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First integrals of the Maxwell–Bloch system

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Abstract. We investigate the analytic, rational and C^1 first integrals of the Maxwell–Bloch system

$$\dot{E} = -\kappa E + gP, \quad \dot{P} = -\gamma_\perp P + gE\triangle, \quad \dot{\triangle} = -\gamma_\parallel(\triangle - \triangle_0) - 4gPE,$$

where $\kappa, \gamma_\perp, g, \gamma_\parallel, \triangle_0$ are real parameters. In addition, we prove this system is rationally non-integrable in the sense of Bogoyavlenskij for almost all parameter values.

Résumé. Nous étudions les premières intégrales analytiques, rationnelles et C^1 du système de Maxwell–Bloch

$$\dot{E} = -\kappa E + gP, \quad \dot{P} = -\gamma_\perp P + gE\triangle, \quad \dot{\triangle} = -\gamma_\parallel(\triangle - \triangle_0) - 4gPE,$$

où $\kappa, \gamma_\perp, g, \gamma_\parallel, \triangle_0$ sont des paramètres réels. En outre, nous prouvons que ce système est non intégrable rationnel dans le sens de Bogoyavlenskij pour presque toutes les valeurs de paramètres.

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1. Introduction and statement of the main results

Consider the Maxwell–Bloch system

$$\dot{E} = -\kappa E + gP, \quad \dot{P} = -\gamma_\perp P + gE\triangle, \quad \dot{\triangle} = -\gamma_\parallel(\triangle - \triangle_0) - 4gPE, \tag{1}$$

which describes the interaction between the coupling of the fundamental cavity mode E, the collective atomic polarization P and the population inversion Δ [1]. Here, $\kappa, \gamma_\perp, g, \gamma_\parallel, \triangle_0$ are real parameters and the dots denote derivatives with respect to the time t. As indicated in [1, 7, 8], it can be used to model Type I laser (He-Ne), Type II laser (Ruby and CO2) and Type III laser (far infrared) in the case of $\gamma_\perp \approx \gamma_\parallel \gg \kappa$, $\gamma_\perp \gg \gamma_\parallel \approx \kappa$ and $\triangle_0$ large enough, respectively. This system has been analyzed as a dynamical system by many researchers, see for instance [5, 7, 13, 18] and the references therein. In this paper, we try to understand its complexity and chaotic properties from the view of integrability and non-integrability.

The case when $g = 0$ is trivial, since system (1) with $g = 0$ is a linear system and is integrable. In what follows, we assume $g \neq 0$. For convenience, as reported in [13], we make a time rescaling $t \to gt$ and rewrite (1) into the following form

$$\dot{x} = -ax + y, \quad \dot{y} = -by + xz, \quad \dot{z} = -c(z - \delta_0) - 4xy, \tag{2}$$

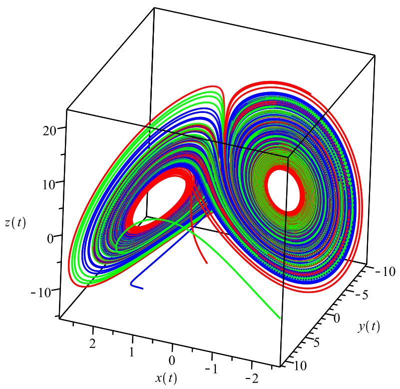
where $x,\ y,\ z$ simply present $E, P, \triangle$ and $a = g^{-1}\kappa,\ b = g^{-1}\gamma_\perp,\ c = g^{-1}\gamma_\parallel$ and $\delta_0 = \triangle_0$. Obviously, system (2) and the Maxwell–Bloch system (1) with $g \neq 0$ admit the same dynamical behavior and integrability properties. We turn to study first integrals of system (2) below.

The associated vector field of system (2) is

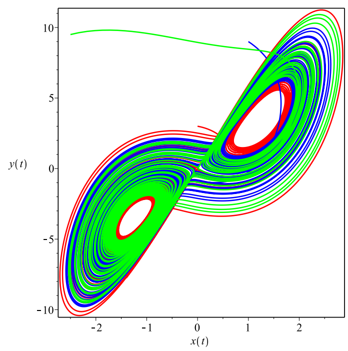
$$X := (-ax + y)\frac{\partial}{\partial x} - (by - xz)\frac{\partial}{\partial y} - \big(c(z - \delta_0) + 4xy\big)\frac{\partial}{\partial z}.$$

Let U be an open set in $\mathbb{C}^3$. A non-constant function $\Phi(x, y, z) \in C(\mathrm{U}, \mathbb{C})$ is called a first integral of system (2) if it stays constant along all solution curves $(x(t), y(t), z(t))$ of (2). If Φ is differentiable, then the definition can be written as $X(\Phi) \equiv 0$. When $\mathrm{U} = \mathbb{C}^3$, the first integral Φ is called a global first integral. When a first integral Φ is a rational (polynomial or analytic) function, we say that Φ is a rational (polynomial or analytic) first integral.

We first consider first integrals and integrability for system (2) in the category of rational functions. System (2) admits chaotic behaviour for a larger range of its parameters. For example, the chaotic attractor of (2) and its projections in the coordinate planes (x, y), (x, z) and (y, z) are shown in Figs. 1 and Figs. 2, see [7] for more details. It should be pointed out that the Maxwell–Bloch system is quite different from the Lorenz system because it is only for $\Delta_0 = (\kappa + 1)(\gamma_{||} + 1)$ that system (1) can be transformed into the Lorenz form. In short, the numerical analysis yields that the dynamics of (2) are complex and in fact chaotic, which inspire us to prove both systems are non-integrable.

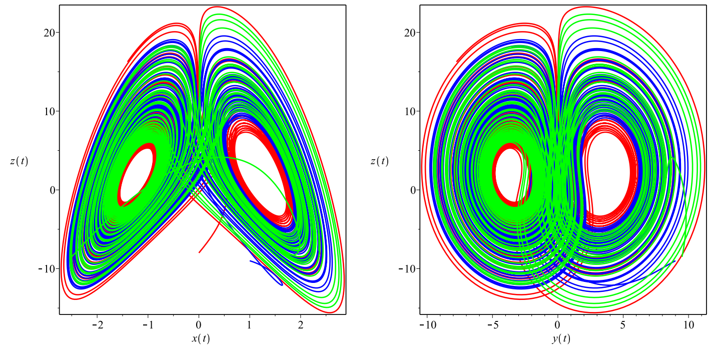


(a)



(b)

Figure 1. (a) chaotic attractor for system (2) with parameter values $a = 2.81, b = 0.64, c = 0.65, \delta_0 = 28$; (b) its projections onto the planes (x, y). (Attracteur chaotique du système (2) avec les valeurs des paramètres $a = 2.81, b = 0.64, c = 0.65, \delta_0 = 28$; (b) ses projections sur les plans (x, y).)



(a) (b)

Figure 2. Projections of the chaotic attractor into the plane (x,z) (a) and into the plane (y,z) (b). (Projections de l'attracteur chaotique dans le plan (x,z) (a) et dans le plan (y,z) (b).)

Recall that an n-dimensional analytic differential system

$$\dot{x} = F(x), \quad x \in \mathscr{M}, \quad t \in \mathbb{C}, \tag{3}$$

is integrable in the sense of Bogoyavlenskij if for some $k \in \{1, \dots n\}$ it admits $(n-k)$ functionally independent first integrals $\Phi_1, \dots, \Phi_{n-k}$ and k vector fields $w_1 = F, \dots, w_k$ such that

$$[w_i, w_j] = 0, \text{ and } w_j[\Phi_i] = 0, \text{ for } 1 \le i \le n-k, \; 1 \le j \le k.$$

This notion of integrability was introduced by Bogoyavlenskij, and is a natural generalization of the Liouvillian integrability from Hamiltonian systems to general dynamical systems [4].

The next result shows that system (2) is rationally non-integrable in the sense of Bogoyavlenskij for almost all parameter values.

Theorem 1. *Assume $c \neq 0$ and $2\sqrt{a^2 - 2ab + b^2 + 4\delta_0}/c$ is not an odd number. Then,*

(i) *System* (2) *does not possess any rational first integral.*
(ii) *System* (2) *is not rationally integrable in the sense of Bogoyavlenskij.*

Proof of Theorem 1 is based on an analysis of the differential Galois group of normal variational equations of (2) along a certain particular solution. Morales–Ruiz, Ramis, Simó, Baider, Churchill, Rod and Singer have applied the differential Galois theory to the non-integrability of Hamiltonian systems and developed the Morales–Ramis theory, see [3, 6, 19, 20] and references therein. In 2010, Ayoul and Zung [2] extended the Morales–Ramis theory to the non-integrability of general dynamical systems by using the so-called cotangent lift.

Next, we deal with the global analytic first integrals of system (2), and this class of first integrals includes polynomial first integrals. There are only very few families of differential equations in which a complete classification of global analytic first integrals is known, see for instance [14, 15, 16].

Theorem 2. *The following statements hold for system* (2).

(i) *If $a = b = c = 0$, it has the two functionally independent polynomial first integrals*

$$\Phi_1 = 2x^2 + z, \quad \Phi_2 = 4y^2 + z^2.$$

(ii) *If $a \neq 0$, $b = c = 0$, the unique global analytic first integrals of* (2) *are of the form $\Phi(4y^2 + z^2)$, where Φ is analytic over $\mathbb{C}$.*

(iii) *If $a \neq 0$, $v := b = c \neq 0$, it has no global analytic first integrals which are analytic in the parameter v in a neighborhood of $v = 0$.*

Theorem 2 (i) can be checked easily by direct computations and we omit its proof.

Finally, we prove the absence of C^1 first integrals for (2) under some conditions.

Proposition 3. *Let $(a,b,c) = (a_0 - b_1\epsilon + a_2\epsilon^2, -a_0 + b_1\epsilon + b_2\epsilon^2, c_1\epsilon + c_2\epsilon^2)$ and $\delta = -a_0^2 - \omega^2$ with $\omega > 0$, $a_0(a_2 + b_2) > 0$, $c_1 \neq 0$ and ϵ a small parameter. Then for $|\epsilon| \neq 0$ sufficiently small,*

(i) *System (2) has a periodic solution γ_ϵ.*

(ii) *System (2) has no C^1 first integral $\Phi(x,y,z)$ in the neighborhood of γ_ϵ such that $\nabla\Phi(x,y,z)$ and $(-ax+y, -by+xz, -c(z-\delta_0)-4xy)$ are linearly independent on the points of γ_ϵ.*

Statement (i) is proved by Cândido et al. [5], and we use their results in the essential way to prove statement (ii).

This paper is divided as follows. Section 2 will provide proofs of Theorems 1. The proof of Theorem 2 and Proposition 3 will be given in Section 3 and Section 4, respectively.

2. Proof of Theorem 1

In this section we first recall two results which are due to the works of Ayoul, Zung [2] and Shi, Li [11, 12], respectively.

Suppose system (3) has a non-equilibrium solution $\psi(t)$. The variational equations along its phase curve Γ have the form

$$\dot{\xi} = T(F)\xi, \quad \xi \in T_\Gamma\mathcal{M}, \tag{4}$$

where $T_\Gamma\mathcal{M}$ is the vector bundle of $T\mathcal{M}$ restricted on Γ. Then, by means of the natural projection π from $T_\Gamma\mathcal{M}$ to the normal bundle $T_\Gamma\mathcal{M}/\Gamma$, we can reduce (4) to the normal variational equations

$$\dot{\eta} = \pi_*(T(F)(\pi^{-1}\eta)), \quad \eta \in T_\Gamma\mathcal{M}/\Gamma. \tag{5}$$

Lemma 4 ([2]). *Assume that system (3) is B-integrable in the meromorphic category in a neighbourhood of a phase curve Γ. Then the identity component of the differential Galois group of the normal variational equations (5) along Γ is Abelian.*

Lemma 5 ([11, 12]). *Assume that system (3) has $m (1 \leq m < n)$ functionally independent meromorphic first integrals in a neighborhood of Γ. Then the Lie algebra $\mathcal{G}$ of the differential Galois group G of equations (5) has m meromorphic invariants, and the identity component G^0 of G has at most $(n-m-1)(n-1)$ generators, i.e.,*

$$G^0 = \{(e^{\mathfrak{X}_1 t_1} \cdot e^{\mathfrak{X}_2 t_2} \ldots e^{\mathfrak{X}_k t_k})^s \mid (t_1, \ldots, t_k) \in \tilde{V} \subset \mathbb{C}^k, s \in \mathbb{N}\},$$

where $\{\mathfrak{X}_1, \ldots, \mathfrak{X}_k\}$ is a basis of $\mathcal{G}$ with $k \leq (n-m-1)(n-1)$, $\tilde{V}$ is a neighborhood of the origin in $\mathbb{C}^k$. Especially,

(i) *If $m = n-1$, i.e., system (3) is completely integrable, then $\mathcal{G} = \{0\}, G^0 = \{\mathbb{1}\}$, where $\mathbb{1}$ denote the identity element of G.*

(ii) *If $m = n-2$, then $\mathcal{G}, G^0$ have at most $n-1$ generators.*

(iii) *If $n = 3$ and $m = 1$, then $\mathcal{G}, G^0$ are solvable.*

System (2) admits an invariant manifold

$$\mathcal{N} = \{(x,y,z) \in \mathbb{C}^3 : x = y = 0\}.$$

Then equations restricted to $\mathcal{N}$ is given by

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = -c(z - \delta_0). \tag{6}$$

Solving (6) yields a particular solution $\psi(t) = (0, 0, \exp(-ct) + \delta_0)$ of system (2). Let Γ be the flow generated by this solution. The variational equations along Γ read

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} -a & 1 & 0 \\ \exp(-ct) + \delta_0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{7}$$

Since the particular solution $\psi(t)$ lies in $\mathcal{N}$, the equations for variables (ξ, η) form a closed subsystem

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -a & 1 \\ \exp(-ct) + \delta_0 & -b \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{8}$$

which is the so-called normal variational equations along Γ. A straightforward computation yields a second-order equation

$$\ddot{\xi} + (a+b)\dot{\xi} + \big(ab - \exp(-ct) - \delta_0\big)\xi = 0, \tag{9}$$

which is equivalent to (8). In order to transform (9) into an equation with rational coefficients, we apply a variable change $\tau = -c\exp(-ct)$. Using

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \ddot{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau} + (\dot{\tau})^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}$$

we can rewrite (9) as

$$\xi'' + p(\tau)\xi' + q(\tau)\xi = 0, \tag{10}$$

where

$$p(\tau) = \frac{c-a-b}{c\tau}, \quad q(\tau) = \frac{abc - \delta_0 c + \tau}{c^3\tau^2},$$

and the prime denotes the derivative with respect to τ.

Further, under the classical change of the dependent variable

$$\xi(\tau) = \chi(\tau)\exp\left(-\frac{1}{2}\int p(\tau)\mathrm{d}\tau\right) = \chi(\tau)|\tau|^{(c-a-b)/2c},$$

(10) becomes

$$\begin{aligned} \chi'' &= \left(\frac{p^2}{4} + \frac{p'}{2} - q\right)\chi \\ &= \left(\frac{c(a^2 - 2ab + b^2 - c^2 + 4\delta_0) - 4\tau}{4c^3\tau^2}\right)\chi. \end{aligned} \tag{11}$$

Note that (11) has a regular point $\tau = 0$ of order two and an irregular singular point $\tau = \infty$ of order one. Hence, we can transform it into the Bessel equation whose differential Galois group has been studied widely. Indeed, we first make the time scale

$$\tau \longrightarrow \omega = \frac{2}{\sqrt{c^3}}\sqrt{\tau},$$

and transform (11) into

$$\omega^2\frac{\mathrm{d}^2\chi}{\mathrm{d}\omega^2} - w\frac{\mathrm{d}\chi}{\mathrm{d}\omega} + \left(\omega^2 - \frac{a^2 - 2ab + b^2 - c^2 + 4\delta_0}{c^2}\right)\chi = 0. \tag{12}$$

Then using again the classical change of the dependent variable

$$\chi = \tilde{\chi}\sqrt{\omega},$$

we transform (12) into the Bessel equation in the reduced form:

$$\frac{\mathrm{d}^2\tilde{\chi}}{\mathrm{d}\omega^2} = \left(\frac{4n^2 - 1}{4\omega^2} - 1\right)\tilde{\chi}, \tag{13}$$

with $n = \sqrt{a^2 - 2ab + b^2 + 4\delta_0}/c$.

For the solvability of the identity component of the differential Galois group associated with (13), we have the following theorem due to Morales–Ruiz [19].

Lemma 6. *The identity component G^0 of* (13) *is solvable if and only if $n+1/2$ belongs to $\mathbb{Z}$.*

Proof of Theorem 1. (i)**.** Let us assume the opposite: assume (2) has a rational first integral, then by Lemma 5, the identity component of the differential Galois group of normal variational equation (9) is solvable. Both (9) and (13) admit the same Liouvillian solvability in the category of rational functions. Hence, the identity component of (13) is also solvable. But by Lemma 6 and assumption of Theorem 1, identity component of (13) is not solvable. This leads to a contradiction.

(ii)**.** Assume (2) is rational integrable in the sense of Bogoyavlenskij. It follows from Lemma 4 that the identity component of the differential Galois group of normal variational equation (9) is Abelian. It is well-known that an Abelian group is solvable. Using the same procedure as above, we get a contradiction. □

3. Proof of Theorem 2

3.1. *Proof of Theorem 2 (ii)*

The following lemma plays an important role in the proof of statement (ii) of Theorem 2.

Lemma 7 **(**[10]**).** *Assume system* (3) *has a singularity at $x=0$, i.e., $F(0)=0$. If system* (3) *admits k functionally independent formal first integrals $\Phi_1(x),\dots,\Phi_k(x)$ in a neighborhood of the origin and the Jacobian matrix $DF(0)$ has $n-k$ eigenvalues $\lambda_1,\dots,\lambda_{n-k}$ satisfying*

$$\sum_{i=1}^{n-k} k_i\lambda_i \neq 0, \text{ for any } k_1,\dots,k_{n-k}\in\mathbb{Z}^+ \text{ with } \sum_{i=1}^{n-k} k_i \geq 1,$$

then all formal first integrals for system (3) *in a neighborhood of $x=0$ are formal series in $\Phi_1,\dots,\Phi_k$.*

Obviously, system (2) with $b=c=0$ becomes

$$\dot{x}=-ax+y,\quad \dot{y}=xz,\quad \dot{z}=-4xy. \tag{14}$$

A direct computation shows that $\Phi_1=4y^2+z^2$ is a global analytic first integral of system (14). Note that system (14) has the curve

$$S=\{(0,0,z):z\in\mathbb{C}\}$$

of singular points. Set $F=(-ax+y,xz,-4xy)$. At the singularity $p_z\in S$, the eigenvalues of $DF(p_z)$ are

$$\lambda_1=0,\ \lambda_2(p_z)=\frac{-a+\sqrt{a^2+4z}}{2},\ \lambda_3(p_z)=\frac{-a-\sqrt{a^2+4z}}{2}.$$

For $z=-1$, we have $\lambda_2(p_{-1})\lambda_3(p_{-1})=1>0$. Due to $a\in\mathbb{R}/\{0\}$, there are the following three cases:

(i) $\lambda_2(p_{-1}),\lambda_3(p_{-1})$ are all positive. In this case, we have

$$k_1\lambda_1+k_2\lambda_2>0,\ \text{ for any } k_1,k_2\in\mathbb{Z}^+\text{with}\quad k_1+k_2>0.$$

(ii) $\lambda_2(p_{-1}),\lambda_3(p_{-1})$ are all negative. In this case, we have

$$k_1\lambda_1+k_2\lambda_2<0,\ \text{ for any } k_1,k_2\in\mathbb{Z}^+\text{with}\quad k_1+k_2>0,$$

(iii) $\lambda_2(p_{-1}),\lambda_3(p_{-1})$ are a pair of conjugate complex numbers, i.e., $\lambda_2(p_{-1})=A+Bi$, $\lambda_3(p_{-1})=A-Bi$ with $A=-a/2\neq 0$ and $B=\sqrt{4-a^2}\in\mathbb{R}$. In this case, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, it is easy to check

$$k_1\lambda_2+k_2\lambda_3=A(k_1+k_2)+B(k_1-k_2)i\neq 0.$$

Now we translate the singular point p_{-1} at the origin of coordinates. Then we can apply Lemma 7 with $n = 3, k = 1$ and $\Phi_1 = 4y^2 + z^2$. Hence, all global analytic first integrals of (14), which is also a form first integrals, are formal series in the variable $4y^2 + z^2$. We complete the proof of statement (ii) of Theorem 2.

3.2. *Proof of Theorem 2 (iii)*

Now we study system (2) with $a \neq 0$ and $b = c = v \neq 0$. Since v is a parameter, we rewrite this system as

$$\dot{x} = -ax + y, \quad \dot{y} = -vy + xz, \quad \dot{z} = -v(z - \delta_0) - 4xy, \quad \dot{v} = 0. \tag{15}$$

It is worth to mention that if a function Φ is a first integral of system (2) with $a \neq 0$ and $b = c = v \neq 0$, then it is also a first integral of system (15), not vice versa. For example, $\Phi = \Phi(v)$ different from a constant is a first integral of (15) but is not a first integral of system (2).

To complete the proof of this theorem, we only need to show that if $\Phi = \Phi(x, y, z, v)$ is a global analytic first integral of system (15), then Φ is a global analytic function in the variable v.

Assume $\Phi = \Phi(x, y, z, v)$ is a global analytic first integral of system (15). Then $\Phi(x, y, z, 0)$ is a global analytic first integral of (2) with $b = c = 0$. It follows from Theorem 2 (ii) that $\Phi(x, y, z, 0) = T(4y^2 + z^2)$, where T is a global analytic function in the variable $4y^2 + z^2$. Using the convergent power series in a neighborhood of $(0, 0, 0, 0)$, we can rewrite $\Phi = T + v\Phi_1$ with $T = T(4y^2 + z^2)$ and $\Phi_1 = \Phi_1(x, y, z, v)$. By definition of first integrals, we have

$$(-ax + y)\frac{\partial(T + v\Phi_1)}{\partial x} + (-vy + xz)\frac{\partial(T + v\Phi_1)}{\partial y} + (-v(z - \delta_0) - 4xy)\frac{\partial(T + v\Phi_1)}{\partial z} = 0$$

So after dividing by v, we have

$$(-ax + y)\frac{\partial\Phi_1}{\partial x} + xz\frac{\partial\Phi_1}{\partial y} - 4xy\frac{\partial\Phi_1}{\partial z} + v\left(-y\frac{\partial\Phi_1}{\partial y} - (z - \delta_0)\frac{\partial\Phi_1}{\partial z}\right) = (8y^2 + 2z^2 - 2\delta_0 z)\frac{\mathrm{d}T}{\mathrm{d}G}. \tag{16}$$

Then, setting $\hat{\Phi}_1(x, y, z) := \Phi_1(x, y, z, 0)$ and restricting (16) to $v = 0$ yields to

$$(-ax + y)\frac{\partial\hat{\Phi}_1}{\partial x} + xz\frac{\partial\hat{\Phi}_1}{\partial y} - 4xy\frac{\partial\hat{\Phi}_1}{\partial z} = (8y^2 + 2z^2 - 2\delta_0 z)\frac{\mathrm{d}T}{\mathrm{d}G}. \tag{17}$$

Evaluating (17) on the points of $(x, y, z) = (0, 0, z)$, we see (17) becomes

$$(2z^2 - 2\delta_0 z)\frac{\mathrm{d}T}{\mathrm{d}G} = 0, \tag{18}$$

which implies $\mathrm{d}T/\mathrm{d}G = 0$, i.e., T is a constant and is denoted by $T = T(0)$. Therefore, from (17), we have

$$(-ax + y)\frac{\partial\hat{\Phi}_1}{\partial x} + xz\frac{\partial\hat{\Phi}_1}{\partial y} - 4xy\frac{\partial\hat{\Phi}_1}{\partial z} = 0. \tag{19}$$

In other words, $\hat{\Phi}_1$ is a global analytic first integral of system (14). Again, due to Theorem 2 (ii), we get that $\hat{\Phi}_1 = \hat{\Phi}_1(4y^2 + z^2)$. Consequently, we have $\Phi_1 = \hat{\Phi}_1(G) + v\Phi_2(x, y, z, v)$. Then, $\Phi = T(0) + v\hat{\Phi}_1(G) + v^2\Phi_2(x, y, z, v)$. By definition of first integrals, we obtain

$$(-ax + y)\frac{\partial\Phi_2}{\partial x} + xz\frac{\partial\Phi_2}{\partial y} - 4xy\frac{\partial\Phi_2}{\partial z} + v\left(-y\frac{\partial\Phi_2}{\partial y} - (z - \delta_0)\frac{\partial\Phi_2}{\partial z}\right) = (8y^2 + 2z^2 - 2\delta_0 z)\frac{\mathrm{d}\hat{\Phi}_1}{\mathrm{d}G}, \tag{20}$$

where $G = 4y^2 + z^2$. Using the same arguments used for T and Φ_1, we can show that $\hat{\Phi}_1 = \hat{\Phi}_1(0)$ and $\Phi_2 = \hat{\Phi}_2(G) + v\Phi_3(x, y, z, v)$. Repeating this procedure inductively, we see that $\Phi = \Phi(v)$.

4. Proof of Proposition 3

In this section, our proof is based on the fact that birth of isolated periodic solutions can be regarded as an obstacle to integrability. Let $\phi(t)$ be a T-periodic solution of system (3). Then the variational equations of (3) along $\phi(t)$ is given by the following linear periodic system

$$\dot{\xi} = A(t)\xi, \; A(t) = \frac{\partial F}{\partial x}\Big|_{x=\phi(t)}. \tag{21}$$

Set $M(t)$ be the fundamental matrix of (21), i.e., $\dot{M} = A(t)M$ and $M(0) = I$. The matrix $M(T)$ is called the monodromy matrix of the periodic solution $\Phi(t)$, and the eigenvalues $\rho_1, \ldots, \rho_n$ of $M(T)$ are called multipliers of this periodic solution.

The following result is due to Poincaré [21], for the proof see [17].

Lemma 8. *Assume system* (3) *has a periodic orbit* γ *having only one multiplier equal to* 1*, then system* (3) *has no* C^1 *first integrals* Φ *defined in a neighborhood of* γ *such that the vectors* $\nabla\Phi(x)$ *and* $F(x)$ *are linearly independent on the points* $x \in \gamma$*.*

Proof of Proposition 3. It was shown in [5, §6.1 and 6.2] that under the assumptions of Theorem 3 system (2) admits a periodic orbit γ_ϵ such that the Poincará map at this periodic orbit has the following two eigenvalues

$$\lambda_1 = \epsilon^2 \frac{2\pi(a_2+b_2)}{\omega} + \mathcal{O}(\epsilon^3),$$
$$\lambda_2 = -\epsilon\frac{2c_1\pi}{\omega} + \epsilon^2 \frac{2\pi(a_0 b_1 c_1 + \omega(c_1^2\pi - c_2\omega))}{\omega^3} + \mathcal{O}(\epsilon^3),$$

where $\mathcal{O}(\epsilon)$ denotes the terms of order ϵ. Therefore, for ϵ sufficiently small we have that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. It is well-known that for multipliers of certain periodic solution of an autonomous differential system, one of the multipliers is always 1, and the remaining multipliers are equal to the eigenvalues of the Poincaré map at this periodic orbit, see [9] for instance. Therefore, by Lemma 8, we see that system (2) has no C^1 first integral $\Phi(x,y,z)$ in the neighborhood of γ_ϵ such that $\nabla\Phi(x,y,z)$ and $(-ax+y, -by+xz, -c(z-\delta_0)-4xy)$ are linearly independent on the points of γ_ϵ. The proof of this result is completed. □

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