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Symmetry of solutions to singular fractional elliptic equations and applications

Symétrie radiale des solutions d'équations elliptiques fractionnaires singulières et quelques applications

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Abstract. In this article, we study the symmetry of positive solutions to a class of singular semilinear elliptic equations whose prototype is

$$(P)\quad \begin{cases} (-\Delta)^s u = \frac{1}{u^\delta} + f(u),\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $0 < s < 1$, $n \geq 2s$, $\Omega = B_r(0) \subset \mathbb{R}^n$, $\delta > 0$, $f(u)$ is a locally Lipschitz function. We prove that classical solutions are radial and radially decreasing (see Theorem 1). The proof uses the moving plane method adapted to the non local setting. We then give two applications of this main result: Theorem 2 establishes the uniform apriori bound for classical solutions in case of polynomial growth nonlinearities whereas Theorem 3 ensures in case of exponential growth nonlinearities the convergence of large solutions with unbounded energy to a singular solution.

Résumé. Dans cet article, nous étudions la symétrie et la monotonie des solutions positives d'une équation elliptique semi-linéaire singulière dont le modèle type est

$$(P)\quad \begin{cases} (-\Delta)^s u = \frac{1}{u^\delta} + f(u),\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

où $0 < s < 1$, $\Omega = B_r(0) \subset \mathbb{R}^n$, $n \geq 2s$, $\delta > 0$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ est localement Lipschitz. Nous démontrons que les solutions classiques de ce problème type sont à symétrie radiale et radialement décroissantes (Théorème 1). Pour cela, nous mettons en oeuvre la méthode du "moving plane". Nous utilisons ensuite ce résultat général de symétrie pour étudier le comportement global de solutions d'équations elliptiques singulières non locales : existence d'estimations a priori uniformes (Théorème 2), convergence de solutions à énergie non bornée vers une solution singulière (Théorème 3).

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Les problèmes faisant intervenir des opérateurs laplaciens fractionnaires apparaissent dans de nombreuses situations physiques comme la combustion, les modèles de cristaux, les modèles de dislocations dans les systèmes mécaniques, plus généralement dans les modèles où la diffusion est singulière et/ou fait intervenir des interactions de longue portée. A ce titre, ils font l'objet depuis de nombreuses années d'une intense et riche recherche. Dans cette note, nous nous intéressons aux propriétés des solutions de ces problèmes lorsqu'ils font intervenir une nonlinéarité singulière. Précisément, nous démontrons que les solutions classiques du problème non local et singulier (P) sont radiales et radialement décroissantes (voir Théorème 1). La preuve utilise la méthode du moving plane. Comme dans le cas d'opérateurs locaux, les principaux ingrédients sont la validité d'un principe de maximum dans des domaines étroits et le principe du maximum fort. Nous soulignons ici que la généralisation de ces outils est loin d'être triviale compte tenu du caractère non local de l'opérateur et de la nature non lipschitzienne du terme source. Pour cela, nou suivons l'approche de [10].

Nous donnons ensuite deux applications à ce résultat général. La première (Théorème 2) combinant le résultat avec un théorème de Liouville établit une borne uniforme pour les solutions classiques quand la nonlinéarité est de croissance sous critique. La seconde étudie le comportement des solutions au voisinage d'une point de bifurcation asymptotique lorsque $n = 1$ et $s = 1/2$ et lorsque la nonlinéarité est de croissance exponentielle. L'occurence d'un profil d'explosion donnée par une solution singulière est démontrée sous certaines conditions (voir Théorème 3). Une étape importante dans la preuve du théorème qui constitue par ailleurs un résultat d'intérêt plus large est l'étude des singularités isolés dans l'esprit du résultat bien connu de Brezis–Lions ([4]) dans le cas local. Dans ce contexte, le Théorème 5 (voir la preuve dans [2]) étend un résultat récent de Chen and Quaas ([5]) dans le cas de nonlinéarités exponentielles.

L'étude de problèmes nonlocaux et singuliers ont fait l'objet de contributions récentes où les questions d'existence, de multiplicité de solutions faibles ainsi que leur régularité (Höldérienne et Sobolev) ont été étudié avec diverses méthodes (voir [1], [3], [9], [11]). Les Théorème 1, 2, 3 et 5 apportent des résultats tout à fait nouveaux sur les propriétés de symétrie et de comportement global des solutions dans le contexte de ces problèmes.

1. Introduction and main results

Nonlocal elliptic equations involving general integral differential operators as fractional Laplacian have been studied for many years by an important number of researchers and a vast amount of work is present in the literature dealing with existence and regularity results. This kind of problems appears in several physical models like combustion, crystals, dislocations in mechanical systems and many other problems where anomalous diffusion or/and interaction with long range come into picture. The study of fractional and singular equations have been investigated more recently. We quote the following references about existence, uniqueness, multiplicity and regularity issues: [1], [3], [9], [11]. In this note, we are interested in the question of symmetry of solutions to the singular problem (P). Precisely we show the following main result:

Theorem 1. *Let $\delta > 0$ and f be a locally Lipschitz function. Then a classical solution u to (P) is radially symmetric and strictly decreasing in $|x|$.*

The proof of Theorem 1 involves the moving plane method adapted in the non local setting. In this regard, as in the local case, we need a maximum principle in narrow domains and a strong maximum principle to hold for equations of the type (P). The extension of these key tools is not straighforward due to the non local nature of $(-\Delta)^s$ and the presence of a singular nonlinearity in the right hand side. Besides this, we will take advantage of monotonicity properties of the nonlinear operator $(-\Delta)^s u - \frac{1}{u^\delta}$ and borrow some "local" maximum principle shown in [10]. Next, we apply this main result in two different situations: Consider the problem

$$(Q) \qquad \begin{cases} (-\Delta)^s u = \mu\left(\frac{1}{u^\delta} + f(u)\right), \; u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where Ω is a bounded domain with C^2 boundary regularity. The first concerns the existence of uniform a priori bound for classical solutions to (Q) when f has a subcritical growth. In the spirit of the work [7], we combine the monotonicity property of solutions near the boundary of Ω and a blow up technique with the help of a Liouville theorem. Precisely we prove:

Theorem 2. *Let $n > 2s$ and $\mu_0 > 0$. Let u be the classical solution of (Q) with $f(u) = u^p$ for $1 < p < \frac{n+2s}{n-2s}$ and $\mu \geq \mu_0$ Then $\|u\|_\infty \leq C_1$ with C_1 depending only on δ, p, Ω, μ_0.*

The second application concerns the asymptotic behaviour of large solutions with respect to the parameter μ. Let $s = \frac{1}{2}$, $n = 1$, $\Omega = B_r(0)$ and $f(u) = h(u)\exp(u^\alpha)$ for some $1 < \alpha \leq 2$ where h satisfies $\lim_{t\to\infty} h(t)e^{-\epsilon t^\alpha} = 0$ and $\lim_{t\to\infty} h(t)e^{\epsilon t^\alpha} = \infty$ for any $\epsilon > 0$. Then we have the following result:

Theorem 3. *Let $\mu_0 > 0$ and u be the classical solution of (Q) for some $\mu \geq \mu_0$. Then for any $\epsilon > 0$, the following holds*

$$\|u\|_{L^\infty(B_r\setminus B_\epsilon)} \leq C_2(\delta, n, \epsilon, \mu_0).$$

In addition, we have the following blow up profile: Let $\{u_k\}$ be a sequence of solutions for the problem (Q) such that $\|u_k\|_{L^\infty(B_r)} \to \infty$, $\mu_k \to \tilde{\mu}$ with $\tilde{\mu} > 0$,

(1) *There exists a singular solution $\tilde{u}$ in $C^s_{loc}(B_r \setminus \{0\})$ such that $u_k - \tilde{u} \to 0$ in $L^\infty_{loc}(B_r \setminus \{0\})$.*
(2) *If $(u_k)_{k\in\mathbb{N}}$ has uniform bounded energy and $F(t) = O(f(t))$ as $t \to \infty$ where $F(t)$ is the antiderivative of f, then $\tilde{\mu} = 0$.*

The proof uses mainly Theorem 1 to get local uniform bound in $\Omega\setminus\{0\}$, the asymptotic behavior of f and Theorem 5 (proved in [2] and extending some results in [5]) to get the behavior of solutions near isolated singularities. In [2], we give further applications.

The paper is organized as follows: In Section 2, we provides the details of the proof of Theorem 1 and in Section 3, we give proofs of Theorems 2 and 3.

2. Proof of Theorem 1

In this section, we prove Theorem 1 concerning the problem (P). First, as in [1], we introduce the notion of solution:

Definition 4. *A function $u \in C_0(\overline{\Omega})$ with $u \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ is said to be a classical solution of (P) if $\inf_K u > 0$ for any compact set $K \subset \Omega$ and*

$$\langle(-\Delta)^s u, \psi\rangle = \int_\Omega u(-\Delta)^s\psi = \mathscr{C}\int_Q \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = \int_\Omega \left(\frac{1}{u^\delta} + f(u)\right)\psi\,\mathrm{d}x$$

for all $\psi \in \tau = \{\phi \mid \phi : \mathbb{R}^n \to \mathbb{R}$, measurable, $(-\Delta)^s\phi \in L^\infty(\Omega)$ and ϕ has compact support in $\Omega\}$.

To prove our main Theorem 1, we use the moving plane method. In this regard, we introduce the following definitions:

Let $A_\lambda := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 = \lambda\}$ and

$$\Sigma_\lambda := \begin{cases} \{x \in \mathbb{R}^n : x_1 < \lambda\} & \text{if } \lambda \leq 0, \\ \{x \in \mathbb{R}^n : x_1 > \lambda\} & \text{if } \lambda > 0 \end{cases}$$

for some $\lambda \in \mathbb{R}$ and $\mathscr{D}_\lambda(x) := (2\lambda - x_1, x_2, \ldots, x_n)$ be the reflection of the point x about A_λ and $v_\lambda(x) := u_\lambda(x) - u(x)$ where $u_\lambda(x) = u(\mathscr{D}_\lambda(x))$.

Proof of Theorem 1. Let u be a classical solution of (P). To prove radial symmetry and strict monotonicity of the solution u, it is enough to prove $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $\lambda \in (-r, r)$, by moving hyperplane A_λ in a fixed direction. Since, if $v_\lambda(x) \geq 0$ for all $\lambda \in (-r, r)$ and $x \in B_r(0)$ holds then we can rotate and move the hyperplane A_λ in the direction close to fixed direction to get the desired result. Since λ is independent from the direction of movement of hyperplane A_λ, so we fix $\nu(x_0) = (1, 0, \ldots, 0)$ (without loss of generality) as the direction of movement of hyperplane A_λ where ν denotes the unit outward normal vector at $x_0 = (r, 0, \ldots, 0) \in \partial B_r(0)$. We divide the proof of above assertion into the following claims:

Claim 1. $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $|\lambda| \in [r_1, r)$ for some $r_1 > 0$.

Suppose that $v_\lambda < 0$ in a region $P \subset \Sigma_\lambda \cap B_r$ for some $r - \epsilon_1 < |\lambda| < r$ and $\epsilon_1 > 0$. Then by using Poincaré inequality and since f is a Lipschitz function with Lipschitz constant C_L in the neighborhood of x_0, we obtain

$$\int_{\mathbb{R}^n} \left((-\Delta)^{\frac{s}{2}} (u - u_\lambda)^+\right)^2 \leq \langle (-\Delta)^s(-v_\lambda), (-v_\lambda)^+ \rangle = \int_{B_r} \left(\frac{1}{u^\delta} - \frac{1}{(u_\lambda)^\delta} + f(u) - f(u_\lambda) \right) (-v_\lambda)^+ \,\mathrm{d}x$$
$$< C_L \int_P ((u - u_\lambda)^+)^2 \,\mathrm{d}x \leq C(\mathrm{diam}(P)) \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{s}{2}} (u - u_\lambda)^+\right)^2 \,\mathrm{d}x.$$

Then by choosing $\epsilon_1 > 0$ small enough such that $C(\mathrm{diam}(P)) \leq 1$, one has $(-v_\lambda)^+ = (u - u_\lambda)^+ = 0$. Now by rotating and moving the hyperplane A_λ in a direction close to the outward normal ν in any neighborhood of $x_0 \in \partial\Omega$ and repeating the above steps by taking into account that $x_0 \in \partial B_r(0)$, $\nu(x_0)$ is arbitrary and by using continuity of solution u we obtain, $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \setminus B_{r_1}(0)$ and $|\lambda| \in [r_1, r)$ for some $r_1 > 0$.

Claim 2. $v_\lambda \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $|\lambda| \in [0, r_1)$.

From Claim 1, we can assume that $\lambda = r_1$ be the smallest value such that $0 \leq r_1 < r$, $v_{r_1} \geq 0$ in $B_r \setminus B_{r_1}$ and satisfies

$$(-\Delta)^s v_{r_1}(x) - \frac{1}{u_{r_1}^\delta(x)} + \frac{1}{u^\delta(x)} = f(u_{r_1}) - f(u) \;\text{ in } B_r \setminus B_{r_1}. \tag{1}$$

Step 1: $\operatorname{ess\,inf}_R v_{r_1} > 0$ **for every compact subset** $R \subset B_r \setminus B_{r_1}$**.** To prove this, we adapt in our situation the maximum principles in nonlocal setting i.e. Proposition 3.5 (maximum principle in narrow domains) and Proposition 3.6 (strong maximum principle) in [10]. Since v_{r_1} is non-trivial in $B_r \setminus B_{r_1}$, then it is enough to prove that $\operatorname{ess\,inf}_{B_{r_0}(x^*)} v_{r_1} > 0$ for all $x^* \in B_r \setminus B_{r_1}$ and r_0 sufficiently small. From Claim 1, $v_{r_1} \geq 0$ and $v_{r_1}(x) = -v_{r_1}(\mathscr{D}_{r_1}(x))$ in Σ_{r_1} then there exists a bounded set $Q \subset \Sigma_{r_1}$ with $x^* \not\in \overline{Q}$ and $\widetilde{\mu} := \inf_Q v_{r_1} > 0$. In the spirit of Lemma 2.1 in [10], we fix r_0 such that $U = B_{2r_0}(x^*)$ and $0 < r_0 < \frac{1}{4}\,\mathrm{dist}(x^*, Q \cup (\mathbb{R}^n \setminus \Sigma_{r_1}))$ and $\lambda_1(U) \geq C_L(f)$ where $C_L(f)$ is the Lipschitz constant of f and $\lambda_1(U)$ is the first eigenvalue of $(-\Delta)^s$ in U. Now, we construct a subsolution of $(-\Delta)^s \widetilde{u} = c(x)\widetilde{u}$ in U where

$$c(x) = \begin{cases} \frac{f(u_{r_1}) - f(u)}{v_{r_1}} - \frac{\delta}{(\theta u + (1-\theta) u_{r_1})^{\delta+1}} & \text{if } v_{r_1} \neq 0, \\ 0 & \text{if } v_{r_1} = 0 \end{cases}$$

for some $\theta \in (0,1)$. Define $k : \mathbb{R}^n \to \mathbb{R}$, $k(x) = m(x) - m(\mathscr{D}_{r_1}(x)) + a[\mathbb{1}_Q(x) - \mathbb{1}_Q(\mathscr{D}_{r_1}(x))]$ with $m \in C_c^2(\mathbb{R}^n)$, $0 \leq m \leq 1$ on $\mathbb{R}^n$, $m(x) = 1$ in $B_{r_0}(x^*)$, $m(x) = 0$ in $\mathbb{R}^n \setminus B_{2r_0}(x^*)$ and satisfies $k(\mathscr{D}_{r_1}(x)) = -k(x)$ on Σ_{r_1}, $k = 0$ in $\Sigma_{r_1} \setminus (U \cup Q)$ and $k = a$ on Q where the choice of a will be fixed later. Then by Proposition 2.3 in [10] we obtain, $\langle (-\Delta)^s m, \psi \rangle \leq C_1 \|\psi\|_{L^1(U)}$ for $\psi \in \tau$, $\psi \geq 0$ and $C_1 = C_1(m)$ independent of ψ. Since $\psi = 0$ in $\mathbb{R}^n \setminus U$, $(U \cap Q) \cup (U \cap \mathscr{D}_{r_1}(Q)) = \emptyset$ and $m(\mathscr{D}_{r_1}(x))\psi(x) = \mathbb{1}_Q(x)\psi(x) = \mathbb{1}_{\mathscr{D}_{r_1}(U)}(x)\psi(x) = 0$ in $\mathbb{R}^n$. Then we have $\langle (-\Delta)^s k, \psi \rangle \leq C_a \|\psi\|_{L^1(U)}$ where

$$C_a := C + \sup_{x \in U} \int_{\mathscr{D}_{r_1}(U)} \frac{1}{|x - y|^2} \,\mathrm{d}y - a \inf_{x \in U} \int_Q \left(\frac{1}{|x - y|^2} - \frac{1}{|x - \mathscr{D}_{r_1}(y)|^2} \right) \mathrm{d}y.$$

Since $|x-y| \le |x-\mathscr{D}_{r_1}(y)|$ for all $x, y \in \Sigma_{r_1}$, $\overline{U} \subset \Sigma_{r_1}$ and then continuity of the function $x \mapsto \int_Q \left(\frac{1}{|x-y|^2} - \frac{1}{|x-\mathscr{D}_{r_1}(y)|^2}\right) \mathrm{d}y$ implies $C_a \le -C_L(f)$, by taking a sufficiently large. Since $v_{r_1} \ge 0$ in U, we obtain k is the required subsolution in U. Then Proposition 3.5 in [10], implies $\widetilde{v}_{r_1}(x) := v_{r_1}(x) - \frac{\tilde{\mu}}{a}k(x) \ge 0$ a.e. in U which further gives $v_{r_1}(x) \ge \frac{\tilde{\mu}}{a}k(x) = \frac{\tilde{\mu}}{a} > 0$ a.e. in $B_{r_0}(x^*)$ and completes the proof of Step 1.

Step 2: $r_1 = 0$**.** To prove this, we proceed by contradiction by assuming $r_1 > 0$. Since r_1 is the smallest value such that $v_{r_1} \ge 0$ in Σ_{r_1}, so we will prove that for a small $\epsilon > 0$ we have $v_{r_1-\epsilon} \ge 0$ in $\Sigma_{r_1-\epsilon}$. This will provide the required contradiction that r_1 is the smallest value. Fix γ (to be determined later) and let $S \Subset \Sigma_{r_1}$ such that $|\Sigma_{r_1} \setminus S| \le \frac{\gamma}{2}$. Then by using Claim 1 and continuity of solution we get $v_{r_1-\epsilon} > 0$ in S for ϵ small enough. Since $v_{r_1-\epsilon}$ satisfies (1) in $\Sigma_{r_1-\epsilon} \setminus S$ then by using $|x-y| \le |x-\mathscr{D}_{r_1-\epsilon}(y)|$ for all $x, y \in \Sigma_{r_1-\epsilon}$, $\mathscr{D}_{r_1-\epsilon}(\mathbb{R}\setminus\Sigma_{r_1-\epsilon}) = \Sigma_{r_1-\epsilon}$ and taking $w := \mathbb{1}_{\Sigma_{r_1-\epsilon}} v^-_{r_1-\epsilon}$ such that $\operatorname{supp}(w) \subset \Sigma_{r_1-\epsilon} \setminus S$ as a test function, then after some straightforward computations we obtain

$$\begin{aligned}\langle(-\Delta)^s w, w\rangle + \langle(-\Delta)^s v_{r_1-\epsilon}, w\rangle &= -2\int_{\Sigma_{r_1-\epsilon}}\int_{\mathbb{R}^n} \frac{w(x)[w(y)+v_{r_1-\epsilon}(y)]}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\ &= -2\int_{\Sigma_{r_1-\epsilon}}\int_{\Sigma_{r_1-\epsilon}} w(x)\left(\frac{v^+_{r_1-\epsilon}(y)}{|x-y|^{n+2s}} - \frac{v_{r_1-\epsilon}(y)}{|x-\mathscr{D}_{r_1-\epsilon}(y)|^{n+2s}}\right)\mathrm{d}y\,\mathrm{d}x \le 0. \quad (2)\end{aligned}$$

Let $\lambda^{r_1}_{1,\epsilon}$ be the first eigenvalue of $(-\Delta)^s$ in $\Sigma_{r_1-\epsilon} \setminus S$ and by mean value theorem together with (2) we get, for some $\theta \in (0,1)$

$$\begin{aligned}\lambda^{r_1}_{1,\epsilon}(\Sigma_{r_1-\epsilon}\setminus S)\int_{\Sigma_{r_1-\epsilon}\setminus S} |v^-_{r_1-\epsilon}|^2\,\mathrm{d}x &\le \langle(-\Delta)^s w, w\rangle \le -\langle(-\Delta)^s v_{r_1-\epsilon}, w\rangle \\ &= \int_{\Sigma_{r_1-\epsilon}\setminus S} \frac{\delta v_{r_1-\epsilon}\mathbb{1}_{\Sigma_{r_1-\epsilon}\setminus S} v^-_{r_1-\epsilon}}{(\theta u + (1-\theta)u_{r_1-\epsilon})^{\delta+1}}\,\mathrm{d}x + \int_{\Sigma_{r_1-\epsilon}\setminus S} (-f(u_{r_1-\epsilon}) + f(u))\mathbb{1}_{\Sigma_{r_1-\epsilon}\setminus S} v^-_{r_1-\epsilon}\,\mathrm{d}x \\ &\le C_L\int_{\Sigma_{r_1-\epsilon}\setminus S} |v^-_{r_1-\epsilon}|^2\,\mathrm{d}x.\end{aligned}$$

Then by Lemma 2.1 in [10] and choosing γ small enough, we obtain $v_{r_1-\epsilon} \ge 0$ in $\Sigma_{r_1-\epsilon}$. Then $r_1 = 0$ and repeating the proof by moving hyperplane A_λ as in Claim 1 we obtain u is radially symmetric. Now Claim 1 gives further the strict monotonicity property. The proof is now complete. □

3. Applications

In the section, we prove Theorem 2 and Theorem 3. The uniform apriori bounds stated in Theorem 2 is obtained by the moving plane method near the boundary and a blow up analysis combined with available Liouville Theorem (see [6]) in the interior of domain.

Proof of Theorem 2. First we suppose that Ω is strictly convex then Claim 1 in Theorem 1 combined with moving plane method gives boundary estimates and when Ω is not strictly convex, we perform Kelvin transform near any boundary point (see [1], [6]). While, for interior estimates, we proceed by blow-up analysis. Precisely, assume there exists a sequence of bounded solution $(u_k)_{k\in\mathbb{N}}$ and a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $M_k = \sup_{y\in\Omega} u_k(y) = u_k(x_k) \to \infty$ as $k \to \infty$. Let λ_k is the sequence of positive numbers (to be determined later) and $y = \frac{x-x_k}{\lambda_k} \in \Omega_k$. From boundary estimates, notice that $\operatorname{dist}(x_k, \partial\Omega) \ge c > 0$ uniformly in k. Define the blow up function $v_k(y) = \lambda_k^{\frac{2s}{p-1}} u_k(x)$ where $\lambda_k^{\frac{2s}{p-1}} M_k = 1$. We noticed that $\lambda_k \to 0$ as $k \to \infty$ (since

$M_k \to \infty$) and for large k, $v_k(y)$ is well defined in $B_{\frac{m}{\lambda_k}}(0)$ and $\sup_{y \in B_{\frac{m}{\lambda_k}}(0)} v_k(y) = v^k(0) = 1$ where $0 < 2m \le \inf_k \operatorname{dist}(x_k, \partial\Omega)$. Accordingly, v_k satisfies

$$(-\Delta)^s v_k = \mu_k \left(\frac{\lambda_k^{\frac{2s(p+\delta)}{p-1}}}{v_k^\delta} + v_k^p \ \text{ in } B_{\frac{m}{\lambda_k}} \right).$$

Now passing to the limits we obtain, $v_k \to v$ in $C^s_{loc}(\mathbb{R}^n)$ and satisfies $(-\Delta)^s v = v^p$ in $\mathbb{R}^n$, $v(0) = 1$ and by using Liouville Theorem (see [6, Theorem 4]), we get a contradiction. □

Now we prove Theorem 3. From Theorem 1 we know that the solutions are radial and radially decreasing, from this we only need to study the behavior near an isolated singularity. For that we exploit the following result proved in [2]:

Theorem 5. *Let $u \in L^1(\Omega)$ be non-negative distributional solution of*

$$(P_s) \begin{cases} (-\Delta)^s u = g(u),\ u \ge 0 & \text{in } \Omega' := \Omega \setminus \{0\}, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $g(u) \in L^p_{loc}(\Omega')$ with $p > \frac{n}{2s}$. Then $g(u) \in L^1(\Omega)$ and there exists $k \ge 0$ such that u is distributional solution of

$$\begin{cases} (-\Delta)^s u = g(u) + k\delta_0,\ u \ge 0, & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3}$$

Proof of Theorem 3. Using Theorem 1, we obtain every classical solution of u of (Q) is radially symmetric and decreasing with respect to $|x|$. Then for every $\epsilon > 0$ there exists $\alpha_1 > 0$ such that for any $x \in B_r \setminus B_\epsilon$, we have a measurable set Z_ϵ satisfying $|Z_\epsilon| \ge \alpha_1$, $Z_\epsilon \subset B_r \setminus B_\epsilon$ and $u(y) \ge u(x)$, $\forall\, y \in B_\epsilon$. Then by multipying ψ_1 (eigenfunction with respect to first eigenvalue μ_1 of $(-\Delta)^s$ in B_r) to the equation satisfied by u, we obtain

$$\mu \int_{B_r} \frac{\psi_1}{u^\delta}\, \mathrm{d}x + \int_{B_r} \exp(u^\alpha)\psi_1\, \mathrm{d}x = \mu_1 \int_{B_r} u\psi_1$$

and for any $m \ge \frac{\mu_1}{\mu}$, there exists a $C > 0$, $mt - C \le \frac{1}{t^\delta} + \exp(t^\alpha)$, $t \in \mathbb{R}^+$. Then by using $u(y) \ge u(x)$, $\forall\, y \in B_\epsilon$ and $|Z_\epsilon| \ge \alpha_1$ it implies that $u(x) \le C_2$ for all $x \in B_r \setminus B_\epsilon$ where C_2 is independent of u. Now we prove the blow up profile. From Theorem 1 and above estimates, we know that $(u_k)_k$ blows up only at 0. We deduce by regularity theory (see [1]) that the sequence $(u_k)_k$ converge to a singular solution u uniformly in $B_r \setminus \{0\}$. From Theorem 5 and the asymptotic growth of f, we prove that $k = 0$ in (3) and u is a singular solution of (Q). Finally assume that $(u_k)_k$ has uniform bounded energy. Then we easily get that u belongs to the energy space and from Moser–Trudinger inequality (see [13] or [8, Lemma 2.1]) and Remark 1.5 in [12] we obtain u is bounded which provides a contradiction and completes the proof. □

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