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Sur la loi faible des grands nombres pour des sommes pondérées de variables aléatoires négativement superadditivement-dépendantes

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Abstract. In this paper, we extend Kolmogorov–Feller weak law of large numbers for maximal weighted sums of negatively superadditive dependent (NSD) random variables. In addition, we make a simulation study for the asymptotic behavior in the sense of convergence in probability for weighted sums of NSD random variables.

Résumé. Dans cet article, nous étendons la loi faible des grands nombres de Kolmogorov–Feller à des sommes pondérées maximales de variables aléatoires négativement superadditivement-dépendantes (NSD). En outre, nous construisons une étude de simulation du comportement asymptotique au sens de la convergence en probabilité pour les sommes pondérées de variables aléatoires NSD.

1. Introduction

Limit theorems, in particular laws of large numbers, play exceedingly important role in probability theory and its applications in mathematical statistics. Among laws of large numbers, weighted laws are studied by many authors. This kind of limit theorems appear in a natural way for random variables with infinite or equal to zero expected value, and are called exact laws of large numbers (see Adler and Matuła [1]). On the other hand different concepts of dependence, with a special emphasis on positive and negative dependence, have been intensively explored in past decades (see Bulinski and Shashkin [5]). In this context, laws of large numbers for positively and negatively dependent random variables are still an attractive area of research. In this paper we will focus on weak laws of large numbers under negative dependence assumptions.

The concept of negatively associated (NA) random variables was introduced by Alam and Saxena [2] and carefully studied by Joag-Dev and Proschan [10] and Block et al. [4]. As pointed out and proved by Joag-Dev and Proschan, a number of well-known multivariate distributions possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability theory. Many investigators discuss applications of NA to probability, stochastic processes and statistics.

Definition 1. *Random variables* $X_1, X_2, \ldots, X_n$ *are said to be NA if for every pair of disjoint subsets* A_1 *and* A_2 *of* $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where the functions f_1 *and* f_2 *are increasing in any variable (or decreasing in any variable) and such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.*

The next dependence notion is negative superadditive dependence, which is weaker than NA. The concept of negatively superadditive-dependent (NSD) random variables was introduced by Hu [8] as follows.

Definition 2. *A random vector* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *is said to be NSD if*

$$\mathbb{E}\phi(X_1, X_2, \ldots, X_n) \leq \mathbb{E}\phi(X_1^*, X_2^*, \ldots, X_n^*)$$

where $X_1^*, X_2^*, \ldots, X_n^*$ *are independent such that* X_i^* *and* X_i *have the same distribution for each* i *and* ϕ *is a superadditive function such that the expectations in the above equation exist. A sequence* $\{X_n, n \geq 1\}$ *of random variables is said to be NSD if for each* $n \geq 1$, *the vector* $(X_1, X_2, \ldots, X_n)$ *is NSD.*

Hu [8] gave an example illustrating that NSD does not imply NA, and posed an open problem whether NA implies NSD. In addition, he provided some basic properties and three structural theorems for NSD random vectors. Christofides and Vaggelatou [6] solved this open problem and indicated that NA implies NSD. Therefore, the NSD structure is an extension of NA structure and it is sometimes more useful. It can be used to get many important probability inequalities.

This is known that a sequence of independent and identically distributed (i.i.d.) random variables satisfies the Kolmogorov–Feller weak law of large numbers (WLLN) i.e.

$$n\mathbb{P}(|X_1| > n) \to 0 \quad \text{if and only if} \quad \frac{\sum_{k=1}^n X_k - n\mathbb{E}X_1 \mathbb{I}[|X_1| \leq n]}{n} \to 0 \quad \text{in probability.}$$

For details on the Kolmogorov–Feller WLLN, its extensions and improvements we refer the reader to Petrov [14], Yuan et al. [16] and Naderi et al. [12].

Jajte [9] studied a large class of summability methods defined as follows: it is said that a sequence $\{X_n, n \geq 1\}$ of r.v.'s is almost surely summable to a r.v. X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^n \frac{1}{h(k)} X_k \to X \text{ a.s., } n \to \infty.$$

For a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables Jajte proved that $\{X_n - \mathbb{E}X_n \mathbb{I}[|X_n| \leq \phi(n)], \ n \geq 1\}$ is almost surely summable to 0 by the method (h, g) if and only if $\mathbb{E}\phi^{-1}(|X_1|) < \infty$ (ϕ^{-1} is inverse of ϕ), where g, h and $\phi(y) = g(y)h(y)$ are functions satisfying some additional conditions. The most up-to-date survey on this matter may be found in Fazekas et al. [7], Naderi et al. [13] and Shen [15].

In what follows we shall use the concept of regularly varying functions (see [3]).

Definition 3. *A measurable function $U : [a, \infty) \to (0, \infty)$, $a \in \mathbb{R}$, is called regularly varying at infinity with exponent ρ, denoted as $U \in \mathscr{RV}(\rho)$, if for all $t > 0$,*

$$\lim_{x \to \infty} \frac{U(tx)}{U(x)} = t^\rho.$$

If $\rho = 0$ then we say that U is slowly varying at infinity and write $U \in \mathscr{SV}$.

Finally, we will recall the concept of stochastic domination, which will be used in the sequel.

Definition 4. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$\mathbb{P}(|X_n| > x) \leq C\mathbb{P}(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

In Section 2 we will present our main results which are devoted to the WLLN in the Kolmogorov–Feller version for weighted sums of NSD random variables. We will study sequences summable by the method (h, g) described above, and we will make a simulation study in Section 3.

Throughout the paper, let C denote a positive constant not depending on n and let $\mathbb{I}(A)$ be the indicator function of the set A.

2. Main Results

We will provide some preliminary facts needed for the proof of our main results. The first two lemmas come from Hu [8].

Lemma 5. *If $X_1, X_2, \ldots, X_n$ are NSD random variables and $g_1, g_2, \ldots, g_n$ are non-decreasing functions, then $g_1(X_1), g_2(X_2), \ldots, g_n(X_n)$ are also NSD random variables.*

Lemma 6. *Let $n \geq 2$ and $X_1, X_2, \ldots, X_n$ be NSD random variables with mean zero and finite second moments. Then for any $\varepsilon > 0$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon\right) \leq \frac{8}{\varepsilon^2}\sum_{i=1}^{n} \operatorname{Var} X_i.$$

The last one is a basic property for stochastic domination. The proof is standard, so we omit the details.

Lemma 7. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\mathbb{E}|X_n|^\alpha \mathbb{I}[|X_n| \leq b] \leq C_1\left[\mathbb{E}|X|^\alpha \mathbb{I}[|X| \leq b] + b^\alpha \mathbb{P}(|X| > b)\right],$$
$$\mathbb{E}|X_n|^\alpha \mathbb{I}[|X_n| > b] \leq C_2 \mathbb{E}|X|^\alpha \mathbb{I}[|X| > b],$$

where C_1 and C_2 are positive constants. It is also obvious that $\mathbb{E}(|X_n|^\alpha) \leq C\mathbb{E}(|X|^\alpha)$.

We start with the assumptions which will be imposed on our weights. Let $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ be nonnegative functions let us put $\phi(y) = g(y)h(y)$. Assume that the following conditions are satisfied:

(A1) h is nondecreasing and ϕ is strictly increasing with $\phi([0,\infty)) = [0,+\infty)$,
(A2) there exists a constant $b > 0$ such that

$$\sum_{i=1}^{n} \frac{1}{h^2(i)} \le b \frac{n}{h^2(n)}.$$

In the following, let $\{X_n, n \ge 1\}$ be a sequence of NSD random variables, which is stochastically dominated by a random variable X. We will also use the notations $m_{i,n} = \mathbb{E}X_i\mathbb{I}[|X_i| \le \phi(n)]$, $M_{i,n}$ is the median of random variable $X_i\mathbb{I}[|X_i| \le \phi(n)]$ and

$$\bar{X}_i = -\phi(n)\mathbb{I}[X_i < -\phi(n)] + X_i\mathbb{I}[|X_i| \le \phi(n)] + \phi(n)\mathbb{I}[X_i > \phi(n)],$$

for $i \ge 1$.

The following lemma will be the main tool for proving our result.

Lemma 8. *Let g, h and ϕ, satisfy the conditions (A1)–(A2) and $\mathbb{P}(|X| > x) \in \mathscr{RV}(\rho)$, for some $\rho > -1$. If $\lim_{n\to\infty} n\mathbb{P}\big(|X| > \phi(n)\big) = 0$, then the following statements hold*

(i) $\lim_{n\to\infty} \max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{\mathbb{E}(\bar{X}_i) - \mathbb{E}(X_i\mathbb{I}[|X_i| \le \phi(n)])}{h(i)} \right| = 0$,

(ii) $\lim_{n\to\infty} \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_i|\mathbb{I}[|X_i| \le \phi(n)])}{h(i)} = 0$,

(iii) $\lim_{n\to\infty} \frac{1}{g^2(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_i|^2\mathbb{I}[|X_i| \le \phi(n)])}{h^2(i)} = 0$,

(iv) $\lim_{n\to\infty} \frac{1}{g^2(n)} \sum_{i=1}^{n} \frac{\phi^2(n)\mathbb{P}(|X_i| > \phi(n))}{h^2(i)} = 0$.

Proof. (i). By Definition 4 and (A2) we get

$$\max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{\mathbb{E}(\bar{X}_i) - \mathbb{E}(X_i\mathbb{I}[|X_i| \le \phi(n)])}{h(i)} \right|$$

$$\le \max_{1 \le k \le n} \frac{1}{g(n)} \sum_{i=1}^{k} \frac{\mathbb{E}(\phi(n)\mathbb{I}[|X_i| > \phi(n)])}{h(i)}$$

$$= \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(\phi(n)\mathbb{I}[|X_i| > \phi(n))}{h(i)} = h(n) \sum_{i=1}^{n} \frac{\mathbb{P}(|X_i| > \phi(n)])}{h(i)}$$

$$\le Ch(n)\mathbb{P}(|X| > \phi(n)) \sum_{i=1}^{n} \frac{1}{h(i)} \le Ch^2(n)\mathbb{P}(|X| > \phi(n)) \sum_{i=1}^{n} \frac{1}{h^2(i)}$$

$$\le Ch^2(n)\mathbb{P}(|X| > \phi(n)) \frac{n}{h^2(n)} = Cn\mathbb{P}(|X| > \phi(n)) \to 0.$$

(ii). According to Lemma 7, (A2) and Theorem 1.5.11 (i) in [3] we obtain

$$\frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_i|\mathbb{I}[|X_i| \le \phi(n)])}{h(i)}$$

$$\le \frac{C\left[\mathbb{E}(|X|\mathbb{I}[|X| \le \phi(n)]) + \phi(n)\mathbb{P}(|X| > \phi(n))\right]}{g(n)} \sum_{i=1}^{n} \frac{1}{h(i)}$$

$$\le \frac{Ch(n)\left[\mathbb{E}(|X|\mathbb{I}[|X| \le \phi(n)]) + \phi(n)\mathbb{P}(|X| > \phi(n))\right]}{g(n)} \sum_{i=1}^{n} \frac{1}{h^2(i)}$$

$$\le \frac{Cn\mathbb{E}(|X|\mathbb{I}[|X| \le \phi(n)])}{\phi(n)} + Cn\mathbb{P}(|X| > \phi(n)) \le 2Cn\mathbb{P}(|X| > \phi(n)) \to 0.$$

(iii). This case is similar to the proof of (3) in [12].

(iv). By Definition 4 and (A2) we get

$$\frac{1}{g^2(n)} \sum_{i=1}^{n} \frac{\phi^2(n)\mathbb{P}(|X_i| > \phi(n))}{h^2(i)} \le Ch^2(n)\mathbb{P}(|X| > \phi(n)) \sum_{i=1}^{n} \frac{1}{h^2(i)}$$
$$\le Cn\mathbb{P}(|X| > \phi(n)) \to 0. \qquad \square$$

Let us state our main result.

Theorem 9. *Let g, h and ϕ, satisfy the conditions (A1)–(A2) and $\mathbb{P}(|X| > x) \in \mathscr{RV}(\rho)$, for some $\rho > -1$. If $\lim_{n\to\infty} n\mathbb{P}(|X| > \phi(n)) = 0$, then*

$$\max_{1 \le k \le n} \frac{1}{g(n)} \sum_{i=1}^{k} \frac{X_i - m_{i,n}}{h(i)} \to 0 \ \text{ in probability, as } n \to \infty. \tag{1}$$

Proof. For $k \ge 1$, define

$$S_k = \sum_{i=1}^{k} \frac{X_i - m_{i,n}}{h(i)}, \ \bar{S}_k = \sum_{i=1}^{k} \frac{\bar{X}_i - m_{i,n}}{h(i)}.$$

It is easy to see that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{1 \le k \le n} |S_k| > \varepsilon g(n) \right) \le \sum_{i=1}^{n} \mathbb{P}(|X_i| > \phi(n)) + \mathbb{P}\left(\max_{1 \le k \le n} |\bar{S}_k| > \varepsilon g(n) \right) =: I + II.$$

It is clear, by Definition 4 and $n\mathbb{P}(|X| > \phi(n)) \to 0$, that $I \to 0$, as $n \to \infty$. By Lemma 8 (i), we see that for sufficiently large n

$$II := \mathbb{P}\left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} \frac{\bar{X}_i - m_{i,n}}{h(i)} \right| > \varepsilon g(n) \right)$$

$$\le \mathbb{P}\left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} \frac{\bar{X}_i - \mathbb{E}(\bar{X}_i)}{h(i)} \right| + \max_{1 \le k \le n} \left| \sum_{i=1}^{k} \frac{\mathbb{E}(\bar{X}_i) - m_{i,n}}{h(i)} \right| > \varepsilon g(n) \right)$$

$$\le \mathbb{P}\left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} \frac{\bar{X}_i - \mathbb{E}(\bar{X}_i)}{h(i)} \right| > \frac{\varepsilon}{2} g(n) \right).$$

It is easily seen, by Lemma 5, that $\{\bar{X}_n - \mathbb{E}(\bar{X}_n), n \ge 1\}$ is also a sequence of NSD random variables. Then, we have by Lemma 6 and Lemma 8 (iii), (iv)

$$II \le \frac{4}{\varepsilon^2 g^2(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(\bar{X}_i^2)}{h^2(i)}$$

$$\le \frac{4C}{\varepsilon^2 g^2(n)} \sum_{i=1}^{n} \frac{\phi^2(n)\mathbb{P}(|X_i| > \phi(n))}{h^2(i)} + \frac{4C}{\varepsilon^2 g^2(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_i|^2 \mathbb{I}[|X_i| \le \phi(n)])}{h^2(i)} \to 0.$$

The proof is complete. $\qquad \square$

Corollary 10. *Let g, h and ϕ, satisfy the conditions (A1)–(A2) and $\mathbb{P}(|X| > x) \in \mathscr{RV}(\rho)$, for some $\rho > -1$. If $\lim_{n\to\infty} n\mathbb{P}(|X| > \phi(n)) = 0$, then*

$$\max_{1 \le k \le n} \frac{1}{g(n)} \sum_{i=1}^{k} \frac{X_i}{h(i)} \to 0 \ \text{ in probability, as } n \to \infty. \tag{2}$$

Proof. Let us observe that,

$$\max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{X_i}{h(i)} \right| \le \max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{X_i - m_{i,n}}{h(i)} \right| + \max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{m_{i,n}}{h(i)} \right|$$

and

$$\max_{1 \le k \le n} \frac{1}{g(n)} \left| \sum_{i=1}^{k} \frac{m_{i,n}}{h(i)} \right| \le \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_i|\mathbb{I}[|X_i| \le \phi(n)])}{h(i)}.$$

Therefore, Theorem 9 and Lemma 8 (ii) imply (2). $\qquad \square$

In the following theorem we change centering constants $m_{i,n}$ to median $M_{i,n}$.

Theorem 11. *Let the conditions of Theorem 9 be satisfied. If additionally $\rho > -1$ and $\lim_{n\to\infty} n^2 \mathbb{P}\left(|X| > \phi(n)\right) = 0$, then*

$$\max_{1\le k\le n} \frac{1}{g(n)} \sum_{i=1}^{k} \frac{X_i - M_{i,n}}{h(i)} \to 0 \text{ in probability, as } n \to \infty. \tag{3}$$

Proof. As we know,

$$\max_{1\le k\le n} \frac{1}{g(n)} \left|\sum_{i=1}^{k} \frac{X_i - M_{i,n}}{h(i)}\right| \le \max_{1\le k\le n} \frac{1}{g(n)} \left|\sum_{i=1}^{k} \frac{X_i - m_{i,n}}{h(i)}\right| + \max_{1\le k\le n} \frac{1}{g(n)} \left|\sum_{i=1}^{k} \frac{m_{i,n} - M_{i,n}}{h(i)}\right|.$$

By the inequality $|m_{i,n} - M_{i,n}| \le \sqrt{\mathrm{Var}(X_i \mathbb{I}[|X_i| \le \phi(n)])}$ (see [11]) we get

$$\max_{1\le k\le n} \frac{1}{g(n)} \left|\sum_{i=1}^{k} \frac{m_{i,n} - M_{i,n}}{h(i)}\right| \le \frac{1}{g(n)} \sum_{i=1}^{n} \frac{|m_{i,n} - M_{i,n}|}{h(i)}$$

$$\le \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\sqrt{\mathrm{Var}(X_i \mathbb{I}[|X_i| \le \phi(n)])}}{h(i)}.$$

Therefore by Theorem 9 and the same argument as used in the proof of Lemma 8 (ii) we get (3). $\square$

Remark 12. The corollaries and examples in [12] can be rewritten for Theorem 9 and Theorem 11.

3. Simulation

In this section, we make a simulation to study numerical performance of WLLN result that was established in Theorem 9. For this purpose we use two examples. In the first one we generate a NSD sequence from the normal distribution, while in the second one from the Lomax distribution. Let us recall that X has the $\mathrm{Lomax}(\eta, \lambda)$ distribution, $\eta, \lambda > 0$ if its distribution function is $F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\eta}$ for $x \ge 0$. Let us observe that $1 - F(x) = \left(1 + \frac{x}{\lambda}\right)^{-\eta}$ is regularly varying with index $-\eta$, therefore to meet the assumptions of Theorem 9 we should require $\eta < 1$. In the following we explain these ways.

Example 13. Let us consider a sequence $\{\eta_n, n \ge 0\}$ of i.i.d. random variables with the standard normal distributions $\mathcal{N}(0,1)$. Let us define a new sequence $\{Y_n, n \ge 1\}$ as follows

$$Y_n = -\eta_{n-1} + \rho \eta_n \tag{4}$$

where $\rho > 0$. The sequence $\{Y_n, n \ge 1\}$ is a moving average sequence, wide sense stationary. Furthermore it is a Gaussian sequence with the same distribution $\mathcal{N}(0, 1 + \rho^2)$. The covariance structure is $\mathrm{Cov}(Y_i, Y_j) = -\rho$ if $|i - j| = 1$ and $\mathrm{Cov}(Y_i, Y_j) = 0$ otherwise. As a negatively correlated gaussian sequence $\{Y_n, n \ge 1\}$ is a NA sequence and in consequence NSD. In fact, the finite-dimensional distributions of this sequence are $(Y_1, \ldots, Y_n) \sim \mathcal{N}_n(\mathbf{0}, \mathbf{R})$, where $\mathbf{R} = [r_{ij}]$ is an $n \times n$ covariance matrix with the diagonal entities $r_{ii} = 1 + \rho^2$ and the off-diagonal ones $r_{ij} = -\rho$, $i = j - 1, j + 1$ and $r_{ij} = 0$, otherwise. The simulation procedure of the sequence $\{Y_n, n \ge 1\}$ is straightforward. Then, for given choices of n, $\alpha, \beta, \varepsilon$ and y_i's as the observation of Y_i's, the value of $s_n := \frac{1}{n^\beta} \sum_{i=1}^{n} \frac{y_i}{i^\alpha}$ are computed and since the sequence $\{Y_n, n \ge 1\}$ are symmetric and therefore $m_{i,n} = 0$ in Theorem 9. By repeating this procedure ($m =$)10000 times, the vector $(s_n^{(1)}, \ldots, s_n^{(m)})$ will be observed. Finally,

$$p_n := \frac{1}{m} \sum_{i=1}^{m} \mathbb{I}\left(|s_n^{(i)}| > \varepsilon\right) \tag{5}$$

is computed as an estimation of $\mathbb{P}(|S_n| > \varepsilon)$. Choosing $X \sim \text{Lomax}(0.5, 1)$, makes the sequence $\{Y_n, n \geq 1\}$ to be dominated by X and $g(t) = t^\beta, h(t) = t^\alpha$ for $\alpha + \beta > 1$, thus the conditions of Theorem 9 are satisfied. The probability p_n in (5) is derived for the following combinations: the correlation $\rho = 0.2$ and $\varepsilon = 0.05$, the sample size $n = 2, 10(50)10010$, and the three scenarios: case 1: $(\alpha_1 = 0.00, \beta_1 = 1.001)$, case 2: $(\alpha_2 = 0.499, \beta_2 = 0.501)$ and case 3: $(\alpha_3 = 0.25, \beta_3 = 0.751)$. Figure 1 exhibits the plot of (n, p_n) for the three cases. It is observed that p_n is a decreasing function of n tending to zero in all cases, thus, Corollary 10 is confirmed. Moreover, Figure 2 shows the boxplot of the s_n for $n = 10, 1000, 5000$. This plot indicates that s_n is distributed around zero-line and its variance will be decreased as n increases.
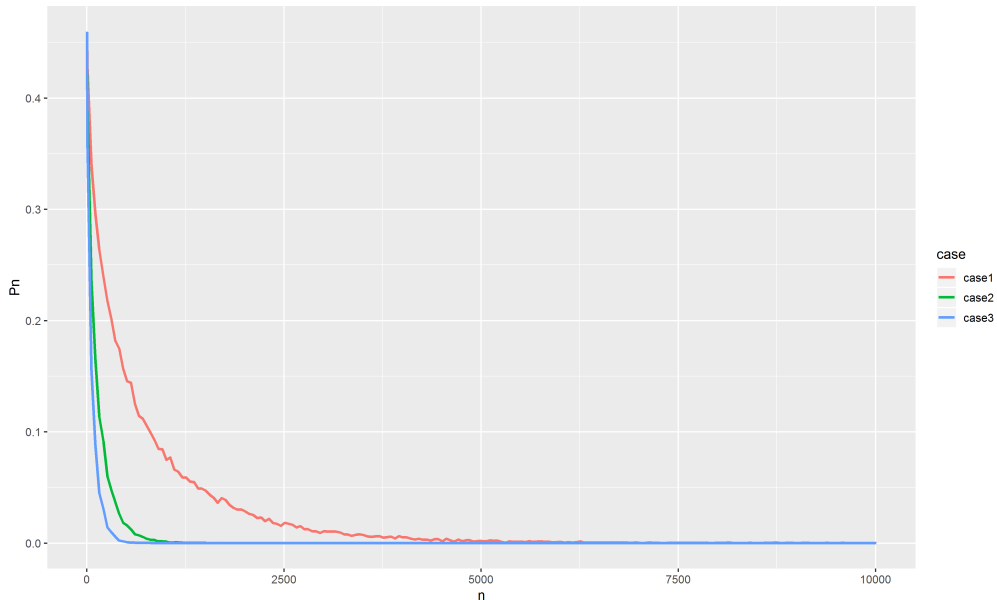

Figure 1. The plot of p_n versus n for Example 13.

Example 14. Now, let X be some random variable with invertible distribution function F, to make further simulations easy we shall choose X to have the Lomax distribution. Then, from the aforementioned sequence $\{Y_n, n \geq 1\}$ we can define a new sequence $\{X_n, n \geq 1\}$ as follows

$$X_n = F^{-1}\left(\Phi\left(\frac{Y_n}{\sqrt{1+\rho^2}}\right)\right),$$

where $\Phi(\cdot)$ stands for the distribution function of $\mathcal{N}(0,1)$. Then $\{X_n, n \geq 1\}$ is still NSD and the random variables X_n are identically distributed, therefore it is dominated by X. Here we consider $(\eta, \lambda) = (0.9, 1)$ The rest of this example is carried out with an almost similar manner as in Example 13, but with some different scenarios for the pairs (α, β) as: case 1: $(\alpha_1 = 2.5, \beta_1 = 1)$, case 2: $(\alpha_2 = 3, \beta_2 = 0.5)$ and case 3: $(\alpha_3 = 3.5, \beta_3 = 0.25)$.

As shown in Figure 3, the results in this example are almost the same as in the former one but with a slower convergence rate.
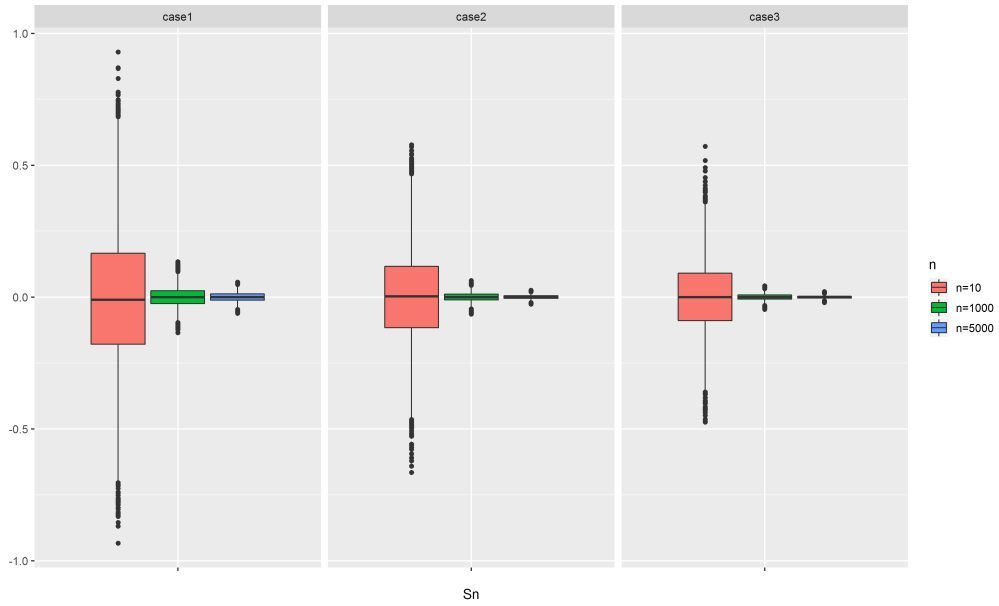
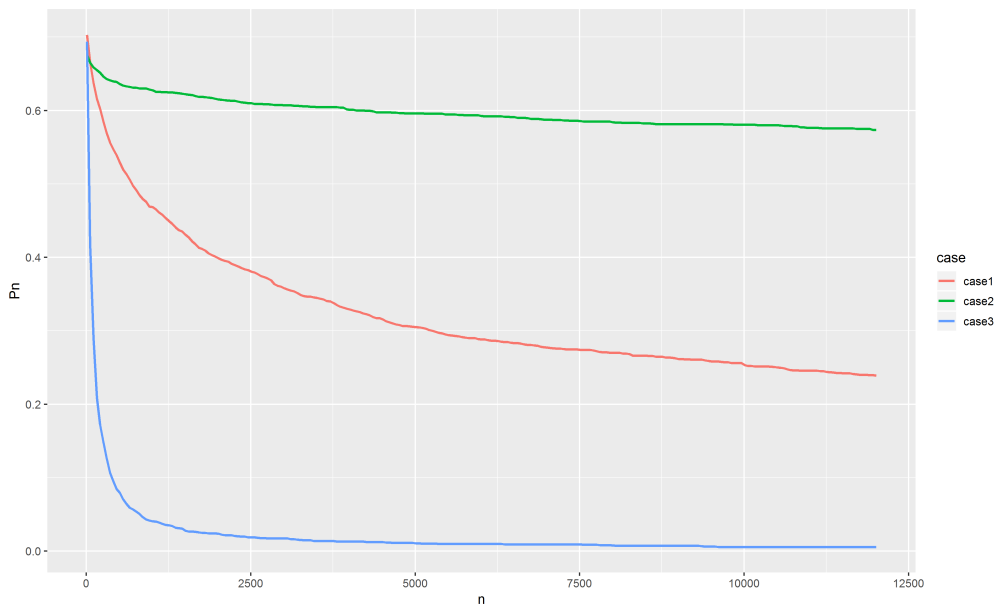**Figure 2.** The boxplot of s_n versus $n = 10, 1000, 5000$ for the three cases mentioned in Example 13.

Figure 3. The plot of p_n versus n for Example 14.

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