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Number Theory / *Théorie des nombres*

# On pairs of equations involving unlike powers of primes and powers of 2

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**Abstract.** In this paper, it is proved that every pair of sufficiently large even integers can be represented by a pair of equations, each containing one prime, one prime square, two prime cubes and 302 powers of 2. This result constitutes a refinement upon that of L. Q. Hu and L. Yang.

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## 1. Introduction

In the 1950s, Linnik [2, 3] proved that each large even integer  $N$  is a sum of two primes and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_{k_1}}, \quad (1)$$

where here and hereafter, the letters  $p$  and  $v$ , with or without subscripts, denote a prime number and a positive integer respectively. The famous Goldbach conjecture implies that  $k_1 = 0$ . The explicit value for the number  $k_1$  was improved by many authors.

In 1999, Liu, Liu and Zhan [5] proved that every sufficiently large even integer  $N$  can be represented in the form

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_{k_2}}, \quad (2)$$

and they also showed that there is a representation of the form (2) for some finite value of  $v_{k_2}$ . The best result now is Zhao [11], who obtained  $k_2 = 39$ .

In 2001, Liu and Liu [4] proved that every large even integer  $N$  can be written as a sum of eight cubes of primes and  $k_3$  powers of 2,

$$N = p_1^3 + p_2^3 + \dots + p_8^3 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_{k_3}}. \quad (3)$$

The acceptable value for the number  $k_3$  was determined by D. Platt and T. Trudgian [11], where  $k_3 = 341$ .

In 2011, Liu and Lü [7] considered the hybrid problem of (1), (2) and (3), i.e.

$$N = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_k}, \tag{4}$$

and proved that every sufficiently large even integer can be written as one prime, one square of primes, two cubes of primes and 161 powers of 2. In 2015, D. Platt and T. Trudgian [10] improved the value of  $k$  to 156, then to 16 by Zhao [12] and finally 15 by Lü [9].

It is of interest to investigate the simultaneous representation of pairs of positive even integers satisfying  $N_2 \gg N_1 > N_2$ , in the form

$$\begin{cases} N_1 = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_k}, \\ N_2 = p_5 + p_6^2 + p_7^3 + p_8^3 + 2^{v_1} + 2^{v_2} + \dots + 2^{v_k}, \end{cases} \tag{5}$$

where  $k$  is a positive integer. In 2017, Hu and Yang [1] proved that for  $k = 455$ , the equations (5) are solvable. In this paper, we obtain a further improvement of the value of  $k$  by giving the following theorem.

**Theorem.** *For  $k = 302$ , the equations (5) are solvable for every pair of sufficiently large positive even integers  $N_1$  and  $N_2$  satisfying  $N_2 \gg N_1 > N_2$ .*

## 2. Notation and Some Preliminary Lemmas

For the proof of the Theorem, in this section we introduce the necessary notation and Lemmas.

Throughout this paper, by  $N_i$  we denote a sufficiently large even integer. In addition, let  $\eta < 10^{-10}$  be a fixed positive constant, and let  $\varepsilon < 10^{-10}$  be an arbitrarily small positive constant not necessarily the same in different formulae. The letter  $p$ , with or without subscripts, is reserved for a prime number. We use  $e(\alpha)$  to denote  $e^{2\pi i \alpha}$  and  $e_q(\alpha) = e(\alpha/q)$ . By  $A \sim B$  we mean that  $B < A \leq 2B$ . We denote by  $(m, n)$  the greatest common divisor of  $m$  and  $n$ . As usual,  $\varphi(n)$  and  $\mu(n)$  denote Euler’s function and the Möbius function respectively. For  $i = 1, 2$ , let

$$\begin{aligned} P_i &= \frac{1}{2} \sqrt{(1-\eta)N_i}, \quad U_i = \frac{1}{2} \left( \frac{\eta N_i}{2} \right)^{\frac{1}{3}}, \quad V_i = \frac{1}{2} \left( \frac{\eta N_i}{2} \right)^{\frac{5}{18}}, \quad L = \frac{\log \left( \frac{N_i}{\log N_i} \right)}{\log 2}, \\ F_i &= F_i(\alpha_i, N_i) = \sum_{p \leq N_i} (\log p) e(\alpha_i p), \quad G_i = G_i(\alpha_i, P_i) = \sum_{p \sim P_i} (\log p) e(\alpha_i p^2), \\ S_i &= S_i(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(\alpha_i p^3), \quad T_i = T_i(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(\alpha_i p^3), \\ H(\alpha_i) &= \sum_{v \leq L} e(\alpha_i 2^v), \quad \mathcal{E}(\lambda) = \{\alpha_i \in (0, 1] : |H(\alpha_i)| \geq \lambda L\}. \end{aligned}$$

For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this purpose, we set

$$Q_{1i} = N_i^{\frac{1}{9} - 2\varepsilon}, \quad Q_{2i} = N_i^{\frac{8}{9} + \varepsilon},$$

and for  $(a_i, q_i) = 1, 1 \leq a_i \leq q_i$ , put

$$\begin{aligned} \mathfrak{M}_i(q_i, a_i) &= \left( \frac{a_i}{q_i} - \frac{1}{q_i Q_{2i}}, \frac{a_i}{q_i} + \frac{1}{q_i Q_{2i}} \right], \quad \mathfrak{M}_i = \bigcup_{1 \leq q_i \leq Q_{1i}} \bigcup_{\substack{a_i=1 \\ (a_i, q_i)=1}}^{q_i} \mathfrak{M}_i(q_i, a_i), \\ \mathfrak{J}_0 &= \left( \frac{1}{Q_{2i}}, 1 + \frac{1}{Q_{2i}} \right], \quad \mathfrak{m}_i = \mathfrak{J}_0 \setminus \mathfrak{M}_i. \end{aligned}$$

Then it follows from orthogonality that

$$\begin{aligned}
 R(N_1, N_2) = & \sum_{\substack{N_1=p_1+p_2^2+p_3^3+p_4^3+2^{v_1}+2^{v_2}+\dots+2^{v_k} \\ N_2=p_5+p_6^2+p_7^3+p_8^3+2^{v_1}+2^{v_2}+\dots+2^{v_k} \\ p_1 \leq N_1, p_2 \sim P_1, p_3 \sim U_1, p_4 \sim V_1, p_5 \leq N_2, \\ p_6 \sim P_2, p_7 \sim U_2, p_8 \sim V_2, v_j \leq L (j=1,2,\dots,k)}} (\log p_1)(\log p_2) \cdots (\log p_8) \\
 & = \int_0^1 \int_0^1 F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k (\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2. \tag{6}
 \end{aligned}$$

Now we state the lemmas required in this paper.

**Lemma 1.** For  $\frac{N_i}{2} < n \leq N_i (i = 1, 2)$ , we have

$$\int_{\mathfrak{M}_i} F_i G_i S_i T_i e(-\alpha n) d\alpha_i = \frac{1}{2 \cdot 3^2} \mathfrak{S}(n) J(n) + O\left(\frac{N_i^{\frac{10}{9}}}{L}\right).$$

Here  $\mathfrak{S}(n)$  is defined as

$$\begin{aligned}
 \mathfrak{S}(n) = & \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q \frac{S_2^*(q, a) S_3^{*2}(q, a) e_q(-an) \mu(q)}{\varphi^4(q)}, \\
 S_k^*(q, a) = & \sum_{\substack{r=1 \\ (r,q)=1}}^q e_q(ar^k),
 \end{aligned}$$

and satisfies  $\mathfrak{S}(n) > 0.24485083$  for  $n \equiv 0 \pmod{2}$ .  $J(n)$  is defined as

$$\begin{aligned}
 J(n) = & \sum_{\substack{n=m_1+m_2+m_3+m_4 \\ m_1 \leq N_i, P_i^2 < m_2 \leq 4P_i^2, \\ U_i^3 < m_3 \leq 8U_i^3, V_i^3 < m_4 \leq 8V_i^3}} (m_2)^{-\frac{1}{2}} (m_3 m_4)^{-\frac{2}{3}},
 \end{aligned}$$

and satisfies

$$N_i^{\frac{10}{9}} \ll J(n) \ll N_i^{\frac{10}{9}}.$$

**Proof.** This can be found in Lemma 2.1 and Section 3 in Liu and Lü [7]. □

**Lemma 2.** For  $\alpha \in \mathfrak{m}_i (i = 1, 2)$ , we have

- (i)  $\max_{\alpha \in \mathfrak{m}_i} |F_i| \ll N_i^{\frac{17}{18} + \varepsilon}$ ,
- (ii)  $\max_{\alpha \in \mathfrak{m}_i} |G_i| \ll N_i^{\frac{4}{9} + \varepsilon}$ ,
- (iii)  $\max_{\alpha \in \mathfrak{m}_i} |S_i| \ll N_i^{\frac{13}{42} + \varepsilon}$ ,
- (iv)  $\max_{\alpha \in \mathfrak{m}_i} |T_i| \ll N_i^{\frac{5}{18} + \varepsilon}$ .

**Proof.** See Lemma 2.5 in Hu and Yang [1]. □

**Lemma 3.** For  $i = 1, 2$ , we have

- (i)  $\int_0^1 |G_i S_i T_i|^2 d\alpha_i \leq 6.4894513 U_i^2 V_i^2$ ,
- (ii)  $\int_0^1 |F_i H^6(2\alpha_i)|^2 d\alpha_i \leq 2.009 \frac{N_i L^{12}}{\log^2 N_i}$ .

**Proof.** For (i), we follow the notation as (3.3) in Zhao [12] and define  $\mathfrak{J}_i(3)$  as

$$\mathfrak{J}_i(3) = \frac{1}{2^2 \cdot 3^4} \sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3 \\ P_i^2 < m_1, n_1^2 \leq 4P_i^2 \\ U_i^3 < m_2, n_2 \leq 8U_i^3 \\ V_i^3 < m_3, n_3 \leq 8V_i^3}} (m_1 n_1)^{-\frac{1}{2}} (m_2 n_2 m_3 n_3)^{-\frac{2}{3}}.$$

Moreover, noting from the fact that

$$\begin{aligned} m_1 &= n_1 + n_2 + n_3 - m_2 - m_3 \\ &\geq n_1 + U^3 + V^3 - 8U^3 - 8V^3 \\ &\geq (1 - 4\eta) n_1, \end{aligned} \tag{7}$$

and

$$\sum_{U_i^3 < m \leq 8U_i^3} m^{-\frac{2}{3}} \sim 3U_i, \quad \sum_{P_i^2 < m \leq 4P_i^2} m^{-1} \sim 2\log 2, \tag{8}$$

we obtain

$$\begin{aligned} \mathfrak{J}_i(3) &\leq \frac{1}{2^2 \cdot 3^4} \sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3 \\ P_i^2 < m_1, n_1 \leq 4P_i^2 \\ U_i^3 < m_2, n_2 \leq 8U_i^3 \\ V_i^3 < m_3, n_3 \leq 8V_i^3}} (1 - 4\eta)^{-\frac{1}{2}} n_1^{-1} (m_2 n_2 m_3 n_3)^{-\frac{2}{3}} \\ &\leq \frac{1 + o(1)}{2^2 \cdot 3^4} (1 + 4\eta) \cdot 2\log 2 \cdot (3U_i)^2 (3V_i)^2 \\ &\leq \left( \frac{\log 2}{2} + o(1) \right) U_i^2 V_i^2. \end{aligned}$$

Thus we deduce from Lemma 3.1 and Lemma 4.1 in Zhao [12] that

$$\int_0^1 |G_i S_i T_i|^2 d\alpha \leq 6.4894513 U_i^2 V_i^2.$$

We then complete the proof of Lemma 3 (i). For (ii), it is Lemma 2.4 in Lü [8]. □

**Lemma 4.** Let  $\Xi(N_i, k) = \{(1 - \eta)N_i \leq n_i \leq N_i : n_i = N_i - 2^{\nu_1} - 2^{\nu_2} - \dots - 2^{\nu_k}\} (i = 1, 2)$ . For  $k \geq 2$  and  $N_1 \equiv N_2 \equiv 0 \pmod{2}$ , we have

$$\sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} J(n_1) J(n_2) \geq 43.407769 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k.$$

**Proof.** The domain of  $J(n_i)$  can be written as

$$\mathfrak{D} = \left\{ (m_1, m_2, m_3, m_4) : \begin{array}{l} m_1 \leq N_i, P_i^2 < m_2 \leq (2P_i)^2, U_i^3 < m_3 \leq (2U_i)^3, \\ V_i^3 < m_3 \leq (2V_i)^3, m_1 = n - m_2 - m_3 - m_4 \end{array} \right\}.$$

Define

$$\mathfrak{D}^* = \{(m_2, m_3, m_4) : P_i^2 < m_2 \leq 3P_i^2, U_i^3 < m_3 \leq (2U_i)^3, V_i^3 < m_3 \leq (2V_i)^3\}.$$

For  $(m_1, \dots, m_6) \in \mathfrak{D}^*$ , we can deduce from  $(1 - \eta)N_i \leq n_i \leq N_i$  that

$$m_1 = n_i - m_2 - m_3 - m_4 \leq N_i.$$

Thus  $\mathfrak{D}^*$  is a subset of  $\mathfrak{D}$ . Then it follows from (8) and Euler–Maclaurin summation that

$$\begin{aligned} \mathfrak{J}(n_i) &\geq \sum_{(m_2, m_3, m_4) \in \mathfrak{D}^*} m_2^{-\frac{1}{2}} m_3^{-\frac{2}{3}} m_4^{-\frac{2}{3}} \\ &\geq \sum_{P_i^2 < m_2 \leq 3P_i^2} m_2^{-\frac{1}{2}} \sum_{U_i^3 < m_3 \leq (2U_i)^3} m_3^{-\frac{2}{3}} \sum_{V_i^3 < m_4 \leq (2V_i)^3} m_4^{-\frac{2}{3}} \\ &\geq 2 \cdot 3 \cdot 3 \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) (1 - \eta)^{\frac{1}{2}} N_i^{\frac{1}{2}} U_i V_i \\ &\geq (9\sqrt{3} - 9) (1 - \eta)^{\frac{1}{2}} N_i^{\frac{1}{2}} U_i V_i. \end{aligned} \tag{9}$$

It follows from (9) and Lemma 4.1 in Liu [6] that

$$\begin{aligned} \sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} J(n_1)J(n_2) &\geq (9\sqrt{3} - 9)^2 (1 - \eta) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 \sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} 1 \\ &\geq (9\sqrt{3} - 9)^2 (1 - \eta) (1 - \varepsilon) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k \\ &\geq 43.407769 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k. \end{aligned} \quad \square$$

**Lemma 5.** *We have*

$$\text{meas}(\mathcal{E}(\lambda)) \ll N_i^{-E(\lambda)}, \quad \text{with } E(0.9457435) > \frac{109}{126} + 10^{-10}.$$

**Proof.** See Table 1 in Platt and Trudgian [10]. □

### 3. Auxiliary Estimates

We initiate our proof by recalling the Farey dissections (6) that

$$\begin{aligned} R(N_1, N_2) &= \int_0^1 \int_0^1 F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &= \left( \int_{\mathfrak{M}_1} + \int_{\mathfrak{m}_1 \cap \mathcal{E}(\lambda)} + \int_{\mathfrak{m}_1 \setminus \mathcal{E}(\lambda)} \right) \left( \int_{\mathfrak{M}_2} + \int_{\mathfrak{m}_2 \cap \mathcal{E}(\lambda)} + \int_{\mathfrak{m}_2 \setminus \mathcal{E}(\lambda)} \right) \\ &\quad F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &= \sum_{s=1}^3 \sum_{t=1}^3 R_{st}, \end{aligned}$$

where  $R_{st}$  denotes the combination of  $s$ -th term in the first bracket and the  $t$ -th term in the second bracket.

**Proposition 6.** *We have*

$$R_{11} \geq 0.008032 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k.$$

**Proof.** By the definition of  $\Xi(N_i, k)$ , Lemma 1 and Lemma 4, we get

$$\begin{aligned} R_{11} &= \int_{\mathfrak{M}_1} F_1 G_1 S_1 T_1 H^k(\alpha_1) e(-\alpha_1 N_1) d\alpha_1 \int_{\mathfrak{M}_2} F_2 G_2 S_2 T_2 H^k(\alpha_2) e(-\alpha_2 N_2) d\alpha_2 \\ &= \sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k)}} \int_{\mathfrak{M}_1} F_1 G_1 S_1 T_1 e(-\alpha_1 n_1) d\alpha_1 \int_{\mathfrak{M}_2} F_2 G_2 S_2 T_2 e(-\alpha_2 n_2) d\alpha_2 \\ &\geq \frac{1}{2^2 \cdot 3^4} \sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k)}} \mathfrak{S}(n_1) \mathfrak{S}(n_2) J(n_1) J(n_2) + O\left(N_1^{\frac{10}{9}} N_2^{\frac{10}{9}} L^{k-1}\right) \\ &\geq \frac{1}{2^2 \cdot 3^4} \times 43.407769 \times (0.24485083)^2 N_1 N_2 U_1 U_2 V_1 V_2 L^k \\ &\geq 0.008032 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k. \end{aligned}$$

This completes the proof of Proposition 6. □

We now introduce three lemmas essential in our proof of the following propositions and the theorem.

**Lemma 7.** *We have*

$$I_{1i} = \int_{\mathfrak{M}_i} |F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i)| d\alpha_i \leq 3.610722 N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}}.$$

**Proof.** It follows from Lemma 3 and Cauchy’s inequality that

$$\begin{aligned} I_{1i} &\leq \left( \max_{\alpha \in \mathfrak{M}_i} |H(2\alpha_i)| \right)^{\frac{k}{2}-6} \int_{\mathfrak{M}_i} |F_i G_i S_i T_i H^6(2\alpha_i)| d\alpha_i \\ &\leq L^{\frac{k}{2}-6} \left( \int_0^1 |G_i S_i T_i|^2 d\alpha_i \right)^{\frac{1}{2}} \left( \int_0^1 |F_i H^6(2\alpha_i)|^2 d\alpha_i \right)^{\frac{1}{2}} \\ &\leq 3.610722 N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}}. \end{aligned}$$
□

**Lemma 8.** *We have*

$$I_{2i} = \int_{\mathfrak{m}_i \cap \mathcal{E}(\lambda)} |F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i)| d\alpha_i \ll N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}-1}.$$

**Proof.** Using the definition of  $\mathcal{E}(\lambda)$ , the trivial bound of  $H(2\alpha_i)$ , Lemma 2 and Lemma 5, we get

$$\begin{aligned} I_{2i} &\ll \max_{\alpha \in \mathfrak{m}_i} |F_i G_i S_i T_i| L^{\frac{k}{2}} \left( \int_{\mathcal{E}(\lambda)} 1 d\alpha_i \right) \\ &\ll N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}-1}. \end{aligned}$$
□

**Lemma 9.** *We have*

$$I_{3i} = \int_{\mathfrak{m}_i \setminus \mathcal{E}(\lambda)} |F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i)| d\alpha_i \leq 3.610722 N_i^{\frac{1}{2}} U_i V_i \lambda^{\frac{k}{2}-6} L^{\frac{k}{2}}.$$

**Proof.** We deduce from the trivial bound  $|H(2\alpha_i)| \leq |H(\alpha_i)| + 2 \leq (1 + o(1))\lambda L$ , Lemma 3 and Cauchy’s inequality that

$$\begin{aligned} I_{3i} &\leq (\lambda L)^{\frac{k}{2}-6} \left( \int_0^1 |G_i S_i T_i|^2 d\alpha_i \right)^{\frac{1}{2}} \left( \int_0^1 |F_i H^6(2\alpha_i)|^2 d\alpha_i \right)^{\frac{1}{2}} \\ &\leq 3.610722 N_i^{\frac{1}{2}} U_i V_i \lambda^{\frac{k}{2}-6} L^{\frac{k}{2}}. \end{aligned}$$

This completes the proof of Lemmas 7–9. □

According to Cauchy's inequality, we have

$$|H(\alpha_1 + \alpha_2)| \leq \sqrt{|H(2\alpha_1)H(2\alpha_2)|}.$$

Now we turn to give an upper bound of  $R_{12}, R_{21}, R_{22}, R_{23}$  and  $R_{32}$ .

**Proposition 10.** *We have*

$$R_{12}, R_{21}, R_{22}, R_{23}, R_{32} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

**Proof.**

$$\begin{aligned} R_{12} &= \int_{\mathfrak{M}_1} \int_{m_2 \cap \mathcal{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\ll I_{11} I_{22} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{10}$$

Similarly to (10), we obtain

$$R_{21} \ll I_{12} I_{21} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

Moreover,

$$\begin{aligned} R_{22} &= \int_{m_1 \cap \mathcal{E}(\lambda)} \int_{m_2 \cap \mathcal{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\ll I_{21} I_{22} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{11}$$

Besides, we have

$$\begin{aligned} R_{23} &= \int_{m_1 \cap \mathcal{E}(\lambda)} \int_{m_2 \setminus \mathcal{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\ll I_{21} I_{32} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{12}$$

Analogously to (12), we get

$$R_{32} \ll I_{22} I_{31} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

Now the proof of Proposition 10 is complete. □

Next we give the estimation for  $R_{13}$  and  $R_{31}$ .

**Proposition 11.** *We have*

$$R_{13}, R_{31} \ll (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k.$$

**Proof.**

$$\begin{aligned} R_{13} &= \int_{\mathfrak{M}_1} \int_{m_2 \setminus \mathcal{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\leq I_{11} I_{32} \leq (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k. \end{aligned}$$

In a similar manner, we get

$$R_{31} \leq I_{12} I_{31} \leq (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k.$$

This completes the proof of Proposition 11. □

It remains to estimate  $R_{33}$ .

**Proposition 12.** *We have*

$$R_{33} \leq (0.3486527(3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 \lambda^{k-12} L^k.$$



**Proof.**

$$R_{33} = \int_{m_1 \setminus \mathcal{E}(\lambda)} \int_{m_2 \setminus \mathcal{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2$$

$$\leq I_{31} I_{32} \leq (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 \lambda^{k-12} L^k. \quad \square$$

#### 4. Proof of Theorem

On combining recalling Propositions 6–12, we arrive at the conclusion that

$$R(N_1, N_2) \geq 0.008032 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k - 2 \times (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k$$

$$- (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 \lambda^{k-12} L^k$$

$$\geq N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k (0.008032 - 2 \times (3.610722)^2 \lambda^{\frac{k}{2}-6} - (3.610722)^2 \lambda^{k-12}) \quad (13)$$

When  $k \geq 302$  and  $\lambda = 0.9457435$ , we get

$$R(N) > 0 \quad (14)$$

for all sufficiently large even integers  $N$ . Now by (14), the proof of the Theorem is completed.

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