{(n)} x^k = n! \frac{x}{(1-x)^{n+1}}.$$

Combining these power series we deduce the following relation:

Proposition 1. *For non-negative integers n and r , we have*

$$\sum_{k=0}^r \binom{k+n}{k} x^k (1-x)^{n+1} = 1 - \sum_{k=0}^n \binom{n+r+1}{k+r+1} x^{r+1+k} (1-x)^{n-k}.$$

The geometric r -Lah polynomials can be generated by

$$\sum_{n=0}^{\infty} \mathfrak{L}_{n,r}(x-1) \frac{t^n}{n!} = \frac{1}{(1-t)^{2r-1}} \frac{1}{1-xt}, \quad |xt(1-t)| < 1. \tag{11}$$

This follows from (10) and (6) and leads to investigate some properties of these polynomials. For instance,

$$\mathfrak{L}_{n,r}(-1) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r (-1)^k k! = (2r-1)^{(n)}, \quad n \geq 0, r \geq 1. \tag{12}$$

Moreover, utilizing the generalized binomial theorem

$$\frac{1}{(1-t)^\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^{(n)}}{n!} t^n, \quad |t| < 1 \tag{13}$$

in (11) yields the following identity:

Proposition 2. *We have*

$$\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+j+1)^{(r)}} = \frac{\binom{n+r-1}{n} / \binom{n+j}{n}}{(n+j+1)^{(r)}}.$$

As another consequence of (11), we now present a new formulation for the *binomial hyperharmonic identity* [39, Corollary 3.1]:

$$h_n^{(r)} = \sum_{k=0}^n \binom{n}{k} \alpha(k, r),$$

where

$$\alpha(k, r) = \begin{cases} h_k^{(r-k)}, & 0 \leq k < r \\ (-1)^{k+\delta_r} (r-1)! / k_r, & k \geq r \end{cases}$$

$\delta_r = 0$ or 1 , according to r is even or odd.

Theorem 3. *For all positive integers n and r ,*

$$\frac{h_n^{(r)}}{n^{(r)}} = \frac{1}{n} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k+1}}{k^{(r)}}.$$

Proof. Let $2r-1 \geq 0$ be an integer. Integrating both sides of (11) with respect to x from 0 to 1, we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 \mathfrak{L}_{n,r}(x-1) dx = -\frac{1}{t} \frac{\ln(1-t)}{(1-t)^{2r-1}}.$$

It is seen from (2) and (10) that

$$\int_0^1 \mathfrak{L}_{n,r}(x-1) dx = \sum_{k=0}^n (-1)^k \left[\begin{matrix} n \\ k \end{matrix} \right]_r \frac{k!}{k+1} = n! h_{n+1}^{(2r-1)}. \tag{14}$$

Hence, (5) completes the proof. □

We now write (11) in the form

$$\sum_{n=0}^{\infty} \mathfrak{L}_{n,r}(x-1) \frac{(1-e^{-t})^n}{n!} e^{-tm} = e^{-t(m-2r+1)} \frac{1}{1+x(e^{-t}-1)}$$

by setting $t \rightarrow 1 - e^{-t}$ and then multiplying both sides by e^{-tm} . We recall the r -geometric polynomials defined by the generating function [22, 32]

$$\sum_{n=0}^{\infty} w_{n,r}(x) \frac{t^n}{n!} = \frac{1}{1-x(e^t-1)} e^{rt}. \tag{15}$$

Utilizing (15) and the generating function of r -Stirling numbers of the second kind [14, Theorem 16]

$$\sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!} e^{rt}, \tag{16}$$

we relate the r -geometric polynomials and geometric r -Lah polynomials as in the following:

Theorem 4. *For all integers $n \geq 1$ and $m + 1 \geq 2r \geq 1$, we have*

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m \mathfrak{L}_{k,r}(x-1) = w_{n,m+1-2r}(-x). \tag{17}$$

It should be noted that for $r = 1/2$ and $x = 1/2$, (17) becomes

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m \frac{k!}{2^k} = E_n(m),$$

upon the use of $\mathfrak{L}_{k,1/2}(-1/2) = k!/2^k$ and $w_{n,m}(-1/2) = E_n(m)$. Here, $E_n(x)$ is the n^{th} Euler polynomial [45, p. 529].

Moreover, integrating both sides of (17) with respect to x from 0 to 1, and using (14) we see that

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m k! h_{k+1}^{(2r-1)} = \int_0^1 w_{n,m+1-2r}(-x) dx. \tag{18}$$

We now integrate (15) with respect to x from 0 to 1 and use the generating function of Bernoulli polynomials [45, p. 529]

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt},$$

to deduce that

$$\int_0^1 w_{n,r}(-x) dx = B_n(r), \tag{19}$$

which is a natural extension of the first identity given in [35, Theorem 1.3]. Thus, (18) and (19) entail the identity (20) given in the following theorem. In addition, applying r -Stirling transform to (20) and then using the well-known formula $B_k(1-x) = (-1)^k B_k(x)$ give (21).

Theorem 5. *For all non-negative integers n, r, m , we have*

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m k! h_{k+1}^{(r)} = B_n(m-r) \tag{20}$$

and

$$\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_m B_k(r) = n! h_{n+1}^{(r+m-1)}. \tag{21}$$

It is worth noting that (20) reduces to the well-known formula

$$\sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^k}{k+1} k! = B_n$$

since $h_{k+1}^{(0)} = 1/(k+1)$ and $B_n(0) = B_n$ is the n^{th} Bernoulli number. Moreover, both of (20) and (21) specialize some formulas in [15, p. 128–129].

4. Identities via harmonic geometric r -Lah polynomials

We continue to present identities for (hyper)harmonic numbers, which follow from the connection between hyperharmonic numbers and polynomials that appear in (8) for $f(x) = -\frac{\ln(1-x)}{(1-x)}$. In this case, using [23, Equation (27)]

$$\frac{d^k}{dx^k} \left(\frac{-\ln(1-x)}{1-x} \right) = k! \frac{H_k - \ln(1-x)}{(1-x)^{k+1}},$$

we deduce that

$$(xD + 2r)^{(n)} \left(\frac{-\ln(1-x)}{1-x} \right) = \frac{1}{1-x} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_r H_k k! \left(\frac{x}{1-x} \right)^k - \frac{\ln(1-x)}{1-x} \mathfrak{L}_{n,r} \left(\frac{x}{1-x} \right).$$

Let ${}_H\mathfrak{L}_{n,r}(x)$ denote the sum in the right-hand side of the above equation, i.e.,

$${}_H\mathfrak{L}_{n,r}(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_r k! H_k x^k, \quad (22)$$

which we call *harmonic geometric r -Lah polynomials*. Considering the generating function of harmonic numbers (2), we arrive at a closed-form evaluation formula for power series involving harmonic numbers.

Theorem 6. For all non-negative integers n, r

$$\sum_{m=0}^{\infty} (m+2r)^{(n)} H_m x^m = \frac{1}{1-x} {}_H\mathfrak{L}_{n,r} \left(\frac{x}{1-x} \right) - \frac{\ln(1-x)}{1-x} \mathfrak{L}_{n,r} \left(\frac{x}{1-x} \right).$$

In particular, for $r = 1/2$, we have

$$\sum_{m=0}^{\infty} \binom{m+n}{n} H_m x^m = \frac{1}{1-x} \sum_{k=0}^n \binom{n}{k} \frac{H_k x^k}{(1-x)^k} - \frac{\ln(1-x)}{(1-x)^{n+1}}.$$

We use (2) and (3) to see that

$$\sum_{m=0}^{\infty} \binom{m+n}{m} (H_{n+m} - H_m) x^m = H_n \sum_{m=0}^{\infty} \binom{m+n}{n} x^m - \frac{1}{1-x} \sum_{k=0}^n \binom{n}{k} \frac{H_k x^k}{(1-x)^k}.$$

We now utilize (3), (13) and the formula [8, Corollary 8]

$$\sum_{k=0}^n \binom{n}{k} H_k \lambda^k = (1+\lambda)^n H_n - \sum_{j=1}^n \frac{1}{j} (1+\lambda)^{n-j}$$

with $\lambda = x/(1-x)$ to obtain a generating function for hyperharmonic numbers with respect to upper index:

Theorem 7. We have

$$\sum_{m=0}^{\infty} h_n^{(m+1)} x^m = \sum_{j=0}^{n-1} \frac{1}{n-j} \left(\frac{1}{1-x} \right)^{j+1}.$$

The above generating function can be equivalently written as

$$\sum_{m=0}^{\infty} h_n^{(m+1)} x^m = \sum_{j=0}^{n-1} \frac{1}{n-j} \sum_{m=0}^{\infty} \frac{(j+1)^{(m)}}{m!} x^m. \quad (23)$$

This yields

$$h_n^{(m+1)} = \sum_{j=1}^n \binom{m+n-j}{m} \frac{1}{j}, \quad (24)$$

which was proved in [4, 24] by different methods.

Moreover, integrate both sides of (23) with respect to x from 0 to x and multiply it by $\frac{1}{x}$. Repeat this procedure for q times to obtain

$$\sum_{m=0}^{\infty} \frac{h_n^{(m+1)}}{(m+1)^q} x^m = \sum_{j=0}^{n-1} \frac{1}{n-j} \sum_{m=0}^{\infty} \frac{(j+1)^{(m)}}{m!} \frac{x^m}{(m+1)^q}.$$

Then we have obtained the following closed-form evaluation formula for an Euler-type sum:

Theorem 8.

$$\sum_{m=0}^{\infty} \frac{h_n^{(m+1)}}{(m+1)^q} x^m = \sum_{j=0}^{n-1} \frac{1}{n-j} \Phi_{j+1}^*(x, q, 1),$$

where

$$\Phi_{\mu}^*(z, s, a) = \sum_{m=0}^{\infty} \frac{\mu^{(m)}}{m!} \frac{z^m}{(m+a)^s}$$

is a generalization of the Hurwitz–Lerch zeta function [29].

From (22), it is seen that

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{L}_{n,r}(x) \frac{t^n}{n!} &\stackrel{(6)}{=} \frac{1}{(1-t)^{2r}} \sum_{k=0}^{\infty} H_k \left(\frac{xt}{1-t} \right)^k \\ &\stackrel{(2)}{=} \frac{1}{(1-t)^{2r-1}} \frac{\ln(1-t) - \ln(1-xt-t)}{1-xt-t}, \left| \frac{xt}{1-t} \right| < 1. \end{aligned}$$

Therefore, we have the generating function for the harmonic geometric r -Lah polynomials

$$\sum_{n=0}^{\infty} {}_H\mathcal{L}_{n,r}(x-1) \frac{t^n}{n!} = \frac{\ln(1-t) - \ln(1-xt)}{(1-t)^{2r-1} (1-xt)}. \tag{25}$$

Comparing (2) and (25), we reach that the harmonic geometric r -Lah polynomials are also closely related with hyperharmonic numbers as

$${}_H\mathcal{L}_{n,r}(-1) = -n! h_n^{(2r-1)}.$$

We now present some binomial-harmonic sums, which are also generalizations of the symmetric formula.

Theorem 9.

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{k+1}}{(k+m+1)^{(r)}} H_{k+m} &= \frac{m!}{(n+m+r)!} [n! h_n^{(r)} - r^{(n)} H_m] \\ &= \frac{r^{(n)} m!}{(n+m+r)!} (H_{n+r-1} - H_{r-1} - H_m). \end{aligned} \tag{26}$$

In particular,

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{k+1}}{(k+1)^{(r)}} H_k &= \frac{h_n^{(r)}}{(n+1)^{(r)}}, \\ \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{k+1}}{k+m+1} H_{k+m} &= \frac{n! m!}{(n+m+1)!} (H_n - H_m). \end{aligned}$$

Proof. By induction on m , it can be shown that

$$\sum_{n=m}^{\infty} \frac{d^m}{dx^m} {}_H\mathcal{L}_{n,r}(x-1) \frac{t^n}{n!} = m! \frac{\ln(1-t) - \ln(1-xt) + H_m}{(1-t)^{2r-1} (1-xt)^{m+1}} t^m.$$

Thus, we have

$$\frac{d^m}{dx^m} \frac{{}_H\mathcal{L}_{n+m,r}(x-1)}{(n+m)!} \Big|_{x=0} = m! \left(-h_n^{(2r-1)} + H_m \frac{(2r-1)^{(n)}}{n!} \right).$$

Hence, (26) follows from (22) and (5). □

To investigate the relation between the harmonic geometric r -Lah polynomials and some other well-known numbers or polynomials, we recall the harmonic r -geometric polynomials, defined by [22]

$${}_H w_{n,r}(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r k! H_k x^k.$$

The following theorem presents a relationship between the harmonic geometric r -Lah, r -geometric and harmonic r -geometric polynomials. The proof is similar to that of Theorem 4, so we omit it.

Theorem 10. For all integers $n \geq 1$ and $m + 1 \geq 2r \geq 1$

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m {}_H \mathfrak{L}_{k,r}(x-1) = n w_{n-1,m-2r+1}(-x) + {}_H w_{n,m-2r+1}(-x). \tag{27}$$

Since

$$w_{n,r}(0) = r^n, {}_H w_{n,r}(0) = 0 \text{ and } {}_H \mathfrak{L}_{n,r}(-1) = -n! h_n^{(2r-1)}, \tag{27} \text{ implies [15, p. 129]}$$

$$\sum_{k=1}^n (-1)^{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m k! h_k^{(r)} = n(m-r)^{n-1}.$$

Theorem 11. For all integers $n, m, r \geq 1$,

$$\begin{aligned} \sum_{k=1}^n \binom{n+1}{k+1} \frac{(-1)^{k+1}}{(k+1)^{(r)}} H_k &= \frac{1}{(n+r)!} \left[\begin{matrix} n+1 \\ 2 \end{matrix} \right]_r \\ &= \frac{1}{2(r-1)!} \left\{ (H_{n+r} - H_{r-1})^2 - (H_{n+r}^{(2)} - H_{r-1}^{(2)}) \right\} \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{1}{n!} \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_m k B_{k-1}(r) \\ = \binom{n+r+m-1}{r+m-2} \left\{ (H_{n+r+m-1} - H_{r+m-2})^2 - (H_{n+r+m-1}^{(2)} - H_{r+m-2}^{(2)}) \right\}, \end{aligned} \tag{29}$$

where

$$H_n^{(2)} = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}.$$

Proof. Integrating both sides of (25) with respect to x from 0 to 1, we have

$$\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} \int_0^1 {}_H \mathfrak{L}_{n,r+s}(x-1) dx = -\frac{1}{2} \frac{\ln^2(1-t)}{(1-t)^{2r+2s-1}}.$$

This and (2) yield

$$\int_0^1 {}_H \mathfrak{L}_{n-1,r+s}(x-1) dx = -\frac{(n-1)!}{2} \sum_{k=1}^{n-1} h_k^{(2r-1)} h_{n-k}^{(2s)}. \tag{30}$$

Therefore, we deduce from (22) and (5) that

$$\sum_{k=1}^n h_k^{(r)} h_{n+1-k}^{(s)} = 2 \sum_{k=1}^n \binom{n+r+s}{k+r+s} \frac{(-1)^{k+1}}{k+1} H_k.$$

The sum on the left-hand side can be evaluated in two ways: The first is to use the Broder's "vertical" exponential generating function for the Stirling numbers of the first kind [14, Theorem 15], which gives

$$\sum_{k=1}^n h_k^{(r)} h_{n+1-k}^{(s)} = \frac{2}{(n+1)!} \left[\begin{matrix} n+1 \\ 2 \end{matrix} \right]_{r+s}.$$

The second is to use the Xu’s relations (2.3) and (2.13) in [50]. In this case we find that

$$\sum_{k=1}^n h_k^{(r)} h_{n+1-k}^{(s)} = \binom{n+r+s}{r+s-1} \left\{ (H_{n+r+s} - H_{r+s-1})^2 - (H_{n+r+s}^{(2)} - H_{r+s-1}^{(2)}) \right\}. \tag{31}$$

These complete the proof of (28).

To prove (29) we first integrate both sides of (27):

$$\sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m \int_0^1 {}_H\mathcal{S}_{k,r}(x-1) dx = n \int_0^1 w_{n-1,m-2r+1}(-x) dx + \int_0^1 {}_Hw_{n,m-2r+1}(-x) dx.$$

The values of the first two integrals are known from (30) and (19). For the third integral, we use the following generating function for the harmonic r -geometric polynomials

$$\sum_{n=0}^{\infty} {}_Hw_{n,r}(x) \frac{t^n}{n!} = \frac{-\ln(1-x(e^t-1))}{1-x(e^t-1)} e^{rt},$$

which follows from (16) and (2). We then deduce that

$$\int_0^1 {}_Hw_{n,r}(-x) dx = -\frac{n}{2} B_{n-1}(r), \quad n \geq 1, r \geq 0,$$

which is a generalization of the second identity given in [35, Theorem 1.3]. Combining all these results give

$$\sum_{k=1}^n (-1)^{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m k! \sum_{l=1}^k h_l^{(r-1)} h_{k+1-l}^{(r)} = n B_{n-1}(m-2r+1).$$

r -Stirling transform implies

$$\sum_{l=1}^n h_l^{(r-1)} h_{n+1-l}^{(r)} = \frac{1}{n!} \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_m k B_{k-1}(2r-m).$$

Hence, (29) follows from (31). □

In particular,

$$\begin{aligned} \sum_{k=0}^n (-1)^{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m k! \left[(H_{k+1})^2 - H_{k+1}^{(2)} \right] &= n B_{n-1}(m-1), \\ \sum_{k=1}^n \frac{h_k^{(r)}}{n+1-k} &= \binom{n+r}{r-1} \left\{ (H_{n+r} - H_{r-1})^2 - (H_{n+r}^{(2)} - H_{r-1}^{(2)}) \right\}, \\ \sum_{k=1}^n \frac{H_k}{n+1-k} &= (H_{n+1})^2 - H_{n+1}^{(2)} = 2 \sum_{k=1}^n \frac{H_k}{k+1}, \end{aligned}$$

where the last equality follows from [47, p. 850].

5. Identities via exponential r -Lah polynomials

In this section, we deal with (8) for $f(x) = e^x$ and present some identities for arising polynomials. We then present some new relations for r -Lah numbers and hyperharmonic numbers.

It is seen from (8) that

$$(xD + 2r)^{(n)} e^x = e^x \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k = e^x \mathcal{L}_{n,r}(x), \tag{32}$$

where

$$\mathcal{L}_{n,r}(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k, \tag{33}$$

which we call exponential r -Lah polynomials. These polynomials are also handled with a different point of view in the recent paper [46]. One can see that the polynomial $\mathcal{L}_{n,r}(x)$ has the following generating function

$$\frac{1}{(1-t)^{2r}} e^{x \frac{t}{1-t}} = \sum_{n=0}^{\infty} \mathcal{L}_{n,r}(x) \frac{t^n}{n!}, \tag{34}$$

which entails

$$\begin{aligned} \mathcal{L}_{n,r+s}(x+y) &= \sum_{k=0}^n \binom{n}{k} \mathcal{L}_{k,r}(x) \mathcal{L}_{n-k,s}(y), \\ \mathcal{L}_{n,r+s}(x) &= \sum_{k=0}^n \binom{n}{k} (2s)^{(n-k)} \mathcal{L}_{k,r}(x) \end{aligned} \tag{35}$$

and the following Theorem 12:

Theorem 12. *We have the following recurrence relations:*

$$\frac{1}{n!} \mathcal{L}_{n,r+\frac{m+1}{2}}(x) = \sum_{k=0}^n \frac{1}{k!} \mathcal{L}_{k,r+\frac{m}{2}}(x) \tag{36}$$

and

$$\mathcal{L}_{n+1,r}(x) = (2n+2r+x) \mathcal{L}_{n,r}(x) - n(2r+n-1) \mathcal{L}_{n-1,r}(x). \tag{37}$$

Proof. Taking m times derivative of (34) with respect to x , we see that

$$\frac{d^m}{dx^m} \mathcal{L}_{n,r}(x) = (n)_m \mathcal{L}_{n-m,r+\frac{m}{2}}(x). \tag{38}$$

On the other hand, we have

$$\sum_{n=1}^{\infty} \frac{d}{dx} \mathcal{L}_{n,r}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{\mathcal{L}_{k,r}(x)}{k!} t^n$$

and then, by (38),

$$\frac{1}{n!} \mathcal{L}_{n,r+\frac{1}{2}}(x) = \sum_{k=0}^n \frac{\mathcal{L}_{k,r}(x)}{k!}.$$

Again we use (38) to obtain (36).

The proof of (37) follows by differentiating (34) with respect to t . □

We have the following formulas for r -Lah numbers.

Corollary 13. *We have*

$$\begin{aligned} \frac{1}{(n+m+1)!} \left[\begin{matrix} n+m+1 \\ l+m+1 \end{matrix} \right]_r &= \sum_{k=l}^n \frac{1}{(k+m)!} \left[\begin{matrix} k+m \\ l+m \end{matrix} \right]_r, \\ \sum_{k=l}^p \frac{k}{(k+m)!} \left[\begin{matrix} k+m \\ l+m \end{matrix} \right]_r &= \frac{p+1}{(p+m+1)!} \left[\begin{matrix} p+m+1 \\ l+m+1 \end{matrix} \right]_r - \frac{1}{(p+m+2)!} \left[\begin{matrix} p+m+2 \\ l+m+2 \end{matrix} \right]_r, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \sum_{k=l}^p \frac{k^2}{(k+m)!} \left[\begin{matrix} k+m \\ l+m \end{matrix} \right]_r &= \frac{(p+1)^2}{(p+m+1)!} \left[\begin{matrix} p+m+1 \\ l+m+1 \end{matrix} \right]_r - \frac{2p+3}{(p+m+2)!} \left[\begin{matrix} p+m+2 \\ l+m+2 \end{matrix} \right]_r \\ &\quad + \frac{1}{(p+m+3)!} \left[\begin{matrix} p+m+3 \\ l+m+3 \end{matrix} \right]_r. \end{aligned}$$

Proof. From (36) and (38), we conclude that

$$\frac{1}{n!} \left[\begin{matrix} n \\ l \end{matrix} \right]_{r+\frac{m+1}{2}} = \sum_{k=l}^n \frac{1}{k!} \left[\begin{matrix} k \\ l \end{matrix} \right]_{r+\frac{m}{2}}$$

and

$$\binom{n+m}{n} \left[\begin{matrix} n \\ k \end{matrix} \right]_{r+\frac{m}{2}} = \left[\begin{matrix} n+m \\ k+m \end{matrix} \right]_r \binom{k+m}{m}, \tag{40}$$

respectively. These formulas give (39).

Summing both sides of (39) over n , we find that

$$\begin{aligned} \frac{1}{(p+m+2)!} \left[\begin{matrix} p+m+2 \\ l+m+2 \end{matrix} \right]_r &= \sum_{k=l}^p \sum_{n=k}^p \frac{1}{(k+m)!} \left[\begin{matrix} k+m \\ l+m \end{matrix} \right]_r \\ &= \frac{p+1}{(p+m+2)!} \left[\begin{matrix} p+m+1 \\ l+m+1 \end{matrix} \right]_r - \sum_{k=l}^p \frac{k}{(k+m)!} \left[\begin{matrix} k+m \\ l+m \end{matrix} \right]_r, \end{aligned}$$

which is the second relation of this corollary. The third relation follows from the second relation by summing over p . □

Appealing to (32) and (33), and noting that $a^{(m+n)} = a^{(m)} (a+m)^{(n)}$, we see that

$$e^x \mathcal{L}_{n+2s,r}(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{r+s} (xD+2r)^{(2s)} (x^k e^x).$$

Using the Taylor expression of e^x in (32) and considering that $(xD+2r)^{(n)} x^k = (k+2r)^{(n)} x^k$ give

$$(xD+2r)^{(n)} (x^k e^x) = x^k e^x \mathcal{L}_{n,r+k/2}(x).$$

Thus, we have obtained the first identity in the following proposition. The second is a consequence of the first and (33).

Proposition 14. For all non-negative integer n ,

$$\mathcal{L}_{n+2s,r}(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{r+s} x^k \mathcal{L}_{2s,r+\frac{k}{2}}(x)$$

and

$$\left[\begin{matrix} n+2s \\ m \end{matrix} \right]_r = \sum_{k=0}^m \left[\begin{matrix} n \\ k \end{matrix} \right]_{r+s} \left[\begin{matrix} 2s \\ m-k \end{matrix} \right]_{r+\frac{k}{2}}.$$

We want to finalize this section giving a connection between the exponential r -Lah polynomials and geometric r -Lah polynomials, namely,

$$\mathcal{L}_{n,r}(x) = \int_0^\infty e^{-\lambda} \mathcal{L}_{n,r}(x\lambda) d\lambda. \tag{41}$$

This connection follows from (33), (10) and the well-known identity

$$\int_0^\infty z^k e^{-z} dz = k!, \quad k \in \mathbb{N}.$$

Then, with the use of (14), we see that this connection leads some identities for the hyperharmonic numbers:

Theorem 15. *We have*

$$(n + 1) h_{n+1}^{(r)} = (n + r) h_n^{(r)} + \frac{r^{(n)}}{n!},$$

$$h_{n+1}^{(r+s)} = \sum_{k=0}^n \binom{n-k+s}{s} h_{k+1}^{(r-1)}$$

and

$$h_{n+1}^{(r+s)} = \sum_{k=0}^{\min(n,s)} \binom{s}{k} h_{n+1-k}^{(r+k)}.$$

Proof. To prove the first identity, we replace x by $x\lambda$ in (37) and multiply both sides by $e^{-\lambda}$. We then integrate with respect to λ from 0 to ∞ , with the use of (41), and obtain that

$$\mathfrak{L}_{n+1,r}(x) = (2n + 2r) \mathfrak{L}_{n,r}(x) - n(n + 2r - 1) \mathfrak{L}_{n-1,r}(x) + \int_0^\infty x\lambda \mathcal{L}_{n,r}(x\lambda) e^{-\lambda} d\lambda. \tag{42}$$

It is clear from (33) that

$$\int_0^\infty x\lambda \mathcal{L}_{n,r}(x\lambda) e^{-\lambda} d\lambda = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^{k+1} (k + 1)!.$$

We now integrate both sides of (42) with respect to x from -1 to 0 and use (14) to deduce that

$$(n + 1)! h_{n+2}^{(2r-1)} = (2n + 2r) n! h_{n+1}^{(2r-1)} - (n + 2r - 1) n! h_n^{(2r-1)} + \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r \frac{(-1)^{k+1}}{k + 2} (k + 1)!. \tag{43}$$

Now utilizing (14), (12) and the following recurrence relation [41, Theorem 3.1]

$$\left[\begin{matrix} n + 1 \\ k \end{matrix} \right]_r = \left[\begin{matrix} n \\ k - 1 \end{matrix} \right]_r + (n + k + 2r) \left[\begin{matrix} n \\ k \end{matrix} \right]_r, \quad 1 \leq k \leq n, \tag{44}$$

we find that

$$\begin{aligned} \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r \frac{(-1)^{k+1}}{k + 2} (k + 1)! &= \sum_{k=1}^{n+1} \left[\begin{matrix} n + 1 \\ k \end{matrix} \right]_r \frac{(-1)^k}{k + 1} k! - \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r (n + 2r + k) \frac{(-1)^k}{k + 1} k! \\ &= (n + 1)! h_{n+2}^{(2r-1)} - (n + 2r - 1) n! h_{n+1}^{(2r-1)} - (2r - 1) (n)!. \end{aligned}$$

Hence, (43) completes the proof of the first identity.

Proofs of the second and the third identities are similar, but for this time we use (35) and

$$\mathcal{L}_{n,r+s}(x) = \sum_{k=0}^n \binom{n}{k} (2s)_{n-k} \mathcal{L}_{k,r+\frac{n-k}{2}}(x), \tag{45}$$

instead of (37), respectively. The relation (45) is a consequence of

$$(xD + 2r)^{(n)} [f(x)g(x)] = \sum_{k=0}^n \binom{n}{k} [(xD)_k f(x)] [(xD + 2r + k)^{(n-k)} g(x)],$$

which follows from (8), (40) and (9). □

It is worth noting that the first identity occurs in the recent paper [26]. Moreover, the identities given by Nyul and Rácz [41] for r -Lah numbers can be easily derived by using the exponential r -Lah polynomials, for instance, the identities (5) and (44) are consequences of (37) and (45), respectively.

6. Skew-hyperharmonic numbers

In this final section, we introduce a generalization of skew-harmonic numbers H_n^- . We then investigate some basic properties of these numbers.

Integrating both sides of (11) with respect to x from -1 to 0 , we see that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-1}^0 \mathcal{L}_{n,r}(x-1) dx = \frac{1}{t} \frac{\ln(1+t)}{(1-t)^{2r-1}}.$$

Since the skew-harmonic numbers have the generating function (see, for example [10])

$$\sum_{n=1}^{\infty} H_n^- t^n = \frac{\ln(1+t)}{(1-t)},$$

we set

$$\sum_{n=1}^{\infty} h_n^{(r)-} t^n = \frac{\ln(1+t)}{(1-t)^r} \tag{46}$$

and call *skew-hyperharmonic numbers* for $h_n^{(r)-}$. In the light of this notation, we find that

$$\int_{-1}^0 \mathcal{L}_{n,r}(x-1) dx = n! h_{n+1}^{(2r-1)-}.$$

This relation gives rise to evaluate some finite summations in terms of skew-hyperharmonic numbers. These results are stated in the following Theorem 16.

Theorem 16. *We have*

$$\begin{aligned} \frac{h_n^{(r)-}}{n^{(r)}} &= \frac{1}{n} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k+1}}{k^{(r)}} (2^k - 1), \\ \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^{(r)}} 2^{k+1} &= \frac{1}{(n+1)^{(r)}} \left[(n+1) h_{n+1}^{(r)-} - (n+r) h_n^{(r)-} + \frac{r^{(n)}}{n!} \right], \\ h_{n+1}^{(r+s)-} &= \sum_{k=0}^n \binom{n-k+s}{n-k} h_{k+1}^{(r-1)-} \end{aligned}$$

and

$$h_{n+1}^{(r+s)-} = \sum_{k=0}^{\min(n,s)} \binom{s}{k} h_{n+1-k}^{(r+k)-}$$

The proof of the first identity is similar to the proof of Theorem 3. The proofs of the other identities are similar to the proof of Theorem 15. So we omit the proofs. Note that in the case $r = 1$, the first identity was recorded in [11].

Particular cases of third identity give counterparts of (1), (24) and [26, p. 20] as follows:

Corollary 17.

$$\begin{aligned} h_n^{(r)-} &= \sum_{k=1}^n h_k^{(r-1)-}, \\ h_n^{(r+1)-} &= \sum_{k=1}^n \binom{n-k+r}{r} \frac{(-1)^{k+1}}{k}, \\ h_n^{(r+1)-} &= \sum_{k=1}^n \binom{n-k+r-1}{r-1} H_k^-. \end{aligned}$$

Finally, we have the following closed formula:

Theorem 18.

$$\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^{(r)}} 2^{-k-1} = \frac{1}{(n+1)^{(r)}} \left[r h_r^{(n+1)^-} - (n+1) h_{r-1}^{(n+2)^-} + \frac{(-1)^r}{2^{n+1}} \right].$$

Proof. It is obvious from (46) that

$$\sum_{k=0}^n \binom{k+r}{k} (-1)^{n-k} = (n+1) h_{n+1}^{(r+1)^-} - (r+1) h_n^{(r+2)^-}, \quad n \geq 1.$$

We combine this and Proposition 1 with $x = -1$ and complete the proof. \square

We conclude this section by noting that skew-hyperharmonic numbers can be discussed from Euler sums point of view. Moreover, non-integer property of them can be examined.

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