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Une inégalité sur la courbure affine minimale par le flot de raccourcissement des courbes

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Abstract. As an application of the affine curve shortening flow, we will prove an inequality for minimum affine curvature of a smooth simple closed curve in the Euclidean plane.

Résumé. Comme application du flot de raccourcissement des courbes, nous prouverons une inégalité sur la courbure affine minimale d'une courbe fermée simple lisse dans le plan euclidien.

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1. Introduction

Let $\gamma \subset \mathbb{R}^2$ be a smooth Jordan curve. From Pestov-Ionin's work [17] on inscribed discs, one has

$$\kappa_{\max} \geq \sqrt{\frac{\pi}{A}}, \tag{1}$$

where $\kappa_{\max}$ and A are the maximum curvature and enclosed area of γ, respectively, and the equality in (1) holds if and only if γ is a circle. An analytical proof of the inequality (1) can be found in [11, Proposition 2.1]. As an application of the curve shortening flow in the plane (see Gage-Hamilton [8], Grayson [9]), Pankrashkin [15] gave the inequality (1) and discussed its equality case. Pankrashkin and Popoff [16] showed that inequality (1) plays a significant role in the study of some eigenvalue problems. Ferone, Nitsch and Trombetti [7] considered the maximal mean curvature of a smooth surface.

Following the work on the curve shortening flow in the plane (see Gage-Hamilton [8], Grayson [9]), Sapiro and Tannenbaum [18] studied the affine curve shortening problem, and the

Grayson type theorem under this flow is obtained by Angenet-Sapiro-Tannenbaum [5]. From a completely different context, the affine curve shortening flow arises from image processing and computer vision (see Alvarez-Cuichard-Lions-Morel [2]), and it was derived by the axiomatic method. A general result concerning this topic can be found in Olver-Sapiro-Tannenbaum [14]. Recently, Ivaki [12] dealt with centro-affine curvature flows on centrally symmetric convex curves and obtained some new inequalities for affine curvature. Andrews researched the affine curve-lengthening flow in the plane (see [4]) and dealt with the affine curve shortening problem for convex hypersurfaces (see [3]). Other aspects of the affine curve shortening flow can be found in Ai-Chou-Wei [1], Angenet-Sapiro-Tannenbaum [5], Chou-Zhou [6], Jiang-Wang-Wei [13], etc., and the literature therein.

In this short paper, we will intend to consider the following question:

Question 1. *Is there a similar inequality as* (1) *for the affine case in the Euclidean plane?*

To give the answer to Question 1, motivated by the work of Pankrashkin [15] and Angenent-Sapiro-Tannenbaum [5], we will show the next theorem which is an inequality for the minimum affine curvature via the affine curve shortening flow.

Theorem 2. *If γ is a smooth simple closed curve on the Euclidean plane, then*

$$\mu_{\min} \leq \left(\frac{\pi}{A}\right)^{\frac{2}{3}}, \tag{2}$$

where $\mu_{\min}$ and A are respectively the minimum affine curvature and enclosed area of γ, and the equality in (2) *holds if and only if γ is an ellipse.*

In this paper, *a simple closed curve* means a closed curve which has no self-intersections.

2. A minimum affine curvature inequality

Let $\mathscr{C} : S^1 \to \mathbb{R}^2$ be a smooth embedded curve with parameter p. A reparametrization of $\mathscr{C}(p)$ to a new parameter s can be performed such that

$$[\mathscr{C}_s, \mathscr{C}_{ss}] = 1 \tag{3}$$

where $[X, Y]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $X, Y \in \mathbb{R}^2$. The relation is invariant under proper affine transformations, and the parameter s is called the *affine arc-length*. Let

$$g(p) = [\mathscr{C}_p, \mathscr{C}_{pp}]^{\frac{1}{3}},$$

the parameter s is explicitly given by

$$s(p) = \int_0^p g(\xi)\mathrm{d}\xi.$$

By differentiating (3), one has

$$[\mathscr{C}_s, \mathscr{C}_{sss}] = 0,$$

which implies that $\mathscr{C}_s$ and $\mathscr{C}_{sss}$ are linearly dependent and there exists μ such that

$$\mathscr{C}_{sss} + \mu\mathscr{C}_s = 0.$$

This equation and (3) lead to

$$\mu = [\mathscr{C}_{ss}, \mathscr{C}_{sss}], \tag{4}$$

and μ is called the *affine curvature*. A more comprehensive account of various aspects of the Affine Differential Geometry can be found in [6, 18, 19].

In the rest of this paper, considering the affine curve shortening flow:

$$\begin{cases} \frac{\partial \mathscr{C}(p,t)}{\partial t} = \mathscr{C}_{ss}(p,t), \\ \mathscr{C}(\cdot,0) = \mathscr{C}_0(\cdot). \end{cases}$$

Since the affine isoperimetric inequality plays a significant role in the Affine Differential Geometry and the affine curve shortening problem, we state it as a independent lemma (see [6, Theorem 4.4]).

Lemma 3 (The affine isoperimetric inequality). *For any closed embedded convex curve γ, one has*

$$\mathscr{L}^3 \le 8\pi^2 A \tag{5}$$

with equality holds if and only if γ is an ellipse, where $\mathscr{L}$ and A are respectively the affine length and enclosed area of γ.

Proof of Theorem 2. If $\mathscr{C}(p)$ is an ellipse of form $(a\cos p, b\sin p)$, where $a, b > 0$, then, by Green's formula and (4), its enclosed area and affine curvature are πab and $(ab)^{-\frac{2}{3}}$, respectively. Hence, the equality in (2) holds.

Set $F = \mu^2$ and $F_{\min}(t) = \min\{F(s,t) \mid s \in [0,\mathscr{L}]\}$. By [18, p. 96 (32)] (see also [6, p. 105 (4.8)]), one has

$$\begin{aligned} \frac{\partial F}{\partial t} &= 2\mu\frac{\partial \mu}{\partial t} = 2\mu\left(\frac{1}{3}\frac{\partial^2 \mu}{\partial s^2} + \frac{4}{3}\mu^2\right) \\ &= \frac{2}{3}\mu\frac{\partial^2 \mu}{\partial s^2} + \frac{8}{3}F^{\frac{3}{2}} \\ &= \frac{1}{3}\frac{\partial^2 F}{\partial s^2} - \frac{1}{6F}\left(\frac{\partial F}{\partial s}\right)^2 + \frac{8}{3}F^{\frac{3}{2}}. \end{aligned}$$

Since $F_{\min}(t)$ is Lipschitz continuous, it is differentiable almost everywhere. Let $\widetilde{s}$ be the point such that $F(\widetilde{s}(t),t) = F_{\min}(t)$. By Hamilton's technique of the maximum principle (see [10, p. 159 3.4.Corollary]),

$$\frac{\mathrm{d}F_{min}(t)}{\mathrm{d}t} \ge \frac{\partial F}{\partial t}(\widetilde{s}(t),t) = \frac{1}{3}\frac{\partial^2 F}{\partial s^2} - \frac{1}{6F}\left(\frac{\partial F}{\partial s}\right)^2 + \frac{8}{3}F^{\frac{3}{2}}$$

and at the point $(\widetilde{s},t)$,

$$\frac{\partial^2 F}{\partial s^2} \ge 0 \quad \text{and} \quad \frac{\partial F}{\partial s} = 0.$$

Hence,

$$\frac{\mathrm{d}F_{min}(t)}{\mathrm{d}t} \ge \frac{8}{3}F_{\min}(t)^{\frac{3}{2}}, \tag{6}$$

which implies that

$$\frac{\mathrm{d}F_{min}(t)^{-\frac{1}{2}}}{\mathrm{d}t} \le -\frac{4}{3}.$$

Integrating the above expression over $[t,\omega)$ yields

$$\frac{1}{\sqrt{F_{\min}(\omega)}} - \frac{1}{\sqrt{F_{\min}(t)}} \le -\frac{4}{3}(\omega - t),$$

where ω is the maximal existence time. From [5, Theorem 15.1], it follows that ω is finite, and $A(t) \to 0$ and $F_{\min}(t) \to \infty$ as $t \to \omega$. By [18, p. 101 (41)] (see also [6, p. 105 (4.9)]), one has

$$A_t = -\mathscr{L},$$

which together with inequality (5) in Lemma 3 yields that

$$A(t)^{\frac{2}{3}} \le \frac{4}{3}\pi^{\frac{2}{3}}(\omega - t). \tag{7}$$

Hence,

$$\frac{1}{\mu_{\min}(t)} \geq \frac{4}{3}(\omega - t) \geq \left(\frac{A(t)}{\pi}\right)^{\frac{2}{3}},$$

which implies the desired result. If this inequality is an equality, then the inequalities (6) and (7) must be equalities, which yields that the equality in (5) holds, that is, they are ellipses. □

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