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# Rational cubic fourfolds in Hassett divisors

# *Cubiques rationnelles de dimension 4 dans les diviseurs de Hassett*

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**Abstract.** We prove that every Hassett's Noether–Lefschetz divisor of special cubic fourfolds contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in the moduli space of smooth cubic fourfolds.

**Résumé.** Nous prouvons que chaque diviseur de Hassett–Noether–Lefschetz de cubiques spéciales de dimension 4 contient une union de trois sous-variétés paramétrant des cubiques rationnelles de dimension 4, de codimension deux dans l'espace de modules des cubiques lisses de dimension 4.

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## 1. Introduction

The rationality problem of smooth cubic fourfolds is one of the most widely open problems in algebraic geometry; we refer to the survey [12] for a comprehensive progress. It has been known that all smooth cubic surfaces are rational since the 19th century. In 1972, Clemens–Griffiths [8] proved that all smooth cubic threefolds are nonrational. For smooth cubic fourfolds, however, the situation is very mysterious. It is expected that a very general smooth cubic fourfold should be nonrational (cf. [10, 11]). Until now, many examples of smooth rational cubic fourfolds are known, but the existence of a smooth nonrational cubic fourfold is still unknown.

Using Hodge theory and lattice theory, Hassett [11] introduced the notion of *special cubic fourfolds* (see Definition 3). Simultaneously, Hassett [11, Theorem 1.0.1] gave a countably infinite list of irreducible divisors $\mathscr{C}_d$ of special cubic fourfolds in the moduli space $\mathscr{C}$ of smooth cubic fourfolds and showed that $\mathscr{C}_d$ is nonempty if and only if $d > 6$ and $d \equiv 0, 2 \pmod 6$. Such a nonempty $\mathscr{C}_d$ is called a *Hassett's Noether–Lefschetz divisor* (for short a *Hassett divisor*).

Currently, there exist two popular point of views toward the rationality of smooth cubic fourfolds and both have associated $K3$ surfaces:

- Hassett's Hodge-theoretic result ( [11, Theorem 5.1.3]): a smooth cubic fourfold $X$ has a Hodge-theoretically associated $K3$ surface if and only if its moduli point $[X] \in \mathscr{C}_d$ for some *admissible value* $d$ (i.e., $d > 6$, $d \equiv 0, 2 \pmod 6$, $4 \nmid d$, $9 \nmid d$ and $p \nmid d$ for any odd prime $p \equiv 2 \pmod 3$));
- Kuznetsov's derived categorical conjecture ( [16, Conjecture 1.1]): a smooth cubic fourfold $X$ is rational if and only if its Kuznetsov component $\mathrm{Ku}(X)$ is derived equivalent to a $K3$ surface (i.e., $\mathrm{Ku}(X)$ is called *geometric*), where $\mathrm{Ku}(X)$ is the right orthogonal to the exceptional collection $\{\mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\}$ in the bounded derived category of coherent sheaves on $X$.

It is important to notice that Kuznetsov's conjecture implies that a very general cubic fourfold is not rational, since for a very general cubic fourfold its Kuznetsov component can not be geometric. Addington–Thomas [2, Theorem 1.1] showed that for a smooth cubic fourfold $X$ if $\mathrm{Ku}(X)$ is geometric then $[X] \in \mathscr{C}_d$ for some admissible $d$, and conversely for any admissible value $d$, the set of cubic fourfolds $[X] \in \mathscr{C}_d$ for which $\mathrm{Ku}(X)$ is geometric is a Zariski open dense subset; see also Huybrechts [13] for the twisted version and a further study. Recently, based on Bridgeland stability conditions on $\mathrm{Ku}(X)$ constructed in [5, Theorem 1.2], Bayer–Lahoz–Macrì–Nuer–Perry–Stellari [4, Corollary 29.7] proved that for any admissible value $d$, $\mathrm{Ku}(X)$ is geometric for *every* $[X] \in \mathscr{C}_d$. So we now know that for a smooth cubic fourfold $X$ its Kuznetsov component $\mathrm{Ku}(X)$ is geometric if and only if $[X] \in \mathscr{C}_d$ for some admissible value $d$. Then one can restate Kuznetsov's conjecture as the following equivalent form.

**Conjecture 1.** *A smooth cubic fourfold $X$ is* rational *if and only if $[X] \in \mathscr{C}_d$ for some admissible value $d$.*

The first three admissible values are $14, 26, 38$. Every cubic fourfold in $\mathscr{C}_{14}$ is rational [7,9]; see also [21, Theorem 2] for a different proof. Based on Kontsevich–Tschinkel [15, Theorem 1], Russo–Staglianò [21, Theorems 4, 7] finally showed that every cubic fourfold in $\mathscr{C}_{26}$ and $\mathscr{C}_{38}$ is rational; see also [20] for the construction of explicit birational maps. So far "if " part of Conjecture 1 has been confirmed only for the three Hassett divisors $\mathscr{C}_{14}, \mathscr{C}_{26}, \mathscr{C}_{38}$. Thus finding rational cubic fourfolds in other Hassett divisors is of interest. The main result of this paper is the following.

**Theorem 2 (=Theorem 9).** *Every Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

The idea of the proof is simple: we first show any two Hassett divisors intersect by Theorem 7, which is of independent interest (for considerations of the intersections among Hassett divisors, see [2,3,7,10] etc.), and finally we consider the intersections $\mathscr{C}_d \cap \mathscr{C}_{14}$, $\mathscr{C}_d \cap \mathscr{C}_{26}$ and $\mathscr{C}_d \cap \mathscr{C}_{38}$ for every Hassett divisor $\mathscr{C}_d$.

After completing this paper, Russo–Staglianò [22] announced the rationality of every cubic fourfold in $\mathscr{C}_{42}$. We remark that our method used for the proof of Theorem 2 also works in this case (in particular, it can be shown that the four intersections $\mathscr{C}_d \cap \mathscr{C}_{14}$, $\mathscr{C}_d \cap \mathscr{C}_{26}$, $\mathscr{C}_d \cap \mathscr{C}_{38}$, $\mathscr{C}_d \cap \mathscr{C}_{42}$ are mutually distinct).

Throughout this paper, we work over the complex number field $\mathbb{C}$.

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## 2. Lattice and Hodge theory for cubic fourfolds

In this section, we collect some known results on Hodge structures and lattices associated with smooth cubic fourfolds. We refer to [6, 11, 12, 14] for more detailed discussions, especially for the Hodge-theoretic aspect, and to [19, 23] for the basics of abstract lattice theory.

The cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by $\mathbb{P}(H^0(\mathbb{P}^5, \mathscr{O}(3))) \cong \mathbb{P}^{55}$. Moreover, the smooth cubic hypersurfaces form a Zariski open dense subset $\mathscr{U} \subset \mathbb{P}^{55}$. Then the moduli space of smooth cubic fourfolds is the quotient space

$$\mathscr{C} := \mathscr{U} // \mathrm{PGL}(6, \mathbb{C})$$

which is a 20-dimensional quasi-projective variety.

Let $X$ be a smooth cubic fourfold. Then the cohomology $H^*(X, \mathbb{Z})$ is torsion-free and the Hodge numbers for the middle cohomology of $X$ are as follows:

$$0 \quad 1 \quad 21 \quad 1 \quad 0.$$

The Hodge–Riemann bilinear relations imply that $H^4(X, \mathbb{Z})$ is a unimodular lattice under the intersection form $(.)$ of signature $(21, 2)$. Furthermore, as abstract lattices, [11, Proposition 2.1.2] implies the middle cohomology and the primitive cohomology

$$L := E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0} \simeq H^4(X, \mathbb{Z})$$

$$L^0 := (h^2)^{\perp} \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2 \simeq H^4_{\mathrm{prim}}(X, \mathbb{Z})$$

where the square of the hyperplane class $h$ is given as $h^2 = (1, 1, 1) \in I_{3,0}$ of which the intersection form is given by the identity matrix of rank 3, $A_2 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ the hyperbolic plane, $E_8$ is the unimodular positive definite even lattice of rank 8. Note that $L^0$ is an even lattice.

**Definition 3 (Hassett [11]).** *A smooth cubic fourfold $X$ is called* special *if it contains an algebraic surface not homologous to a complete intersection.*

The integral Hodge conjecture holds for smooth cubic fourfolds ( [25, Theorem 18] or see also [4, Corollary 29.8] for a new proof). Thus, a smooth cubic fourfold $X$ is *special* if and only if the rank of the positive definite lattice

$$A(X) := H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

is at least 2.

**Definition 4 (Hassett [11]).** *A* labelling *of a special cubic fourfold consists of a positive definite rank two saturated (i.e. the quotient group $A(X)/K$ is torsion free) sublattice*

$$K \subset A(X) \ \ \textit{such that} \ \ h^2 \in K,$$

*and its discriminant $d$ is the determinant of the intersection form on $K$.*

In [11], Hassett defined $\mathscr{C}_d$ as the set of special cubic fourfolds $X$ with labelling of discriminant $d$. Moreover, Hassett [11, Theorem 1.0.1] showed that $\mathscr{C}_d \subset \mathscr{C}$ is an irreducible divisor and is nonempty if and only if

$$d > 6 \quad \text{and} \quad d \equiv 0, 2 \pmod 6. \tag{$\star$}$$

The following proposition is a generalization of [11, Theorems 1.0.1].

**Proposition 5 ( [12, Proposition 12 and p. 43]).** *Fix a positive definite lattice $M$ of rank $r \geq 2$ admitting a saturated embedding*

$$M \subset L \ \ \textit{such that} \ \ h^2 \in M.$$

*We denote by $\mathscr{C}_M \subset \mathscr{C}$ the smooth cubic fourfolds $X$ admitting algebraic classes with this lattice structure*

$$h^2 \in M \subset A(X) \subset L.$$

*Then $\mathscr{C}_M$ has codimension $r-1$ and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$, provided $\mathscr{C}_M$ is nonempty. Moreover, $\mathscr{C}_M$ is nonempty if and only if there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or $K_6$, where $K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $K_6 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.*

This proposition is crucial for our purpose, so we sketch a proof for the convenience of readers.

**Sketch of proof.** Suppose $\mathscr{C}_M$ is nonempty. If $K_6 \subset M$ is a sublattice with $h^2 \in K_6$, then there is a smooth cubic fourfold $X$ such that $A(X) \cap \langle h^2 \rangle^\perp$ contains an element $r$ with $(r.r) = 2$ and this contradicts Voisin [24, Section 4, Proposition 1]; furthermore, Hassett [11, Theorem 4.4.1] excludes the case when $K_2 \subset M$ is a sublattice with $h^2 \in K_2$.

Conversely, suppose that there exists no rank two sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or $K_6$. Since the signature of $L$ is $(21,2)$ and $M \subset L$ is positive definite, by a standard argument, one can always find $\omega \in L \otimes_{\mathbb{Z}} \mathbb{C}$ such that

$$(\omega.\omega) = 0, \ (\omega.\overline{\omega}) < 0 \text{ and } L \cap \omega^\perp = M.$$

According to the description of the image of the period map for cubic fourfolds (Laza [17, Theorem 1.1] and Looijenga [18, Theorem 4.1]), one has a smooth cubic fourfold $X$ and an isometry $\phi : H^4(X, \mathbb{Z}) \xrightarrow{\simeq} L$ mapping the square of the hyperplane class to $h^2 \in L$ and a generator of $H^{3,1}(X)$ to $\omega$. Thus $M = A(X)$ and hence $\mathscr{C}_M$ contains $[X]$ and nonempty. $\square$

In the rest of the text, we will frequently use the following lemma to check the nonemptyness condition in the Proposition 5.

**Lemma 6.** *Let $M \subset L$ be a positive definite saturated sublattice and $h^2 \in M$. Then the following three conditions are equivalent:*

(i) *there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or $K_6$;*
(ii) *there exists no $r \in M$ such that $(r.r) = 2$ (i.e., $M$ does not represent 2);*
(iii) *for any $0 \neq x \in M$, $(x.x) \geq 3$.*

*In particular, if $M$ satisfies one of the three equivalent conditions, then $\emptyset \neq \mathscr{C}_M \subset \mathscr{C}_{M'}$ for any saturated sublattice $M' \subset M \subset L$ such that $h^2 \in M'$.*

**Proof.** First of all, (ii) $\Rightarrow$ (i) is clear since both $K_2$ and $K_6$ represent 2.

Secondly, (i) $\Rightarrow$ (ii). Suppose that there exists $r \in M$ such that $(r.r) = 2$. We denote by $K \subset M$ the sublattice generated by $h^2$ and $r$. Hence, the Gram matrix of $K$ with respect to the basis $(h^2, r)$ is

$$\begin{pmatrix} (h^2.h^2) & (h^2.r) \\ (r.h^2) & (r.r) \end{pmatrix} = \begin{pmatrix} 3 & a \\ a & 2 \end{pmatrix}.$$

Replacing $r$ by $-r$ if necessary, we may and will assume $a \geq 0$. Since $K$ is positive definite, we have $a^2 < 6$ and thus $a = 0,1,2$. If $a = 0$ (resp. 2), then $K$ is isometric to $K_6$ (resp. $K_2$), contradiction. If $a = 1$, then $h^2 - 3r \in (h^2)^\perp = L^0$ and $((h^2 - 3r).(h^2 - 3r)) = 15$, an odd number, contradicting to the fact $L^0$ is even.

Finally, clearly (iii) implies (ii). Conversely, we show (ii) implies (iii). By hypothesis, we may assume that there is $r \in M$ with $(r.r) = 1$. Then let $K \subset M$ be the sublattice generated by $h^2$ and $r$. Hence, the Gram matrix of $K$ with respect to the basis $(h^2, r)$ is

$$\begin{pmatrix} 3 & a \\ a & 1 \end{pmatrix}$$

where $a = (h^2.r)$. Replacing $r$ by $-r$ if necessary, we may and will assume $a \geq 0$. Since $K$ is positive definite, we have $a^2 < 3$ and thus $a = 0,1$. If $a = 0$, then $r \in (h^2)^\perp = L^0$ and $(r.r) = 1$, an odd number, contradicting to the fact $L^0$ is even. If $a = 1$, then $K$ is isometric to $K_2$ and $K$ represents 2, contradiction. □

## 3. Intersections of Hassett divisors

In this section, we prove Theorem 2 (=Theorem 9) and discuss some related results (Theorem 7 and Theorem 13).

Firstly, we setup some notations for latter use. Let

$$L = E_8^{\oplus 2} \oplus U_1 \oplus U_2 \oplus I_{3,0},$$

where $U_1$ and $U_2$ are two copies of $U$. The standard basis of $U$ consists of isotropic elements $e, f$ with $(e.f) = 1$. We denote the standard basis of $U_i$ by $e_i, f_i$, $i = 1,2$, and denote by $h^2$ the element $(1,1,1) \in I_{3,0} \subset L$.

We will use the following theorem, an interesting result for itself, to prove Theorem 9.

**Theorem 7.** *Any two Hassett divisors intersect, i.e.,* $\mathscr{C}_{d_1} \cap \mathscr{C}_{d_2} \neq \emptyset$ *for any two integers* $d_1$ *and* $d_2$ *satisfying* $(\star)$. *Moreover, there exists a smooth cubic fourfold* $X$ *and a codimension-two subvariety* $\mathscr{C}_{A(X)} \subset \mathscr{C}$ *such that* $[X] \in \mathscr{C}_{A(X)} \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ *and* $A(X)$ *is a rank 3 lattice with discriminant* $d_1 d_2/3$, *except if both* $d_1$ *and* $d_2$ *are* $\equiv 2 \pmod 6$, *in which case the discriminant is* $(d_1 d_2 - 1)/3$.

**Proof.** By definition, an integer $d$ satisfies $(\star)$ if $d > 6$ and $d \equiv 0,2 \pmod 6$. Therefore, the proof is divided into three cases:

**Case 1:** $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 0 \pmod 6$**.** Suppose $d_1 = 6n_1$, $d_2 = 6n_2$ and $n_1, n_2 \geq 2$. We consider the rank 3 lattice

$$M := \langle h^2, \alpha_1, \alpha_2 \rangle \subset L$$

generated by $h^2$, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of $M$ with respect to the basis $(h^2, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2n_1 & 0 \\ 0 & 0 & 2n_2 \end{pmatrix}.$$

Therefore, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. In addition, for any nonzero $v = xh^2 + y\alpha_1 + z\alpha_2 \in M$, where $x, y, z$ are integers, we have

$$(v.v) = 3x^2 + 2n_1 y^2 + 2n_2 z^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers $x, y, z$ is nonzero. Hence, the embedding $M \subset L$ satisfies Lemma 6 (iii). Thus, by Lemma 6 and Proposition 5, $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\operatorname{disc}(A(X)) = d_1 d_2/3$. Moreover, we consider the sublattices

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

with discriminant $d_1$, and

$$K_{d_2} := \langle h^2, \alpha_2 \rangle \subset M$$

with discriminant $d_2$. Clearly, both $K_{d_1}$ and $K_{d_2}$ are saturated sublattices of $M$. Applying Lemma 6 and Proposition 5 again, we obtain $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we want.

**Case 2:** $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 2 \pmod 6$**.** Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1, \alpha_2 + (0,0,1) \rangle \subset L$$

where $(0,0,1) \in I_{3,0}$. Then the Gram matrix of $M$ with respect to the basis $(h^2, \alpha_1, \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2n_1 & 0 \\ 1 & 0 & 2n_2+1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. Furthermore, for any nonzero $v = xh^2 + y\alpha_1 + z(\alpha_2 + (0,0,1)) \in M$, we get

$$(v.v) = 2x^2 + 2n_1y^2 + 2n_2z^2 + (x+z)^2 \geq 3$$

since $n_1 \geq 2$, $n_2 \geq 1$ and at least one of the integers $x, y, z$ is nonzero. Hence, by Lemma 6 and Proposition 5, we conclude that $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\operatorname{disc}(A(X)) = d_1d_2/3$. Similarly, we consider the sublattices:

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

of discriminant $d_1$, and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

of discriminant $d_2$. Again Lemma 6 and Proposition 5 imply $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted.

**Case 3: $d_1 \equiv 2 \pmod 6$ and $d_2 \equiv 2 \pmod 6$.** Assume $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1) \rangle \subset L$$

here $(0,1,0) \in I_{3,0}$. Then the Gram matrix of $M$ with respect to the basis $(h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2n_1+1 & 0 \\ 1 & 0 & 2n_2+1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. For any nonzero $v = xh^2 + y(\alpha_1 + (0,1,0)) + z(\alpha_2 + (0,0,1)) \in M$, we obtain

$$(v.v) = x^2 + 2n_1y^2 + 2n_2z^2 + (x+y)^2 + (x+z)^2 \geq 3$$

since $n_1, n_2 \geq 1$ and at least one of the integers $x, y, z$ is nonzero. Hence, Lemma 6 and Proposition 5 concludes that $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\operatorname{disc}(A(X)) = (d_1d_2 - 1)/3$. Moreover, we consider

$$K_{d_1} := \langle h^2, \alpha_1 + (0,1,0) \rangle \subset M$$

with discriminant $d_1$ and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

with discriminant $d_1$. By Lemma 6 and Proposition 5, we obtain $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. As a consequence, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted. This finishes the proof of Theorem 7. □

**Remark 8.** Note that it has been known for 20 years that $\mathscr{C}_8 \cap \mathscr{C}_{14} \neq \emptyset$ (Hassett [10]) and proved more recently that $\mathscr{C}_8$ intersects every Hassett divisor (Addington–Thomas [2, Theorem 4.1]). It is also shown that $\mathscr{C}_8 \cap \mathscr{C}_{14}$ has five irreducible components ( [3, 7]). Moreover, [7, p. 166] has mentioned that $\mathscr{C}_{14}$ intersects many other divisors $\mathscr{C}_d$, however, it is not obvious to see which Hassett divisors intersect with $\mathscr{C}_{14}$.

Consequently, Theorem 7 not only generalizes [2, Theorem 4.1] but also implies that $\mathscr{C}_{14}$ intersects all Hassett divisors. Because of the same reason, we may conclude the main result of the current paper.

**Theorem 9 (=Theorem 2).** *Every Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

**Proof.** Applying Theorem 7 to the pairs of integers $(d_1, d_2) = (d, 14)$, $(d, 26)$, $(d, 38)$. Then there exist three smooth cubic fourfolds $X_1$, $X_2$ and $X_3$ such that

$$[X_1] \in \mathscr{C}_{A(X_1)} \subset \mathscr{C}_d \cap \mathscr{C}_{14} \subset \mathscr{C}_d,$$
$$[X_2] \in \mathscr{C}_{A(X_2)} \subset \mathscr{C}_d \cap \mathscr{C}_{26} \subset \mathscr{C}_d,$$
$$[X_3] \in \mathscr{C}_{A(X_3)} \subset \mathscr{C}_d \cap \mathscr{C}_{38} \subset \mathscr{C}_d,$$

where $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$, and $\mathscr{C}_{A(X_3)}$ are subvarieties of codimension-two in $\mathscr{C}$. Here $A(X_1)$, $A(X_2)$ and $A(X_3)$ are three different rank 3 lattices of discriminants:

- if $d \equiv 0 \pmod 6$, then $\operatorname{disc}(A(X_1)) = 14d/3$, $\operatorname{disc}(A(X_2)) = 26d/3$ and $\operatorname{disc}(A(X_3)) = 38d/3$;
- if $d \equiv 2 \pmod 6$, then $\operatorname{disc}(A(X_1)) = (14d-1)/3$, $\operatorname{disc}(A(X_2)) = (26d-1)/3$ and $\operatorname{disc}(A(X_3)) = (38d-1)/3$.

By definition of $\mathscr{C}_{A(X_i)}$ (see Proposition 5), a smooth cubic fourfold $[X] \in \mathscr{C}_{A(X_i)}$ only if there exists a saturated embedding $A(X_i) \subset A(X)$. Since $A(X_1)$, $A(X_2)$ and $A(X_3)$ are rank 3 lattices of different discriminants, it follows that there is no saturated embedding $A(X_i) \subset A(X_j)$ if $i \neq j$. Therefore, $[X_i] \notin \mathscr{C}_{A(X_j)}$ if $i \neq j$ and $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$, and $\mathscr{C}_{A(X_3)}$ are three different subvarieties of codimension-two in $\mathscr{C}$.

Moreover, since every smooth cubic fourfold in $\mathscr{C}_{14}$, $\mathscr{C}_{26}$ and $\mathscr{C}_{38}$ is rational ( [7, 21]), so every smooth cubic fourfold in $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$ and $\mathscr{C}_{A(X_3)}$ is rational. Therefore, $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$ and $\mathscr{C}_{A(X_3)}$ are three different codimension-two subvarieties which parametrize rational cubic fourfolds. This completes the proof of Theorem 9. □

Our main result also motivates the following natural question:

**Question 10.** *Suppose that $d$ satisfies $(\star)$ and $d$ is not an admissible value. Does the Hassett divisor $\mathscr{C}_d$ contain a union of countably infinite codimension-two subvarieties in $\mathscr{C}$ parametrizing rational cubic fourfolds?*

The answer to Question 10 has already been known for $\mathscr{C}_8$ and $\mathscr{C}_{18}$ ( [1, 10]).

**Corollary 11.** *The answer to Question 10 is yes if the "if" part of Conjecture 1 holds.*

Returning to Conjecture 1, as a by-product of Theorem 9 (=Theorem 2), we have the following.

**Corollary 12.** *For every admissible value $d$, the Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

To obtain more information about the Hassett divisors, it is of importance to notice that Addington–Thomas [2, Theorem 4.1] showed that for any $d$ satisfying $(\star)$ there exists a cubic fourfold $[X] \in \mathscr{C}_8 \cap \mathscr{C}_d$ such that $[X] \in \mathscr{C}_{d'}$ for some admissible value $d'$. Even if it is conjectured to be rational, however, it is still unknown whether such a $X$ is rational or not. Using the idea of the proof of Theorem 7 and Theorem 9, we obtain a generalization of [2, Theorem 4.1].

**Theorem 13.** *If $d_1$ and $d_2$ satisfy $(\star)$, then $\mathscr{C}_{14} \cap \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ contains a codimension-three subvariety in $\mathscr{C}$ parametrizing rational cubic fourfolds.*

**Proof.** Analogously to the proof of Theorem 7, we only need to consider three cases:

**Case 1.** Given $d_1 = 6n_1$ and $d_2 = 6n_2$ with $n_1, n_2 \geq 2$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1, \alpha_2\rangle \subset L$$

where $v = (3,1,0) \in I_{3,0} \subset L$, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of $M$ with respect to the basis $(h^2, v, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & 10 & 0 & 0 \\ 0 & 0 & 2n_1 & 0 \\ 0 & 0 & 0 & 2n_2 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. For any nonzero $v = x_1 h^2 + x_2 v + x_3 \alpha_1 + x_4 \alpha_2 \in M$, we have

$$(v.v) = 2(x_1 + 2x_2)^2 + x_1^2 + 2x_2^2 + 2n_1 x_3^2 + 2n_2 x_4^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers $x_i$ is nonzero $(i = 1,2,3,4)$. Hence, Lemma 6 and Proposition 5 conclude that $\mathscr{C}_M$ is nonempty and has codimension 3. In addition, we consider the lattices $K_{14} = \langle h^2, v\rangle$ and $K_{d_i} := \langle h^2, \alpha_i\rangle \subset M$ with discriminant $d_i$. By Lemma 6 and Proposition 5, we obtain $\mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and also $\mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $\emptyset \neq \mathscr{C}_M \subset \mathscr{C}_{14} \cap \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted, since every cubic fourfold in $\mathscr{C}_{14}$ is rational.

Since Case 2 and Case 3 are the same as Case 1, we just give the main ingredients and left the details to the interested readers.

**Case 2.** Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1, \alpha_2 + (0,0,1)\rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, v\rangle$, $K_{d_1} := \langle h^2, \alpha_1\rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0,0,1)\rangle \subset M$.

**Case 2.** Given $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1)\rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, v\rangle$, $K_{d_1} := \langle h^2, \alpha_1 + (0,1,0)\rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0,0,1)\rangle \subset M$. $\square$

## References


[1] N. Addington, B. Hassett, Y. Tschinkel, A. Várilly-Alvarado, "Cubic fourfolds fibered in sextic del Pezzo surfaces", *Am. J. Math.* **141** (2019), no. 6, p. 1479-1500.

[2] N. Addington, R. Thomas, "Hodge theory and derived categories of cubic fourfolds", *Duke Math. J.* **163** (2014), no. 10, p. 1885-1927.

[3] A. Auel, M. Bernardara, M. Bolognesi, A. Várilly-Alvarado, "Cubic fourfolds containing a plane and a quintic del Pezzo surface", *Algebr. Geom.* **1** (2014), no. 2, p. 181-193.

[4] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, P. Stellari, "Stability conditions in families", https://arxiv.org/abs/1902.08184v1, 2019.

[5] A. Bayer, M. Lahoz, E. Macrì, P. Stellari, "Stability conditions on Kuznetsov components", https://arxiv.org/abs/1703.10839v2, 2019.

[6] A. Beauville, R. Donagi, "La variété des droites d'une hypersurface cubique de dimension 4", *C. R. Math. Acad. Sci. Paris* **301** (1985), p. 703-706.

[7] M. Bolognesi, F. Russo, G. Staglianò, "Some loci of rational cubic fourfolds", *Math. Ann.* **373** (2019), no. 1-2, p. 165-190.

[8] C. H. Clemens, P. Griffiths, "The intermediate Jacobian of the cubic threefold", *Ann. Math.* **95** (1972), p. 281-356.

[9] G. Fano, "Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del $4^0$ ordine", *Comment. Math. Helv.* **15** (1943), p. 71-80.

[10] B. Hassett, "Some rational cubic fourfolds", *J. Algebr. Geom.* **8** (1999), no. 1, p. 103-114.

[11] ———, "Special cubic fourfolds", *Compos. Math.* **120** (2000), no. 1, p. 1-23.

[12] ———, "Cubic fourfolds, K3 surfaces, and rationality questions", in *Rationality problems in algebraic geometry*, Lecture Notes in Mathematics, vol. 2172, Springer, 2016, p. 29-66.
[13] D. Huybrechts, "The K3 category of a cubic fourfold", *Compos. Math.* **153** (2017), no. 3, p. 586-620.
[14] ———, "Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories", in *Birational Geometry of Hypersurfaces*, Lecture Notes of the Unione Matematica Italiana, vol. 26, Springer, 2019, p. 165-198.
[15] M. Kontsevich, Y. Tschinkel, "Specialization of birational types", *Invent. Math.* **217** (2019), no. 2, p. 415-432.
[16] A. Kuznetsov, "Derived categories of cubic fourfolds", in *Cohomological and geometric approaches to rationality problems. New Perspectives*, Progress in Mathematics, vol. 282, Birkhäuser, 2010, p. 219-243.
[17] R. Laza, "The moduli space of cubic fourfolds via the period map", *Ann. Math.* **172** (2010), no. 1, p. 673-711.
[18] E. Looijenga, "The period map for cubic fourfolds", *Invent. Math.* **177** (2009), no. 1, p. 213-233.
[19] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications", *Math. USSR, Izv.* **14** (1980), p. 103-167.
[20] F. Russo, G. Staglianò, "Explicit rationality of some cubic fourfolds", https://arxiv.org/abs/1811.03502v1, 2018.
[21] ———, "Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds", *Duke Math. J.* **168** (2019), no. 5, p. 849-865.
[22] ———, "Trisecant Flops, their associated K3 surfaces and the rationality of some Fano fourfolds", https://arxiv.org/abs/1909.01263v2, 2019.
[23] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer, 1973.
[24] C. Voisin, "Théorème de Torelli pour les cubiques de $\mathbb{P}^5$", *Invent. Math.* **86** (1986), p. 577-601.
[25] ———, "Some aspects of the Hodge conjecture", *Jpn. J. Math.* **2** (2007), no. 2, p. 261-296.